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# A going down theorem for Grothendieck Chow motives

Charles De Clercq\*

**Abstract.** Let  $X$  be a geometrically split, geometrically irreducible variety over a field  $F$  satisfying Rost nilpotence principle. Consider a field extension  $E/F$  and a finite field  $\mathbb{F}$ . We provide in this note a motivic tool giving sufficient conditions for so-called outer motives of direct summands of the Chow motive of  $X_E$  with coefficients in  $\mathbb{F}$  to be lifted to the base field. This going down result has been used S. Garibaldi, V. Petrov and N. Semenov to give a complete classification of the motivic decompositions of projective homogeneous varieties of inner type  $E_6$  and to answer a conjecture of Rost and Springer.

## Introduction

Throughout this note  $F$  will be the base field and by an  $F$ -variety we will mean a smooth, projective scheme over  $F$ . Given an  $F$ -variety  $X$ , we denote by  $\mathrm{Ch}(X)$  the Chow group  $\mathrm{CH}(X) \otimes_{\mathbb{Z}} \mathbb{F}$  of cycles on  $X$  modulo rational equivalence with coefficients in a finite field  $\mathbb{F}$ . We write  $\mathrm{Ch}(\overline{X})$  for the colimit of all  $\mathrm{Ch}(X_K)$ , where  $K$  runs through all field extensions  $K/F$  and if  $X$  is integral we denote by  $F(X)$  its function field.

For any field extension  $L/F$ , an element lying in the image of the natural morphism of  $\mathrm{Ch}(X_L) \rightarrow \mathrm{Ch}(\overline{X})$  is called  $L$ -rational. The image of any correspondence  $\alpha \in \mathrm{Ch}(X_L)$  under the canonical morphism  $\mathrm{Ch}(X_L) \rightarrow \mathrm{Ch}(\overline{X})$  is denoted by  $\overline{\alpha}$ . An  $F$ -variety  $X$  is geometrically split if the Grothendieck Chow motive of  $X_{\overline{F}} = X \times_{\mathrm{Spec}(F)} \mathrm{Spec}(\overline{F})$  with coefficients in  $\mathbb{F}$  is isomorphic to a finite direct sum of Tate motives, for an algebraic closure  $\overline{F}/F$ . The variety  $X$  satisfies the Rost nilpotence principle with coefficients in  $\mathbb{F}$  if for any field extensions  $L/E/F$  the kernel of the restriction map  $\mathrm{res}_{L/E} : \mathrm{End}(M(X_E)) \rightarrow \mathrm{End}(M(X_L))$  consists of nilpotents.

As shown in [1], any projective homogeneous  $F$ -variety under the action of a semisimple affine algebraic group is geometrically split and satisfies the Rost nilpotence principle. It follows by [2, Corollary 35] (see also [7, Corollary 2.6]) that the Grothendieck Chow motive of these varieties with coefficients in  $\mathbb{F}$  decomposes in an essentially unique way as a direct sum of indecomposable motives. The study of these decompositions have already shown to be very fruitful (see [6], [7], [11]).

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The notion of *upper* motives, previously defined by Vishik in the context of quadrics in [11], was further developed by Karpenko in [7] to describe the indecomposable motives lying in the motivic decomposition of projective homogeneous varieties. If  $X$  is a homogeneous  $F$ -variety,  $E/F$  a field extension and the upper motive of  $M(X_E)$  is a direct summand of another motive  $M_E$ , [11, Theorem 4.15] and [6, Proposition 4.6] give sufficient conditions for the upper motive of  $X$  to be a direct summand of  $M$ . The purpose of the present note is to push these ideas further. We define the notions of *upper*, *lower* and *outer* direct summands of a direct summand  $N$  of the motive of a geometrically split  $F$ -variety. We then show some lifting property of outer summands of  $N_E$  to the base field with the following result.

**Theorem 1.** Let  $N$  be a direct summand of the motive (with coefficients in  $\mathbb{F}$ ) of a geometrically split, geometrically integral  $F$ -variety  $X$  satisfying the Rost nilpotence principle with coefficients in  $\mathbb{F}$  and  $M$  a twisted direct summand of the motive of another  $F$ -variety  $Y$ . Assume that there is a field extension  $E/F$  such that

1. every  $E(X)$ -rational cycle in  $\text{Ch}(\overline{X \times Y})$  is  $F(X)$ -rational;
2. the motive  $N_E$  has an indecomposable outer direct summand which is also a direct summand of the motive  $M_E$ .

Then the motive  $N$  has an outer direct summand which is also a direct summand of  $M$ .

Theorem 1 allows one to descend outer motives of direct summands projective homogeneous varieties which appear on some field extension  $E/F$  of the base field. This generalizes [6, Proposition 4.6], one of the key ingredients in the proof of [6, Theorem 1.1], replacing the whole motive of a variety  $X$  by a direct summand. To replace  $X$  by an arbitrary direct summand, one needs to construct explicitly the rational cycles to get an outer summand defined over  $F$ , and thus theorem 1 gives a new proof of [6, Proposition 4.6]. Note that assumption 1 of theorem 1 holds if the field extension  $E(X)/F(X)$  is unirational, i.e. if there is a field extension  $L/E(X)$  such that  $L/F(X)$  is purely transcendental.

The following particular case of theorem 1 was used by Garibaldi, Petrov and Semenov in [5] to both determine all the motivic decompositions of homogeneous  $F$ -varieties of inner type  $E_6$  and prove a conjecture of Rost and Springer in [5].

**Corollary.** ([5, Proposition 3.2]) Let  $X$  and  $Y$  be two projective homogeneous  $F$ -varieties for a semisimple affine algebraic group, and let  $M$  and  $N$  be direct summands of the motives of  $Y$  and  $X$  respectively with coefficients in  $\mathbb{F}$ . Assume that  $N_{F(Y)}$  is an indecomposable direct summand of  $M_{F(Y)}$  and  $Y$  has an  $F(X)$ -point. Then  $N$  is a direct summand of  $M$ .

*Proof.* Setting  $E = F(Y)$ , the field extension  $E(X)/F(X)$  is purely transcendental, hence assumption 1) of theorem 1 holds.  $\square$

# 1 Grothendieck Chow motives

Our main reference for the construction of the category of Grothendieck Chow motives over  $F$  with coefficients in  $\mathbb{F}$  is [3, §63-65].

Let  $X$  and  $Y$  be two  $F$ -varieties and  $X = \coprod_{k=1}^n X_k$  be the decomposition of  $X$  as a disjoint union of irreducible components with respective dimension  $d_1, \dots, d_n$ . For any integer  $i$  the group of *correspondences* between  $X$  and  $Y$  of degree  $i$  with coefficients in  $\mathbb{F}$  is defined by  $\text{Corr}_i(X, Y) = \coprod_{k=1}^n \text{Ch}_{d_k+i}(X_k \times Y)$ . We now consider the category  $\text{C}(F; \mathbb{F})$  whose objects are pairs  $X[i]$ , where  $X$  is an  $F$ -variety and  $i$  is an integer. Morphisms are defined in terms of correspondences by  $\text{Hom}_{\text{C}(F; \mathbb{F})}(X[i], Y[j]) = \text{Corr}_{i-j}(X, Y)$ . For any correspondences  $f : X[i] \rightsquigarrow Y[j]$  and  $g : Y[j] \rightsquigarrow Z[k]$  in  $\text{Mor}(\text{C}(F; \mathbb{F}))$  the composite  $g \circ f : X[i] \rightsquigarrow Z[k]$  is defined by

$$g \circ f = ({}^X p_Y^Z)_* (({}^{X \times Y} p_Z)^*(f) \cdot (p_X^{Y \times Z})^*(g)) \quad (*)$$

where  ${}^U p_V^W : U \times V \times W \rightarrow U \times W$  is the natural projection.

The category  $\text{C}(F; \mathbb{F})$  is preadditive and its additive completion  $\text{CR}(F; \mathbb{F})$  is the category of correspondences over  $F$  with coefficients in  $\mathbb{F}$ , which has a structure of tensor additive category given by  $X[i] \otimes Y[j] = (X \times Y)[i + j]$ . The category  $\text{CM}(F; \mathbb{F})$  of Grothendieck Chow motives with coefficients in  $\mathbb{F}$  is the pseudo-abelian envelope of the category  $\text{CR}(F; \mathbb{F})$ . Its objects are couples  $(X, \pi)$ , where  $X$  is an object of the category  $\text{CR}(F; \mathbb{F})$ , and  $\pi \in \text{End}(X)$  is a projector (i.e.  $\pi \circ \pi = \pi$ ). Morphisms are given by  $\text{Hom}_{\text{CM}(F; \mathbb{F})}((X, \pi), (Y, \rho)) = \rho \circ \text{Hom}_{\text{CR}(F; \mathbb{F})}(X, Y) \circ \pi$  and the objects of  $\text{CM}(F; \mathbb{F})$  are called *motives*. For any  $F$ -variety  $X$  the motives  $(X[i], \Gamma_{id_X})$  (where  $\Gamma_{id_X}$  is the graph of the identity of  $X$ ) will be denoted  $X[i]$  and  $X[0]$  is the motive of  $X$ . The motives  $\mathbb{F}[i] = \text{Spec}(F)[i]$  are the *Tate motives*.

**Lemma 1.** Let  $(X, \pi)$  be a direct summand of the motive of an  $F$ -variety  $X$ . A motive  $M$  is a direct summand of  $(X, \pi)$  if and only if  $M$  is isomorphic to  $(X, \rho)$ , for some projector  $\rho$  satisfying  $\pi \circ \rho \circ \pi = \rho$ .

*Proof.* Since  $\text{End}((X, \pi)) = \pi \circ \text{Ch}_{\dim(X)}(X \times X) \circ \pi$ , any projector  $\rho$  in  $\text{End}((X, \pi))$  satisfies  $\pi \circ \rho \circ \pi = \rho$ .  $\square$

**Definition 1.** Let  $M \in \text{CM}(F; \mathbb{F})$  be a motive and  $i$  an integer. The  $i$ -dimensional Chow group  $\text{Ch}_i(M)$  of  $M$  is defined by  $\text{Hom}_{\text{CM}(F; \mathbb{F})}(\mathbb{F}[i], M)$ . The  $i$ -codimensional Chow group  $\text{Ch}^i(M)$  of  $M$  is defined by  $\text{Hom}_{\text{CM}(F; \mathbb{F})}(M, \mathbb{F}[i])$ .

For any field extension  $E/F$  and any correspondence  $\alpha : X[i] \rightsquigarrow Y[j]$  the pull-back of  $\alpha$  along the natural morphism  $(X \times Y)_E \rightarrow X \times Y$  will be denoted  $\alpha_E$ . If  $N = (X, \pi)[i]$  is a twisted motivic direct summand of  $X$ , the motive  $(X_E, \pi_E)[i]$  will be denoted  $N_E$ .

Finally the category  $\text{CM}(F; \mathbb{F})$  is endowed with a duality functor. If  $X$  and  $Y$  are two  $F$ -varieties and  $\alpha \in \text{Ch}(X \times Y)$  is a correspondence, the image of  $\alpha$  under the exchange isomorphism  $X \times Y \rightarrow Y \times X$  is denoted  ${}^t\alpha$ . The *duality functor* is the additive functor  $\dagger : \text{CM}(F; \mathbb{F}) \rightarrow \text{CM}(F; \mathbb{F})$  determined by the formula  $M(X)[i]^\dagger = M(X)[- \dim(X) - i]$  and such that for any correspondence  $\alpha : X[i] \rightsquigarrow Y[j]$ ,  $\alpha^\dagger = {}^t\alpha$ .

## 2 Direct summands of geometrically split $F$ -varieties

Throughout this section we consider a geometrically split  $F$ -variety  $X$  and  $E/F$  a splitting field of  $X$ . By [8, Proposition 1.5] the pairing

$$\Psi : \begin{array}{ccc} \text{Ch}(X_E) \times \text{Ch}(X_E) & \longrightarrow & \mathbb{F} \\ (\alpha, \beta) & \longmapsto & \text{deg}(\alpha \cdot \beta) \end{array}$$

is non degenerate hence gives rise to an isomorphism of  $\mathbb{F}$ -modules between  $\text{Ch}(X_E)$  and its dual space  $\text{Hom}_{\mathbb{F}}(\text{Ch}(X_E), \mathbb{F})$  given by  $\alpha \mapsto \Psi(\alpha, \cdot)$ . The dual basis of a homogeneous basis  $(x_k)_{k=1}^n$  of  $\text{Ch}(X_E)$  with respect of  $\Psi$  is the basis  $(x_k^*)_{k=1}^n$  of  $\text{Ch}(X_E)$  such that for any  $1 \leq i, j \leq n$ ,  $\Psi(x_i, x_j^*) = \delta_{ij}$ , where  $\delta_{ij}$  is the Kronecker symbol. By definition of the composition  $(*)$  in  $\text{CM}(F; \mathbb{F})$ , if  $y$  (resp.  $y'$ ) lies in  $\text{Ch}(Y)$  (resp.  $\text{Ch}(Y')$ ) for two other  $F$ -varieties  $Y$  and  $Y'$  and if  $(i, j)$  are two integers, the composition of the correspondences  $x_i \times y \in \text{Ch}(X_E \times Y)$  and  $y' \times x_j^* \in \text{Ch}(Y' \times X_E)$  is given by

$$(x_i \times y) \circ (y' \times x_j^*) = \delta_{ij}(y' \times y) \in \text{Ch}(Y' \times Y). \quad (1)$$

Note that the Kunneth decomposition holds in  $\text{Ch}(X_E \times Y)$  and  $\text{Ch}(Y' \times X_E)$  in the view of [3, Proposition 64.3], since  $X_E$  is split, and thus the cycles of  $\text{Ch}(X_E \times Y)$  and  $\text{Ch}(Y' \times X_E)$  may always be written that way.

**Upper, lower and outer motives.** Let  $\pi \in \text{Ch}_{\dim(X)}(X \times X)$  be a non-zero projector and  $N = (X, \pi)$  the associated summand of the motive of  $X$ . The *base* of  $N$  is the set  $\mathcal{B}(N) = \{i \in \mathbb{Z}, \text{Ch}_i(N_E) \text{ is not trivial}\}$ . The *bottom* of  $N$  (denoted by  $b(N)$ ) is the least integer of  $\mathcal{B}(N)$  and the *top* of  $N$  (denoted by  $t(N)$ ) is the greatest integer of  $\mathcal{B}(N)$ . We now introduce the notion of upper and lower direct summands of  $N$ , previously introduced by Vishik in the context of the motives of quadrics in [11, Definition 4.6].

**Definition 2.** Let  $N$  be a direct summand of the twisted motive of a geometrically split  $F$ -variety and  $M$  a motivic direct summand of  $N$ . We say that

1.  $M$  is *upper* in  $N$  if  $b(M) = b(N)$ ;
2.  $M$  is *lower* in  $N$  if  $t(M) = t(N)$ ;
3.  $M$  is *outer* in  $N$  if  $M$  is both lower and upper in  $N$ .

**Remark 1.** Keeping the same  $F$ -variety  $X$  and any direct summand  $N = (X, \pi)$ , consider a homogeneous basis  $(x_k)_{k=1}^n$  of  $\text{Ch}(X_E)$  and its dual basis  $(x_k^*)_{k=1}^n$ . The base, bottom and top of  $N$  can be easily determined by the decomposition

$$\pi_E = \sum_{i,j=1}^n \pi_{i,j}(x_i \times x_j^*)$$

noticing that  $\mathcal{B}(N) = \{\dim(x_i), \pi_{i,j} \neq 0 \text{ for some } j\}$ .

**Lemma 2.** Let  $N$  be a motivic direct summand of a geometrically split  $F$ -variety and  $M$  a direct summand of  $N$ . Then  $M$  is lower in  $N$  (resp. upper in  $N$ ) if and only if the dual motive  $M^\dagger$  is upper in  $N^\dagger$  (resp.  $M^\dagger$  is lower in  $N^\dagger$ ).

*Proof.* For any motive  $O$  and for any integer  $i$ ,  $\text{Ch}^i(O^\dagger) = \text{Ch}_{-i}(O)$ . It follows that  $b(O^\dagger) = -t(O)$  and  $t(O^\dagger) = -b(O)$ .  $\square$

**The Krull-Schmidt property.** Let  $\mathbf{C}$  be a pseudo-abelian category and  $\mathfrak{C}$  be the set of the isomorphism classes of objects of  $\mathbf{C}$ . We say that the category  $\mathbf{C}$  satisfies the *Krull-Schmidt property* if the monoid  $(\mathfrak{C}, \oplus)$  is free. The Krull-Schmidt property holds for the motives of geometrically split  $F$ -varieties satisfying the Rost nilpotence principle in  $\text{CM}(F; \mathbb{F})$  by [7, Corollary 2.6].

**Proof of the main result.** In order to prove theorem 1, we will need the following lemma, which will allow us to construct explicitly the rational cycles lifting outer motives to the base field.

**Lemma 3.** Let  $N$  be a motivic direct summand of a geometrically split, geometrically irreducible  $F$ -variety  $X$  satisfying the Rost nilpotence principle and  $M$  a twisted direct summand of an  $F$ -variety  $Y$ . Assume the existence of a field extension  $E/F$  such that

1. any  $E(X)$ -rational cycle in  $\text{Ch}(\overline{X \times Y})$  is  $F(X)$ -rational;
2. there are two correspondences  $\alpha : N_E \rightsquigarrow M_E$  and  $\beta : M_E \rightsquigarrow N_E$  such that  $\beta \circ \alpha$  is a projector and  $(X_E, \beta \circ \alpha)$  is a lower direct summand of  $N_E$ .

Then there are two correspondences  $\gamma : N \rightsquigarrow M$  and  $\delta : M_E \rightsquigarrow N_E$  such that  $(X_E, \delta \circ \gamma_E)$  is a direct summand of  $N_E$  which contains all lower indecomposable direct summands of  $(X_E, \beta \circ \alpha)$ . Furthermore if  $\overline{\beta}$  is  $F$ -rational, then  $\overline{\delta}$  is also  $F$ -rational.

*Proof.* We may assume by lemma 1 that  $M=(Y, \rho)[i]$  and  $N=(X, \pi)$ . We construct explicitly the two correspondences  $\gamma$  and  $\delta$ . Since  $E(X)$  is a field extension of  $E$ ,  $\overline{\alpha}$  is  $E(X)$ -rational, hence  $F(X)$ -rational by assumption 1. Let  $L/F$  be a field extension. Then the morphism  $\text{Spec}(F(X_L)) \rightarrow X_L$  induces a pull-back morphism  $\varepsilon^* : \text{Ch}(\overline{X \times Y \times X}) \rightarrow \text{Ch}(\overline{(X \times Y)_{F(X)}})$  which maps  $F$ -rational cycles onto  $F(X)$ -rational cycles by [3, Corollary 57.11], and so there is a cycle  $\alpha_1 \in \text{Ch}(X \times Y \times X)$  such that  $\varepsilon^*(\overline{\alpha}_1) = \overline{\alpha}$ . Since  $\varepsilon^*$  maps any homogeneous cycle  $\sum_i x_i \times y_i \times 1$  to  $\sum_i x_i \times y_i$  and vanishes on homogeneous cycles whose codimension on the third factor is strictly positive, we have  $\overline{\alpha}_1 = \overline{\alpha} \times 1 + \dots$  where ” $\dots$ ” is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y \times X})$  with strictly positive codimension on the third factor.

We now look at  $\alpha_1$  as a correspondence  $X \rightsquigarrow Y \times X$  and consider the cycle  $\alpha_2 = \alpha_1 \circ \pi$ . By formula 1 we have

$$\overline{\alpha}_2 = (\overline{\alpha} \times 1) \circ \overline{\pi} + \dots,$$

where ” $\dots$ ” is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y \times X})$  with dimension at most  $t(N)$  on the first factor (since these terms come from the first factors

of  $\bar{\pi}$ ) and strictly positive codimension on the third factor (since these terms come from the third factors of  $\bar{\alpha}_1 - \bar{\alpha} \times 1$ ). Finally considering the pull-back of the morphism  $\Delta : X \times Y \rightarrow X \times Y \times X$  induced by the diagonal embedding  $X$  and setting  $\alpha_3 = \Delta^*(\alpha_2)$ , we have

$$\bar{\alpha}_3 = \bar{\alpha} \circ \bar{\pi} + \dots$$

where " $\dots$ " stands for a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times Y})$  with dimension strictly lesser than  $t(N)$  on the first factor.

Composing with  $\bar{\pi} \circ \bar{\beta}$  on the left and  $\bar{\pi}$  on the right, we get that

$$\bar{\pi} \circ \bar{\beta} \circ \bar{\alpha}_3 \circ \bar{\pi} = \bar{\pi} \circ \bar{\beta} \circ \bar{\alpha} \circ \bar{\pi} + \xi$$

where  $\xi$  is a linear combination of homogeneous cycles of strictly lesser dimension than  $t(N)$  on the first factor since they come from the first factors of  $\bar{\alpha}_3 \circ \bar{\pi} - \bar{\alpha} \circ \bar{\pi}$ . The correspondence  $\beta \circ \alpha$  defines a direct summand of the motive  $N_E$  and thus by lemma 1

$$\bar{\pi} \circ \bar{\beta} \circ \bar{\alpha}_3 \circ \bar{\pi} = \bar{\beta} \circ \bar{\alpha} + \xi.$$

By formula 1,  $\bar{\beta} \circ \bar{\alpha} \circ \xi$ ,  $\xi \circ \xi$  and  $\xi \circ \bar{\beta} \circ \bar{\alpha}$  are linear combinations of homogeneous cycles of dimension strictly lesser than  $t(N)$  on the first factor. Repeating the same procedure and since  $k \circ h$  is a projector, we see that for any integer  $n$

$$(\bar{\pi} \circ \bar{\beta} \circ \bar{\alpha}_3 \circ \bar{\pi})^n = \bar{\beta} \circ \bar{\alpha} + \dots \quad (2)$$

where " $\dots$ " is a linear combination of homogeneous cycles in  $\text{Ch}(\overline{X \times X})$  with dimension on the first factor strictly lesser than  $t(N)$ . Since the direct summand  $(X, \beta \circ \alpha)$  is lower, all these correspondences are non-zero and by [7, Corollary 2.2] an appropriate power  $(\pi_E \circ \beta \circ (\alpha_3)_E \circ \pi_E)^{n_0}$  is a projector. If we set  $\gamma = \rho \circ \alpha_3 \circ \pi$  and  $\delta = (\pi_E \circ \beta \circ (\alpha_3)_E \circ \pi_E)^{n_0-1} \circ \pi_E \circ \beta$ , we see that  $\bar{\delta}$  is  $F$ -rational if  $\bar{\beta}$  is  $F$ -rational. The correspondence  $\delta \circ \gamma_E$  is a projector which defines a direct summand of  $N_E$  by lemma 1.

Consider the decomposition  $\bar{\beta} \circ \bar{\alpha} = \sum_{i,j=1}^s p_{ij}(x_i \times x_j^*)$  of  $\bar{\beta} \circ \bar{\alpha}$  with respect to a basis  $(x_i)_{i=1}^s$  of  $\text{Ch}(\overline{X})$ . By formula 2, the decomposition of  $\bar{\delta} \circ \bar{\gamma}$  in  $(x_i \times x_j^*)_{i,j=1}^s$  has a non-zero coefficient for any couple  $(i, j)$  such that  $p_{ij}$  is non zero and  $\dim(x_i) = t((X_E, \beta \circ \alpha))$ . The Krull-Schmidt property and remark 1 then imply that any lower indecomposable direct summand of  $(X_E, \beta \circ \alpha)$  is a direct summand of  $(X, \delta \circ \gamma_E)$ .  $\square$

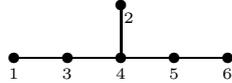
We now show how we can derive the proof of theorem 1 from the rational cycles constructed in lemma 3. To lift the outer motive to the base field  $F$ , we apply lemma 3 and the duality functor twice in order produce two correspondences which are defined on the base field.

*Proof of Theorem 1.* Let  $O = (X_E, \kappa)$  be an outer indecomposable direct summand of  $N_E$  which is also a direct summand of  $M_E$ . We prove theorem 1 by applying lemma 3 once, then the duality functor and finally lemma 3 another time to get all our correspondences defined over the base field  $F$ .

Since  $O$  is a direct summand of  $M_E$ , there are two correspondences  $\alpha : N_E \rightsquigarrow M_E$  and  $\beta : M_E \rightsquigarrow N_E$  such that  $\beta \circ \alpha = \kappa$ . Moreover  $O$  is lower in  $N_E$ , so lemma 3 justifies the existence of two other correspondences  $\alpha' : N \rightsquigarrow M$  and  $\beta' : M_E \rightsquigarrow N_E$  such that  $O_2 = (X_E, \beta' \circ \alpha'_E)$  is a direct summand of  $N_E$ , and the motive  $O_2$  is outer in  $N_E$  since it contains  $O$ . The dual motive  $O_2^\dagger = (X_E, {}^t\alpha'_E \circ {}^t\beta')$  is therefore outer in  $N_E^\dagger$  by lemma 2 and is a direct summand of the dual motive  $M_E^\dagger$ . Twisting these three motives by  $\dim(X)$ , we can apply lemma 3 again. The correspondence  $\overline{{}^t\alpha'}$  is  $F$ -rational, so lemma 3 gives two correspondences  $\gamma : N^\dagger \rightsquigarrow M^\dagger$  and  $\delta : M^\dagger \rightsquigarrow N^\dagger$  such that the motive  $(X_E, \delta_E \circ \gamma_E)$  is both an outer direct summand of  $N^\dagger$  (since it contains the dual motive  $O^\dagger$ ) and a direct summand of  $M^\dagger$ . Transposing again, the motive  $(X, {}^t\gamma \circ {}^t\delta)$  is an outer direct summand of  $N$  and a direct summand of  $M$ .  $\square$

### 3 Motivic decompositions for groups of inner type $E_6$

The purpose of this section is to discuss the complete classification of the motivic decompositions of projective homogeneous varieties of inner type  $E_6$ , which is achieved in [5]. Let  $G$  be an algebraic group of inner type  $E_6$  and  $X$  a projective  $G$ -homogeneous variety. We choose the following numbering of the Dynkin diagram  $G$ .



The results of [9] show that in the case where  $X$  is *generically split*, any indecomposable summand of the  $\mathbb{F}_p$ -motive of  $X$  is isomorphic to a shift of the upper motive  $\mathcal{R}_p(G)$  of the variety of Borel subgroups of  $G$ . Furthermore, the structure of the motives  $\mathcal{R}_p(G)$  is determined in [9] in terms of the so-called  $J$ -invariant modulo  $p$  of  $G$ .

The  $J$ -invariant was first introduced by Vishik in [12] in the context of quadratic forms. Petrov, Semenov and Zainoulline define in [9] the notion of  $J$ -invariant modulo  $p$  of an arbitrary semisimple algebraic group  $G$ , denoted by  $J_p(G)$ , which is an  $r$ -tuple of integers  $(j_1, \dots, j_r)$  given by the rational cycles in  $\text{Ch}(\overline{G})$ . By [9, Table 4.13], the  $J$ -invariant modulo 3 of a semisimple adjoint algebraic group of inner type  $E_6$  is  $(j_1, j_2)$ , with  $0 \leq j_1 \leq 2$  and  $0 \leq j_2 \leq 1$ .

Another invariant attached to  $G$  is the Tits index, which consists of the data of the Dynkin diagram of  $G$  with some vertices being circled. The complete classification of the Tits indices of type  $E_6$ , provided in [10], is as follows :



Let  $\Theta$  be a subset of the vertices of the Dynkin diagram of  $G$  and  $X_\Theta$  a projective  $G$ -homogeneous variety of type  $\Theta$ . By [9, Table 3.6] and a case by case analysis of the above Tits indices, the variety  $X_\Theta$  is either split or generically split if  $p \neq 3$ ,  $\Theta \neq \{2\}, \{4\}$

or  $\{2, 4\}$  and if  $j_1 = 0$ . Finally, the results of [1] and [5, §8] imply that the understanding of the motivic decompositions of  $X_{\{2\}}$ ,  $X_{\{4\}}$ , and  $X_{\{2,4\}}$  is reduced to the study of the upper motive of  $X_2$  in  $\text{CM}(F; \mathbb{F}_3)$ , which is denoted by  $M_{j_1, j_2}$ .

Using theorem 1, Garibaldi, Petrov and Semenov provide some restrictions on the Poincaré polynomial of the upper motive of  $X$ , if  $X$  is an anisotropic projective homogeneous variety satisfying some technical assumptions (see [5, Proposition 7.6]). Assuming that  $J_3(G)=(1, 0)$ , they observe that although the variety  $X_{\{2\}}$  satisfies all those technical assumptions, the Poincaré polynomial of its upper motive does not match with the conclusion of [5, Proposition 7.6]. In particular  $X_{\{2\}}$  has a 0-cycle of degree coprime to 3, and  $M_{1,0}$  is the Tate motive  $\mathbb{F}_3$ .

Furthermore, the authors deduce from the fact that  $M_{1,0}$  is the Tate motive that the  $J$ -invariant modulo 3 of  $G$  cannot be  $(2, 0)$  (see [5, Corollary 8.10]). Indeed, if  $J_3(G)=(2, 0)$  and  $\text{SB}(3, A)$  is the Severi-Brauer variety of right ideals of reduced dimension 3 in the Tits algebra of  $G$ , then  $J_3(G_{F(\text{SB}(3, A))})=(1, 0)$ . In particular,  $X_2$  has a zero-cycle of degree coprime to 3 over the function field of  $\text{SB}(3, A)$ . It follows that the upper motive of  $X_2$  would be isomorphic to the upper motive of  $\text{SB}(3, A)$ , and thus the canonical 3-dimensions of  $X_2$  and  $\text{SB}(3, A)$  would be equal, a contradiction.

The authors use similar techniques to provide isotropy criteria for projective homogeneous varieties. They consider several varieties which satisfy the technical assumptions of [5, Proposition 7.6] without fulfilling its conclusion, and thus have a zero cycle of degree 3 (or a rational point). These examples include projective homogeneous varieties for orthogonal group with application to the isotropy of varieties of type  $E_7$  and varieties of type  $E_8$  (see [5, Lemma 10.15, 10.21]). They also produce with these techniques isotropy criteria for projective homogeneous varieties in terms of the Rost invariant (see [5, Proposition 10.18] for type  $E_7$  and [5, Propositions 10.22] for type  $E_8$ ).

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