Arithmetic Nullstellensatz and Nonstandard Methods

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Abstract. In this study we find height bounds for polynomial rings over integral domains. We apply nonstandard methods and hence our constants will be ineffective. Furthermore we consider unique factorization domains and possible bounds for valuation rings and arithmetical functions.

1. Introduction

The arithmetic version of the Nullstellensatz states that if $f_1, ..., f_s$ belong to $\mathbb{Z}[X_1, ..., X_n]$ without a common zero in $\mathbb{C}$, then there exist a in $\mathbb{Z} \setminus \{0\}$ and $g_1, ..., g_s$ in $\mathbb{Z}[X_1, ..., X_n]$ such that $a = f_1g_1 + ... + f_sg_s$. Finding degree and height bounds for a and $g_1, ..., g_s$ has received continuous attention using computational methods. By $\deg f$, we mean the total degree of the polynomial f in several variables. T. Krick, L. M. Pardo and M. Sombra [9] prove that: If $f_1, ..., f_s$ are as above with $D := \max_i \deg(f_i)$ and $H := \max_i h(f_i)$ where $h(f_i) =$ logarithm of the maximum module of its coefficients, then there exist $a \in \mathbb{Z} \setminus \{0\}$ and $g_1, ..., g_s \in \mathbb{Z}[X_1, ..., X_n]$ such that

(i) $a = f_1g_1 + ... + f_sg_s$
(ii) $\deg(g_i) \leq 4nD^n$
(iii) $h(a), h(g_i) \leq 4n(n+1)D^n(H + \log s + (n+7)\log(n+1)D)$.

This result is sharp and efficient. For similar results we refer the reader to [1, 2].

On the other hand finding bounds in mathematics using nonstandard extensions have been studied often, for example: Given a field K, if $f_0, f_1, ..., f_s$ in $K[X_1, ..., X_n]$ all have degree less than D and f_0 in $\langle f_1, ..., f_s \rangle$, then $f_0 = \sum_{i=1}^{s} f_ih_i$ for certain h_i whose degrees are bounded by a constant $C = C(n, D)$ depending only on n and D. This result

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was first established in a paper of G. Hermann [8] using algorithmic tools. Then the same result was proved by L. van den Dries and K. Schmidt [5] using nonstandard methods, and they paved the way for how nonstandard methods can be used for such bounds. Their work in [5] influenced us to apply nonstandard methods in order to prove the existence of bounds for the complexity of the coefficients of h_i as above by taking $f_0 = 1$. We also define an abstract height function which generalizes the absolute value function and measures the complexity of the coefficients of polynomials over $R[X_1, ...X_n]$, where R is an integral domain. We will generalize the result of [9] to any integral domain and height function and furthermore our constant c_2 for the height function does not depend on R or s, but it is ineffective. We assume that all rings are commutative with unity. Moreover throughout this article R stands for an integral domain and K for its field of fractions. The symbol $h = h_R$ denotes a height function on R which will be defined in the next section. We prove the following theorem:

Main Theorem. *Let R be a ring with a height function. For all $n \geq 1$, $D \geq 1$, $H \geq 1$ there are two constants $c_1(n, D)$ and $c_2(n, D, H)$ such that if $f_1, ..., f_s$ in $R[X_1, ...X_n]$ have no common zero in K^{alg} with $\deg(f_i) \leq D$ and $h(f_i) \leq H$, then there exist nonzero a in R and $h_1, ..., h_s$ in $R[X_1, ...X_n]$ such that*

(i) $a = f_1h_1 + ... + f_sh_s$
(ii) $\deg(h_i) \leq c_1$
(iii) $h(a), h(h_i) \leq c_2$.

2. Preliminaries

2.1. **Height Function.** Let $\theta : \mathbb{N} \to \mathbb{N}$ be a function. We say that

$$h : R \to [0, \infty)$$

is a height function of θ-type if for any x and y in R with $h(x) \leq n$ and $h(y) \leq n$, then both $h(x + y) \leq \theta(n)$ and $h(xy) \leq \theta(n)$. We say that h is a height function on R if h is a height function of θ-type for some $\theta : \mathbb{N} \to \mathbb{N}$.

We can extend the height function h to the polynomial ring $R[X_1, ...X_n]$ by

$$h\left(\sum_{\alpha} a_\alpha X^\alpha\right) = \max_{\alpha} h(a_\alpha).$$

Note that this extension does not have to be a height function, it is just an extension of functions. Now we give some examples of height functions.

Examples: For the following examples of height functions, one can take $\theta(n) = (n+1)^2$.

- If $(R, |\cdot|)$ is an absolute valued ring then $h(x) = |x|$ is a height function. Moreover $h(x) = |x| + 1$ and $h(x) = \max(1, |x|)$ are also height functions on R.
- The degree function on $R[X_1, ..., X_n]$ is a height function.
- Let O be the set of algebraic integers. For α in O we define its Mahler measure as

  $$M(\alpha) = \prod_{|\alpha_j| \geq 1} |\alpha_j|$$

  where α_j is a Galois conjugate of α. Then the logarithmic height

  $$h(\alpha) = \frac{\log M(\alpha)}{d}$$

  is a height function on O, where $d = \deg(\alpha)$. It is not known whether there exists an absolute constant $c > 1$ such that if $M(\alpha) > 1$ then $M(\alpha) \geq c$. This question was posed by D. Lehmer [10] around 1933. The best known example of the smallest Mahler measure greater than 1 so far was also given by Lehmer: if α is a root of the polynomial

  $$X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$$

  then $M(\alpha) \approx 1.17628$. For detailed results on Mahler measure and Lehmer's problem, see [14].
- Let λ be a positive real number. On $\mathbb{Z}[X]$, define

  $$h(a_0 + a_1 X + ... + a_k X^k) = \sum_{i=0}^{k} |a_i| \lambda^i.$$

  Then this is a height function on $\mathbb{Z}[X]$.

- Let $h : R \to [0, \infty)$ be a function such that the sets
$$A_n = \{x \in R : h(x) \leq n\}$$
are all finite for all $n \geq 1$. Then h is a height function of θ-type where $\theta(n) = \max_{x,y \in A_n} \{h(x + y) + h(xy)\}$.
- The p-adic valuation on $\mathbb{Z}$ is not a height function. Note that 1 and $p^n - 1$ are not divisible by p but but their sum is divisible by p^n.

2.2. **Nonstandard Extensions and Height Function.** Now we define a nonstandard extension following [7].

Definition 2.1 (Nonstandard Extension of a Set)**.** Let $\mathbb{M}$ be a nonempty set. A nonstandard extension of $\mathbb{M}$ consists of a mapping that assigns a set ${}^*A$ to each A in $\mathbb{M}^m$ for all $m \geq 0$, such that ${}^*\mathbb{M}$ is non-empty and the following conditions are satisfied for all $m, n \geq 0$:

(E1) The mapping preserves Boolean operations on subsets of $\mathbb{M}^m$: if $A \subseteq \mathbb{M}^m$, then ${}^*A \subseteq ({}^*\mathbb{M})^m$; if $A; B \subseteq \mathbb{M}^m$, then ${}^*(A \cup B) = {}^*A \cup {}^*B$, ${}^*(A \cap B) = {}^*A \cap {}^*B$ and ${}^*(A \setminus B) = {}^*A \setminus {}^*B$.

(E2) The mapping preserves basic diagonals: if $1 \leq i < j \leq m$ and $\Delta = \{(x_1, ..., x_m) \in \mathbb{M}^m : x_i = x_j\}$ then ${}^*\Delta = \{(x_1, ..., x_m) \in ({}^*\mathbb{M})^m : x_i = x_j\}$.

(E3) The mapping preserves Cartesian products: if $A \subseteq \mathbb{M}^m$ and $B \subseteq \mathbb{M}^n$, then ${}^*(A \times B) = {}^*A \times {}^*B$. (We regard $A \times B$ as a subset of $\mathbb{M}^{m+n}$.)

(E4) The mapping preserves projections that omit the final coordinate: let π denote projection of $n+1$-tuples on the first n coordinates; if $A \subseteq \mathbb{M}^{n+1}$ then ${}^*(\pi(A)) = \pi({}^*A)$.

The set ${}^*\mathbb{M}$ will denote the nonstandard extension of $\mathbb{M}$. For example, an ultrapower of $\mathbb{M}$ which respect to a nonprincipal ultrafilter on $\mathbb{N}$ is a proper nonstandard extension of $\mathbb{M}$. Subsets of ${}^*\mathbb{M}$ of the form ${}^*A$ for some subset A of $\mathbb{M}$ are called internal. Not every subset of ${}^*\mathbb{M}$ need to be internal. We list the basic properties of nonstandard extensions with no proof.

- For each $n \geq 0$, ${}^*(\mathbb{M}^n) = ({}^*\mathbb{M})^n$ and ${}^*\emptyset = \emptyset$.
- For any $A, B \subseteq \mathbb{M}^n$, ${}^*A = {}^*B$ iff $A = B$.

- For each $x \in \mathbb{M}$, the set ${}^*\{x\}$ has exactly one element.
  Using the properties above, we can embed $\mathbb{M}$ into ${}^*\mathbb{M}$. So without loss of generality we may assume that $\mathbb{M}$ is a subset of ${}^*\mathbb{M}$. Moreover, if $A \subseteq \mathbb{M}^n$ then ${}^*A \cap \mathbb{M}^n = A^n$, in particular, $A \subseteq {}^*A$. Also every function on A extends to a function on ${}^*A$. The new function is denoted by ${}^*f$, but without confusion we write f instead. Lastly we give the most important property of nonstandard extensions.
- **Transfer formula:** The two sets $\mathbb{M}$ and ${}^*\mathbb{M}$ satisfy the same first order sentences. Moreover if $\phi(v_1, ..., v_n)$ is a formula over $\mathbb{M}$ and $B = \{(x_1, ..., x_m) \in \mathbb{M}^n : \phi(x_1, ..., x_n) \text{ is true in } \mathbb{M}^n\}$ then ${}^*B = \{(x_1, ..., x_m) \in {}^*\mathbb{M}^n : {}^*\phi(x_1, ..., x_n) \text{ is true in } {}^*\mathbb{M}^n\}$, where ${}^*\phi(v_1, ..., v_n)$ is the corresponding formula of $\phi(v_1, ..., v_n)$.

The notion of a nonstandard extension and its properties can be generalized to many sorted structures. This will be significant for the definition of the height function which takes values in $\mathbb{R}$. By a structure we mean a set equipped with some functions and relations on it. For example, a ring is a structure with addition and multiplication. A subset of a structure $\mathbb{M}$ which is given by a first order formula is called a definable subset of $\mathbb{M}$. We say that a structure $\mathbb{M}$ is $\aleph_1$-saturated if whenever a collection of definable subsets $(A_i)_{i\in I}$ whose parameters come from a countable set satisfies the finite intersection property (that means for any finite subset I_0 of I we have $\bigcap_{i\in I_0} A_i$ is not empty) then $\bigcap_{i\in I} A_i$ is not empty.
We assume all nonstandard extensions are $\aleph_1$-saturated. Let

$${}^*(K[X_1, ...X_n])$$

be a proper nonstandard extension of $K[X_1, ...X_n]$. For instance an ultrapower of $K[X_1, ...X_n]$ which respect to a nonprincipal ultrafilter on $\mathbb{N}$ is $\aleph_1$-saturated. Ultraproducts of structures automatically become $\aleph_1$-saturated. Note that ${}^*(R[X_1, ...X_n])$, ${}^*R$ and ${}^*K$ are internal sets. The height function h on $R[X_1, ...X_n]$ extends to ${}^*(R[X_1, ...X_n])$ which takes values in ${}^*\mathbb{R}$ though this extension is no longer a height function if h is unbounded. Moreover it satisfies the same first order properties as h. In particular if x, y in ${}^*R$ with $h(x) \leq n$ and $h(y) \leq n$, where $n \in {}^*\mathbb{N}$, then we have both $h(x + y) \leq \theta(n)$ and $h(xy) \leq \theta(n)$. Note

that ${}^*K[X_1, ...X_n] \subsetneq {}^*(K[X_1, ...X_n])$. Define

$$R_{fin} = \{x \in {}^*R : h(x) \in \mathbb{R}_{fin}\}$$

where $\mathbb{R}_{fin} = \{x \in {}^*\mathbb{R} : |x| < n \text{ for some } n \in \mathbb{N}\}$ and ${}^*\mathbb{R}$ is a nonstandard extension of $\mathbb{R}$. The elements in ${}^*\mathbb{R} \setminus \mathbb{R}$ are called infinite.

By the properties of the height function, if there is a height function on R, we see that R_{fin} is a subring of ${}^*R$ and it contains R. The next lemma shows when R_{fin} is internal.

Lemma 2.2. *The set R_{fin} is an internal subset of ${}^*R$ if and only if the height function on R is bounded.*

Proof. Suppose $R_{fin} = {}^*A$ for some subset A of R. First we show that the height function on A must be bounded. To see this, if there is a sequence $(a_n)_n$ in A such that $\lim_{n\to\infty} h(a_n) = \infty$, then by saturation there is an element in ${}^*A$ whose height is infinite. This contradicts the fact that all the elements in R_{fin} have bounded height. So the height function on A is bounded. Therefore the height function on ${}^*A$ is also bounded. However since R_{fin} contains R, the height function on R must be bounded. Conversely if the height function on R is bounded, then we have $R_{fin} = {}^*R$ and so R_{fin} is internal. □

Now we fix some more notations. Put $L = Frac(R_{fin})$ which is a subfield of ${}^*K$. Note that ${}^*K$ is the fraction field of ${}^*R$. Also we fix some algebraic closure K^{alg} of K.

For more detailed information about Nonstandard Analysis and Model Theory, the reader might consult [6], [7] and [11]. In fact of being a height function is very related to the set R_{fin}. The following proposition is the nonstandard point of view definition of a height function. However it is ineffective, i.e. it does not provide the θ-type of the height function.

Proposition 2.3. *A function $h : R \to [0, \infty)$ is a height function on R if and only if R_{fin} is a subring of ${}^*R$.*

Proof. We have seen that if h is a height function then R_{fin} is a subring. Conversely suppose R_{fin} is a subring and h is not a height function.

This means there is some $N \in \mathbb{N}$ such that we have two sequences (r_n) and (s_n) in R with $h(r_n) \leq N$ and $h(s_n) \leq N$, but $\lim_{n\to\infty} h(r_n \star s_n) = \infty$, where the binary operation $\star$ means either addition or multiplication. By saturation, we get two elements r and s in ${}^*R$ such that $h(r) \leq N$, $h(s) \leq N$ but $h(r \star s)$ is infinite. This contradicts the fact that R_{fin} is a subring. □

2.3. **Faithfulness and degree bounds.** In this subsection, we list some results from commutative algebra and in particular about faithful extension of modules. We refer the reader to [3], [12] or [13]. Moreover we give the results in [5] that lead to the existence of the constant c_1.

Lemma 2.4. *Let F be a field and $f_1, ..., f_s \in F[X_1, ...X_n]$. Then $1 \in \langle f_1, ..., f_s\rangle$ if and only if $f_1, ..., f_s$ have no common zeros in F^{alg}.*

Proof. $\Rightarrow$: Clear.
$\Longleftarrow$: By Hilbert's Nullstellensatz, there are $g_1, ..., g_s \in F^{alg}[X_1, ..., X_n]$ such that $1 = f_1g_1 + ... + f_sg_s$. This is a linear system when we consider the coefficients of all the polynomials. Therefore $1 = f_1Y_1 + ... + f_sY_s$ has a solution in F^{alg}. Now by the Gauss-Jordan Theorem, this linear system has a solution in F. So there are $h_1, ..., h_s \in F[X_1, ..., X_n]$ such that

$$1 = f_1h_1 + ... + f_sh_s.$$

□

Definition 2.5. Let A and B be two rings and $A \subseteq B$. We say that B is a faithful extension of A, if the ideal BI is proper in B whenever $I \subset A$ is a proper ideal.

Lemma 2.6. *Let A and B be two rings. Suppose $A \subseteq B$ and B is a faithful extension of A. If $a, a_1, ..., a_k$ are in A and the linear equation*

$$a_1x_1 + ... + a_kx_k = a$$

has a solution in B, then it has a solution in A.

Lemma 2.7. *Let $F \subseteq F_1$ be a field extension. Then the extension $F[X_1, ...X_n] \subseteq F_1[X_1, ...X_n]$ is faithful.*

Proof. Let $I \subset F[X_1, ...X_n]$ be a proper ideal. Then since I is finitely generated, $I = \langle f_1, ...f_s\rangle$ for some $f_1, ..., f_s \in F[X_1, ...X_n]$. By (2.4),

$f_1, ..., f_s$ have a common zero in F^{alg}. Since we may assume $F^{alg} \subseteq {F_1}^{alg}$, there is a common zero of $f_1, ..., f_s$ in ${F_1}^{alg}$. So by (2.4) again, $IF_1[X_1, ...X_n] \neq F_1[X_1, ...X_n]$. □

For the following Lemma see [5, 1.8].

Lemma 2.8. *The extension* ${}^*K[X_1, ...X_n] \subset {}^*(K[X_1, ...X_n])$ *is faithful.*

Using (2.8), we can obtain the existence of the constant c_1. The original proof in [5] also uses the concept of flatness to prove the existence of the constant c_1. For the details see [5, 1.11].

Theorem 2.9. *If* $f_0, f_1, ..., f_s$ *in* $K[X_1, ..., X_n]$ *all have degree less than* D *and* f_0 *is in* $\langle f_1, ..., f_s \rangle$*, then* $f_0 = \sum_{i=1}^{s} f_i h_i$ *for certain* h_i *whose degrees are bounded by a constant* $c_1 = c_1(n, D)$ *depending only on* n *and* D*.*

2.4. **UFD with the p-property.**

Definition 2.10. We say that R is a UFD with the p-property if R is an unique factorization domain endowed with an absolute value such that every unit has absolute value 1 and if there are primes p and q satisfying

$$|p| < 1 < |q|,$$

then there is another prime r non-associated to p with $|r| < 1$.

Examples

- $\mathbb{Z}$ is a UFD with the p-property whose primes have absolute value bigger than 1.
- $\mathbb{Z}_p$ (p-adic integers) is a UFD with the p-property whose only prime has absolute value $1/p$.
- Let $\gamma \in (0, 1)$ be a transcendental number. Then the ring $S = \mathbb{Z}[\gamma]$ is a unique factorization domain since it is isomorphic to $\mathbb{Z}[X]$ and its units are only 1 and -1. We put the usual absolute value on S. Then S has infinitely many primes p with $|p| < 1$ and infinitely many primes q with $|q| > 1$. So S is a UFD with the p-property.

Lemma 2.11. *Suppose R is a UFD with the p-property. If there are primes p and q with $|p| < 1 < |q|$, then there are infinitely many non-associated primes with absolute value strictly less than 1 and infinitely many non-associated primes with absolute value strictly bigger than 1.*

Proof. We know there are at least two non-associated primes with absolute value less than 1. Let $p_1, ..., p_k$ (for $k \geq 2$) be non-associated primes with absolute value less than 1. Put $A = p_1...p_k$. Now choose m large enough such that $\left| \sum_{i=1}^{k} (A/p_i)^m \right| < 1$. Since this element is not a unit, it must be divisible by a prime whose absolute value strictly less than 1. This gives us a new prime. For the second part, given $q_1, ..., q_k$ primes of absolute value larger than 1, for large n the element ${q_1}^n q_2...q_k + 1$ provides a new prime that has absolute value greater than 1. □

3. Proof of the Main Theorem

In this section we will give the proof of the Main Theorem.

Main Theorem. *Let R be a ring with a height function of θ-type. For all $n \geq 1$, $D \geq 1$, $H \geq 1$ there are two constants $c_1(n, D)$ and $c_2(n, D, H, \theta)$ such that if $f_1, ..., f_s$ in $R[X_1, ...X_n]$ have no common zero in K^{alg} with $\deg(f_i) \leq D$ and $h(f_i) \leq H$, then there exist nonzero a in R and $h_1, ..., h_s$ in $R[X_1, ...X_n]$ such that*

(i) $a = f_1h_1 + ... + f_sh_s$
(ii) $\deg(h_i) \leq c_1$
(iii) $h(a), h(h_i) \leq c_2$
(iv) *If R is a UFD with the p-property and $h(x) = |x|$ is the absolute value on R, then we can choose a such that $\gcd(a, a_1, ..., a_m) = 1$ where $a_1, ..., a_m$ are all elements that occur as some coefficient of some h_i.*

Remark 3.1. The constant c_1 does not depend on s because the vector space

$$V(n, D) = \{f \in K[X_1, ...X_n] : \deg(f) \leq D\}$$

is finite dimensional over K. In fact the dimension is $q(n,D)=\binom{n+D}{n}$. Given $1=f_1h_1+...+f_sh_s$, we may always assume $s\leq q=q(n,D)$ because if $s>q$ then $f_1,...,f_s\in V(n,D)$ are linearly dependent over K. Assume first that $r\leq q$ many of them are linearly independent. Therefore the other terms $f_{r+1},...,f_s$ can be written as a linear combination of $f_1,...,f_r$ over K. Thus the equation $1=f_1h_1+...+f_sh_s$ may be transformed into another equation $1=f_1g_1+...+f_rg_r$. Consequently if $1\in\langle f_1,...,f_s\rangle$, then $1\in\langle f_{i_1},...,f_{i_r}\rangle$ where $r\leq q$ and $i_j\in\{1,...,s\}$. Hence, we can always assume $s=q$. Similarly the constant c_2 does not depend on s. Moreover, none of the constants depend on R.

Remark 3.2. If R is a ring with absolute value which has arbitrarily small nonzero elements, then we can multiply both sides of the equation

$$a=f_1h_1+...+f_sh_s$$

by some small $\epsilon\in R$. Therefore the height bound c_2 can be taken 1 and the result becomes trivial. Note that (iv) in the Main Theorem prevents us from doing this if there are no small units in R. However if there is a unit u with $|u|<1$, then multiplying both sides of the equation with powers of u the height can be made small again. So the interesting case is when there are no small units which is equivalent to all the units having absolute value 1. Note also that if $|ab|<1$ then $|a|$ can be very big and $|b|$ can be very small. So cancellation can make the height larger if there are sufficiently small and big elements in the ring. Thus for the equation

$$a=f_1h_1+...+f_sh_s,$$

simply dividing by $\gcd(a,a_1,...,a_m)$ may not work in order to obtain (iv) in the Main Theorem.

Proof of the Main Theorem: By Theorem 2.9, the constant c_1 exists and it only depends on n and D. Now we prove the existence of the constant c_2. Assume n, D and H are given and there is no bound c_2. Therefore for every $m\geq 1$ there exists an integral domain R_m with a height function ht_m of θ-type and $f_1,...,f_s$ in $R_m[X_1,...X_n]$ with $\deg f_i\leq D$ and $ht_m(f_i)\leq H$ witnessing to this. Thus in the field of fractions K_m of R_m, there exist $g_1,...,g_s$ in $K_m[X_1,...,X_m]$ with

$\deg g_i \leq c_1$ and

$$1 = f_1 h_1 + ... + f_s g_s,$$

but for all $h_1, ..., h_s \in K_m[X_1, ..., X_n]$ with $\deg h_i \leq c_1$,

$$1 = f_1 h_1 + ... + f_s h_s$$

implies $\max_j ht_m(a_j) > m$ where $a_j \in R_m$ is an element that occurs as a numerator or denominator of some h_i. Set

$$V_m(n, A) = \{f \in K_m[X_1, ..., X_n] : \deg(f) \leq A\}$$

where $A \in \mathbb{N}$. By Remark 3.1, this is a finite dimensional vector space over K and the dimension is $q(n, A)$. So we can consider its elements as a finite tuple over K_m and any element of $V_m(n, A)$ is of the form $(a_1, ..., a_{q(n,A)})$.

Also put $V_{R_m}(n, A) := V_m(n, A) \cap R_m[X_1, ..., X_n]$. Note that the height of a tuple in $V_{R_m}(n, A)$ is the maximum of its coordinates. Our language contains a symbol h for the height function and also the ring operations. Consider the formula $\phi_m(v_1, ..., v_s)$:

$$\exists a_1^1 \exists b_1^1 ... \exists a_{q(n,c_1)}^1 \exists b_{q(n,c_1)}^1 ... \exists a_1^s \exists b_1^s ... \exists a_{q(n,c_1)}^s \exists b_{q(n,c_1)}^s$$

$$\left(\left(\bigwedge_{i=1}^{s} (h(v_i) \leq H) \right) \wedge \left(1 = \sum_{i=1}^{s} v_i \left(\frac{a_1^i}{b_1^i}, ..., \frac{a_{q(n,c_1)}^i}{b_{q(n,c_1)}^i} \right) \right) \right)$$

$$\wedge \Bigg(\forall d_1^1 \forall e_1^1 ... \forall d_{q(n,c_1)}^1 \forall e_{q(n,c_1)}^1 ... \forall d_1^s \forall e_1^s ... \forall d_{q(n,c_1)}^s \forall e_{q(n,c_1)}^s$$

$$\left(\left(\bigwedge_i (\max_j (h(d_j^i), h(e_j^i)) \leq m) \right) \rightarrow \left(1 \neq \sum_{i=1}^{s} v_i \left(\frac{d_1^i}{e_1^i}, ..., \frac{d_{q(n,c_1)}^i}{e_{q(n,c_1)}^i} \right) \right) \right).$$

Note that this formula can be seen as a formula in R_m by seeing each v_i as the tuple of variables representing the polynomial f_i. We see that $R_m \models \psi_m$ where $\psi_m = \exists v_1 ... \exists v_s \phi_m(v_1, ..., v_s)$. By compactness there is an integral domain R with a height function h_R of θ-type that satisfies all ψ_m. Now we consider the set of formulas

$$p(v_1, ..., v_s) = \{\phi_m(v_1, ..., v_s) : m = 1, 2, 3...\}.$$

This is a set of formulas over ${}^*V_R(n, D)$ using countably many parameters. This set is finitely consistent, so by saturation there is a

realization $f_1, ..., f_s$ in ${}^*(R[X_1, ..., X_n])$. But according to $p(v_1, ..., v_s)$, the polynomials $f_1, ..., f_s$ are in $R_{fin}[X_1, ..., X_n]$ and their degrees are less than D. Furthermore the linear system

$$f_1Y_1 + ... + f_sY_s = 1$$

has a solution in ${}^*K[X_1, ..., X_n]$ (because bounded degree polynomials of ${}^*(K[X_1, ..., X_n])$ are in ${}^*K[X_1, ..., X_n]$) but not in $L[X_1, ..., X_n]$. This contradicts (2.6) because the extension $L[X_1, ..., X_n] \subset {}^*K[X_1, ..., X_n]$ is faithful by (2.7).

Hence we know that given $f_1, ...f_s \in R[X_1, .., X_n]$ with no common zeros in K^{alg} with $\deg(f_i) \leq D$ and $h(f_i) \leq H$, there are $h_1, ..., h_s$ in $K[X_1, ...X_n]$ such that $1 = f_1h_1 + ... + f_sh_s$ and $\deg(h_i) \leq c_1(n, D)$. Moreover $s \leq q(n, D)$ and $h(e) \leq c_3(n, D, H, \theta)$ where $e \in R$ is an element which occurs as a numerator or denominator for some coefficient of some h_i. Let $b_1, ..., b_m$ be all the elements in R that occur as a denominator for some coefficient of some h_i. Note that $m = m(n, D) \leq q^2$ depends on n and D only. Also we know that $h(b_i) \leq c_3$. Put

$$a = b_1...b_m.$$

By the multiplicative properties of the height function, we get $h(a) \leq c_4(n, D, H, \theta)$ for some c_4. Now we see that

$$a = \sum_{i=1}^{s} f_i(ah_i),$$

f_i and ah_i are in $R[X_1, .., X_n]$ and $\deg(ah_i) = \deg(h_i) \leq c_1$. Moreover, again by the multiplicative properties of the height function, we have $h(ah_i) \leq c_5(n, D, H, \theta)$. Now take $c_2 = \max(c_4, c_5)$. Therefore we obtain (i), (ii) and (iii).

Now we prove (iv). Assume R is a UFD with the p-property. We need to choose a such that $\gcd(a, a_1, ..., a_m) = 1$ where $a_1, ..., a_m$ are all elements that occur as some coefficient of some h_i. If all the primes in R have absolute value bigger than 1 or smaller than 1, then we can divide both sides of the equation

$$a = f_1h_1 + f_2h_2 + ... + f_sh_s$$

by $\gcd(a, a_1, ..., a_m)$ and get the result because if all the primes in R have absolute value bigger than 1, then cancellation makes the height

smaller and if all the primes in R have absolute value less than 1 then height is bounded by 1. The remaining case is when there are primes of absolute value bigger than 1 and primes of absolute value smaller than 1. By (2.11), there are infinitely many primes with absolute value strictly less than 1. Now choose a prime p such that $|p| < 1$ and p does not divide a. Let d be the greatest common divisor of all coefficients of f_1 and f_2. Then, the coefficients of f_1/d and f_2/d have no common divisor. On the other hand, since there are both small and large elements in the ring, d can be very small and so f_1/d and f_2/d may have very large absolute values. Thus choose a natural number k such that $p^k f_1/d$ and $p^k f_2/d$ have absolute value less than 1. Put $v = c_1(n, D) + 1$. Then we have

$$0 = f_1(X_1{}^v p^k f_2/d) + f_2(-X_1{}^v p^k f_1/d).$$

Therefore we obtain that

$$a = f_1(h_1 + X_1{}^v p^k f_2/d) + f_2(h_2 - X_1{}^v p^k f_1/d) + ... + f_s h_s$$

$$= f_1 g_1 + f_2 g_2 + ... + f_s g_s$$

where $\deg g_i \leq D(c_1 + 1) = c(n, D)$ and $h(g_i) \leq c_2$. Observe that $\gcd(a, a_1, ..., a_m) = 1$ where $a_1, ..., a_m$ are all elements that occur as some coefficient of some g_i. □

4. Concluding Remarks and Further Discussion of the Main Theorem

In this section we discuss the Main Theorem in terms of unique factorization domains, valuations and some arithmetical functions. Also we give some counter examples for the Main Theorem for non-height functions.

4.1. **UFD with the 1-property.**

Definition 4.1. We say that R is a UFD with the 1-property if R is an unique factorization domain endowed with an absolute value such that every unit has absolute value 1 and there is only one prime p of absolute value less than 1 and infinitely many primes q of absolute value greater than 1.

Example: Let R be an unique factorization domain and p be a prime in R. Put the p-adic absolute value on R with $|p|_p = 1/2$. Let $c > 1$ be any real number. On $R[X]$ we define

$$|a_0 + a_1X + ... + a_kX^k| = \max_i c^i|a_i|_p.$$

Then $R[X]$ is a UFD with the 1-property whose only small prime is p.

We proved the Main Theorem for UFD with the p-property. Thus the remaining case is when R is a UFD with the 1-property.

Proposition 4.2. *Let R be a UFD with the 1-property. Then we cannot ensure the correctness of (iii) and (iv) simultaneously in the Main Theorem.*

Proof. Let p be the unique small prime in R of absolute value less than 1. Let B be an element in R of absolute value very big which is coprime to p. Choose m minimal such that $|p^mB| \leq c_2$. Set $f_1 = p^{2m+1} + p^{2m}X$ and $f_2 = p^mB - p^mBX$. Clearly f_1 and f_2 have no common zero since

$$p^{2m}B(p + 1) = Bf_1 + p^mf_2$$

and p is not -1. Whenever we write $a = f_1h_1 + f_2h_2$, we get that p^m divides h_2 and B divides h_1. Also we have that $p^{2m}B$ divides a. Moreover if q is coprime to pB which divides h_1 then q also divides h_2, so it divides a. Now suppose $|h_i| \leq c_2$ for $i = 1, 2$. Furthermore we may assume all the prime divisors of a, h_1 and h_2 are among the prime divisors of pB. Since B divides h_1, we see that p^m divides h_1 since p is the unique small prime in R. Thus p^m divides a, h_1 and h_2. Therefore, in order to satisfy (iv) in the Main Theorem, we need to divide a, h_1 and h_2 by p^m. So the absolute value of h_1/p^m becomes very large.

□

4.2. **Valuations.**

Definition 4.3. A valuation v on an integral domain R is a function $v : R \to \Gamma \cup \{\infty\}$ from R into an ordered abelian group Γ that satisfies the followings:

(i) $v(a) = \infty$ if and only if $a = 0$
(ii) $v(xy) = v(x) + v(y)$
(iii) $v(x + y) \geq \min(v(x), v(y))$.

Here ∞ is some element that is bigger than every element in Γ.

For a nonzero polynomial in n-variable we define its valuation as follows:

$$v\left(\sum_{\alpha} a_\alpha X^\alpha\right) = \max_{\alpha}\{v(a_\alpha) : a_\alpha \neq 0\}.$$

Note that this may not be a valuation that satisfies the three conditions above. Take $R = \mathbb{Z}$ and as a valuation we put a p-adic valuation for some prime p. Set $f_1 = 1+X+(1-p^m)X^2$ and $f_2 = X^3$ where m is some large integer. Then the valuations of f_1 and f_2 are 0 and clearly they have no common zero in $\mathbb{C}$. One can see that 1 is a linear combination of f_1 and f_2 and so every integer is. However, whenever we write $a = f_1h_1 + f_2h_2$ where a is nonzero, then h_1 must have degree bigger than 2 and the first three coefficients of h_1 are uniquely determined: if $h_1(x) = b_0 + b_1X + b_2X^2 + ... + b_kX^k$ then automatically we have $b_0 = a$, $b_1 = -a$ and $b_2 = ap^m$. So the valuation of b_2 can be very large. The main nonstandard reason behind this is the fact that

$$R_{vfin} = \{x \in {}^*R : v(x) \in \mathbb{R}_{fin}\} \cup \{0\}$$

is not a ring, because for nonstandard $N \in {}^*\mathbb{N}$ the elements $p^N - 1$ and 1 is in R_{vfin} but not their sum. Therefore by (2.3), we know that the p-adic valuation on $\mathbb{Z}$ is not a height function.

If we take $g_1 = p^m - 1 + X$ and $g_2 = 1 - X$ then they have no common zero and whenever we write $a = g_1h_1 + g_2h_2$, then h_1 and h_2 must have the same degree and same leading coefficient. This implies that p^m divides a which means that valuation of a can be very big even if the valuations of g_1 and g_2 are 0.

A valuation is called trivial if for all nonzero x we have $v(x) = 0$. We say that a valuation is a height function if the set R_{vfin} is a subring. In fact we can determine when a valuation is a height function.

Lemma 4.4. *A valuation v on R is a height function if and only if it is trivial.*

Proof. If the valuation is trivial then clearly it is a height function. Conversely is v is not trivial, then it is unbounded. So by saturation

there is an element a in ${}^*R$ whose valuation is unbounded. Then

$$v(a-1)=0$$

because if two elements have different valuation then the valuation of their sum is the minimum of their valuations. So the elements $a-1$ and 1 are in R_{vfin}, but not their sum. □

4.3. **Arithmetical Functions.** Now we discuss some arithmetical functions and which of them are height functions.

Definition 4.5. A function $g:\{1,2,3,...\}\to\mathbb{C}$ is called an arithmetical function.

Every arithmetical function g extends to $\mathbb{Z}$ by defining $g(n)=g(-n)$ and $g(0)=0$. Such a function on $\mathbb{Z}$ is called an arithmetical function on $\mathbb{Z}$. Similarly for an an arithmetical function g on $\mathbb{Z}$, we extend it to $\mathbb{Z}[X]$ by

$$g(a_0+a_1X+...+a_kX^k)=\max_i g(a_i).$$

Let ${}^*\mathbb{Z}$ be a proper nonstandard extension of $\mathbb{Z}$. Note that

$$\mathbb{Z}_{fin}=\{x\in{}^*\mathbb{Z}:|x|<n \text{ for some } n\in\mathbb{N}\}=\mathbb{Z}.$$

For an arithmetical function g, we define

$$\mathbb{Z}_{gfin}=\{x\in{}^*\mathbb{Z}:|g(x)|<n \text{ for some } n\in\mathbb{N}\}.$$

By (2.3), $|g|$ is a height function if and only if $\mathbb{Z}_{gfin}$ is a subring. Now we give some examples of arithmetical functions.

Examples:

- $\varphi(n)=|\{1\le k\le n:(k,n)=1\}|$
- $\pi(n)=$ number of primes less than n
- $d(n)=$ number of divisors of n
- $\omega(n)=$ number of distinct prime factors of n.

Lemma 4.6. *Let g be an arithmetical function and assume that*

$$\lim_{n\to\infty} g(n)=\infty.$$

Then $|g|$ is a height function.

Proof. If N is an infinite number in ${}^*\mathbb{Z}$ then $g(N)$ is also infinite. This shows that $\mathbb{Z}_{gfin}=\mathbb{Z}_{fin}=\mathbb{Z}$ which is a subring of ${}^*\mathbb{Z}$. Hence by (2.3), $|g|$ is a height function on $\mathbb{Z}$. □

Lemma 4.7. *For all $n \geq 1$, we have $\frac{\sqrt{n}}{2} \leq \varphi(n)$.*

Proof. Since $\varphi(n) = \prod_{p|n} n(1 - \frac{1}{p})$, we get $\varphi(n) \geq \frac{n}{2^{\omega(n)}} \geq \frac{n}{d(n)}$. Finally since $d(n) \leq 2\sqrt{n}$, we get the result. □

Corollary 4.8. *The functions $\pi(n)$ and $\varphi(n)$ are height functions.*

Proof. Since there are infinitely many primes and $\frac{\sqrt{n}}{2} \leq \varphi(n)$, these two functions are height functions. □

For the other two functions $d(n)$ and $\omega(n)$, they take small values when n is a prime number.

Fact: Every sufficiently large odd integer can be written as a sum of three primes. This was proved by I. M. Vinogradov. For more about this theorem, we refer the reader to [4].

Lemma 4.9. *The functions $d(n)$ and $\omega(n)$ are not height functions.*

Proof. By three primes theorem and the transfer formula, there is an odd infinite N in ${}^*\mathbb{Z}$ which can be written as a sum of three primes in ${}^*\mathbb{P}$ where $\mathbb{P}$ is the set of all primes. Furthermore we can choose N such that $\omega(N)$ is infinite. This shows that the sets $\mathbb{Z}_{\omega fin}$ and $\mathbb{Z}_{dfin}$ are not closed under addition. So by (2.3), they cannot be height functions on $\mathbb{Z}$. □

The next two Corollaries are also true for the function $\omega(n)$. For simplicity, we just give the proofs for the divisor function.

Corollary 4.10. *There exist a natural number A and two sequences $\{a_n\}$ and $\{b_n\}$ in $\mathbb{N}$ such that $d(a_n) \leq A$ and $d(b_n) \leq A$ but*

$$\lim_{n\to\infty} d(a_n + b_n) = \infty.$$

Corollary 4.11. *The Main Theorem is not true for the function $d(n)$.*

Proof. Set $f_1 = a_n + X + {b_n}^2 X^2$ and $f_2 = X^3$ where a_n and b_n are as in (4.10). Then $d(f_1)$ and $d(f_2)$ are bounden by A^2 and they have no common zero in $\mathbb{C}$. However, whenever we write $a = f_1 h_1 + f_2 h_2$ where a is nonzero, then h_1 must have degree bigger than 2 and the first three coefficients of h_1 are uniquely determined: if $h_1(x) = c_0 + c_1 X + c_2 X^2 + ... + c_k X^k$ then automatically we have $c_0 = a$, $c_1 = -a_n a$

and $c_2 = a(a_n - b_n)(a_n + b_n)$. Hence $d(c_2)$ can be very large. Moreover if we put $g_1 = a_n + X$ and $g_2 = b_n - X$ then they have no common zero. However, whenever we write $a = g_1h_1 + g_2h_2$, then $d(a) \geq d(a_n + b_n)$. Thus a has many divisors although $d(g_1)$ and $d(g_2)$ are bounded by A. □

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