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AMPLE THOUGHTS

DANIEL PALACÍN AND FRANK O. WAGNER

ABSTRACT. Non- n -ampleness as defined by Pillay [20] and Evans [6] is preserved under analysability. Generalizing this to a more general notion of Σ -ampleness, this gives an immediate proof for all simple theories of a weakened version of the Canonical Base Property (CBP) proven by Chatzidakis [5] for types of finite SU-rank. This is then applied to the special case of groups.

1. INTRODUCTION

Recall that a partial type π over a set A in a simple theory is *one-based* if for any tuple \bar{a} of realizations of π and any $B \supseteq A$ the canonical base $\text{Cb}(\bar{a}/B)$ is contained in the bounded closure $\text{bdd}(\bar{a}A)$. In other words, forking dependence is either trivial or behaves as in modules: Any two sets are independent over the intersection of their bounded closures. One-basedness implies that the forking geometry is particularly well-behaved; for instance one-based groups are bounded-by-abelian-by-bounded. The principal result in [26] is that one-basedness is preserved under analyses (i.e. iterative approximations by some other types): a type analysable in one-based types is itself one-based. This generalized earlier results of Hrushovski [12] and Chatzidakis [5]. One-basedness is the first level in a hierarchy of possible geometric behaviour of forking independence first defined by Pillay [20] and slightly modified by Evans [6], n -ampleness, modelled on the behavior of flags in n -space. Not 1-ample means one-based; not 2-ample is equivalent to a notion previously introduced by Hrushovski [13], CM-triviality. Fields

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are n -ample for all $n < \omega$, as is the non-abelian free group [17]. In [20] Pillay defines n -ampleness locally for a single type and shows that a superstable theory of finite Lascar rank is non n -ample if and only if all its types of rank 1 are; his proof implies that in such a theory, a type analysable in non n -ample types is again non n -ample.

We shall give a definition of n -ampleness for invariant families of partial types, and generalize Pillay's result to arbitrary simple theories. Note that for $n = 1$ this gives an alternative proof of the main result in [26]. Since for types of infinite rank the algebraic (bounded) closure used in the definition is not necessarily appropriate (for a regular type p one might, for instance, replace it by p -closure), we also generalize the notion to Σ -closure for some \emptyset -invariant collection of partial types (thought of as small), giving rise to the notion of n - Σ -ample. This may for instance be applied to consider ampleness modulo types of finite SU-rank, or modulo supersimple types. Readers not interested in this additional generality are invited to simply replace Σ -closure by bounded closure. However, this will only marginally shorten the proofs. As an immediate Corollary of the more general version, we shall derive a weakened version of the Canonical Base Property CBP [23] shown by Chatzidakis [5], where analysability replaces internality in the definition. We also give a version appropriate for supersimple theories. Finally, we deduce that in a simple theory with enough regular types, a hyperdefinable group modulo its approximate centre is analysable in the family of non one-based regular types; the group modulo a normal nilpotent subgroup is almost internal in that family. This can be thought of as a general version of the properties of one-based groups mentioned above.

Our notation is standard and follows [25]. Throughout the paper, the ambient theory will be simple, and we shall be working in \mathfrak{M}^{heq} , where \mathfrak{M} is a sufficiently saturated model of the ambient theory. Thus tuples are tuples of hyperimaginaries, and $\text{dcl} = \text{dcl}^{heq}$.

2. INTERNALITY AND ANALYSABILITY

For the rest of the paper Σ will be an \emptyset -invariant family of partial types. Recall first the definitions of internality, analysability, foreignness and orthogonality.

Definition 2.1. Let π be a partial type over A . Then π is

- (almost) Σ -internal if for every realization a of π there is $B \downarrow_A a$ and a tuple \bar{b} of realizations of types in Σ based on B , such that $a \in \text{dcl}(B\bar{b})$ (or $a \in \text{bdd}(B\bar{b})$, respectively).
- Σ -analysable if for any realization a of π there are $(a_i : i < \alpha) \in \text{dcl}(Aa)$ such that $\text{tp}(a_i/A, a_j : j < i)$ is Σ -internal for all $i < \alpha$, and $a \in \text{bdd}(A, a_i : i < \alpha)$.

A type $\text{tp}(a/A)$ is *foreign* to Σ if $a \not\downarrow_{AB} \bar{b}$ for all $B \downarrow_A a$ and \bar{b} realizing types in Σ over B .

Finally, $p \in S(A)$ is *orthogonal* to $q \in S(B)$ if for all $C \supseteq AB$, $a \models p$, and $b \models q$ with $a \downarrow_A C$ and $b \downarrow_B C$ we have $a \downarrow_C b$.

So p is foreign to Σ if p is orthogonal to all completions of partial types in Σ , over all possible parameter sets.

The following lemmas and their corollaries are folklore, but we add some precision about non-orthogonality.

Lemma 2.2. *Suppose $a \downarrow b$ and $a \not\downarrow_b c$. Let $(b_i : i < \omega)$ be an indiscernible sequence in $\text{tp}(b)$ and put $p_b = \text{tp}(c/b)$. Then p_{b_i} is non-orthogonal to p_{b_j} for all $i, j < \omega$.*

Proof: We prolong the sequence to have length α . As $a \downarrow b$ and $(b_i : i < \alpha)$ is indiscernible, by [25, Theorem 2.5.4] we may assume $ab \equiv ab_i$ for all $i < \alpha$ and $a \downarrow (b_i : i < \alpha)$. Let $B = (b_i : i < \omega)$, so $(b_i : \omega \leq i < \alpha)$ is independent over B and $a \downarrow B$. Choose c_i with $b_i c_i \equiv_a bc$ and

$$c_i \downarrow_{ab_i} (b_j : j < \alpha)$$

for all $\omega \leq i < \alpha$. Then $ab_i c_i \downarrow_B (b_j : j \neq i)$ for all $\omega \leq i < \alpha$. By indiscernability, if p_{b_i} were orthogonal to p_{b_j} for some $i \neq j$, then they would be orthogonal for all $i \neq j$. As $c_i \downarrow_{b_i} (b_j : j \neq i)$, the sequence $(b_i c_i : \omega \leq i < \alpha)$ would be independent over B . However, $a \not\downarrow_B b_i c_i$ for all $\omega \leq i < \alpha$, contradicting the boundedness of weight of $\text{tp}(a/B)$. \square

Lemma 2.3. *Suppose $a \downarrow b$ and $a' = \text{Cb}(bc/a)$. Let \mathcal{P} be the family of $\text{bdd}(\emptyset)$ -conjugates of $\text{tp}(c/b)$ non-orthogonal to $\text{tp}(c/b)$. Then $a' \in \text{bdd}(a)$ is \mathcal{P} -internal and $\text{bdd}(ab) \cap \text{bdd}(bc) \subseteq \text{bdd}(a'b)$.*

Proof: If $a \downarrow bc$ then $a' \in \text{bdd}(\emptyset)$ and $\text{bdd}(ab) \cap \text{bdd}(bc) = \text{bdd}(b)$, so there is nothing to show. Assume $a \not\downarrow_b c$. Clearly $a' \in \text{bdd}(a)$; as $bc \downarrow_{a'} a$ we get $c \downarrow_{a'b} a$ and hence $\text{bdd}(ab) \cap \text{bdd}(bc) \subseteq \text{bdd}(a'b)$. Let $(b_i c_i : i < \omega)$ be a Morley sequence in $\text{Lstp}(bc/a)$ with $b_0 c_0 = bc$. Then $a' \in \text{dcl}(b_i c_i : i < \omega)$; since $b \downarrow a$ we get $(b_i : i < \omega) \downarrow a$, whence

$(b_i : i < \omega) \perp a'$. So a' is internal in $\{\text{tp}(c_i/b_i) : i < \omega\}$. Finally, $\text{tp}(c_i/b_i)$ is non-orthogonal to $\text{tp}(c/b)$ for all $i < \omega$ by Lemma 2.2. \square

Corollary 2.4. *If $a \perp b$ and $\text{tp}(c/b)$ is (almost) Σ -internal, then $\text{Cb}(bc/a)$ is (almost) Σ -internal. The same statement holds with analysable instead of internal.*

Proof: Let $d \perp_b c$ and \bar{e} realize partial types in Σ over bd such that $c \in \text{dcl}(bd\bar{e})$ (or $c \in \text{bdd}(bd\bar{e})$, respectively). We may take $d\bar{e} \perp_{bc} a$. Then $d \perp_b ac$, whence $a \perp bd$. So $\text{Cb}(bd\bar{e}/a)$ is Σ -internal by Lemma 2.3. But $a \perp_{bc} d\bar{e}$ and $c \in \text{dcl}(bd\bar{e})$ implies $\text{Cb}(bc/a) \in \text{dcl}(\text{Cb}(bd\bar{e}/a))$; similarly $c \in \text{bdd}(bd\bar{e})$ implies $\text{Cb}(bc/a) \in \text{bdd}(\text{Cb}(bd\bar{e}/a))$.

The proof for Σ -analysability is analogous. \square

Definition 2.5. Two partial types π_1 and π_2 are *perpendicular*, denoted $\pi_1 \perp \pi_2$, if for any set A containing their domains and any tuple $\bar{a}_i \models \pi_i$ for $i = 1, 2$ we have $\bar{a}_1 \perp_A \bar{a}_2$.

For instance, orthogonal types of rank 1 are perpendicular.

Corollary 2.6. *Suppose $a \perp b$, and $a_0 \in \text{bdd}(ab)$ is (almost) Π -internal over b for some b -invariant family Π of partial types. Let Π' be the family of $\text{bdd}(\emptyset)$ -conjugates π' of partial types $\pi \in \Pi$ with $\pi' \not\perp \pi$. Then there is (almost) Π' -internal $a_1 \in \text{bdd}(a)$ with $a_0 \in \text{bdd}(a_1b)$. The same statement holds with analysable instead of internal.*

Proof: If $\text{tp}(a_0/b)$ is Π -internal, there is $c \perp_b a_0$ and \bar{e} realizing partial types in Π over bc such that $a_0 \in \text{dcl}(bc\bar{e})$; we choose them with $c\bar{e} \perp_{ba_0} a$. So $c \perp_b a$, whence $a \perp bc$. Furthermore, we may assume that $e \not\perp_{bc} a$ for all $e \in \bar{e}$, since otherwise $ec \perp_b a_0$ and we may just include e in c , reducing the length of \bar{e} . Now $a_0 \in \text{bdd}(abc) \cap \text{bdd}(bc\bar{e})$, so by Lemmas 2.2 and 2.3 there is Π' -internal $a_1 \in \text{bdd}(a)$ with $a_0 \in \text{bdd}(bca_1)$. Since $a \perp_b c$ implies $a_0 \perp_{a_1b} c$, we get $a_0 \in \text{bdd}(a_1b)$.

For the almost internal case, we replace definable by bounded closure; for the analysability statement we iterate, adding a_1 to the parameters. \square

To finish this section, a decomposition lemma for almost internality.

Lemma 2.7. *Let $\Sigma = \bigcup_{i < \alpha} \Sigma_i$, where $(\Sigma_i : i < \alpha)$ is a collection of pairwise perpendicular \emptyset -invariant families of partial types. If $\text{tp}(a/A)$ is almost Σ -internal, then there are $(a_i : i < \alpha)$ interbounded over A with a such that $\text{tp}(a_i/A)$ is Σ_i -internal for $i < \alpha$.*

Clearly, if a is a finite imaginary tuple, we only need finitely many a_i .

Proof: By assumption there is $B \perp_A a$ and some tuples $(b_i : i < \alpha)$ such that b_i realizes partial types in Σ_i over B , with $a \in \text{bdd}(B, b_i : i < \alpha)$. Let $a_i = \text{Cb}(Bb_i/Aa)$. Then $a_i \in \text{bdd}(Aa)$ and $\text{tp}(a_i/A)$ is Σ_i -internal by Corollary 2.4.

Put $\bar{a} = (a_i : i < \alpha)$. Then $a \perp_{Aa_i} Bb_i$ implies $a \perp_{B\bar{a}} b_i$; since $b_i \perp_{Ba} (b_j : j \neq i)$ by perpendicularity we obtain $b_i \perp_{B\bar{a}} (a, b_j : j \neq i)$ for all $i < \alpha$. Hence $(a, b_i : i < \alpha)$ is independent over $B\bar{a}$, and in particular

$$a \perp_{B\bar{a}} (b_i : i < \alpha).$$

Since $a \in \text{bdd}(B, b_i : i < \alpha)$ we get $a \in \text{bdd}(B\bar{a})$; as $a \perp_A B$ implies $a \perp_{A\bar{a}} B$ we obtain $a \in \text{bdd}(A\bar{a})$. \square

3. Σ -CLOSURE, Σ_1 -CLOSURE AND A THEORY OF LEVELS

In his proof of Vaught's conjecture for superstable theories of finite rank [3], Buechler defines the first level $\ell_1(a)$ of an element a of finite Lascar rank as the set of all $b \in \text{acl}^{eq}(a)$ internal in the family of all types of Lascar rank one; higher levels are defined inductively by $\ell_{n+1}(a) = \ell_1(a/\ell_n(a))$. The notion has been studied by Prerna Bihani Juhlin in her thesis [1] in connection with a reformulation of the canonical base property. We shall generalise the notion to arbitrary simple theories.

Definition 3.1. For an ordinal α the α -th Σ -level of a over A is defined inductively by $\ell_0^\Sigma(a/A) = \text{bdd}(A)$, and for $\alpha > 0$

$$\ell_\alpha^\Sigma(a/A) = \{b \in \text{bdd}(aA) : \text{tp}(b/\bigcup_{\beta < \alpha} \ell_\beta(a/A)) \text{ is almost } \Sigma\text{-internal}\}.$$

Finally, we shall write $\ell_\infty^\Sigma(a/A)$ for the set of all hyperimaginaries $b \in \text{bdd}(aA)$ such that $\text{tp}(b/A)$ is Σ -analysable.

Remark 3.2. Clearly, $\text{tp}(a/A)$ is Σ -analysable if and only if $\ell_\infty^\Sigma(a/A) = \text{bdd}(aA)$ if and only if $\ell_\alpha^\Sigma(a/A) = \text{bdd}(aA)$ for some ordinal α , and the minimal such α is the minimal length of a Σ -analysis of a over A .

Lemma 3.3. *If $a \perp b$, then $\ell_\alpha^\Sigma(ab) = \text{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b))$ for any α .*

Proof: Let $c = \ell_\alpha^\Sigma(ab)$. Clearly, $\ell_\alpha^\Sigma(a)\ell_\alpha^\Sigma(b) \subseteq c$. Conversely, put $a_0 = \text{Cb}(bc/a)$. Then $\text{tp}(a_0)$ is internal in the family of $\text{bdd}(\emptyset)$ -conjugates

of $\text{tp}(c/b)$ by Corollary 2.4; since even $\text{tp}(c)$ is Σ -analysable in α steps, so is $\text{tp}(a_0)$. Thus $a_0 \subseteq \ell_\alpha^\Sigma(a)$. Now $bc \downarrow_{a_0} a$ implies

$$c \downarrow_{\ell_\alpha^\Sigma(a)b} a,$$

whence $c \subseteq \text{bdd}(\ell_\alpha^\Sigma(a), b)$. By symmetry, $c \subseteq \text{bdd}(\ell_\alpha^\Sigma(b), a)$, that is,

$$\ell_\alpha^\Sigma(ab) \subseteq \text{bdd}(\ell_\alpha^\Sigma(a), b) \cap \text{bdd}(\ell_\alpha^\Sigma(b), a).$$

On the other hand, $a \downarrow b$ yields $a \downarrow_{\ell_\alpha^\Sigma(a)\ell_\alpha^\Sigma(b)} b$. Thus,

$$\text{bdd}(\ell_\alpha^\Sigma(a), b) \cap \text{bdd}(\ell_\alpha^\Sigma(b), a) = \text{bdd}(\ell_\alpha^\Sigma(a), \ell_\alpha^\Sigma(b)),$$

whence the result. \square

Corollary 3.4. *If $(a_i : i \in I)$ is an \emptyset -independent sequence, then $\ell_\alpha^\Sigma(a_i : i \in I) = \text{bdd}(\ell_\alpha^\Sigma(a_i) : i \in I)$.*

Proof: Let $c = \ell_\alpha^\Sigma(a_i : i \in I)$ and set $b_J = \text{Cb}(c/a_i : i \in J)$ for each finite $J \subseteq I$. Note for each finite $J \subseteq I$ that $\text{tp}(b_J)$ is Σ -analysable in α steps. Thus $b_J \subseteq \ell_\alpha^\Sigma(a_i : i \in J)$. On the other hand,

$$\ell_\alpha^\Sigma(a_i : i \in J) = \text{bdd}(\ell_\alpha^\Sigma(a_i) : i \in J)$$

by Lemma 3.3 and induction, since $J \subseteq I$ is finite. Therefore

$$c \downarrow_{(\ell_\alpha^\Sigma(a_i) : i \in I)} (a_i : i \in I)$$

by the finite character of forking, whence $c \subseteq \text{bdd}(\ell_\alpha^\Sigma(a_i) : i \in I)$. \square

We shall see that the first level governs domination-equivalence.

Definition 3.5. An element a Σ -dominates an element b over A , denoted $a \triangleright_A^\Sigma b$, if for all c such that $\text{tp}(c/A)$ is Σ -analysable, $a \downarrow_A c$ implies $b \downarrow_A c$. Two elements a and b are Σ -domination-equivalent over A , denoted $a \sqsubseteq_A^\Sigma b$, if $a \triangleright_A^\Sigma b$ and $b \triangleright_A^\Sigma a$. If Σ is the set of all types, it is omitted.

The following generalizes a theorem by Buechler [3, Proposition 3.1] for finite Lascar rank.

Theorem 3.6. *Let Σ' be an \emptyset -invariant family of partial types.*

- (1) a and $\ell_1^\Sigma(a/A)$ are Σ -domination-equivalent over A .
- (2) If $\text{tp}(a/A)$ is Σ -analysable, then a and $\ell_1^\Sigma(a/A)$ are domination-equivalent over A .
- (3) If $\text{tp}(a/A)$ is $\Sigma \cup \Sigma'$ -analysable and foreign to Σ' , then a and $\ell_1^\Sigma(a/A)$ are domination-equivalent over A .

Proof: (1) Since $\ell_1^\Sigma(a/A) \in \text{bdd}(Aa)$, clearly a dominates (and Σ -dominates) $\ell_1^\Sigma(a/A)$ over A .

For the converse, suppose $\text{tp}(b/A)$ is Σ -analysable and $b \not\downarrow_A a$. Consider a sequence $(b_i : i < \alpha)$ in $\text{bdd}(Ab)$ such that $\text{tp}(b_i/A, b_j : j < i)$ is Σ -internal for all $i < \alpha$ and $b \in \text{bdd}(A, b_i : i < \alpha)$. Since $a \not\downarrow_A b$ there is a minimal $i < \alpha$ such that $a \not\downarrow_{A, (b_j : j < i)} b_i$. Put

$$a' = \text{Cb}(b_j : j \leq i/Aa) \in \text{bdd}(Aa).$$

Then $\text{tp}(a'/A)$ is Σ -internal by Corollary 2.4, and $a' \subseteq \ell_1^\Sigma(a/A)$. Clearly $a' \not\downarrow_A (b_j : j \leq i)$, whence $a' \not\downarrow_A b$ and finally $\ell_1^\Sigma(a/A) \not\downarrow_A b$. This shows (1).

(2) follows from (3) setting $\Sigma' = \emptyset$.

(3) Suppose $b \not\downarrow_A a$. We may assume that $b = \text{Cb}(a/Ab)$, so $\text{tp}(b/A)$ is $(\Sigma \cup \Sigma')$ -analysable. Thus

$$b \not\downarrow_A \ell_1^{\Sigma \cup \Sigma'}(a/A)$$

by (1). Now $\text{tp}(\ell_1^{\Sigma \cup \Sigma'}(a/A)/A)$ is foreign to Σ' since $\text{tp}(a/A)$ is; it is hence almost Σ -internal. Therefore $\ell_1^{\Sigma \cup \Sigma'}(a/A) \subseteq \ell_1^\Sigma(a/A)$ and so $b \not\downarrow_A \ell_1^\Sigma(a/A)$. \square

Remark 3.7. If $\text{tp}(a/A)$ is Σ_0 -analysable and Σ_1 is a subfamily of Σ_0 such that $\text{tp}(a/A)$ remains Σ_1 -analysable, then

$$\ell_1^{\Sigma_1}(a/A) \subseteq \ell_1^{\Sigma_0}(a/A) \subseteq \text{bdd}(aA)$$

and $\ell_1^{\Sigma_1}(a/A)$ and $\ell_1^{\Sigma_0}(a/A)$ are both domination-equivalent to a over A . In fact it would be sufficient to have Σ_1 such that $\text{tp}(\ell_1^{\Sigma_0}(a/A)/A)$ is Σ_1 -analysable.

Question 3.8. When is there a minimal (boundedly closed) $a_0 \in \text{bdd}(aA)$ domination-equivalent with a over A ?

If T has finite SU-rank, one can take $a_0 \in \text{bdd}(aA) \setminus \text{bdd}(A)$ with $SU(a_0/A)$ minimal possible.

Definition 3.9. • A type $\text{tp}(a/A)$ is Σ -flat if $\ell_1^\Sigma(a/A) = \ell_2^\Sigma(a/A)$.

It is A -flat if it is Σ -flat for all A -invariant Σ . It is flat if for all $B \supseteq A$ every nonforking extension to B is B -flat. A theory T is flat if all its types are.

- A type $p \in S(A)$ is A -ultraflat if it is almost internal in any A -invariant family of partial types it is non-foreign to. It is ultraflat if for any $B \supseteq A$ every nonforking extension to B is B -ultraflat.

Flatness and ultraflatness are clearly preserved under non-forking extensions and non-forking restrictions, and under adding and forgetting parameters.

Remark 3.10. If $\text{tp}(a/A)$ is Σ -flat, then $\ell_\alpha^\Sigma(a/A) = \ell_1^\Sigma(a/A)$ for all $\alpha > 0$. Clearly, ultraflat implies flat.

Example.

- Generic types of fields or definably simple groups interpretable in a simple theory are ultraflat.
- Types of Lascar rank 1 are ultraflat.
- If there is no boundedly closed set between $\text{bdd}(A)$ and $\text{bdd}(aA)$, then $\text{tp}(a/A)$ is A -ultraflat.
- In a small simple theory there are many A -ultraflat types over finite sets A , as the lattice of boundedly closed subsets of $\text{bdd}(aA)$ is scattered for finitary aA .

Next we shall prove that any type internal in a family of Lascar rank one types is also flat.

Lemma 3.11. *It $\text{tp}(a/A)$ is flat (ultraflat), then so is $\text{tp}(a_0/A)$ for any $a_0 \in \text{bdd}(Aa)$.*

Proof: Consider a set B extending A with $B \downarrow_A a_0$; we may assume that $B \downarrow_{Aa_0} a$, whence $B \downarrow_A a$.

Firstly, the flat case is clear since $\ell_\alpha^\Sigma(a_0/B) = \ell_\alpha^\Sigma(a/B) \cap \text{bdd}(Ba_0)$ for any $\alpha > 0$ and for any B -invariant family Σ . Assume now $\text{tp}(a/A)$ is ultraflat, $a_0 \in \text{bdd}(Aa)$ and $\text{tp}(a_0/B)$ is not foreign to some B -invariant family Σ . Then $\text{tp}(a/B)$ is not foreign to Σ , hence almost Σ -internal, as is $\text{tp}(a_0/B)$. \square

Corollary 3.12. *If $\text{tp}(a/A)$ is almost internal in a family of types of Lascar rank one, then it is flat.*

Proof: Assume there is some $B \downarrow_A a$ and some tuple \bar{b} of realizations of types of Lascar rank one over B such that $a \subseteq \text{bdd}(B\bar{b})$. We may assume \bar{b} is an independent sequence over B since all its elements have SU-rank one. Hence \bar{b} is an independent sequence over any $C \supseteq B$ with $C \downarrow_B \bar{b}$, so $\text{tp}(\bar{b}/B)$ is flat by Corollary 3.4. Thus, $\text{tp}(a/B)$ is flat by Lemma 3.11, and so is $\text{tp}(a/A)$. \square

Question 3.13. Is every (finitary) type in a small simple theory non-orthogonal to a flat type?

Question 3.14. Is every type in a supersimple theory non-orthogonal to a flat type?

Problem 3.15. Construct a flat type which is not ultraflat.

We shall now recall the definitions and properties of Σ -closure from [24, Section 4.0] in the stable and [25, Section 3.5] in the simple case (where it is called P -closure: our Σ corresponds to the collection of all P -analysable types which are co-foreign to P). Buechler and Hoover [2, Definition 1.2] redefine such a closure operator in the context of superstable theories and reprove some of the properties [2, Lemma 2.5].

Definition 3.16. The Σ -closure $\Sigma\text{cl}(A)$ of a set A is the collection of all hyperimaginaries a such that $\text{tp}(a/A)$ is Σ -analysable.

Remark 3.17. We think of partial types in Σ as small. We always have $\text{bdd}(A) \subseteq \Sigma\text{cl}(A)$; equality holds if Σ is the family of all bounded types. Other useful examples for Σ are the family of all types of SU -rank $< \omega^\alpha$ for some ordinal α , the family of all supersimple types in a properly simple theory, or the family of p -simple types of p -weight 0 for some regular type p , giving rise to Hrushovski's p -closure [10].

Fact 3.18. *The following are equivalent:*

- (1) $\text{tp}(a/A)$ is foreign to Σ .
- (2) $a \downarrow_A \Sigma\text{cl}(A)$.
- (3) $a \downarrow_A \text{dcl}(aA) \cap \Sigma\text{cl}(A)$.
- (4) $\text{dcl}(aA) \cap \Sigma\text{cl}(A) \subseteq \text{bdd}(A)$.

Proof: The equivalence of (1), (2) and (3) is [25, Lemma 3.5.3]; the equivalence (3) \Leftrightarrow (4) is obvious. \square

Unless it equals bounded closure, Σ -closure has the size of the monster model and thus violates the usual conventions. The equivalence (2) \Leftrightarrow (3) can be used to cut it down to some small part.

Fact 3.19. *Suppose $A \downarrow_B C$. Then $\Sigma\text{cl}(A) \downarrow_{\Sigma\text{cl}(B)} \Sigma\text{cl}(C)$. More precisely, for any $A_0 \subseteq \Sigma\text{cl}(A)$ we have $A_0 \downarrow_{B_0} \Sigma\text{cl}(C)$, where $B_0 = \text{dcl}(A_0B) \cap \Sigma\text{cl}(B)$. In particular, $\Sigma\text{cl}(AB) \cap \Sigma\text{cl}(BC) = \Sigma\text{cl}(B)$.*

Proof: This is [25, Lemma 3.5.5]; the second clause follows from Fact 3.18. \square

Lemma 3.20. *Suppose $C \subseteq A \cap B \cap D$ and $AB \downarrow_C D$.*

- (1) *If $\Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(C)$, then $\Sigma\text{cl}(AD) \cap \Sigma\text{cl}(BD) = \Sigma\text{cl}(D)$.*

- (2) If $\text{bdd}(A) \cap \Sigma\text{cl}(B) = \text{bdd}(C)$, then $\text{bdd}(AD) \cap \Sigma\text{cl}(BD) = \text{bdd}(D)$.

Proof: (1) This is [25, Lemma 3.5.6], which in turn adapts [18, Fact 2.4].

- (2) This is similar to (1). By Fact 15

$$\Sigma\text{cl}(BD) \quad \downarrow \quad AB ; \\ \Sigma\text{cl}(B) \cap \text{dcl}(AB)$$

since $AD \downarrow_A AB$ we obtain

$$\text{Cb}(\text{bdd}(AD) \cap \Sigma\text{cl}(BD)/AB) \subseteq \text{bdd}(A) \cap \Sigma\text{cl}(B) = \text{bdd}(C).$$

Hence

$$\text{bdd}(AD) \cap \Sigma\text{cl}(BD) \quad \downarrow \quad AB \\ C$$

and by transitivity

$$\text{bdd}(AD) \cap \Sigma\text{cl}(BD) \quad \downarrow \quad ABD, \\ D$$

whence the result. \square

The following lemma tells us that we can actually find a set C with $\Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(C)$ as in Lemma 3.20(1), even though the Σ -closures have the size of the monster model.

Lemma 3.21. *Let $C = \text{bdd}(AB) \cap \Sigma\text{cl}(A) \cap \Sigma\text{cl}(B)$. Then $\Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(C)$.*

Proof: Consider $e \in \Sigma\text{cl}(A) \cap \Sigma\text{cl}(B)$ and put $f = \text{Cb}(e/AB)$. Then $e \downarrow_f AB$; since $\text{tp}(e/A)$ is Σ -analysable, so is $\text{tp}(e/f)$, and $e \in \Sigma\text{cl}(f)$. If I is a Morley sequence in $\text{tp}(e/AB)$, then $f \in \text{dcl}(I)$. However, since e is Σ -analysable over A and over B , so is I , whence f . Hence

$$f \in \text{bdd}(AB) \cap \Sigma\text{cl}(A) \cap \Sigma\text{cl}(B) = C.$$

The result follows. \square

However, for considerations such as the canonical base property, one should like to work with the first level of the Σ -closure rather than with the full closure operator.

Definition 3.22. The Σ_1 -closure of A is given by

$$\Sigma_1\text{cl}(A) = \ell_1^\Sigma(\Sigma\text{cl}(A)/A) = \{b : \text{tp}(b/A) \text{ is almost } \Sigma\text{-internal}\}.$$

Unfortunately, unless $\text{tp}(\Sigma\text{cl}(A)/A)$ is Σ -flat, Σ_1 -closure is not a closure operator, as $\Sigma_1\text{cl}(\Sigma_1\text{cl}(A)) \supset \Sigma_1\text{cl}(A)$.

Lemma 3.23. *Suppose $A \downarrow_B C$ with $B \subseteq A \cap C$. Then*

$$\Sigma_1 \text{cl}(A) \downarrow_{\Sigma_1 \text{cl}(B)} C.$$

More precisely, $\Sigma_1 \text{cl}(A) \downarrow_{\Sigma_1 \text{cl}(B) \cap \text{bdd}(C)} C$.

Proof: Consider $a \in \Sigma_1 \text{cl}(A)$ and put $c = \text{Cb}(Aa/C)$. Then $\text{tp}(c/B)$ is almost Σ -internal by Corollary 2.4, and $c \in \text{bdd}(C) \cap \Sigma_1 \text{cl}(B)$. \square

Question 3.24. If $A \downarrow_B C$, is $\Sigma_1 \text{cl}(A) \downarrow_{\Sigma_1 \text{cl}(B)} \Sigma_1 \text{cl}(C)$?

4. Σ -AMPLENESS AND WEAK Σ -AMPLENESS

Let Φ and Σ be \emptyset -invariant families of partial types.

Definition 4.1. Φ is n - Σ -ample if there are tuples a_0, \dots, a_n , with a_n a tuple of realizations of partial types in Φ over some parameters A , such that

- (1) $a_n \not\downarrow_{\Sigma \text{cl}(A)} a_0$;
- (2) $a_{i+1} \downarrow_{\Sigma \text{cl}(Aa_i)} a_0 \dots a_{i-1}$ for $1 \leq i < n$;
- (3) $\Sigma \text{cl}(Aa_0 \dots a_{i-1} a_i) \cap \Sigma \text{cl}(Aa_0 \dots a_{i-1} a_{i+1}) = \Sigma \text{cl}(Aa_0 \dots a_{i-1})$ for $0 \leq i < n$.

Remark 4.2. Pillay [20] requires $a_n \downarrow_{Aa_i} a_0 \dots a_{i-1}$ for $1 \leq i < n$ in item (2). We follow the variant proposed by Evans and Nübling [6] which seems more natural and which implies

$$a_n \dots a_{i+1} \downarrow_{\Sigma \text{cl}(Aa_i)} a_0 \dots a_{i-1}.$$

Lemma 4.3. *If Σ' is a Σ -analysable family of partial types, then n - Σ -ample implies n - Σ' -ample, and in particular n -ample.*

Proof: As in [20, Remark 3.7] we replace a_i by

$$a'_i = \text{Cb}(a'_n \dots a'_{i+1} / \Sigma \text{cl}(Aa_i))$$

for $i < n$, where $a'_n = a_n$. Then

$$a'_n \dots a'_{i+1} \downarrow_{a'_i} \Sigma \text{cl}(Aa_i) \quad \text{and} \quad a'_n \dots a'_{i+1} \downarrow_{\Sigma \text{cl}(Aa_i)} \Sigma \text{cl}(Aa_0 \dots a_i)$$

by Fact 3.18, whence

$$a'_n \dots a'_{i+1} \downarrow_{a'_i} \Sigma \text{cl}(Aa_0 \dots a_i).$$

Put $A' = \Sigma\text{cl}(A) \cap \text{bdd}(Aa'_0)$. Then $A \subseteq A' \subseteq \Sigma\text{cl}(A)$, whence $\Sigma\text{cl}(A) = \Sigma\text{cl}(A')$, and $a'_0 \downarrow_{A'} \Sigma\text{cl}(A)$. Now $a_n \not\downarrow_{\Sigma\text{cl}(A')} a_0$ implies $a'_n \not\downarrow_{\Sigma\text{cl}(A)} a'_0$, whence $a'_n \not\downarrow_{\Sigma'\text{cl}(A)} a'_0$. Clearly $a'_{i+1} \downarrow_{a'_i} \Sigma\text{cl}(Aa_0 \dots a_i)$ implies

$$a'_{i+1} \downarrow_{\Sigma'\text{cl}(A'a'_i)} a'_0 \dots a'_{i-1}$$

for $1 \leq i < n$. Finally,

$$A'a'_0 \dots a'_i a'_{i+1} \downarrow_{\Sigma'\text{cl}(A'a'_0 \dots a'_{i-1})} \Sigma\text{cl}(Aa_0 \dots a_{i-1})$$

yields

$$\Sigma'\text{cl}(A'a'_0 \dots a'_{i-1} a'_{i+1}), \Sigma'\text{cl}(A'a'_0 \dots a'_i) \downarrow_{\Sigma'\text{cl}(A'a'_0 \dots a'_{i-1})} \Sigma\text{cl}(Aa_0 \dots a_{i-1}),$$

so

$$\Sigma'\text{cl}(A'a'_0 \dots a'_{i-1} a'_{i+1}) \cap \Sigma'\text{cl}(A'a'_0 \dots a'_i) \subseteq \Sigma\text{cl}(Aa_0 \dots a_{i-1})$$

implies

$$\Sigma'\text{cl}(A'a'_0 \dots a'_{i-1} a'_{i+1}) \cap \Sigma'\text{cl}(A'a'_0 \dots a'_i) \subseteq \Sigma'\text{cl}(A'a'_0 \dots a'_{i-1}).$$

□

This also shows that in Definition 4.1 one may require a_0, \dots, a_{n-1} to lie in Φ^{heq} , and $a_{i+1} \downarrow_{a_i} \Sigma\text{cl}(Aa_0 \dots a_i)$.

Remark 4.4. [20, Lemma 3.2 and Corollary 3.3] If a_0, \dots, a_n witness n - Σ -ampleness over A , then $a_n \not\downarrow_{\Sigma\text{cl}(Aa_0 \dots a_{i-1})} a_i$ for all $i < n$. Hence a_i, \dots, a_n witness $(n-i)$ - Σ -ampleness over $Aa_0 \dots a_{i-1}$. Thus n - Σ -ample implies i - Σ -ample for all $i \leq n$.

Remark 4.5. It is clear from the definition that even though Φ might be a complete type p , if p is not n - Σ -ample, neither is any extension of p , not only the non-forking ones.

For $n = 1$ and $n = 2$ there are alternative definitions of non- n - Σ -ampleness:

Definition 4.6. (1) Φ is Σ -based if $\text{Cb}(a/\Sigma\text{cl}(B)) \subseteq \Sigma\text{cl}(aA)$ for any tuple a of realizations of partial types in Φ over some parameters A and any $B \supseteq A$.
(2) Φ is Σ -CM-trivial if $\text{Cb}(a/\Sigma\text{cl}(AB)) \subseteq \Sigma\text{cl}(A, \text{Cb}(a/\Sigma\text{cl}(AC)))$ for any tuple a of realizations of partial types in Φ over some parameters A and any $B \subseteq C$ such that $\Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) = \Sigma\text{cl}(AB)$.

Lemma 4.7. (1) Φ is Σ -based if and only if Φ is not 1- Σ -ample.

(2) Φ is Σ -CM-trivial if and only if Φ is not 2- Σ -ample.

Proof: (1) Suppose Φ is Σ -based and consider a_0, a_1, A with $\Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \Sigma\text{cl}(A)$. Put $a = a_1$ and $B = Aa_0$. By Σ -basedness

$$\text{Cb}(a/\Sigma\text{cl}(B)) \subseteq \Sigma\text{cl}(Aa) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(A).$$

Hence $a \downarrow_{\Sigma\text{cl}(A)} \Sigma\text{cl}(B)$, whence $a_1 \downarrow_{\Sigma\text{cl}(A)} a_0$, so Φ is not 1- Σ -ample.

Conversely, if Φ is not Σ -based, let a, A, B be a counterexample. Put $a_0 = \text{Cb}(a_1/\Sigma\text{cl}(B))$ and $a_1 = a$. Then $a_0 \notin \Sigma\text{cl}(Aa_1)$. Now take

$$A' = \text{bdd}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1).$$

Then $\Sigma\text{cl}(A'a_0) \cap \Sigma\text{cl}(A'a_1) = \Sigma\text{cl}(A')$ by Lemma 3.21.

Suppose $a_1 \downarrow_{\Sigma\text{cl}(A')} a_0$. Since $\Sigma\text{cl}(A') \subseteq \Sigma\text{cl}(Aa_0) \subseteq \Sigma\text{cl}(B)$ we have $a_1 \downarrow_{a_0} \Sigma\text{cl}(A')$. As $a_0 = \text{Cb}(a_1/\Sigma\text{cl}(B))$, this implies

$$a_0 \subseteq \Sigma\text{cl}(A') \subseteq \Sigma\text{cl}(Aa_1),$$

a contradiction. Hence a_0, a_1, A' witness 1- Σ -ampleness of Φ .

(2) Suppose Φ is Σ -CM-trivial and consider a_0, a_1, a_2, A with

$$\begin{aligned} \Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1) &= \Sigma\text{cl}(A), \\ \Sigma\text{cl}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0a_2) &= \Sigma\text{cl}(Aa_0), \quad \text{and} \\ a_2 &\downarrow_{\Sigma\text{cl}(Aa_1)} a_0. \end{aligned}$$

Put $a = a_2$, $B = a_0$ and $C = a_0a_1$. Then

$$a_2 \downarrow_{\Sigma\text{cl}(Aa_1)} \Sigma\text{cl}(Aa_0a_1),$$

so $\text{Cb}(a/\Sigma\text{cl}(AC)) \subseteq \Sigma\text{cl}(Aa_1)$. Moreover

$$\Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) = \Sigma\text{cl}(AB),$$

whence by Σ -CM-triviality

$$\begin{aligned} \text{Cb}(a/\Sigma\text{cl}(AB)) &\subseteq \Sigma\text{cl}(A, \text{Cb}(a/AC)) \cap \Sigma\text{cl}(AB) \\ &\subseteq \Sigma\text{cl}(Aa_1) \cap \Sigma\text{cl}(Aa_0) = \Sigma\text{cl}(A). \end{aligned}$$

Hence $a_2 \downarrow_{\Sigma\text{cl}(A)} a_0$, so Φ is not 2- Σ -ample.

Conversely, if Φ is not Σ -CM-trivial, let a, A, B, C be a counterexample. Put

$$\begin{aligned} a_0 &= \text{Cb}(a/\Sigma\text{cl}(AB)), \quad a_1 = \text{Cb}(a/\Sigma\text{cl}(AC)), \quad a_2 = a, \\ A' &= \text{bdd}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1) \subseteq \Sigma\text{cl}(AB). \end{aligned}$$

Then $a_2 \downarrow_{\Sigma\text{cl}(A'a_1)} a_0$ and $a_0 \notin \Sigma\text{cl}(Aa_1)$; by Lemma 3.21

$$\Sigma\text{cl}(A'a_0) \cap \Sigma\text{cl}(A'a_1) = \Sigma\text{cl}(A').$$

Moreover, $a_2 \downarrow_{a_0} \Sigma\text{cl}(AB)$ implies

$$\Sigma\text{cl}(A'a_0a_2) \downarrow_{\Sigma\text{cl}(A'a_0)} \Sigma\text{cl}(AB).$$

Thus

$$\begin{aligned} \Sigma\text{cl}(A'a_0a_2) \cap \Sigma\text{cl}(A'a_0a_1) &\subseteq \Sigma\text{cl}(ABa) \cap \Sigma\text{cl}(AC) \\ &= \Sigma\text{cl}(AB) \cap \Sigma\text{cl}(A'a_0a_2) = \Sigma\text{cl}(A'a_0). \end{aligned}$$

Suppose $a_2 \downarrow_{\Sigma\text{cl}(A')} a_0$. Since $a_2 \downarrow_{a_0} \Sigma\text{cl}(A')$ we obtain

$$a_0 = \text{Cb}(a/\Sigma\text{cl}(AB)) = \text{Cb}(a/a_0\Sigma\text{cl}(A')) \subseteq \Sigma\text{cl}(A') \subseteq \Sigma\text{cl}(Aa_1),$$

a contradiction. Hence a_0, a_1, a_2, A' witness 2- Σ -ampleness of Φ . \square

In our definition of Σ -ampleness, we only consider the type of a_n over a Σ -closed set, namely $\Sigma\text{cl}(A)$. This seems natural since the idea of Σ -closure is to work *modulo* Σ . However, sometimes one needs a stronger notion which takes care of all types. Let us first look at $n = 1$ and $n = 2$.

- Definition 4.8.**
- Φ is *strongly Σ -based* if $\text{Cb}(a/B) \subseteq \Sigma\text{cl}(aA)$ for any tuple a of realizations of partial types in Φ over some A and any $B \supseteq A$.
 - Φ is *strongly Σ -CM-trivial* if $\text{Cb}(a/AB) \subseteq \Sigma\text{cl}(A, \text{Cb}(a/AC))$ for any tuple a of realizations of partial types in Φ over some A and any $B \subseteq C$ with $\Sigma\text{cl}(ABa) \cap \text{bdd}(AC) = \text{bdd}(AB)$.

Remark 4.9. $\text{Cb}(a/\Sigma\text{cl}(B)) \subseteq \text{bdd}(\text{Cb}(a/B), a) \cap \Sigma\text{cl}(\text{Cb}(a/B))$.

Proof: By Fact 3.19 the independence $a \downarrow_{\text{Cb}(a/B)} B$ implies

$$a \downarrow_{\text{dcl}(a, \text{Cb}(a/B)) \cap \Sigma\text{cl}(\text{Cb}(a/B))} \Sigma\text{cl}(B).$$

The result follows. \square

Conjecture. $\text{Cb}(a/B) \subseteq \Sigma\text{cl}(\text{Cb}(a/\Sigma\text{cl}(B)))$.

If this conjecture were true, strong and ordinary Σ -basedness and Σ -CM-triviality would obviously coincide. Since we have not been able to show this, we weaken our definition of ampleness.

Definition 4.10. Φ is *weakly n - Σ -ample* if there are tuples a_0, \dots, a_n , where a_n is a tuple of realizations of partial types in Φ over A , with

- (1) $a_n \not\downarrow_A a_0$.
- (2) $a_{i+1} \downarrow_{Aa_i} a_0 \dots a_{i-1}$ for $1 \leq i < n$.
- (3) $\text{bdd}(Aa_0 \dots a_{i-1}a_i) \cap \Sigma\text{cl}(Aa_0 \dots a_{i-1}a_{i+1}) = \text{bdd}(Aa_0 \dots a_{i-1})$
for $i < n$.

Note that (3) implies that $\text{tp}(a_i/Aa_0 \dots a_{i-1})$ is foreign to Σ by Fact 3.18 for all $i < n$, and so is $\text{tp}(a_i/Aa_{i-1})$ by (2). If Σ is the family of bounded partial types, then weak and ordinary n - Σ -ampleness just equal n -ampleness.

Lemma 4.11. *An n - Σ -ample family of types is weakly n - Σ -ample. If Σ' is Σ -analysable, then a weakly n - Σ -ample family is weakly n - Σ' -ample, and in particular n -ample.*

Proof: If a_0, \dots, a_n witness n - Σ -ampleness over A , we put $a'_n = a_n$,

$$a'_i = \text{Cb}(a'_n \dots a'_{i+1}/\Sigma\text{cl}(Aa_i)) \subseteq \Sigma\text{cl}(Aa_i) \quad \text{for } n > i$$

and

$$A' = \text{bdd}(Aa'_0) \cap \Sigma\text{cl}(Aa'_1) \subseteq \Sigma\text{cl}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \Sigma\text{cl}(A).$$

As in Lemma 4.3 we have for $i < n$

$$a'_n \dots a'_{i+1} \downarrow_{a'_i} \Sigma\text{cl}(Aa_0 \dots a_i).$$

For $0 < i < n$ we obtain $a'_{i+1} \downarrow_{A'a'_i} a'_0 \dots a'_{i-1}$; moreover

$$\begin{aligned} & \text{bdd}(A'a'_0 \dots a'_{i-1}a'_i) \cap \Sigma\text{cl}(A'a'_0 \dots a'_{i-1}a'_{i+1}) \\ & \subseteq \Sigma\text{cl}(A'a'_0 \dots a'_{i-1}a'_i) \cap \Sigma\text{cl}(A'a'_0 \dots a'_{i-1}a'_{i+1}) \\ & \subseteq \Sigma\text{cl}(Aa_0 \dots a_{i-1}a_i) \cap \Sigma\text{cl}(Aa_0 \dots a_{i-1}a_{i+1}) \\ & = \Sigma\text{cl}(Aa_0 \dots a_{i-1}). \end{aligned}$$

But then $a'_i \downarrow_{A'a'_0 \dots a'_{i-1}} \Sigma\text{cl}(Aa_0 \dots a_{i-1})$ yields

$$\text{bdd}(A'a'_0 \dots a'_{i-1}a'_i) \cap \Sigma\text{cl}(A'a'_0 \dots a'_{i-1}a'_{i+1}) = \text{bdd}(A'a'_0 \dots a'_{i-1}),$$

while $\text{bdd}(A'a'_0) \cap \Sigma\text{cl}(A'a'_1) = \text{bdd}(A')$ follows from the definition of A' . Finally $a_n \not\downarrow_{\Sigma\text{cl}(A)} a_0$ implies $a'_n \not\downarrow_{\Sigma\text{cl}(A)} a'_0$, whence $a'_n \not\downarrow_{A'} a'_0$ as $\text{tp}(a'_0/A')$ is foreign to Σ and $\Sigma\text{cl}(A) = \Sigma\text{cl}(A')$.

The second assertion is clear, since $\Sigma'\text{cl}(A) \subseteq \Sigma\text{cl}(A)$ for any set A . \square

This also shows that in Definition 4.10 one may require a_0, \dots, a_{n-1} to lie in Φ^{heq} .

Lemma 4.12. (1) Φ is strongly Σ -based iff Φ is not weakly 1- Σ -ample.

(2) Φ is strongly Σ -CM-trivial iff Φ is not weakly 2- Σ -ample.

Proof: This is similar to the proof of Lemma 4.7, so we shall be concise.

(1) Suppose Φ is strongly Σ -based and consider a_0, a_1, A with

$$\text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \text{bdd}(A).$$

Put $a = a_1$ and $B = Aa_0$. By strong Σ -basedness

$$\text{Cb}(a/B) \subseteq \Sigma\text{cl}(Aa) \cap \text{bdd}(B) = \text{bdd}(A),$$

whence $a_1 \downarrow_A a_0$, so Φ is not weakly 1- Σ -ample.

Conversely, if Φ is not strongly Σ -based, let a, A, B be a counterexample. Put $a_0 = \text{Cb}(a_1/B)$ and $a_1 = a$. Then $a_0 \notin \Sigma\text{cl}(Aa_1)$. Now take $A' = \text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1)$. Clearly $A' = \text{bdd}(A'a_0) \cap \Sigma\text{cl}(A'a_1)$. Suppose $a_1 \downarrow_{A'} a_0$. Since $a_0 = \text{Cb}(a_1/B)$ implies $a_1 \downarrow_{a_0} A'$, we obtain

$$a_0 \subseteq \text{bdd}(A') \subseteq \Sigma\text{cl}(Aa_1),$$

a contradiction. Hence a_0, a_1, A' witness weak 1- Σ -ampleness of Φ .

(2) Suppose Φ is strongly Σ -CM-trivial and consider a_0, a_1, a_2, A with

$$\text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1) = \text{bdd}(A),$$

$$\text{bdd}(Aa_0a_1) \cap \Sigma\text{cl}(Aa_0a_2) = \text{bdd}(Aa_0), \quad \text{and}$$

$$a_2 \downarrow_{Aa_1} a_0.$$

Put $a = a_2$, $B = a_0$ and $C = a_0a_1$. Then $\text{Cb}(a/AC) \subseteq \text{bdd}(Aa_1)$. Moreover

$$\Sigma\text{cl}(ABa) \cap \text{bdd}(AC) = \text{bdd}(AB),$$

whence by strong Σ -CM-triviality

$$\begin{aligned} \text{Cb}(a/AB) &\subseteq \Sigma\text{cl}(A, \text{Cb}(a/AC)) \cap \text{bdd}(AB) \\ &\subseteq \Sigma\text{cl}(Aa_1) \cap \text{bdd}(Aa_0) = \text{bdd}(A). \end{aligned}$$

Hence $a_2 \downarrow_A a_0$, so Φ is not 2- Σ -ample.

Conversely, if Φ is not strongly Σ -CM-trivial, let a, A, B, C be a counterexample. Put

$$a_0 = AB, \quad a_1 = \text{Cb}(a/AC), \quad a_2 = a,$$

$$A' = \text{bdd}(Aa_0) \cap \Sigma\text{cl}(Aa_1).$$

Then $a_2 \downarrow_{A'a_1} a_0$ and $\text{Cb}(a_2/AB) \notin \Sigma\text{cl}(Aa_1) = \Sigma\text{cl}(A'a_1)$; moreover

$$\text{bdd}(A'a_0) \cap \Sigma\text{cl}(A'a_1) = \text{bdd}(A').$$

Clearly

$$\begin{aligned} \Sigma\text{cl}(A'a_0a_2) \cap \text{bdd}(A'a_0a_1) &\subseteq \Sigma\text{cl}(ABa) \cap \text{bdd}(AC) \\ &= \text{bdd}(AB) = \text{bdd}(A'a_0). \end{aligned}$$

Suppose $a_2 \downarrow_{A'} a_0$. Then $\text{Cb}(a_2/AB) \in \text{bdd}(A') \subseteq \Sigma\text{cl}(Aa_1)$, a contradiction. Hence a_0, a_1, a_2, A' witness weak 2- Σ -ampleness of Φ . \square

Lemma 4.13. *If Φ is not (weakly) n - Σ -ample, neither is the family of \emptyset -conjugates of $\text{tp}(a/A)$ for any $a \in \Sigma\text{cl}(\bar{a}A)$, where \bar{a} is a tuple of realizations of partial types in Φ over A .*

Proof: Suppose the family of \emptyset -conjugates of $\text{tp}(a/A)$ is n - Σ -ample, as witnessed by a_0, \dots, a_n over some parameters B . There is a tuple \bar{a} of realizations of partial types in Φ over some \emptyset -conjugates of A inside B such that $a_n \in \Sigma\text{cl}(\bar{a}B)$; we may choose it such that

$$\bar{a} \downarrow_{a_n B} a_0 \dots a_{n-1}.$$

Then $\bar{a} \downarrow_{a_{n-1}a_n B} a_0 \dots a_{n-2}$, and hence

$$\bar{a} \downarrow_{\Sigma\text{cl}(a_{n-1}a_n B)} a_0 \dots a_{n-2}.$$

As $a_n \downarrow_{\Sigma\text{cl}(a_{n-1}B)} a_0 \dots a_{n-2}$ implies

$$\Sigma\text{cl}(a_{n-1}a_n B) \downarrow_{\Sigma\text{cl}(a_{n-1}B)} a_0 \dots a_{n-2}$$

by Fact 3.19, we get

$$\bar{a} \downarrow_{\Sigma\text{cl}(a_{n-1}B)} a_0 \dots a_{n-2}.$$

We also have $\bar{a} \downarrow_{a_0 \dots a_{n-2} a_n B} a_{n-1}$, whence

$$(1) \quad \Sigma\text{cl}(a_0 \dots a_{n-2} \bar{a} B) \downarrow_{\Sigma\text{cl}(a_0 \dots a_{n-2} a_n B)} \Sigma\text{cl}(a_0 \dots a_{n-2} a_{n-1} B);$$

since Σ -closure is boundedly closed,

$$\begin{aligned} \Sigma\text{cl}(a_0 \dots a_{n-2} \bar{a} B) \cap \Sigma\text{cl}(a_0 \dots a_{n-2} a_{n-1} B) \\ \subseteq \Sigma\text{cl}(a_0 \dots a_{n-2} a_n B) \cap \Sigma\text{cl}(a_0 \dots a_{n-2} a_{n-1} B) \\ = \Sigma\text{cl}(a_0 \dots a_{n-2} B). \end{aligned}$$

Finally, $\bar{a} \downarrow_{\Sigma\text{cl}(B)} a_0$ would imply $\Sigma\text{cl}(\bar{a}B) \downarrow_{\Sigma\text{cl}(B)} a_0$ by Fact 3.19, and hence $a_n \downarrow_{\Sigma\text{cl}(B)} a_0$, a contradiction. Thus $\bar{a} \not\downarrow_{\Sigma\text{cl}(B)} a_0$, and $a_0, \dots, a_{n-1}, \bar{a}$ witness n - Σ -ampleness of Φ over B , a contradiction.

Now suppose a_0, \dots, a_n witness weak n - Σ -ampleness over B , and choose \bar{a} as before. Then easily $\bar{a}a_n \downarrow_{Ba_{n-1}} a_0 \dots a_{n-2}$, yielding (2) from the definition. Moreover, equation (1) implies

$$\begin{aligned} \Sigma\text{cl}(a_0 \dots a_{n-2}\bar{a}B) \cap \text{bdd}(a_0 \dots a_{n-2}a_{n-1}B) \\ \subseteq \Sigma\text{cl}(a_0 \dots a_{n-2}a_nB) \cap \text{bdd}(a_0 \dots a_{n-2}a_{n-1}B) \\ = \text{bdd}(a_0 \dots a_{n-2}B). \end{aligned}$$

Finally suppose $\bar{a} \downarrow_B a_0$. Since $\text{tp}(a_0/B)$ is foreign to Σ , so is $\text{tp}(a_0/B\bar{a})$. Then $a_0 \downarrow_{B\bar{a}} \Sigma\text{cl}(B\bar{a})$ by Fact 3.18, whence $a_0 \downarrow_B a_n$, a contradiction. Thus $\bar{a} \not\downarrow_B a_0$, and $a_0, \dots, a_{n-1}, \bar{a}$ witness weak n - Σ -ampleness of Φ over B , again a contradiction. \square

Lemma 4.14. *Suppose $B \downarrow_A a_0 \dots a_n$. If a_0, \dots, a_n witness (weak) n - Σ -ampleness over A , they witness (weak) n - Σ -ampleness over B .*

Proof: Clearly $B \downarrow_{a_0 \dots a_{i-1}A} a_0 \dots a_{i+1}A$, so Lemma 3.20 yields

$$\Sigma\text{cl}(Ba_0 \dots a_{i-1}a_i) \cap \Sigma\text{cl}(Ba_0 \dots a_{i-1}a_{i+1}) = \Sigma\text{cl}(Ba_0 \dots a_{i-1})$$

in the ordinary case, and

$$\text{bdd}(Ba_0 \dots a_{i-1}a_i) \cap \Sigma\text{cl}(Ba_0 \dots a_{i-1}a_{i+1}) = \text{bdd}(Ba_0 \dots a_{i-1})$$

in the weak case, for all $i < n$.

Next, $a_{i+1} \downarrow_{Aa_0 \dots a_i} B$, whence $a_{i+1} \downarrow_{\Sigma\text{cl}(Aa_0 \dots a_i)} \Sigma\text{cl}(Ba_i)$ by Lemma 3.19. Now $a_{i+1} \downarrow_{\Sigma\text{cl}(Aa_i)} a_0 \dots a_{i-1}$ implies $a_{i+1} \downarrow_{\Sigma\text{cl}(Aa_i)} \Sigma\text{cl}(Aa_0 \dots a_i)$, whence

$$a_{i+1} \downarrow_{\Sigma\text{cl}(Ba_i)} a_0 \dots a_{i-1}$$

for $1 \leq i < n$ by transitivity. In the weak case, $a_{i+1} \downarrow_{Aa_i} a_0 \dots a_{i-1}$ implies $a_{i+1} \downarrow_{Aa_i} Ba_0 \dots a_{i-1}$ by transitivity, whence $a_{i+1} \downarrow_{Ba_i} a_0 \dots a_{i-1}$.

Finally, $a_n \downarrow_{\Sigma\text{cl}(A)} \Sigma\text{cl}(B)$ by Fact 3.19, so $a_n \downarrow_{\Sigma\text{cl}(B)} a_0$ would imply $a_n \downarrow_{\Sigma\text{cl}(A)} a_0$, a contradiction. Hence $a_n \not\downarrow_{\Sigma\text{cl}(B)} a_0$. In the weak case, $a_n \downarrow_A B$ and $a_n \not\downarrow_A a_0$ yield directly $a_n \not\downarrow_B a_0$. \square

Lemma 4.15. *Let Ψ be an \emptyset -invariant family of types. If Φ and Ψ are not (weakly) n - Σ -ample, neither is $\Phi \cup \Psi$.*

Proof: Suppose $\Phi \cup \Psi$ is weakly n - Σ -ample, as witnessed by $a_0, \dots, a_n = bc$ over some parameters A , where b and c are tuples of realizations of partial types in Φ and Ψ , respectively. As Ψ is not n - Σ -ample, we

must have $c \downarrow_A a_0$. Put $a'_0 = \text{Cb}(bc/a_0A)$. Then $\text{tp}(a'_0/A)$ is internal in $\text{tp}(b/A)$ by Corollary 2.4. Put

$$a'_n = \text{Cb}(a'_0/a_nA).$$

Then $\text{tp}(a'_n/A)$ is $\text{tp}(a'_0/A)$ -internal and hence $\text{tp}(b/A)$ -internal. Note that $a_n \not\downarrow_A a_0$ implies $a_n \not\downarrow_A a'_0$, whence

$$a'_n \not\downarrow_A a'_0 \quad \text{and} \quad a'_n \not\downarrow_A a_0.$$

Moreover $a'_n \in \text{bdd}(Aa_n)$, so $a_0, \dots, a_{n-1}, a'_n$ witness weak n - Σ -ampleness over A .

As $\text{tp}(a'_n/A)$ is $\text{tp}(b/A)$ -internal, there is $B \downarrow_A a'_n$ and a tuple \bar{b} of realizations of $\text{tp}(b/A)$ with $a'_n \in \text{dcl}(B\bar{b})$. We may assume

$$B \downarrow_{Aa'_n} a_0 \dots a_{n-1},$$

whence $B \downarrow_A a_0 \dots a_{n-1}a'_n$. Hence $a_0, \dots, a_{n-1}, a'_n$ witness weak n - Σ -ampleness over B by Lemma 4.14. As $a'_n \in \text{dcl}(B\bar{b})$, this contradicts non weak n - Σ -ampleness of Φ by Lemma 4.13.

The proof in the ordinary case is analogous, replacing A by $\Sigma\text{cl}(A)$. \square

Corollary 4.16. *For $i < \alpha$ let Φ_i be an \emptyset -invariant family of partial types. If Φ_i is not (weakly) n - Σ -ample for all $i < \alpha$, neither is $\bigcup_{i < \alpha} \Phi_i$.*

Proof: This just follows from the local character of forking and Lemma 4.15. \square

Lemma 4.17. *If the family of \emptyset -conjugates of $\text{tp}(a/A)$ is not (weakly) n - Σ -ample and $a \downarrow A$, then $\text{tp}(a)$ is not (weakly) n - Σ -ample.*

Proof: Suppose $\text{tp}(a)$ is (weakly) n - Σ -ample, as witnessed by a_0, \dots, a_n over some parameters B , where $a_n = (b_i : i < k)$ is a tuple of realizations of $\text{tp}(a)$. For each $i < k$ choose $B_i \downarrow_{b_i}(B, a_0 \dots a_n, B_j : j < i)$ with $B_i b_i \equiv Aa$. Then $B_i \downarrow b_i$, whence $(B_i : i < k) \downarrow Ba_0 \dots a_n$. Then a_0, \dots, a_n witness (weak) n - Σ -ampleness over $(B, B_i : i < k)$ by Lemma 4.14, a contradiction, since $\text{tp}(b_i/B_i)$ is an \emptyset -conjugate of $\text{tp}(a/A)$ for all $i < k$. \square

Remark 4.18. In fact, in the above Lemma it suffices to merely assume that the single type $\text{tp}(a/A)$ is not (weakly) n - Σ -ample in the theory $T(A)$, using Corollary 4.16. It follows that ampleness is preserved under adding and forgetting parameters.

Corollary 4.19. *Let Ψ be an \emptyset -invariant family of types. If Ψ is Φ -internal and Φ is not (weakly) n - Σ -ample, neither is Ψ .*

Proof: Immediate from Lemmas 4.13 and 4.17. \square

Theorem 4.20. *Let Ψ be an \emptyset -invariant family of types. If Ψ is Φ -analysable and Φ is not (weakly) n - Σ -ample, neither is Ψ .*

Proof: Suppose Ψ is n - Σ -ample, as witnessed by a_0, \dots, a_n over some parameters A , where a_n is a tuple of realizations of Ψ . Put $a'_n = \ell_1^\Phi(a_n/\Sigma\text{cl}(A) \cap \text{bdd}(Aa_n))$. Then a_n and a'_n are domination-equivalent over $\Sigma\text{cl}(A) \cap \text{bdd}(Aa_n)$ by Theorem 3.6; moreover a_n and hence a'_n are independent of $\Sigma\text{cl}(A)$ over $\Sigma\text{cl}(A) \cap \text{bdd}(Aa_n)$ by Fact 3.18, so a_n and a'_n are domination-equivalent over $\Sigma\text{cl}(A)$. Thus a_0, \dots, a'_n witness non- Σ -ampleness over A , contradicting Corollary 4.19.

For the weak case we put $a'_n = \ell_1^\Phi(a_n/A)$. So a_n and a'_n are domination-equivalent over A , whence $a'_n \not\perp_A a_0$. Thus a_0, \dots, a'_n witness weak non- Σ -ampleness over A , contradicting again Corollary 4.19. \square

5. ANALYSABILITY OF CANONICAL BASES

As an immediate Corollary to Theorem 4.20, we obtain the following:

Theorem 5.1. *Suppose every type in T is non-orthogonal to a regular type, and let Σ be the family of all n -ample regular types. Then T is not weakly n - Σ -ample.*

Proof: A non n -ample type is not weakly Σ -ample by Lemma 4.11. So all regular types are not weakly n - Σ -ample. But every type is analysable in regular types by the non-orthogonality hypothesis. \square

Corollary 5.2. *Suppose every type in T is non-orthogonal to a regular type. Then $\text{tp}(\text{Cb}(a/b)/a)$ is analysable in the family of all non one-based regular types, for all a, b .*

Proof: This is just Theorem 5.1 for $n = 1$. \square

Note that a forking extension of a non one-based regular type of infinite rank may be one-based.

Remark 5.3. In fact, the proof shows more. For every type p let $\Sigma(p)$ be the collection of types in Σ not foreign to p . Then $\text{tp}(\text{Cb}(a/b)/a)$ is analysable in $\Sigma(\text{tp}(\text{Cb}(a/b)))$. In particular, if $\text{tp}(a)$ or $\text{tp}(b)$ has rank less than ω^α , so does $\text{tp}(\text{Cb}(a/b))$. Hence $\text{tp}(\text{Cb}(a/b)/a)$ is analysable in the family of all non one-based regular types of rank less than ω^α .

Corollary 5.2 is due to Zoé Chatzidakis for types of finite SU-rank in simple theories [5, Proposition 1.10]. In fact, she even obtains $\text{tp}(\text{Cb}(a/b)/\text{bdd}(a) \cap \text{bdd}(b))$ to be analysable in the family of non one-based types of rank 1, and to decompose in components, each of which is analysable in a non-orthogonality class of regular types. In infinite rank, one has to work modulo types of smaller rank. So let Σ_α be the collection of all partial types of SU-rank $< \omega^\alpha$, and \mathcal{P}_α be the family of non Σ_α -based types of SU-rank ω^α . Note that the Σ_α -based types of SU-rank ω^α are precisely the locally modular types of SU-rank ω^α .

Theorem 5.4. *Let $b = \text{Cb}(a/\Sigma_\alpha\text{cl}(b))$ be such that $\text{SU}(b) < \omega^{\alpha+1}$ for some ordinal α and let $A = \Sigma_\alpha\text{cl}(b) \cap \Sigma_\alpha\text{cl}(a)$. Then $\text{tp}(b/A)$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable.*

Proof: Firstly, if $a \in \Sigma_\alpha\text{cl}(b)$ then $a = b \in A$. Similarly, if $b \in \Sigma_\alpha\text{cl}(a)$ then $b \in A$; in both cases $\text{tp}(b/A)$ is trivially $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable. Hence we may assume $a \notin \Sigma_\alpha\text{cl}(b)$ and $b \notin \Sigma_\alpha\text{cl}(a)$.

Suppose towards a contradiction that the result is false and consider a counterexample a, b with $\text{SU}(b)$ minimal modulo ω^α and then $\text{SU}(b/\Sigma_\alpha\text{cl}(a))$ being maximal modulo ω^α . Note that this implies

$$\omega^\alpha \leq \text{SU}(b/a) \leq \text{SU}(b/A) \leq \text{SU}(b) < \omega^{\alpha+1}.$$

Clearly (after adding parameters) we may assume $A = \Sigma_\alpha\text{cl}(\emptyset)$. Then for any c the type $\text{tp}(c)$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable iff $\text{tp}(c/A)$ is.

Claim. *We may assume $a = \text{Cb}(b/\Sigma_\alpha\text{cl}(a))$.*

Proof of Claim: Put $\tilde{a} = \text{Cb}(b/\Sigma_\alpha\text{cl}(a))$ and $\tilde{b} = \text{Cb}(\tilde{a}/\Sigma_\alpha\text{cl}(b))$. Then $\tilde{a} \in \Sigma_\alpha\text{cl}(a)$ and $a \downarrow_{\tilde{a}} b$. Hence $\Sigma_\alpha\text{cl}(b) = \Sigma_\alpha\text{cl}(\tilde{b})$ by [25, Lemma 3.5.8], and $\text{tp}(\tilde{b})$ is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable either. Thus the pair \tilde{a}, \tilde{b} also forms a counterexample. Moreover, $\text{SU}(b)$ equals $\text{SU}(\tilde{b})$ modulo ω^α and $\text{SU}(b/\Sigma_\alpha\text{cl}(a)) = \text{SU}(b/\Sigma_\alpha\text{cl}(\tilde{a}))$ equals $\text{SU}(\tilde{b}/\Sigma_\alpha\text{cl}(\tilde{a}))$ modulo ω^α . \square

Since a is definable over a finite part of a Morley sequence in $\text{Lstp}(b/a)$ by supersimplicity of $\text{tp}(b)$, we see that $\text{SU}(a) < \omega^{\alpha+1}$. On the other hand, $a \notin \Sigma_\alpha\text{cl}(b)$ implies $\text{SU}(a/b) \geq \omega^\alpha$.

Let $\hat{a} \subseteq \text{bdd}(a)$ and $\hat{b} \subseteq \text{bdd}(b)$ be maximal $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable. Then $a \notin \Sigma_\alpha\text{cl}(\hat{a})$ and $b \notin \Sigma_\alpha\text{cl}(\hat{b})$, and $\text{tp}(a/\hat{a})$ and $\text{tp}(b/\hat{b})$ are foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$. Since $\text{Cb}(\hat{a}/b)$ and $\text{Cb}(\hat{b}/a)$ are $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable, we

obtain

$$a \downarrow_{\hat{a}} \hat{b} \quad \text{and} \quad b \downarrow_{\hat{b}} \hat{a}.$$

Claim. $\text{tp}(b/\hat{b})$ and $\text{tp}(a/\hat{a})$ are both Σ_α -based.

Proof of Claim: Let Φ be the family of Σ_α -based types of SU-rank ω^α . Then $\text{tp}(a/\hat{a})$ is $(\Sigma_\alpha \cup \mathcal{P}_\alpha \cup \Phi)$ -analysable, but foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$. Put $a_0 = \ell_1^\Phi(a/\hat{a})$ and $b_0 = \ell_1^\Phi(b/\hat{b})$. Then $a \sqsubseteq_{\hat{a}} a_0$ and $b \sqsubseteq_{\hat{b}} b_0$ by Lemma 3.6(3); as $a \downarrow_{\hat{a}} \hat{b}$ and $b \downarrow_{\hat{b}} \hat{a}$ we even have $a \sqsubseteq_{\hat{a}\hat{b}} a_0$ and $b \sqsubseteq_{\hat{a}\hat{b}} b_0$. Since $a \not\downarrow_{\hat{a}\hat{b}} b$ we obtain $a_0 \not\downarrow_{\hat{a}\hat{b}} b_0$. Moreover, $\text{tp}(a_0/\hat{a})$ and $\text{tp}(b_0/\hat{b})$ are Σ_α -based by Theorem 4.20 (or [26, Theorem 11]).

On the other hand, as $a_0 \not\downarrow_{\hat{b}} b_0$, we see that $b' = \text{Cb}(a_0/\Sigma_\alpha \text{cl}(b_0))$ is not contained in \hat{b} and hence is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable. So a_0, b' is another counterexample; by minimality of SU-rank b and b' have the same SU-rank modulo ω^α , whence $b \in \Sigma_\alpha \text{cl}(b_0)$. Hence $\text{tp}(b/\hat{b})$ is Σ_α -based, as is $\text{tp}(a/\hat{a})$ since $a = \text{Cb}(b/a)$ and $a \downarrow_{\hat{a}} \hat{b}$. \square

Claim. $\Sigma_\alpha \text{cl}(a, \hat{b}) = \Sigma_\alpha \text{cl}(b, \hat{a}) = \Sigma_\alpha \text{cl}(a, b)$.

Proof of Claim: As $\text{tp}(a/\hat{a})$ is Σ_α -based, we have

$$a \downarrow_{\Sigma_\alpha \text{cl}(a) \cap \Sigma_\alpha \text{cl}(\hat{a}b)} \hat{a}b,$$

whence

$$\Sigma_\alpha \text{cl}(a) \downarrow_{\Sigma_\alpha \text{cl}(a) \cap \Sigma_\alpha \text{cl}(\hat{a}b)} b$$

by Fact 3.19. Thus $a = \text{Cb}(b/\Sigma_\alpha \text{cl}(a)) \in \Sigma_\alpha \text{cl}(\hat{a}b)$. Similarly $b \in \Sigma_\alpha \text{cl}(\hat{b}a)$. \square

Let now $(b)^\wedge (b_j : j < \omega)$ be a Morley sequence in $\text{tp}(b/a)$ and let \hat{b}_j represent the part of b_j corresponding to \hat{b} . Then $(\hat{b}_j : j < \omega)$ is a Morley sequence in $\text{tp}(\hat{b}/\hat{a})$ since $a \downarrow_{\hat{a}} \hat{b}$. As $\text{SU}(\hat{b}) < \infty$ there is some minimal $m < \omega$ such that $\hat{a} = \text{Cb}(\hat{b}/\hat{a}) \in \Sigma_\alpha \text{cl}(\hat{b}, \hat{b}_j : j < m)$. Then $m > 0$, as otherwise $\Sigma_\alpha \text{cl}(b) = \Sigma_\alpha \text{cl}(\hat{a}, b) \ni a$, which is impossible. Moreover, $a \in \Sigma_\alpha \text{cl}(\hat{a}, b_j)$ for all $j < m$ by invariance and hence, $a \in \Sigma_\alpha \text{cl}(\hat{b}, b_j : j < m)$.

Put $b' = \text{Cb}(b_j : j < m/\Sigma_\alpha \text{cl}(b))$. Then $(b_j : j < m) \downarrow_{b'\hat{b}} \Sigma_\alpha \text{cl}(b)$, so by Fact 3.19

$$\Sigma_\alpha \text{cl}(\hat{b}, b_j : j < m) \downarrow_{\Sigma_\alpha \text{cl}(b', \hat{b})} \Sigma_\alpha \text{cl}(b).$$

Then $a \downarrow_{\Sigma_\alpha \text{cl}(b', \hat{b})} \Sigma_\alpha \text{cl}(b)$, so $b = \text{Cb}(a/\Sigma_\alpha \text{cl}(b)) \in \Sigma_\alpha \text{cl}(b', \hat{b})$. As $b \notin \Sigma_\alpha \text{cl}(\hat{b})$ we obtain $b' \notin \Sigma_\alpha \text{cl}(\hat{b})$.

Claim. $\text{tp}(b'/\Sigma_\alpha \text{cl}(b') \cap \Sigma_\alpha \text{cl}(b_j : j < m))$ is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable.

Proof of Claim: Note first that $(b_j : j < m) \downarrow_a b$ implies

$$\Sigma_\alpha \text{cl}(b_j : j < m) \downarrow_{\Sigma_\alpha \text{cl}(a)} \Sigma_\alpha \text{cl}(b)$$

by Fact 3.19, whence

$$\Sigma_\alpha \text{cl}(b') \cap \Sigma_\alpha \text{cl}(b_j : j < m) \subseteq \Sigma_\alpha \text{cl}(b) \cap \Sigma_\alpha \text{cl}(a) = \Sigma_\alpha \text{cl}(\emptyset).$$

As $b \in \Sigma_\alpha \text{cl}(b', \hat{b})$ and $\text{tp}(b/\hat{b})$ is not $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable, neither is $\text{tp}(b'/\hat{b})$, nor *a fortiori* $\text{tp}(b'/\Sigma_\alpha \text{cl}(\emptyset))$. \square

As $b' = \text{Cb}(b_j : j < m/\Sigma_\alpha \text{cl}(b'))$, the pair $(b_j : j < m), b'$ forms another counterexample. By minimality $\text{SU}(b)$ equals $\text{SU}(b')$ modulo ω^α , which implies $\Sigma_\alpha \text{cl}(b) = \Sigma_\alpha \text{cl}(b')$.

As $\text{tp}(b_j/\hat{b}_j)$ is foreign to $\Sigma_\alpha \cup \mathcal{P}_\alpha$ and \hat{b} is $(\Sigma_\alpha \cup \mathcal{P}_\alpha)$ -analysable, we obtain $\hat{b} \downarrow_{(\hat{b}_j : j < m)} (b_j : j < m)$ and hence by Fact 3.19

$$\hat{b} \downarrow_{\Sigma_\alpha \text{cl}(\hat{b}_j : j < m)} \Sigma_\alpha \text{cl}(b_j : j < m).$$

On the other hand, as $\hat{a} \in \Sigma_\alpha \text{cl}(\hat{b}, \hat{b}_j : j < m)$ but $\hat{a} \notin \Sigma_\alpha \text{cl}(\hat{b}_j : j < m)$ by minimality of m , we get

$$\text{SU}(\hat{b}/\Sigma_\alpha \text{cl}(\hat{b}_j : j < m)) >_\alpha \text{SU}(\hat{b}/\hat{a}, \Sigma_\alpha \text{cl}(\hat{b}_j : j < m)),$$

where the index α indicates modulo ω^α .

Moreover, as $\hat{b} \downarrow_{\hat{a}} a$ we get $\hat{b} \downarrow_{\Sigma_\alpha \text{cl}(\hat{a})} \Sigma_\alpha \text{cl}(a)$, i.e. $\text{SU}(\hat{b}/\Sigma_\alpha \text{cl}(\hat{a})) = \text{SU}(\hat{b}/\Sigma_\alpha \text{cl}(a))$. Since $\Sigma_\alpha \text{cl}(b) = \Sigma_\alpha \text{cl}(b')$ and $b \in \Sigma_\alpha \text{cl}(a\hat{b})$ we obtain

$$\begin{aligned} \text{SU}(b'/\Sigma_\alpha \text{cl}(b_j : j < m)) &=_\alpha \text{SU}(b/\Sigma_\alpha \text{cl}(b_j : j < m)) \\ &\geq_\alpha \text{SU}(\hat{b}/\Sigma_\alpha \text{cl}(b_j : j < m)) =_\alpha \text{SU}(\hat{b}/\Sigma_\alpha \text{cl}(\hat{b}_j : j < m)) \\ &>_\alpha \text{SU}(\hat{b}/\hat{a}, \Sigma_\alpha \text{cl}(\hat{b}_j : j < m)) =_\alpha \text{SU}(\hat{b}/\Sigma_\alpha \text{cl}(\hat{a})) \\ &=_\alpha \text{SU}(\hat{b}/\Sigma_\alpha \text{cl}(a)) =_\alpha \text{SU}(b/\Sigma_\alpha \text{cl}(a)), \end{aligned}$$

contradicting the maximality of $\text{SU}(b/\Sigma_\alpha \text{cl}(a))$ modulo ω^α . This finishes the proof. \square

As a corollary we obtain Chatzidakis' Theorem for the finite SU-rank case, apart from the decomposition in orthogonal components:

Corollary 5.5. *Let $b = \text{bdd}(\text{Cb}(a/b))$ be such that $\text{SU}(b) < \omega$. Then $\text{tp}(b/\text{bdd}(b) \cap \text{bdd}(a))$ is analysable in the family of all non one-based types of SU-rank 1.*

6. APPLICATIONS AND THE CANONICAL BASE PROPERTY

In this section and the next, Σ^{nob} will be the family of non one-based regular types (seen as partial types). For the applications one would like (and often has) more than mere strongly Σ^{nob} -basedness of canonical bases:

Definition 6.1. A supersimple theory T has the *Canonical Base Property CBP* if $\text{tp}(\text{Cb}(a/b)/a)$ is almost Σ^{nob} -internal for all a, b .

Remark 6.2. In other words, in view of Corollary 5.2 a theory has the CBP if for all a, b the type $\text{tp}(\text{Cb}(a/b)/a)$ is Σ^{nob} -flat.

It had been conjectured that all supersimple theories of finite rank have the CBP, but Hrushovski has constructed a counter-example [14].

Remark 6.3. Chatzidakis has shown for types of finite SU-rank that the CBP implies that even $\text{tp}(\text{Cb}(a/b)/\text{bdd}(a) \cap \text{bdd}(b))$ is almost Σ^{nob} -internal [5, Theorem 1.15].

Example. The CBP holds for types of finite rank in

- Differentially closed fields in characteristic 0 [23].
- Generic difference fields [23, 5].
- Compact complex spaces [4, 7, 22].

Moreover, in those cases we have a good knowledge of the non one-based types.

Kowalski and Pillay [16, Section 4] have given some consequences of strongly Σ -basedness in the context of groups. In fact, they work in a theory with the CBP, but they remark that their results hold, with Σ -analysable instead of *almost Σ -internal*, in all stable strongly Σ -based theories.

Fact 6.4. *Let G be an \emptyset -hyperdefinable strongly Σ -based group in a stable theory.*

- (1) *If $H \leq G$ is connected with canonical parameter c , then $\text{tp}(c)$ is Σ -analysable.*
- (2) *$G/Z(G)$ is Σ -analysable.*

An inspection of their proof shows that mere simplicity of the ambient theory is sufficient, replacing centers by approximate centers and connectivity by local connectivity. Recall that the *approximate center* of a group G is

$$\tilde{Z}(G) = \{g \in G : [G : C_G(g)] < \infty\}.$$

A subgroup $H \leq G$ is *locally connected* if for all group-theoretic or model-theoretic conjugates H^σ of H , if H and H^σ are commensurate, then $H = H^\sigma$. Locally connected subgroups and their cosets have canonical parameters; every subgroup is commensurable with a unique minimal locally connected subgroup, its *locally connected component*. For more details about the approximate notions, the reader is invited to consult [25, Definition 4.4.9 and Proposition 4.4.10].

Proposition 6.5. *Let G be an \emptyset -hyperdefinable strongly Σ -based group in a simple theory.*

- (1) *If $H \leq G$ is locally connected with canonical parameter c , then $\text{tp}(c)$ is Σ -analysable.*
- (2) *$G/\tilde{Z}(G)$ is Σ -analysable.*

Proof: (1) Take $h \in H$ generic over c and $g \in G$ generic over c, h . Let d be the canonical parameter of gH . Then $\text{tp}(gh/g, c)$ is the generic type of gH , so d is interbounded with $\text{Cb}(gh/g, c)$. By strongly Σ -basedness, $\text{tp}(d/gh)$ is Σ -analysable. But $c \in \text{dcl}(d)$, so $\text{tp}(c/gh)$ is Σ -analysable, as is $\text{tp}(c)$ since $c \perp gh$.

- (2) For generic $g \in G$ put

$$H_g = \{(x, x^g) \in G \times G : x \in G\},$$

and let H_g^{lc} be the locally connected component of H_g . Then $g\tilde{Z}(G)$ is interbounded with the canonical parameter of H_g^{lc} , so $\text{tp}(g\tilde{Z}(G))$ is Σ -analysable, as is $G/\tilde{Z}(G)$. \square

Theorem 6.6. *Let G be an \emptyset -hyperdefinable strongly Σ -based group in a simple theory. If G is supersimple or type-definable, there is a normal nilpotent \emptyset -hyperdefinable subgroup N such that G/N is almost Σ -internal. In particular, a supersimple or type-definable group G in a simple theory has a normal nilpotent hyperdefinable subgroup N such that G/N is almost Σ^{nob} -internal.*

Proof: $G/\tilde{Z}(G)$ is Σ -analysable by Proposition 6.5. Hence there is a continuous sequence

$$G = G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots \triangleright G_\alpha \triangleright \tilde{Z}(G)$$

of normal \emptyset -hyperdefinable subgroups such that successive quotients $Q_i = G_i/G_{i+1}$ are Σ -internal for all $i < \alpha$, and $G_\alpha/\tilde{Z}(G)$ is bounded.

Now G acts on every quotient Q_i . Let

$$N_i = \{g \in G : [Q_i : C_{Q_i}(g)] < \infty\}$$

be the approximate stabilizer of Q_i in G , again an \emptyset -hyperdefinable subgroup. If $(q_j : j < \kappa)$ is a long independent generic sequence in Q_i and g, g' are two elements of G which have the same action on all q_j for $j < \kappa$, there is some $j_0 < \kappa$ with $q_{j_0} \perp g, g'$. Since $g^{-1}g'$ stabilizes q_{j_0} it lies in N_i , and gN_i is determined by the sequence $(q_j, q_j^g : j < \kappa)$. Thus G/N_i is Q_i -internal, whence Σ -internal.

Put $N = \bigcap_{i < \alpha} N_i$. Since $\prod_{i < \alpha} G/N_i$ projects definably onto G/N , the latter quotient is also Σ -internal. In order to finish it now suffices to show that N is virtually nilpotent. In particular, we may assume that N is \emptyset -connected.

Consider the approximate ascending central series $\tilde{Z}_i(N)$. Note that N centralizes $G_\alpha/\tilde{Z}(G)$ by \emptyset -connectivity. Moreover, N approximately stabilizes all quotients $(G_i \cap N)/(G_{i+1} \cap N)$. Hence, if $G_{i+1} \cap N \leq \tilde{Z}_j(N)$, then $G_i \cap N \leq \tilde{Z}_{j+1}(N)$. If G is supersimple, we may assume that all the Q_i are unbounded, so α is finite and $N = \tilde{Z}_{\alpha+2}(N)$. In the type-definable case, note that $\tilde{Z}_i(N)$ is relatively \emptyset -definable by [25, Lemma 4.2.6]. So by compactness the least ordinal α_i with $G_{\alpha_i} \cap N \leq \tilde{Z}_i(N)$ must be a successor ordinal, and $\alpha_{i+1} \leq \alpha_i - 1 < \alpha_i$. Hence the sequence must stop and there is $k < \omega$ with $N = \tilde{Z}_k(N)$. But then N is nilpotent by [25, Proposition 4.4.10.3]. \square

Remark 6.7. In a similar way one can show that if G acts definably and faithfully on a Σ -analysable group H and H is supersimple or type-definable, then there is a hyperdefinable normal nilpotent subgroup $N \triangleleft G$ such that G/N is almost Σ -internal.

7. FINAL REMARKS

We have seen that for (weak) Σ -ampleness only the first level of an element is important. However, the difference between strong Σ^{nob} -basedness and the CBP is precisely the possible existence of a second (or higher) Σ^{nob} -level of $\text{Cb}(a/b)$ over a , i.e. its non Σ^{nob} -flatness.

One might be tempted to try to prove the CBP replacing Σ^{nob} -closure by Σ_1^{nob} -closure. In fact it is possible to define a corresponding notion of Σ_1 -ampleness, and to prove an analogue of Theorem 4.20. However,

since Σ_1 -closure is not a closure operator, the equivalence between Σ_1^{nob} -basedness (i.e. the CBP) and non $1\text{-}\Sigma_1^{nob}$ -ampleness breaks down. So far we have not found a way around this.

A possible approach to circumvent the failure of the CBP in general could be to use Theorem 6.6 in the applications, rather than establish the CBP for particular theories and use Fact 6.4 (or Proposition 6.5), but we have not looked into this.

Finally, it might be interesting to look for a variant of ampleness which does take all levels into account, as one might hope to obtain stronger structural consequences.

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