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Abstract

In this contribution we investigate the problem of simultaneous observer based sensor fault reconstruction and state estimation of a class of linear time-varying (LTV) systems that are skew-symmetric models. The main features concern the use of a bank of observers to detect and isolate faulty sensors and in the same time provide unbiased state estimation. On the other hand, we introduced a switching gain technique to deal with singular points. Stability analysis is achieved thanks to the Barbalat's lemma and without solving the well-known time-varying Sylvester equation. The proposed approach is extended to more general LTV systems of any order.

Keywords : Linear time-varying systems; state estimation; fault reconstruction; residual generation; sensor fault.

1 Introduction

Sensors are needed in almost all processes to ensure safe and efficient operations on the process plant. The implementation of instrument fault detection and identification is then necessary for more reliable process plants with higher efficiency. The first step of fault detection is the residual generation to provide an analytical redundancy. The most common approach is the observer-based residual generation (see for example [13], [5] and the references therein) where the output estimation error is used as residual whereas a bank of observers is required for the fault localization. For linear time invariant (LTI) systems, the residual is obtained by the use of the Luenberger observer or the Kalman filter in the stochastic case. However, for linear time-varying (LTV) systems, the application of classical results requires uniform complete observability and the resolution of Riccati or Lyapunov matrix differential equation (see [4], [22], [24], [27]).

In [11], robust fault detection schemes for LTI systems have been extended to LTV systems. An adaptive observer for MIMO LTV systems to jointly estimate the state and an unknown constant parameter has been proposed by Zhang in [27]. The proposed design requires the resolution of a differential matrix equation and is applied to the residual generation of LTV systems affected by a fault actuator or fault system ([25]). Observers are also used in fault tolerant control (FTC) to achieve some closed-loop performances in spite of faults (see for example [8], [20] and [26]).

In this contribution, we focus on the simultaneous observer based sensor diagnosis and state estimation for a class of LTV systems due to the fact that, in some cases, it is necessary to jointly estimate the state and the fault for both the diagnosis and the control reconfiguration (see [6], [7] and [1]).

Motivations behind our work come first from recent results on state estimation of aerial vehicles [3], [17], [18]. Indeed, we investigate here skew symmetric models in a general framework. Those may represent a large class of mechanical systems (see for example [12], [19] and [21]) such as satellite, aircraft or robot systems and output measurements $y(t)$ may represent accelerations. In [15] and [16] the authors proposed simultaneous state and sensor fault estimation for nonlinear and bilinear systems. In [17] and [18], the joint speed and sensor faults estimation of unmanned aerial vehicles was investigated. Motivated also by

these previous works, our approach is based on simultaneous state and sensor fault estimation through a straightforward bank of time-varying observers without using the singular state-space framework (see for example [2] and [23]). Indeed, we propose an useful and simple observer without the use of differential time-varying Riccati equations. This represents one of the main contributions of this paper where the observer gain matrix is computed directly from the system matrices. For diagnosis purposes we have to treat singular points that arise in the observation errors dynamics. For doing so, we introduce a switching gain technique to assure stability of the proposed approach. Stability conditions, based on the use of Barbalat's lemma, are deduced and easily checkable.
This paper is organized as follows : Section 2 introduces the problem formulation. The observer design without sensor faults is presented in Section 3. The simultaneous state estimation and sensor fault estimation is developed in section 4. An extension of the proposed approach to more general LTV systems of order n is given in section 5.

2 Problem statement

To reconstruct sensors fault signals, it is necessary to estimate the state of the system by using the measurements supplied by not failing sensors (see [15] and [16]). This approach thus supposes that healthy sensors are known *a priori*. In our case several problems arise : the sensors give the measurement of the time derivatives of the state ; it is supposed that there is at least one healthy sensor but we do not know *a priori* which sensor is failing or not. We thus suggest to use a bank of generators of residuals to discern if there are failing sensors and then to use the correct measures to estimate the state of the system. The third stage consists of estimating sensors' fault signals. Consider the class of LTV systems of the form

$$(\Sigma)\begin{cases} \dot{x}(t) &= A(t)x(t)+b(t) \\ y(t) &= \dot{x}(t)+f(t) \end{cases} \tag{1}$$

where $x(t) \in R^3$ is the state vector, $y(t) \in R^3$ is the measured output, $b(t) \in R^3$ is a known function and $A(t) = -A^T(t)$ is a time-varying skew-symmetric matrix. $f(t) = \begin{pmatrix} f_1(t) & f_2(t) & f_3(t) \end{pmatrix}^T$ is the time-varying sensor fault vector. We assume that at every time t, only one sensor is faulty and only additive time-varying faults occurring abruptly or incipiently are considered. The class of considered faults includes bias, drifts and variations of the sensor gain (see [15] and [23]). Notice that all skew-symmetric matrices of dimension 3 can be written as

$$A(t) = \begin{pmatrix} 0 & r(t) & -q(t) \\ -r(t) & 0 & p(t) \\ q(t) & -p(t) & 0 \end{pmatrix} \tag{2}$$

and verify $\operatorname{rank}(A(t)) = 2 \quad \text{if } A(t) \neq 0$.
For the bank of state and sensor fault observers and residual generators (SFORG) design, the following assumption is made.

Assumption 1. *Matrices A, $\dot{A}$ and $\ddot{A}$ are bounded.*

The aim of this contribution is the synthesis of a bank of SFORG. The first step that consists on the design of an observer to estimate the state of the system (1) is given in the following section.

3 Observer design with two measurements and without fault

In this section, we assume that two components of the measurement vector are available for each observer.

Consider the following observers for $i = 1, 2, 3$

$$\dot{\hat{x}}^i(t) = N_i(t)\hat{x}^i(t) + M_i(t)b(t) + K_i(t)C_i y(t) \tag{3}$$

where $\hat{x}^i(t)$ is an estimate of $x(t)$ and the three measurement selection matrices are given by

$$C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \; C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The following lemma gives the observer design for $i = 1$ (similar results can be obtained for $i = 2$ and $i = 3$).

Lemma 1. *Assume that $f(t) \equiv 0$ and $\lim_{t\to+\infty} p(t) \neq 0$. If assumption 1 holds and if the time-varying matrices $N_1(t)$, $M_1(t)$ and $K_1(t)$ in relation (3) are chosen as*

$$\begin{aligned} N_1(t) &= A(t) - K_1(t)C_1A(t) & (4a)\\ M_1(t) &= I - K_1(t)C_1 & (4b)\\ K_1(t) &= L_1(t) + \kappa_1 A^T(t)C_1^T & (4c) \end{aligned}$$

$$\textit{with} \quad L_1(t) = \begin{pmatrix} -q(t)/\alpha_1(t) & -r(t)/\alpha_1(t) \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

$$\textit{where } \alpha_1(t) = p(t) \textit{ if } |p(t)| > \nu_1 \textit{ else } \alpha_1(t) = \nu_1$$

and κ_1 and ν_1 are strictly positive tuning parameters and if

$$\operatorname{rank}\begin{pmatrix} C_1\dot{A}(t) \\ C_1A(t) \end{pmatrix} = 3 \tag{6}$$

then the observation error $\epsilon^1(t) = x(t) - \hat{x}^1(t)$ is asymptotically stable.

Remark 1. For $i = 2$ and $i = 3$, in lemma 1, equation (5) becomes

$$L_2(t) = \begin{pmatrix} 1 & 0 \\ -p(t)/\alpha_2(t) & -r(t)/\alpha_2(t) \\ 0 & 1 \end{pmatrix} \tag{7}$$

where $\alpha_2(t) = q(t)$ if $|q(t)| > \nu_2$ else $\alpha_2(t) = \nu_2$

$$L_3(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -p(t)/\alpha_3(t) & -q(t)/\alpha_3(t) \end{pmatrix} \tag{8}$$

where $\alpha_3(t) = r(t)$ if $|r(t)| > \nu_3$ else $\alpha_3(t) = \nu_3$. $\square$

Proof. Using equations (1) with $f(t) \equiv 0$ and (3) the observation error dynamics can be written as

$$\begin{aligned} \dot{\epsilon}^1(t) &= N_1(t)\epsilon^1(t) + (A(t) - N_1(t) - K_1(t)C_1A(t))x(t) \\ &+ (I - M_1(t) - K_1(t)C_1)b(t) \end{aligned} \tag{9}$$

It is easy to see that the unbiasedness conditions

$$\begin{aligned} A(t) - N_1(t) - K_1(t)C_1A(t) &\equiv 0 & (10a)\\ I - M_1(t) - K_1(t)C_1 &\equiv 0 & (10b) \end{aligned}$$

are satisfied if matrices $N_1(t)$, $M_1(t)$ are chosen as in (4a) and (4b).
Using the expression of $K_1(t)$ and choosing $L_1(t)$ as in (4c) and (5) respectively yield

$$N_1(t) = \kappa_1 \left(-A^T(t)C_1^TC_1A(t) + \lambda_1\tilde{A}_1(t)\right) \tag{11}$$

with

$$\tilde{A}_1(t) = \begin{pmatrix} 0 & r(t)\theta_1(t) & -q(t)\theta_1(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{12a}$$

$$\theta_1(t) = \left(1 - \frac{p(t)}{\nu_1}\right), \tag{12b}$$

$$\lambda_1 = \frac{1}{\kappa_1} \quad \text{if } |p(t)| \leq \nu_1 \quad \text{else } \lambda_1 = 0 \tag{12c}$$

Finally the observation error (9) becomes

$$\dot{\epsilon}^1(t) = \kappa_1 \left(\overline{N}_1(t) + \lambda_1 \tilde{A}_1(t)\right) \epsilon(t) \tag{13}$$

with $\overline{N}_1(t) = -A^T(t)C_1^T C_1 A(t)$. The stability analysis of the observation error is performed by considering that

$$\dot{\epsilon}^1(t) = \kappa_1 \overline{N}_1(t)\epsilon^1(t) \tag{14}$$

is the nominal system and $\dot{\epsilon}^1(t) = N_1(t)\epsilon^1(t)$ is the perturbed system *i.e.* for the case $|p(t)| \leq \nu_1$ or $\lambda_1 = \frac{1}{\kappa_1}$. One can see that the origin $\epsilon^1 = 0$ is an equilibrium point for both nominal system (14) and perturbed system (13).

First consider the case $|p(t)| > \nu_1$ or equivalently $\lambda_1 = 0$ and let $V(\epsilon^1) = {\epsilon^1}^T \epsilon^1$ be a Lyapunov function candidate. The time derivative of V along the observation error dynamics (14) leads to

$$\dot{V}(t) = -2\kappa_1 \|C_1 A(t)\epsilon^1(t)\|^2 \ \leq 0 \tag{15}$$

It follows that $\epsilon^1(t)$ is bounded.

Now using Barbalat's lemma and assumption 1 it follows that $\dot{V}(\epsilon^1(t)) \to 0$ as $t \to +\infty$. Then, from relation (15) $C_1 A(t)\epsilon^1(t) \to 0$ as $t \to +\infty$.

In order to use again Barbalat's lemma, we compute the time derivatives of the function $\varphi(t) = C_1 A(t)\epsilon^1(t)$:

$$\dot{\varphi}(t) = (C_1 \dot{A}(t) - \kappa_1 C_1 A(t) A^T(t) C_1^T C_1 A(t))\epsilon^1(t)$$

Using again Barabalat's lemma and assumption 1, it follows that

$$\dot{\varphi}(t) = (C_1 \dot{A}(t) - \kappa_1 C_1 A(t) A^T(t) C_1^T C_1 A(t))\epsilon^1(t) \to 0$$

as $t \to +\infty$. Now since $\dot{\varphi}(t) \to 0$ and $C_1 A(t)\epsilon^1(t) \to 0$ then $C_1 \dot{A}(t)\epsilon^1(t) \to 0$ *i.e.*

$$\begin{pmatrix} C_1 \dot{A}(t) \\ C_1 A(t) \end{pmatrix} \epsilon^1(t) \to 0 \text{ as } t \to +\infty \tag{16}$$

then $\epsilon^1(t) \to 0$ by using the rank condition (6). This ends the proof for the case $|p(t)| > \nu_1$.

For the case where $|p(t)| \leq \nu_1$ during a finite time interval $[t_0, t_1]$ and $\lim_{t \to +\infty} p(t) \neq 0$, the trajectory of the perturbed system is closed to the trajectory of the nominal system by using a theorem in [9] (Theorem 4.2, pp. 86). □

Remark 2. Parameter ν_1 may be chosen sufficiently small to decrease the time interval $[t_0, t_1]$ and κ_1 sufficiently large such that the perturbed system is closed to the nominal system and to increase the estimation error convergence rate.

4 Simultaneous sensor fault diagnosis and state estimation

To estimate simultaneously the state $x(t)$ and the sensors fault $f(t)$, the proposed approach is composed of a bank of three observers. Each observer used two components of the measurement vector for the state estimation and the residual generation. The third component is used to reconstruct the sensor fault signal. Indeed, the proposed SFORG has the following form for $i = 1, 2, 3$

$$(\mathcal{O}_i) \begin{cases} \dot{\hat{x}}^i(t) &= N_i(t)\hat{x}^i(t) + M_i(t)b(t) + K_i(t)Y^i(t) \\ \hat{f}_i(t) &= \overline{Y}^i(t) - \overline{C}_i(t)\dot{\hat{x}}^i(t) \\ \rho_i(t) &= Q_i(t)\dot{\hat{x}}^i(t) + P_i(t)Y^i(t) \end{cases} \tag{17}$$

with

$$\begin{aligned} Y^i(t) &= C_i\dot{x}(t) + \overline{f}^i(t) \\ \overline{Y}^i(t) &= \overline{C}_i\dot{x}(t) + f_i(t) \end{aligned}$$

$\overline{f}^1(t) = \begin{pmatrix} f_2(t) & f_3(t) \end{pmatrix}^T$, $\overline{f}^2(t) = \begin{pmatrix} f_1(t) & f_3(t) \end{pmatrix}^T$, $\overline{f}^3(t) = \begin{pmatrix} f_1(t) & f_2(t) \end{pmatrix}^T$ and

$$\begin{aligned} C_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ C_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ C_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ \overline{C}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},\ \overline{C}_2 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix},\ \overline{C}_3 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

where $\hat{x}^i(t)$ and $\hat{f}_i(t)$ are the estimates of the state vector $x(t)$ and the sensor fault vector component $f_i(t)$ respectively and $\rho_i(t)$ is the i^{th} residual.

Assumption 2. *The following rank conditions are simultaneously verified*

$$\operatorname{rank}\begin{pmatrix} C_i\dot{A}(t) \\ C_iA(t) \end{pmatrix} = 3 \text{ for } i = 1\dots3 \tag{18}$$

Theorem 1. *Assume that assumptions 1 and 2 hold and the matrices $N_i(t)$, $M_i(t)$ and $K_i(t)$ are given by (4) by replacing index "1" by "i" and matrices Q_i and P_i are chosen as*

$$Q_i = -\beta_i C_i \tag{19a}$$
$$P_i = \beta_i I \tag{19b}$$

and matrices $L_i(t)$ are given by relations (5), (7) and (8). Then the following statements hold.

1. *If $\overline{f}^i(t) \equiv 0$ then the dynamics of the observation error $\epsilon^i(t) = x(t) - \hat{x}^i(t)$ is asymptotically stable at the origin and $\rho_i(t) \to 0$ and $\hat{f}_i(t) \to f_i(t)$ as $t \to +\infty$,*

2. *If $\overline{f}^i(t) \neq 0$ i.e. $f_j(t) \neq 0$ for $j \neq i$ and $f_j(t) \neq \dot{\sigma}_j(t)$ with $\sigma(t)$ solution of $\dot{\sigma}(t) = A(t)\sigma(t)$ then $\rho_i(t) \neq 0$ (σ_j is the j^{th} component of σ),*

where ν_i, κ_i and β_i are strictly positive tuning parameters.

Remark 3. The first statement in Theorem 1 concerns the transient response of the observation error and the residual in the absence of fault $\overline{f}^i(t)$ and the second the effect of fault $\overline{f}^i(t)$ on the residual $\rho_i(t)$. Then for the fault isolation problem, the Theorem 1 can be used as follows. After a transient response of the observation error and in absence of fault $\overline{f}^i(t)$ the residual $\rho_i(t)$ is zero. Then after this transient response the residual $\rho_i(t)$ is nonzero if and only if the fault $\overline{f}^i(t)$ is nonzero and $f_j(t) \neq \dot{\sigma}_j(t)$. These results are in accordance for example with definition 2 in Pertew *et al.* ([14]). □

Proof. Using equations (1) and (17) and if matrices $N_i(t)$, $M_i(t)$ and $K_i(t)$ are chosen as in (4) then the observation error $\epsilon^i(t) = x(t) - \hat{x}^i(t)$ dynamics, the i^{th} residual $\rho_i(t)$ and the sensor fault estimation error $\epsilon^i_f(t) = f_i(t) - \hat{f}_i(t)$ can be written as :

$$\dot{\epsilon}^i(t) = N_i(t)\epsilon^i(t) - \left(L_i(t) + \kappa_i A^T(t)C_i^T\right)\overline{f}^i(t) \tag{20}$$

$$\begin{aligned}\rho_i(t) &= \beta_i\left(C_i\epsilon^i(t)+\overline{f}^i(t)\right) &(21)\\ \epsilon_f^i(t) &= -\overline{C}_i\dot{\epsilon}^i(t) &(22)\end{aligned}$$

Using the expression of $K_i(t)$ and choosing $L_i(t)$ as in (5) yield

$$N_i(t)=\kappa_i\left(\lambda_i\tilde{A}_i(t)-A^T(t)C_i^TC_iA(t)\right) \tag{23}$$

where

$$\begin{aligned}\lambda_1 &= \frac{1}{\kappa_1}\text{ if }|p(t)|\leq\nu_1 &(24a)\\ \lambda_2 &= \frac{1}{\kappa_2}\text{ if }|q(t)|\leq\nu_2 &(24b)\\ \lambda_3 &= \frac{1}{\kappa_3}\text{ if }|r(t)|\leq\nu_3 &(24c)\\ \text{else }\lambda_i &= 0 &(24d)\end{aligned}$$

and

$$\tilde{A}_1(t)=\begin{pmatrix}0 & r(t)\theta_1(t) & -q(t)\theta_1(t)\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\ \theta_1(t)=1-\frac{p(t)}{\nu_1}$$

$$\tilde{A}_2(t)=\begin{pmatrix}0 & 0 & 0\\ -r(t)\theta_2(t) & 0 & p(t)\theta_2(t)\\ 0 & 0 & 0\end{pmatrix},\ \theta_2(t)=1-\frac{q(t)}{\nu_2}$$

$$\tilde{A}_3(t)=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ q(t)\theta_3(t) & -p(t)\theta_3(t) & 0\end{pmatrix},\ \theta_3(t)=1-\frac{r(t)}{\nu_3}$$

Finally the observation error becomes

$$\dot{\epsilon}^i(t) = \kappa_i\left(\lambda_i\tilde{A}_i(t)-A^T(t)C_i^TC_iA(t)\right)\epsilon^i(t)-\left(L_i(t)+\kappa_iA^T(t)C_i^T\right)\overline{f}^i(t) \tag{26}$$

Matrices $N_i(t)$ depend on parameters λ_i and in the following we consider that the nominal case is given by $\lambda_i=\bar{\lambda}=0$. In what follows, the proof of the nominal case is made in two steps.

- In the first step, we show that if $\overline{f}^i(t)\equiv 0$ and for the nominal case (*i.e.* $\lambda_i=0$) then the observation error $\epsilon^i(t)=x(t)-\hat{x}^i(t)$ dynamics is asymptotically stable the i^{th} residual $\rho_i(t)$ and the sensor fault estimation error $\epsilon_f^i(t)$ converge to 0.

- The second step is devoted to the case $\overline{f}^i(t)\neq 0$.

First consider the case $\overline{f}^i(t)\equiv 0$ and $\lambda_i=0$. Then the observation error becomes :

$$\dot{\epsilon}^i(t)=-\kappa_iA^T(t)C_i^TC_iA(t)\epsilon^i(t) \tag{27}$$

Then using lemma 1, one can see that the observation error is asymptotically stable. Now using relations (21), (22) and (27), the residual and the sensor fault observation error can be written as :

$$\rho_i(t)=-\beta_i\kappa_iC_iA^T(t)C_i^TC_iA(t)\epsilon^i(t) \tag{28}$$
$$\epsilon_f^i(t)=\kappa_i\overline{C}_iA^T(t)C_i^TC_iA(t)\epsilon^i(t) \tag{29}$$

Thanks to relations (28) and (29), it is easy to see that if $\overline{f}^i(t)\equiv 0$ then $\epsilon_f^i(t)\to 0$ and $\rho_i(t)\to 0$ as $t\to+\infty$ since $\epsilon^i(t)\to 0$ as $t\to+\infty$ and $A(t)$ is bounded. This ends the first step of the proof.

Now consider the case $\overline{f}^i(t) \neq 0$ or equivalently at least one of $f_j(t) \neq 0$ for $j \neq i$. The residual can be written as :

$$\rho_i(t) = S_i(t)\epsilon^i(t) + T_i(t)\overline{f}^i(t) \tag{30}$$

with $S_i(t) = -\kappa_i\beta_i C_i A^T(t) C_i^T C_i A(t)$ and $T_i(t) = -\beta_i C_i(L_i(t) + \kappa_i A^T(t)C_i^T) + \beta_i I = -\beta_i\kappa_i C_i A^T(t) C_i^T$ which lead to

$$\begin{aligned}
S_1(t) &= -\kappa_1\beta_1 p(t)\begin{pmatrix} -q(t) & p(t) & 0 \\ -r(t) & 0 & p(t)\end{pmatrix} \\
S_2(t) &= -\kappa_2\beta_2 q(t)\begin{pmatrix} q(t) & -p(t) & 0 \\ 0 & -r(t) & q(t)\end{pmatrix} \\
S_3(t) &= -\kappa_3\beta_3 r(t)\begin{pmatrix} r(t) & 0 & -p(t) \\ 0 & r(t) & -q(t)\end{pmatrix} \\
T_1(t) &= -\kappa_1\beta_1\begin{pmatrix} 0 & -p(t) \\ p(t) & 0\end{pmatrix} \\
T_2(t) &= -\kappa_2\beta_2\begin{pmatrix} 0 & q(t) \\ -q(t) & 0\end{pmatrix} \\
T_3(t) &= -\kappa_3\beta_3\begin{pmatrix} 0 & -r(t) \\ r(t) & 0\end{pmatrix}.
\end{aligned}$$

Assume that there exists $\left(\overline{f}_p^i(t) \neq 0, \epsilon_p^i(t)\right)$ such that $\rho_i(t) = S_i(t)\epsilon_p^i(t) + T_i(t)\overline{f}_p^i(t) = 0$ then :

$$\overline{f}_p^i(t) = -T_i^{-1}(t)S_i(t)\epsilon_p^i(t) = -C_i A\epsilon_p^i(t) \tag{31}$$

Inserting (31) in (26) with $\lambda_i = 0$ yields :

$$\dot{\epsilon}_p^i(t) = A(t)\epsilon_p^i(t) \tag{32}$$

By inserting (32) in (31) and developing relation (31) for $i = 1$ to 3 we can characterize all the non detectable sensor failures $f_{pk}(t)$ which verify :

$$f_{pk}(t) = \dot{\sigma}_k(t)$$

such that $\sigma(t)$ is a solution of (32) and $\sigma_k(t)$ is the k^{th} component of $\sigma(t)$. More precisely, if there exists $f_j(t) \neq 0$ for $j \neq i$ and $f_j(t) \neq \dot{\sigma}_j(t)$ with $\sigma(t)$ solution of $\dot{\sigma}(t) = A(t)\sigma(t)$ then $\rho_i(t) \neq 0$. Notice that the residual ρ_i cannot be taken in consideration when $\lambda_i \neq 0$. This ends the proof. □

Remark 4. The possibility that a sensor fault is of the form $f_{pk}(t) = \dot{\sigma}_k(t)$ with $\sigma(t)$ solution of $\dot{\sigma}(t) = A(t)\sigma(t)$ is rather not very probable. We can thus detect, isolate and identify the majority of time-varying sensor faults. □

Remark 5. For the case of a single sensor failure f_i, the observer $(\mathcal{O}_i)$ gives an estimation of the state and the fault and the residual $\rho_i(t)$ vanishes to 0. In addition the other residuals are different from zero. □

Remark 6. The proposed approach allows to detect, isolate and estimate the fault in the case of single sensor failure. In case where two or three sensors are failing, we can only detect the presence of failure because all the residuals are different from and do not converge to zero. □

Remark 7. In the time-invariant case, *i.e.* if p, q and r are constant, the eigenvalues of A are $s_1 = 0$, $s_2 = j\sqrt{p^2+q^2+r^2}$ and $s_3 = -j\sqrt{p^2+q^2+r^2}$. The pair (C_iA, A) is not detectable since the eigenspace associated to the eigenvalue $s_1 = 0$ is not observable. Thus there does not exist an observer in the time-invariant case. □

5 General scheme to high order LTV systems

This section is devoted to the extension of the proposed approach to the general case of higher order LTV systems (Σ) where $A(t) \in \mathbb{R}^{n\times n}$ is a time-varying matrix not necessarily skew-symmetric. We introduce first a sufficient condition for the solvability of the unbiasedness condition (10) and we give a formulation to compute the gain matrix L_i. In the second part, we generalize the observability condition (6) which is a sufficient condition for the existence of the observer.

5.1 On the existence of solution $L_i(t)$ of unbiasedness condition (10)

Using the unbiasedness condition (10) with $K_i(t)$ given by (4c) one obtains

$$A(t) - N_i(t) - \left(L_i(t) + \kappa_i A^T(t)C_i^T\right) C_i A(t) \equiv 0 \tag{33}$$

If the matrix $N_i(t)$ is chosen as

$$N_i(t) = -\kappa_i A^T(t) C_i^T C_i A(t) \tag{34}$$

then the unbiasedness condition (33) or (10) is reduced to

$$A(t) - L_i(t) C_i A(t) \equiv 0 \tag{35}$$

Relation (35) is solvable if and only if (see [10])

$$\operatorname{rank}\left(C_i A(t)\right) = \operatorname{rank}\left[(C_i A(t))^T \quad A^T(t)\right]^T \tag{36}$$

Condition (36) is then a sufficient condition for the existence of the observer.

Remark 8. For the case where C_i is full row rank matrix with only one '1' in each row, the condition (36) is equivalent to

$$\operatorname{rank}\left(C_i A(t)\right) = \operatorname{rank} A(t) \tag{37}$$

If the rank condition (36) is satisfied, then a solution $L_i(t)$ of (35) is given by

$$L_i(t) = A(t)\left(C_i A(t)\right)^{\dagger} \tag{38}$$

where $(C_i A(t))^{\dagger}$ is any generalized inverse of $C_i A(t)$ (see [10]). Notice that if $\operatorname{rank}(C_i) = 2$ then there exists at least one matrix C_i such that the condition (36) is verified since $\operatorname{rank}(A(t)) = \operatorname{rank}(C_i A(t)) = 2$ for skew-symmetric matrices of order $n = 3$.

5.2 Generalisation of the observability conditions (6) and (18)

The proposed approach can be extended to the general case of n^{th} order LTV systems by replacing Assumption 1 and Lemma 1 as follows.

Assumption 3. *Matrices A, $\dot{A}$, $\ddot{A}$, $\ldots A^{(r)}$ are bounded where $A^{(r)}$ is the r^{th} time derivative of $A(t)$.*

The following lemma is a generalization of lemma 1.

Lemma 2. *Assume that $f(t) \equiv 0$. If the following statements are satisfied :*

1. *assumption 3 holds,*
2. *matrices C_1 and A verify rank condition (36),*
3. *the following rank condition is satisfied*

$$\operatorname{rank}\left(A(t)^T C_1^T \quad | \quad \dot{A}(t)^T C_1^T \quad | \quad \ldots \quad | \quad A^{(r)}(t)^T C_1^T\right)^T = n \tag{39}$$

4. the time-varying matrices $N_1(t)$, $M_1(t)$ and $K_1(t)$ are given by (4) and $L_1(t)$ by relation (38) where κ_1 is a strictly positive tuning parameter

then the observation error $\epsilon^1(t) = x(t) - \hat{x}^1(t)$ is asymptotically stable.

The proof is similar to the case of Lemma 1 and is then omitted.
Now, it is easy to see that the following assumption is sufficient for the existence of a bank of SFORG

Assumption 4. *The following rank conditions are simultaneously satisfied*

$$\operatorname{rank}\left(A(t)^T C_1^T \quad | \quad \dot{A}(t)^T C_1^T \quad | \quad \dots \quad | \quad A^{(r)}(t)^T C_1^T\right)^T = n \tag{40}$$

for $i = 1 \dots n$.

6 Conclusion

In this note, we addressed the problem of state estimation of third order skew symmetric time-varying systems under a sensor diagnosis procedure. One of the main features is that the resolution of the Riccati equation or the Lyapunov like differential equation is not required for the observer gain design. The proposed approach consists of the design of a bank of SFORG to detect and reconstruct sensor faults and simultaneously to estimate the state of the system. The main contributions concern the stability analysis of the observation errors and the fact of highlighting all trajectories of faults that can be detected. Furthermore, an extension to more general LTV systems of any order was established. The proposed approach was applied to design a bank SFORG for the translation dynamics of an UAV with respect to the body frame. These simulations gave satisfactory results but due to the lack of space are not presented here.

This work is supported by the French National Centre for Scientific Research, the ANR, under the number ANR 09 SECU 12.

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