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# CARTAN SUBGROUPS OF GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

ELÍAS BARO, ERIC JALIGOT, AND MARGARITA OTERO

ABSTRACT. We prove that groups definable in o-minimal structures have Cartan subgroups, and only finitely many conjugacy classes of such subgroups. We also delineate with precision how these subgroups cover the ambient group.

## 1. INTRODUCTION

If  $G$  is an arbitrary group, a subgroup  $Q$  of  $G$  is called a *Cartan subgroup* (in the sense of Chevalley) if it satisfies the two following conditions:

- (1)  $Q$  is nilpotent and maximal with this property among subgroups of  $G$ .
- (2) For any subgroup  $X \leq Q$  which is normal in  $Q$  and of finite index in  $Q$ , the normalizer  $N_G(X)$  of  $X$  in  $G$  contains  $X$  as a finite index subgroup.

The purely group-theoretic definition of a Cartan subgroup as above was designed by Chevalley in order to capture critical properties of very specific subgroups of Lie groups.

In connected reductive algebraic groups over algebraically closed fields and in connected compact real Lie groups, Cartan subgroups correspond typically to centralizers of maximal tori and it is well known that they are connected. It is however worth emphasizing at the outset that in real Lie groups Cartan subgroups need not be connected in general, a point also noticed by Chevalley in the introduction of [7, Chapitre VI]: “*Il convient de noter que les groupes de Cartan de  $G$  ne sont en général pas connexes.*” The diagonal subgroup of  $\mathrm{SL}_2(\mathbb{R})$  is maybe the first example of a nonconnected Cartan subgroup that one should bear in mind. Most of the difficulties for the study of these subgroups in the past, notably in the early work of Cartan, have been this failure of connectedness. This is something that will eventually need considerable attention in the present paper as well.

We are going to study Cartan subgroups from the model-theoretic point of view of groups definable in an o-minimal structure, that is a first-order structure  $\mathcal{M} = \langle M, \leq, \dots \rangle$  equipped with a total, dense, and without end-points definable order  $\leq$  and such that every definable subset of  $M$  is a boolean combination of intervals with end-points in  $M \cup \{\pm\infty\}$ . The most typical example of an o-minimal structure is of course the ordered field  $\mathbb{R}$  of the reals, but there are richer o-minimal structures,

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such as the field of the reals equipped in addition with the exponential function [35].

In order to deal with the non-connectedness of Cartan subgroups in general, we will use the following notion. If  $G$  is a group definable in an arbitrary structure  $\mathcal{M}$ , then we say that it is *definably connected* if and only if it has no proper subgroup of finite index definable in the sense of  $\mathcal{M}$ . Now, a subgroup of a group  $G$  definable in  $\mathcal{M}$  is called a *Carter* subgroup of  $G$  if it is definable and definably connected (in the sense of  $\mathcal{M}$  as usual), and nilpotent and of finite index in its normalizer in  $G$ . All the notions of definability depend on a ground structure  $\mathcal{M}$ , which in the present paper will typically be an o-minimal structure. The notion of a Carter subgroup first appeared in the case of finite groups as *nilpotent and selfnormalizing* subgroups. A key feature is that, in the case of finite solvable groups, they exist and are conjugate [6]. For infinite groups, the notion we are adopting here, incorporating definability and definable connectedness, comes from the theory of groups of finite Morley rank. That theory is another classical branch of group theory in model theory, particularly designed at generalizing algebraic groups over algebraically closed fields. We note that the selfnormalization from the finite case becomes an almost selfnormalization property, and indeed the finite group  $N_G(Q)/Q$  associated to a Carter subgroup  $Q$  typically generalizes the notion of the *Weyl group* relative to  $Q$ . This is something that will also make perfect sense here in the case of groups definable in o-minimal structures.

We will see shortly in Section 2 that for groups definable in o-minimal structures, and actually for groups with the mere descending chain condition on definable subgroups, there is an optimal correspondence between Cartan subgroups and Carter subgroups: the latter ones are exactly the definably connected components of the former ones. In particular Cartan subgroups are automatically definable subgroups, a point not following from the definition of Chevalley in general, but which is always going to be true here.

In Sections 3-6 we will relate Cartan and Carter subgroups to a well behaved notion of dimension for sets definable in an o-minimal structure, notably to *weak genericity* (having maximal dimension) or to *largeness* (having smaller codimension). We will mainly develop their *generous* analogs, where one actually considers the weak genericity or the largeness of the union of conjugates of a given set. The technics and results here will be substantial adaptations and generalizations from [19, 20] in the finite Morley rank case, and our arguments for Cartan and Carter subgroups of groups definable in o-minimal structure will highly depend on dimensional computations and generosity arguments. We will make such dimensional computations in a rather axiomatic framework, essentially with the mere existence of a definable and additive dimension, since they apply as such in many other contexts (groups of finite Morley rank, groups in supersimple theories of finite rank, groups definable over the  $p$ -adics...).

Our main result can be summarized as follows.

**Theorem 1.** *Let  $G$  be a group definable in an o-minimal structure. Then Cartan subgroups of  $G$  exist, are definable, and fall into finitely many conjugacy classes.*

Our proof of Theorem 1 will also strongly depend on the main structural theorem about groups definable in o-minimal structures. It says in essence that any definably connected group  $G$  definable in an o-minimal structure is, modulo a largest normal solvable (and definable) subgroup  $R(G)$ , a direct product of finitely many definably

simple groups which are essentially “known” as groups of Lie type. Hence our proof will consist in an analysis of the interplay between these definably simple factors and the relevant definably connected solvable subgroups of  $G$ . Results specific about groups definable in an o-minimal structure which are used here will be reviewed in Section 7.

A large part of the work will thus be concerned with the case of definably connected solvable groups. In this case we will make a strong use of the previously mentioned largeness and generosity arguments. Mixing them with more algebraic inductive arguments inspired by [13] in the finite Morley rank case, we will obtain the following result in Section 8.

**Theorem 40** *Let  $G$  be a definably connected solvable group definable in an o-minimal structure. Then Cartan subgroups of  $G$  exist and are conjugate, and they are definably connected and selfnormalizing. Moreover, they are largely generous in the following strong sense: for any Cartan subgroup  $Q$ , the (definable) set of elements of  $Q$  contained in a unique conjugate of  $Q$  is large in  $Q$  and largely generous in  $G$ .*

A definably connected group is *semisimple* if it has a finite center and modulo that center abelian normal subgroups are trivial. Semisimplicity is a first-order property, and the main theorem about groups definable in o-minimal structures actually says that any such semisimple group with a trivial center is a direct product of definably simple groups, with each factor a “known” group of Lie type modulo certain elementary equivalences. We will review certain facts more or less classical about Cartan subgroups of Lie groups in Section 9. In Section 10 we will transfer the theory of Cartan subgroups of Lie groups to definably simple groups and get a quite complete description of Cartan subgroups of definably simple groups definable in o-minimal structures.

In Section 11 we will elaborate further on the definably simple case to get a similarly quite complete description of Cartan subgroups of semisimple groups definable in o-minimal structures, obtaining the following general theorem.

**Theorem 62 (lite)** *Let  $G$  be a definably connected semisimple group definable in an o-minimal structure. Then  $G$  has definable Cartan subgroups and the following holds.*

- (1)  $G$  has only finitely many conjugacy classes of Cartan subgroups.
- (2) If  $Q_1$  and  $Q_2$  are Cartan subgroups and  $Q_1^\circ = Q_2^\circ$ , then  $Q_1 = Q_2$ .
- (3) If  $Q$  is a Cartan subgroup, then  $Z(G) \leq Q$ ,  $Q' \leq Z(G)$ , and  $Q^\circ \leq Z(Q)$ .
- (4) If  $Q$  is a Cartan subgroup and  $a \in Q$ , then  $aQ^\circ$  is weakly generous.
- (5) The union of all Cartan subgroups, which is definable by (1), is large.

The general case of a definably connected group  $G$  definable in an o-minimal structure will be considered in Section 12. In this case we have both  $G$  not solvable and not semisimple, or in other words

$$G/R^\circ(G) \neq 1 \text{ and } R^\circ(G) \neq 1.$$

In that case Theorem 1 follows rapidly from Theorems 40 and 62, but some natural questions will remain without answer here. The most important one is maybe the following: if  $Q$  is a Cartan subgroup of  $G$ , is it the case that  $QR^\circ(G)/R^\circ(G)$  is

a Cartan subgroup of the semisimple quotient  $G/R^\circ(G)$ ? This question is indeed equivalent to the fact that Cartan subgroups of  $G/R^\circ(G)$  are exactly of the form  $QR^\circ(G)/R^\circ(G)$  for some Cartan subgroup  $Q$  of  $G$ . We will only manage to prove that for a Cartan subgroup  $Q$  of  $G$ , the group  $QR^\circ(G)/R^\circ(G)$  is a finite index subgroup of a Cartan subgroup of  $G/R^\circ(G)$ , obtaining in particular the expected lifting for the corresponding Carter subgroups. Getting the exact lifting of Cartan subgroups seems to be related to interesting new problems of representation theory in a definable context. In any case, we will mention all what we managed to prove on the correlations between Cartan subgroup of  $G$  and of  $G/R^\circ(G)$ , trying also to work with a not necessarily definably connected ambient group  $G$  when possible. We will conclude in Section 13 with further comments on certain specialized topics, including algebraic or compact factors, Weyl groups relative to the various Cartan subgroups, and parameters.

In this paper definability always means definability with parameters. We refer to [24] for a complete introduction to groups definable in o-minimal structures. We insist that everything is done here for groups *definable* (as opposed to interpretable) in an arbitrary o-minimal structure. This is because the theory of groups in o-minimal structure has been developed in this slightly restricted context since [30], where it is shown that definable groups can be equipped with a nice definable manifold structure making them topological groups. An arbitrary o-minimal structure does not eliminate imaginaries in general, but any group definable in an arbitrary o-minimal structure eliminates imaginaries, and actually has definable choice functions in a very strong sense [9, Theorem 7.2]. In particular, imaginaries coming from a group definable in an o-minimal structure will always be considered as definable in the sequel, and can be equipped with a finite dimension as any definable set. We refer to [32, Chapter 4] or [30] for the dimension of sets definable in o-minimal structures.

Since we already gave the organization of the paper, let us immediately enter into its core.

## 2. CARTAN SUBGROUPS AND CARTER SUBGROUPS

We first consider the relations between Cartan and Carter subgroups of groups definable in o-minimal structures. Actually, by [30, Remark 2.13], such groups satisfy the *descending chain condition* on definable subgroups (*dcc* for short), and we will analyze these relations in the more natural context of groups with the *dcc*. Throughout the present section,  $G$  is a group definable in a structure  $\mathcal{M}$  and definability may refer to  $\mathcal{M}^{\text{eq}}$ , and we say that it satisfies the *dcc* if any strictly descending chain of definable subgroups is stationary after finitely many steps. Notice that the *dcc* always pass to quotients by definable normal subgroups.

We first list some general facts needed in the sequel.

**Fact 2.** [1, Fact 3.1] *Let  $G$  be a definably connected group.*

- (a) *Any definable action of  $G$  on a finite set is trivial.*
- (b) *If  $Z(G)$  is finite, then  $G/Z(G)$  is centerless.*

In a group with the *dcc*, any subset  $X$  is contained in a smallest definable subgroup  $H(X)$  called the *definable hull* of  $X$ : take  $H(X)$  to be the intersection of all definable subgroups of  $G$  containing  $X$ .

**Fact 3.** [1, 3.3 & 3.4] *Let  $G$  be a group with the *dcc* and  $X$  a subset of  $G$ .*

- (a) If  $X$  is  $K$ -invariant for some subset  $K$  of  $G$ , then  $H(X)$  is  $K$ -invariant as well.
- (b) If  $X$  is a nilpotent subgroup of  $G$ , then  $H(X)$  is nilpotent of the same nilpotency class.

We now mention an infinite version of the classical *normalizer condition* in finite nilpotent groups.

**Lemma 4.** *Let  $G$  be a nilpotent group with the dcc on definable subgroups, or merely such that each definable subgroup has a definably connected definable subgroup of finite index. If  $H$  is a definable subgroup of infinite index in  $G$ , then  $N_G(H)/H$  is infinite.*

*Proof.* For instance, one may argue formally as in [31, Proposition 1.12].  $\square$

**Lemma 5.** *Let  $G$  be a group with the dcc.*

- (a) If  $Q$  is a maximal nilpotent subgroup of  $G$ , then  $Q$  is definable.
- (a') If  $Q$  is a Cartan subgroup of  $G$ , then  $Q$  is definable and  $Q^\circ$  is a Carter subgroup of  $G$ .
- (b) If  $Q$  is a Carter subgroup of  $G$ , then  $Q$  is contained in a maximal nilpotent subgroup  $\tilde{Q}$  of  $G$ , and any such subgroup  $\tilde{Q}$  is a Cartan subgroup of  $G$  with  $[\tilde{Q}]^\circ = Q$ .

*Proof.* (a). By Fact 3(b).

(a').  $Q$  is definable by item (a). Since  $Q^\circ$  is a normal subgroup of  $Q$  of finite index in  $Q$ ,  $Q^\circ$  is a finite index subgroup of  $N_G(Q^\circ)$ , and  $Q^\circ$  is a Carter subgroup of  $G$ .

(b). A definable nilpotent subgroup  $H$  containing  $Q$  must satisfy  $H^\circ = Q$  by Lemma 4, and thus  $H \leq N_G(H^\circ) = N_G(Q)$ . Now Fact 3(b) implies that any nilpotent subgroup  $H$  containing  $Q$  satisfies  $Q \leq H \leq N_G(Q)$ . Since  $N_G(Q)/Q$  is finite, there are maximal such subgroups, proving our first claim.

Now fix any such maximal nilpotent subgroup  $\tilde{Q}$ . It is definable by item (a) and we have already seen that  $Q = [\tilde{Q}]^\circ$ , and  $\tilde{Q} \leq N_G([\tilde{Q}]^\circ) = N_G(Q)$ . We now check that  $\tilde{Q}$  is a Cartan subgroup. Let  $X$  be any normal subgroup of finite index of  $\tilde{Q}$ . We first observe that  $H^\circ(X) = Q$ : since  $\tilde{Q}$  is definable we get  $H^\circ(X) \leq [\tilde{Q}]^\circ = Q$ , and since  $H^\circ(X)$  must have finite index in  $\tilde{Q}$  we get the desired equality. Now by Fact 3(a)  $N_G(X)$  normalizes  $H^\circ(X) = Q$ , so  $X \leq N_G(X) \leq N_G(Q)$ . Since  $X$  has finite index in  $\tilde{Q}$  and  $\tilde{Q}$  has finite index in  $N_G(Q)$ ,  $X$  has finite index in  $N_G(Q)$ , and in particular  $X$  has finite index in  $N_G(X)$ .  $\square$

Applying Lemma 5, we have thus that in groups definable in o-minimal structures Carter subgroups are *exactly* the definably connected components of Cartan subgroups, with the latter ones always definable. We also note that Lemma 5(a) gives the automatic definability of unipotent subgroups in many contexts, but that such unipotent subgroups are in general not almost selfnormalizing. We also note that if  $Q$  is a maximal nilpotent subgroup, then it is a Cartan subgroup if and only if  $Q^\circ$  is a Carter subgroup, by Lemma 5. Finally, a selfnormalizing Carter subgroup must be a Cartan subgroup by Lemma 5(b), and a definably connected Cartan subgroup must be a Carter subgroup.

Definably connected nilpotent groups definable in o-minimal structures are divisible by [9, Theorem 6.10], so it is worth bearing in mind that the following always applies in groups definable in o-minimal structures.

**Fact 6.** [1, Lemma 3.10] *Let  $G$  be a nilpotent group with the dcc and such that  $G^\circ$  is divisible. Then  $G = B * G^\circ$  (central product) for some finite subgroup  $B$  of  $G$ .*

When Fact 6 applies, one can strengthen Lemma 5(b) as follows. Again the following statement is valid in groups definable in an o-minimal structure, because they cannot contain an infinite increasing chain of definably connected subgroups (by the existence of a well behaved notion of dimension [24, Corollary 2.4]).

**Lemma 7.** *Let  $G$  be a group with the dcc. Assume that definably connected definable nilpotent subgroups of  $G$  are divisible, and that  $G$  contains no infinite increasing chain of such subgroups. Then any definably connected definable nilpotent subgroup of  $G$  is contained in a maximal nilpotent subgroup of  $G$ .*

*Proof.* Let  $N$  be a definably connected nilpotent subgroup of  $G$ . By assumption,  $N$  is contained in a definably connected definable nilpotent subgroup  $N_1$  which is maximal for inclusion. It suffices to show that  $N_1$  is then contained in a maximal nilpotent subgroup of  $G$ , and by Fact 3(b) we may consider only definable nilpotent subgroups containing  $N_1$ . It suffices then to show that any strictly increasing chain of definable nilpotent subgroups  $N_1 < N_2 < \dots$  is stationary after finitely many steps.

Assume towards a contradiction that  $N_1 < N_2 < \dots$  is such an infinite increasing chain of definable nilpotent subgroups. Recall that  $N_1 = N_1^\circ$ , and notice also that  $N_i^\circ = N_1$  for each  $i$ , since  $N_1$  is maximal subject to being definably connected and containing  $N$ . By Fact 6, each  $N_i$  has the form  $B_i * N_1$  for some finite subgroup  $B_i \leq N_i$ , and in particular  $N_i \leq C_G(N_1) \cdot N_1$ . We may thus replace  $G$  by the definable subgroup  $C_G(N_1) \cdot N_1$ .

Let  $X$  be the union of the groups  $N_i$ . Working modulo the normal subgroup  $N_1$ , we have an increasing chain of finite nilpotent groups. Now  $X/N_1$  is a periodic locally nilpotent group with the dcc on centralizers, and by [5, Theorem A] it is nilpotent-by-finite. Replacing  $X$  by a finite index subgroup of  $X$  if necessary, we may thus assume  $X/N_1$  nilpotent and infinite. Since  $G = C_G(N_1) \cdot N_1$ , the nilpotency of  $X/N_1$  and of  $N_1$  forces  $X$  to be nilpotent (of nilpotency class bounded by the sum of that of  $X/N_1$  and  $N_1$ ). Replacing  $X$  by  $H(X)$ , we may now assume with Fact 3(b) that  $X$  is a definable nilpotent subgroup containing  $N_1$  as a subgroup of infinite index. Then  $N_1 < X^\circ$ , a contradiction to the maximality of  $N_1$ .  $\square$

Before moving ahead, it is worth mentioning concrete examples of Cartan subgroups of real Lie groups to be kept in mind in the present paper. In  $\mathrm{SL}_2(\mathbb{R})$  there are up to conjugacy two Cartan subgroups, the subgroup of diagonal matrices  $Q_1 \simeq \mathbb{R}^\times$ , noncompact and not connected, with corresponding Carter subgroup  $Q_1^\circ \simeq \mathbb{R}^{>0}$ , and  $Q_2 = \mathrm{SO}_2(\mathbb{R})$  isomorphic to the circle group, compact and connected and hence also a Carter subgroup. More generally, and referring to [21, p.141-142] for more details, the group  $\mathrm{SL}_n(\mathbb{R})$  has up to conjugacy  $\lfloor \frac{n}{2} \rfloor + 1$  Cartan subgroups

$$Q_j \simeq [\mathbb{C}^\times]^{j-1} \times [\mathbb{R}^\times]^{n-2j+1} \text{ where } 1 \leq j \leq \lfloor \frac{n}{2} \rfloor + 1,$$

unless  $Q_{\frac{n}{2}+1} \simeq [\mathbb{C}^\times]^{\frac{n}{2}-1} \times \mathrm{SO}_2(\mathbb{R})$  if  $n = 2(j-1)$ .

We will need the following lemma relating the center to Cartan and Carter subgroups. For any group  $G$  we define the iterated centers  $Z_n(G)$  as follows:  $Z_0(G) = \{1\}$  and by induction  $Z_{n+1}(G)$  is the preimage in  $G$  of the center  $Z(G/Z_n(G))$  of  $G/Z_n(G)$ .

**Lemma 8.** *Let  $G$  be a group and for  $n \geq 0$  let  $Z_n := Z_n(G)$ .*

- (a) *If  $Q$  is a Cartan subgroup of  $G$ , then  $Z_n \leq Q$  and  $Q/Z_n$  is a Cartan subgroup of  $G/Z_n$ , and conversely every Cartan subgroup of  $G/Z_n$  has this form.*
- (b) *If  $G$  satisfies the dcc, then Carter subgroups of  $G/Z_n$  are exactly subgroups of the form  $Q^\circ Z_n/Z_n$ , for  $Q$  a Cartan subgroup of  $G$ .*

*Proof.* We may freely use the fact that the preimage in  $G$  of a nilpotent subgroup of  $G/Z_n$  is nilpotent.

(a). Clearly  $Z_n \leq Q$  by maximal nilpotence of  $Q$ . Clearly also,  $Q/Z_n$  is nilpotent maximal in the quotient  $\overline{G} = G/Z_n$ . Let  $\overline{X}$  be a normal subgroup of finite index of  $\overline{Q} = Q/Z_n$ , for some subgroup  $X$  of  $G$  containing  $Z_n$ . The preimage in  $G$  of  $N_{\overline{G}}(\overline{X})$  normalizes  $X$ , which clearly is normal and has finite index in  $Q$ . Since  $Q$  is a Cartan subgroup of  $G$ , we easily get that  $\overline{X}$  has finite index in  $N_{\overline{G}}(\overline{X})$ .

Conversely, let  $Q$  be a subgroup of  $G$  containing  $Z_n$  such that  $Q/Z_n$  is a Cartan subgroup of  $\overline{G} = G/Z_n$ . Clearly  $Q$  has to be maximal nilpotent in  $G$ . Let  $X$  be a normal finite index subgroup of  $Q$ .  $N_G(X)$  normalizes  $\overline{X}$  modulo  $Z_n$ , so it must contain  $\overline{X}$  as a finite index subgroup, and then  $X$  is also a finite index subgroup of  $N_G(X)$ .

(b). By item (a) Cartan subgroups of  $G/Z_n$  are exactly of the form  $Q/Z_n$  for a Cartan subgroup  $Q$  of  $G$  containing  $Z_n$ . So Carter subgroups of  $G/Z_n$  are by Lemma 5 exactly of the form  $[Q/Z_n]^\circ = Q^\circ Z_n/Z_n$ , for  $Q$  a Cartan subgroup of  $G$ .  $\square$

Finally, we will also use the following lemma describing Cartan subgroups of central products.

**Lemma 9.** *Let  $G = G_1 * \dots * G_n$  be a central product of finitely many and pairwise commuting groups  $G_i$ . Then Cartan subgroups of  $G$  are exactly of the form  $Q_1 * \dots * Q_n$  where each  $Q_i$  is a Cartan subgroup of  $G_i$ .*

*Proof.* It suffices to prove our claim for  $n = 2$ . For  $i = 1$  and  $2$  and  $X$  an arbitrary subset of  $G$ , let  $\pi_i(X) = \{g \in G_i \mid \exists h \in G_{i+1} gh \in X\}$ , where the indices  $i$  are of course considered modulo 2. It is clear that when  $X$  is a subgroup of  $G$ ,  $\pi_i(X)$  is a subgroup  $G_i$ . If  $X$  is nilpotent (of nilpotency class  $k$ ), then  $\pi_i(X)$  is nilpotent (of nilpotency class at most  $k + 1$ ): it suffices to consider  $G/G_{i+1}$  and to use the fact that  $G_1 \cap G_2 \leq Z(G_i)$ .

Let  $Q$  be a Cartan subgroup of  $G_1 * G_2$ . Since  $Q \leq \pi_1(Q) * \pi_2(Q)$ , the maximal nilpotence of  $Q$  forces equality. Now it is clear that each  $\pi_i(Q)$  is maximal nilpotent in  $G_i$ , by maximal nilpotence of  $Q$  again. Let now  $X$  be a normal subgroup of  $\pi_1(Q)$  of finite index. Then  $N_{G_1}(X) * \pi_2(Q)$  normalizes  $X * \pi_2(Q)$  and as the latter is a normal subgroup of finite index in  $Q$  one concludes that  $X$  has finite index in  $N_{G_1}(X)$ . Hence  $\pi_1(Q)$  is a Cartan subgroup of  $G_1$ . Similarly,  $\pi_2(Q)$  is a Cartan subgroup of  $G_2$ .

Conversely, let  $Q$  be a subgroup of  $G$  of the form  $Q_1 * Q_2$  for some Cartan subgroups  $Q_i$  of  $G_i$ . Since each  $Q_i$  is maximal nilpotent in  $G_i$  it follows, considering projections as above, that  $Q$  is maximal nilpotent in  $G$ . Let now  $X$  be a normal subgroup of  $Q$  of finite index. Then  $\pi_i(N_G(X))$  normalizes the normal subgroup of finite index  $\pi_i(X)$  of  $Q_i$ . Since  $Q_i$  is a Cartan subgroup of  $G_i$  it follows that  $\pi_i(X)$  has finite index in  $\pi_i(N_G(X))$ . Finally, since  $X \leq \pi_1(X) * \pi_2(X) \leq Q$ , we get that  $X$  has finite index in  $N_G(X)$ .  $\square$

The special case of a direct product in Lemma 9 has also been observed in [7, Chap. VI, §4, Prop. 3].

**Corollary 10.** *Let  $G = G_1 \times \cdots \times G_n$  be a direct product of finitely many groups  $G_i$ . Then Cartan subgroups of  $G$  are exactly of the form  $Q_1 \times \cdots \times Q_n$  where each  $Q_i$  is a Cartan subgroup of  $G_i$ .*

### 3. DIMENSION AND UNIONS

In this section we work with a structure such that each nonempty definable set is equipped with a dimension in  $\mathbb{N}$  satisfying the following axioms for any nonempty definable sets  $A$  and  $B$ .

- (A1) (**Definability**) If  $f$  is a definable function from  $A$  to  $B$ , then the set  $\{b \in B \mid \dim(f^{-1}(b)) = m\}$  is definable for every  $m$  in  $\mathbb{N}$ .
- (A2) (**Additivity**) If  $f$  is a definable function from  $A$  to  $B$ , whose fibers have constant dimension  $m$  in  $\mathbb{N}$ , then  $\dim(A) = \dim(\text{Im}(f)) + m$ .
- (A3) (**Finite sets**)  $A$  is finite if and only if  $\dim(A) = 0$ .
- (A4) (**Monotonicity**)  $\dim(A \cup B) = \max(\dim(A), \dim(B))$ .

In an o-minimal structure, definable sets are equipped with a finite dimension satisfying all these four axioms, by [32, Chapter 4] or [30]. Hence our reader only interested in groups definable in o-minimal structures may read all the following dimensional computations in the restricted context of such groups. But, as mentioned in the introduction, such computations are relevant in other contexts as well (groups of finite Morley rank, groups in supersimple theories of finite rank, groups definable over the p-adics...), and thus we will proceed with the mere axioms A1-4.

Axioms A2 and A3 guarantee that if  $f$  is a definable bijection between two definable sets  $A$  and  $B$ , then  $\dim(A) = \dim(B)$ . Axiom A4 is a strong form of monotonicity in the sense that  $\dim(A) \leq \dim(B)$  whenever  $A \subseteq B$ .

**Definition 11.** *Let  $\mathcal{M}$  be a first-order structure equipped with a dimension  $\dim$  on definable sets and  $X \subseteq Y$  two definable sets. We say that  $X$  is:*

- (a) weakly generic in  $Y$  whenever  $\dim(X) = \dim(Y)$ .
- (b) generic in  $Y$  whenever  $Y$  is a definable group covered by finitely many translates of  $X$ .
- (c) large in  $Y$  whenever  $\dim(Y \setminus X) < \dim(Y)$ .

Clearly, genericity and largeness both imply weak genericity when the dimension satisfies axioms A1-4. If  $G$  is a group definable in an o-minimal structure and  $X$  is a large definable subset of  $G$ , then  $X$  is generic: see [30, Lemma 2.4] for a proof by compactness, and [25, Section 5] for a proof with precise bounds on the number of translates needed for genericity. In the sequel we are only going to use dimensional computations, hence the notions of weak genericity and of largeness. We are not going to use the notion of genericity (which is imported from the theory of stable groups in model theory), but we will make some apparently quite new remarks on genericity and Cartan subgroups in real Lie groups (Remark 56 below).

Our arguments for Cartan subgroups in groups definable in o-minimal structures will highly depend on computations of the dimension of their unions in the style of [19], and to compute the dimension of a union of definable sets we adopt the following geometric argument essentially due to Cherlin.

Assume from now on that  $X_a$  is a uniformly definable family of definable sets, with  $a$  varying in a definable set  $A$  and such that  $X_a = X_{a'}$  if and only if  $a = a'$ . We

have now a combinatorial geometry, where the set of points is  $U := \bigcup_{a \in A} X_a$ , the set of lines is the set  $\{X_a \mid a \in A\}$  in definable bijection with  $A$ , and the incidence relation is the natural one. The set of *flags* is then defined to be the subset of couples  $(x, a)$  of  $U \times A$  such that  $x \in X_a$ . By projecting the set of flags on the set of points, one sees with axiom A1 that for any  $r$  such that  $0 \leq r \leq \dim(A)$ , the set

$$U_r := \{x \in U \mid \dim(\{a \in A \mid x \in X_a\}) = r\}$$

is definable. In particular, each subset of the form  $[X_a]_r := X_a \cap U_r$ , i.e., the set of points  $x$  of  $X_a$  such the set of lines passing through  $x$  has dimension  $r$ , is definable as well.

**Proposition 12.** *In a structure equipped with a dimension satisfying axioms A1-2, let  $X_a$  be a uniformly definable family of sets, with  $a$  varying in a definable set  $A$  and such that  $X_a = X_{a'}$  if and only if  $a = a'$ . Suppose, for some  $r$  such that  $0 \leq r \leq \dim(A)$ , that  $[X_a]_r$  is nonempty and that  $\dim([X_a]_r)$  is constant as  $a$  varies in  $A$ . Then*

$$\dim(U_r) + r = \dim(A) + \dim([X_a]_r).$$

*Proof.* One can consider the definable subflag associated to  $U_r = [\bigcup_{a \in A} X_a]_r$  in the point/line incidence geometry described above. By projecting this definable set on the set of points and on the set of lines respectively, one finds using axiom A2 of the dimension the desired equality as in [19, §2.3].  $\square$

Given a permutation group  $(G, \Omega)$  and a subset  $X$  of  $\Omega$ , we denote by  $N(X)$  and by  $C(X)$  the *setwise* and the *pointwise* stabilizer of  $X$  respectively, that is  $G_{\{X\}}$  and  $G_{(X)}$  in a usual permutation group theory notation. We also denote by  $X^G$  the set  $\{x^g \mid (x, g) \in X \times G\}$ , where  $x^g$  denotes the image of  $x$  under the action of  $g$ , as in the case of an action by conjugation. Subsets of the form  $X^g$  for some  $g$  in  $G$  are also called *G-conjugates* of  $X$ . Notice that the set  $X^G$  can be seen, alternatively, as the union of  $G$ -orbits of elements of  $X$ , or also as the union of  $G$ -conjugates of  $X$ . When considering the action of a group on itself by conjugation, as we will do below, all these terminologies and notations are the usual ones, with  $N(X)$  and  $C(X)$  the *normalizer* and the *centralizer* of  $X$  respectively.

We shall now apply Proposition 12 in the context of permutation groups in a way much reminiscent of [20, Fact 4]. For that purpose we will need that the dimension is well defined on certain imaginaries, and for that purpose we will make the simplifying assumption that the theory considered eliminates such specific imaginaries. We recall that groups definable in o-minimal structures eliminate all imaginaries by [9, Theorem 7.2], so these technical assumptions will always be verified in this context. (And our arguments are also valid in any context where the dimension is well defined and compatible in the relevant imaginaries.) For any quotient  $X/\sim$  associated to an equivalence relation  $\sim$  on a set  $X$ , we call *transversal* any subset of  $X$  intersecting each equivalence class in exactly one point.

**Corollary 13.** *Let  $(G, \Omega)$  be a definable permutation group in a structure equipped with a dimension satisfying axioms A1-3,  $X$  a definable subset of  $\Omega$  such that  $G/N(X)$  (right cosets) has a definable transversal  $A$ . Suppose that, for some  $r$  between 0 and  $\dim(A)$ , the definable subset  $X_r := \{x \in X \mid \dim(\{a \in A \mid x \in X^a\}) = r\}$  is nonempty. Then*

$$\dim(X_r^G) = \dim(G) + \dim(X_r) - \dim(N(X)) - r.$$

*Proof.* We can apply Proposition 12 with the uniformly definable family of  $G$ -conjugates of  $X$ , which is parametrized as  $\{X^a \mid a \in A\}$  since  $A$  is a definable transversal of  $G/N(X)$ . Notice that the sets  $[X^a]_r$  are in definable bijection, as pairwise  $G$ -conjugates, and hence all have the same dimension. Notice also that  $\dim(A) = \dim(G) - \dim(N(X))$  by the additivity of the dimension and its invariance under definable bijections.  $\square$

The following corollary, which is crucial in the sequel, can be compared to [20, Corollary 5].

**Corollary 14.** *Assume furthermore in Corollary 13 that the dimension satisfies axiom A4, and that  $\dim(G) = \dim(\Omega)$  and  $\dim(X) \leq \dim(N(X))$ . Then*

$$\dim(X^G) = \dim(\Omega) \text{ if and only if } \dim(X_0) = \dim(N(X)) (= \dim(X)).$$

*In this case,  $X_0^G$  is large in  $X^G$ .*

*Proof.* If  $\dim(X^G) = \dim(\Omega)$ , then one has for some  $r$  as in Corollary 13 that  $\dim(X_r^G) = \dim(\Omega)$  by axiom A4, and then

$$0 \leq r = \dim(X_r) - \dim(N(X)) \leq \dim(X) - \dim(N(X)) \leq 0$$

by monotonicity of the dimension, showing that all these quantities are equal to 0. In particular  $r = 0$ , and  $\dim(X_0) = \dim(N(X))$ . Conversely, if  $\dim(X_0) = \dim(N(X))$ , then  $\dim(X_0^G) = \dim(G) = \dim(\Omega)$  by Corollary 13.

Assume now the equivalent conditions above are satisfied. The first part of the proof above shows that  $\dim(X_r^G) = \dim(X^G) (= \dim(\Omega))$  can occur only for  $r = 0$ . Hence  $X_0^G$  is large in  $X^G$  by axiom A4 again.  $\square$

**Remark 15.** *In general it seems one cannot conclude also that  $X_0$  is large in  $X$  in Corollary 14. One could imagine the (bizarre) configuration in which  $\dim(X_r) = \dim(X)$  for some  $r > 0$ ; in this case  $\dim(X_r^G) = \dim(\Omega) - r$ .*

In the remainder we will always consider the action of a group  $G$  on itself by conjugation, so the condition  $\dim(G) = \dim(\Omega)$  will always be met in Corollary 14. Then we can apply Corollary 14 with  $X$  any normalizing coset of a definable subgroup  $H$  of  $G$ , as commented in [20, page 1064]. More generally, we now see that we can apply it simultaneously to finitely many such cosets. We first elaborate on the notion of *generosity* defined in [19] and [20] in the finite Morley rank case.

**Definition 16.** *Let  $X$  be a definable subset of a group  $G$  definable in a structure equipped with a dimension satisfying axioms A1-4. We say that  $X$  is*

- (a) *weakly generous in  $G$  whenever  $X^G$  is weakly generic in  $G$ .*
- (b) *generous in  $G$  whenever  $X^G$  is generic in  $G$ .*
- (c) *largely generous in  $G$  whenever  $X^G$  is large in  $G$ .*

**Corollary 17.** *Suppose  $H$  is a definable subgroup of a group  $G$  definable in a structure equipped with a dimension satisfying axioms A1-4, and suppose  $W$  is a finite subset of  $N(H)$  such that  $G/N(WH)$  has a definable transversal. Then  $WH$  is weakly generous in  $G$  if and only if*

$$\dim([WH]_0) = \dim(N(WH)).$$

*In this case,  $[WH]_0^G$  is large in  $[WH]^G$ , and  $\dim([WH]_0) = \dim(WH) = \dim(H) = \dim(N(WH))$ .*

*Proof.* Let  $X = WH$ . Since  $W$  is finite,  $X$  is definable. In order to apply Corollary 14, one needs to check that  $\dim(X) \leq \dim(N(X))$ . Of course, the subgroup  $H$  normalizes each coset  $wH$ , for each  $w \in W \subseteq N(H)$ , and in particular  $H \leq N(WH)$ . We get thus that  $\dim(X) = \dim(WH) = \dim(H) \leq \dim(N(WH)) = \dim(N(X))$ .

Now Corollary 14 gives our necessary and sufficient condition, and the largeness of  $[WH]_0^G$  in  $[WH]^G$ . It also gives  $\dim(X_0) = \dim(X) = \dim(N(X))$ . We have seen already that  $\dim(X) = \dim(H)$ .  $\square$

The following lemma is a fundamental trick below.

**Lemma 18.** *Let  $G$  be a group definable in a structure equipped with a dimension satisfying axioms A1-4 and with the dcc. Let  $X$  be a definable subset of  $G$ ,  $X_0$  the subset of elements of  $X$  contained in only finitely many  $G$ -conjugates of  $X$ , and  $U$  a definable subset of  $X$  such that  $U \cap X_0 \neq \emptyset$ . Then  $N^\circ(U) \leq N(X)$ .*

*Proof.* As in [19, Lemma 3.3], essentially via Fact 2(a).  $\square$

#### 4. COSETS ARGUMENTS

Corollary 17 will be used at the end of this paper in certain arguments reminiscent of a theory of Weyl groups from [20]. Since such specific arguments follow essentially from Corollary 17 we insert here, as a warm up, a short section devoted to them.

**Theorem 19.** *Let  $G$  be a group definable in a structure equipped with a dimension satisfying axioms A1-4 and with the dcc,  $H$  a weakly generous definable subgroup of  $G$ , and  $w$  an element normalizing  $H$  and such that  $G/N(H)$  has a definable transversal. Then one the following must occur:*

- (a) *The coset  $wH$  is weakly generous in  $G$ , or*
- (b) *The definable set  $\{h^{w^{n-1}}h^{w^{n-2}} \dots h \mid h \in H\}$  is not large in  $H$  for any multiple  $n$  of the (necessarily finite) order of  $w$  modulo  $H$ . If  $w$  centralizes  $H$ , then  $\{h^n \mid h \in H\}$  is not large in  $H$ .*

*Proof.* We proceed essentially as in [20, Lemmas 11-12]. Assume  $wH$  not weakly generous. In particular  $w \in N(H) \setminus H$  since  $H$  is weakly generous by assumption. By Corollary 17,  $H_0$  is weakly generic in  $N(H)$ ; in particular  $H$  has finite index in  $N(H)$ . Of course,  $N(wH) \leq N(H)$  since  $H = \{ab^{-1} : a, b \in wH\}$ , and one sees then that  $N(wH)$  is exactly the preimage in  $N(H)$  of the centralizer of  $w$  modulo  $H$ . To summarize,  $H \leq N(wH) \leq N(H)$ , with  $N(H)/H$  finite. In particular  $w$  has finite order modulo  $H$ . Notice also at this stage that  $G/N(wH)$  has a definable transversal (of the form  $AX$  where  $X$  is a definable transversal of  $G/N(H)$  and  $A$  is a definable transversal of the finite quotient  $N(H)/N(wH)$ ). Since we assume  $wH$  not weakly generous, Corollary 17 implies that  $[wH]_0$  is not weakly generic in  $wH$ . In other words, the (definable) set of elements of the coset  $wH$  contained in infinitely many  $G$ -conjugates of  $wH$  is large in  $wH$ .

Assume towards a contradiction  $\{h^{w^{n-1}}h^{w^{n-2}} \dots h \mid h \in H\}$  large in  $H$  for  $n$  a multiple of the finite order of  $w$  modulo  $H$ . Let  $\phi : wh \mapsto (wh)^n$  denote the definable map, from  $wH$  to  $H$ , consisting of taking  $n$ -powers. As

$$\phi(wH) = w^n \cdot \{h^{w^{n-1}}h^{w^{n-2}} \dots h \mid h \in H\}$$

our contradictory assumption forces that  $\phi(wH)$  must be large in  $H$ .

Then  $H_0 \cap \phi(wH)$  must be weakly generic in  $H$ . Since the dimension can only get down when taking images by definable functions,  $\phi^{-1}(H_0 \cap \phi(wH))$  necessarily has to be weakly generic in the coset  $wH$ . Therefore one finds an element  $x$  in the intersection of this preimage with the large subset  $[wH] \setminus [wH]_0$  of elements of  $wH$  contained in infinitely many  $G$ -conjugates of  $wH$ . Now since  $w^n \in H$  and  $N(wH)$  has finite index in  $N(H)$  it follows that  $\phi(x) = x^n$  belongs to infinitely many  $G$ -conjugates of  $H$ , a contradiction since  $\phi(x)$  belongs to  $H_0$ . This proves our main statement in case (b).

For our last remark in case (b), notice that when  $w$  centralizes  $H$  one has  $\{h^{w^{n-1}}h^{w^{n-2}} \dots h \mid h \in H\} = \{h^n \mid h \in H\}$ .  $\square$

**Corollary 20.** *Suppose additionally in Theorem 19 that  $w$  has order  $n$  modulo  $H$  and that  $H$  is  $n$ -divisible ( $n \geq 1$ ). Then one of the following must occur:*

- (a) *The coset  $wH$  is weakly generous in  $G$ , or*
- (b)  *$C_H(w)$  is a proper subgroup of  $H$ .*

*Proof.* Suppose that both alternatives fail. Then  $\{h^n \mid h \in H\}$  is not large in  $H$  by Theorem 19, a contradiction since this set is  $H$  by  $n$ -divisibility.  $\square$

The following corollary of Theorem 19 will be particularly adapted in the sequel to Cartan subgroups of groups definable in o-minimal structures.

**Corollary 21.** *Suppose additionally in Theorem 19 that  $H$  is definably connected and divisible and that  $\langle w \rangle H$  is nilpotent. Then the coset  $wH$  is weakly generous in  $G$ .*

*Proof.* This is clear if  $w$  is in  $H$ , so we may assume  $w \in N(H) \setminus H$ . As above  $w$  has finite order modulo  $H = H^\circ$ . By *dcc* of the ambient group and [1, Lemma 3.10], the coset  $wH$  contains a torsion element which commutes with  $H = H^\circ$ , and thus we may assume  $C_H(w) = H$ . By divisibility of  $H = H^\circ$ ,  $\{h^n \mid h \in H\} = H$  is large in  $H$ , and by Theorem 19 the coset  $wH$  must be weakly generous in  $G$ .  $\square$

We will also use the following more specialized results in the same spirit, which apply as usual to nilpotent groups definable in o-minimal structures by [9, Theorem 6.10].

**Lemma 22.** *Let  $H$  be a nilpotent divisible group definable in a structure equipped with a dimension satisfying axioms A1-4, with the *dcc*, and with no infinite elementary abelian  $p$ -subgroups for any prime  $p$ . Let  $\phi$  be the map consisting of taking  $n$ -th powers for some  $n \geq 1$ . If  $X$  is a weakly generic definable subset of  $H$ , then  $\phi(X)$  is weakly generic as well.*

*Proof.* Considering the dimension, it suffices to show that  $\phi$  has finite fibers. Suppose  $a^n = b^n$  for some elements  $a$  and  $b$  in  $H$ . If  $aZ(H) = bZ(H)$ , then our assumption forces, with  $a$  fixed, that  $b$  can only vary in a finite set, as desired. Hence, working in  $H/Z(H)$ , it suffices to show that  $a^n = b^n$  implies  $a = b$ . But by [1, Lemma 3.10(a')] all definable sections of  $H/Z(H)$  are torsion-free, and our claim follows easily by induction on the nilpotency class of  $H/Z(H)$ .  $\square$

**Corollary 23.** *Let  $Q$  be a nilpotent group definable in a structure equipped with a dimension satisfying axioms A1-4, with the *dcc*, and with no infinite elementary abelian  $p$ -subgroups for any prime  $p$ . Suppose  $Q^\circ$  divisible, and let  $a \in Q$ ,  $n$  a multiple of the order of  $a$  modulo  $Q^\circ$ , and  $\phi$  the map consisting of taking  $n$ -th powers. If  $X$  is a weakly generic definable subset of  $aQ^\circ$ , then  $\phi(X)$  is a weakly generic subset of  $Q^\circ$ .*

*Proof.* By [1, Lemma 3.9], we may assume that  $a$  centralizes  $Q^\circ$ . Now for any  $x \in Q^\circ$  we have  $\phi(ax) = a^n x^n$ . Hence, if  $x$  varies in a weakly generic definable subset  $X$  of  $Q^\circ$ , then  $\phi(ax)$  also by Lemma 22 in  $H = Q^\circ$ .  $\square$

## 5. GENEROSITY AND LIFTING

In the present section we study the behaviour of weak or large generosity when passing to quotients by definable normal subgroups. We continue with the mere axioms A1-4 of Section 3 for the dimension, and with the existence of definable transversal for certain imaginaries to ensure that their dimensions is also well defined. As above, everything applies in particular to groups definable in o-minimal structures.

**Proposition 24.** *Let  $G$  be a group definable in a structure equipped with a dimension satisfying axioms A1-4,  $N$  a definable normal subgroup of  $G$ ,  $H$  a definable subgroup of  $G$  containing  $N$ , and  $Y$  a definable subset of  $H$  large in  $H$ . Suppose also that  $G/N$  and  $G/N(H \setminus Y^H)$  have definable transversals.*

- (a) *If  $H/N$  is weakly generous in  $G/N$ , then  $Y$  is weakly generous in  $G$ .*
- (b) *If  $H/N$  is largely generous in  $G/N$ , then  $Y$  is largely generous in  $G$ .*

*Proof.* First note that  $H^G$  is a union of cosets of  $N$ , since  $N \leq H$  and  $N \trianglelefteq G$ . Hence the weak (resp. large) generosity of  $H/N$  in  $G/N$  forces the weak (resp. large) generosity of  $H$  in  $G$ . In any case,  $\dim(H^G) = \dim(G)$ .

Replacing  $Y$  by  $Y^H$  if necessary, we may assume  $H \leq N(Y)$  and  $Y$  large in  $H$ .

**Claim 25.** *Let  $Z = H \setminus Y$ . Then  $Z^G$  cannot be weakly generic in  $H^G$ .*

*Proof.* Suppose  $Z^G$  weakly generic in  $H^G$ . Then  $\dim(Z^G) = \dim(H^G) = \dim(G)$ . Since  $Z \subseteq H \subseteq N_G(Z)$ , Corollary 14 yields  $\dim(Z) = \dim(N_G(Z))$ . In particular  $\dim(Z) = \dim(H)$ , a contradiction to the largeness of  $Y$  in  $H$ .  $\square$

(a). Since  $\dim(H^G) = \dim(G)$  and  $H^G = Y^G \cup Z^G$ , Claim 25 yields  $\dim(Y^G) = \dim(G)$ .

(b). In this case  $H^G$  is large in  $G$ . Since  $G = (G \setminus H^G) \sqcup (H^G \setminus Y^G) \sqcup Y^G$ , Claim 25 now forces  $Y^G$  to be large in  $G$ .  $\square$

**Corollary 26.** *Assume that  $G$ ,  $N$ ,  $H$ , and  $Y$  are as in Proposition 24, and that  $Y = Q^H$  for some largely generous definable subgroup  $Q$  of  $H$ .*

- (a) *If  $H/N$  is weakly generous in  $G/N$ , then so is  $Q$  in  $G$*
- (b) *If  $H/N$  is largely generous in  $G/N$ , then so is  $Q$  in  $G$ .*

*Proof.* It suffices to apply Proposition 24 with  $Y = Q^H$ , noticing that  $Y^G = Q^G$ .  $\square$

**Corollary 27.** *Assume furthermore that  $Q$  is a Carter subgroup of  $H$  in Corollary 26, and that  $N_G(Q)/Q$  has a definable transversal. Then, in both cases (a) and (b),  $Q$  is a Carter subgroup of  $G$ .*

*Proof.* By definition,  $Q$  is definable, definably connected, and nilpotent. So it suffices to check that  $Q$  is a finite index subgroup of  $N_G(Q)$ . But in any case, it follows from the weak generosity of  $Q$  in  $G$  given in Corollary 26 and from Corollary 17 that  $\dim(Q) = \dim(N_G(Q))$ . Now axiom A3 applies.  $\square$

## 6. WEAKLY GENEROUS NILPOTENT SUBGROUPS

In the present section we shall rework arguments from [19] concerning weakly generous Carter subgroups. Throughout the section,  $G$  is a group definable in a structure with a dimension satisfying axioms A1-4, and with the *dcc*. As in the preceding sections, everything applies in particular to groups definable in an o-minimal structure.

**Lemma 28.** *Let  $G$  be a group definable in a structure with a dimension satisfying axioms A1-4, and with the *dcc*. Let  $H$  be a definable subgroup of  $G$  such that  $N^\circ(H) = H^\circ$ ,  $H_0$  the set of elements of  $H$  contained in only finitely many conjugates of  $H$ , and  $N$  a definable nilpotent subgroup of  $G$  such that  $N \cap H_0$  is nonempty. Then  $N^\circ \leq H^\circ$ .*

*Proof.* Let  $U = N \cap H$ . By assumption  $U \cap H_0$  is nonempty, so by Lemma 18  $N^\circ(U) \leq N^\circ(H) = H^\circ$ . In particular,  $N_N^\circ(U) \leq (N \cap H)^\circ = U^\circ$ , which shows that  $U$  has finite index in  $N_N(U)$ . Now Lemma 4 shows that  $U$  must have finite index in  $N$ , and in particular  $U^\circ = N^\circ$ . Hence,  $N^\circ = (N \cap H)^\circ \leq H^\circ$ .  $\square$

**Corollary 29.** *Let  $G$  be a group definable in a structure with a dimension satisfying axioms A1-4, and with the *dcc*. Let  $Q$  be a definable nilpotent weakly generous subgroup of  $G$  such that  $G/N(Q)$  has a definable transversal, and let  $Q_0$  denote the set of elements of  $Q$  contained in only finitely many conjugates of  $Q$ . Then:*

- (a) *For any definable nilpotent subgroup  $N$  such that  $N \cap Q_0 \neq \emptyset$ , we have  $N^\circ \leq Q^\circ$ .*
- (b) *For any  $g$  in  $G$  such that  $Q_0 \cap Q^g \neq \emptyset$ , we have that  $Q^\circ = [Q^\circ]^g$ .*

*Proof.* (a). As  $Q$  is weakly generous, we have  $N^\circ(Q) = Q^\circ$  by Corollary 17. Hence Lemma 28 gives  $N^\circ \leq Q^\circ$ . (b). Item (a) applied with  $N = Q^g$  yields  $[Q^\circ]^g = [Q^g]^\circ \leq Q^\circ$ . Now applying Lemma 4 shows that  $[Q^\circ]^g$  cannot be of infinite index in  $Q^\circ$  (as otherwise we would contradict that  $N^\circ(Q) = Q^\circ$ ), and thus  $[Q^\circ]^g = Q^\circ$ .  $\square$

**Corollary 30.** *Suppose in addition in Corollary 29 that  $Q$  is a Carter subgroup of  $G$ . Then, for any  $g \in Q_0$  and any definably connected definable nilpotent subgroup  $N$  containing  $g$ , we have  $N \leq Q$ . In particular,  $Q$  is the unique maximal definably connected definable nilpotent subgroup containing  $g$ , and the distinct conjugates of  $Q_0$  are indeed disjoint, forming thus a partition of a weakly generic subset of  $G$ .*

*Proof.* It suffices to apply Corollary 29.  $\square$

As a result one also obtains the following general theorem, which can be compared to the main result of [19].

**Theorem 31.** *Let  $G$  be a group definable in a structure with a dimension satisfying axioms A1-4, and with the *dcc*. Then  $G$  has at most one conjugacy class of largely generous Carter subgroups  $Q$  such that  $G/N(Q)$  has a definable transversal. If such a Carter subgroup exists, then the set of elements contained in a unique conjugate of that Carter subgroup is large in  $G$ .*

*Proof.* Let  $P$  and  $Q$  be two largely generous Carter subgroups of  $G$ . We want to show that  $P$  and  $Q$  are conjugate. We have  $P_0^G$  and  $Q_0^G$  large in  $G$  by Corollary 17. Since the intersection of two large sets is nontrivial (and in fact large as well), we get that  $P_0^G \cap Q_0^G$  is nonempty, so after conjugation we may thus assume  $P_0 \cap Q_0$  nonempty. But then Corollary 30 gives  $P = Q$ .

Our last claim follows also from Corollary 30.  $\square$

7. ON GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

We shall now collect results specific to groups definable in o-minimal structures which are needed in the sequel. We recall that groups definable in o-minimal structures satisfy the *dcc* on definable subgroups [30, Remark 2.13], and o-minimal structures are equipped with a dimension satisfying axioms A1-4 considered in the previous sections [32, Chapter 4]. As commented before, we can freely apply all the results of the preceding sections to the specific case of groups definable in an o-minimal structure. We also recall that all the technical assumptions on the existence of transversals in Sections 3-6 are satisfied, since groups definable in o-minimal structures eliminate all imaginaries by [9, Theorem 7.2]. As mentioned already in the introduction, we consider only groups  $G$  definable in an o-minimal structure, but [9, Theorem 7.2] also allows one to consider any group of the form  $K/L$ , where  $L \trianglelefteq K \leq G$  are definable subgroups, as definable.

**Fact 32.** [1, §6] *Let  $G$  be a group definable in an o-minimal structure, with  $G^\circ$  solvable, and  $A$  and  $B$  two definable subgroups of  $G$  normalizing each other. Then  $[A, B]$  is definable, and definably connected whenever  $A$  and  $B$  are.*

Any group  $G$  definable in an o-minimal structure has a largest normal nilpotent subgroup  $F(G)$ , which is also definable [1, Fact 3.5], and a largest normal solvable subgroup  $R(G)$ , which is also definable [1, Lemma 4.5].

**Fact 33.** *Let  $G$  be a definably connected solvable group definable in an o-minimal structure.*

- (a) [9, Theorem 6.9]  $G'$  is nilpotent.
- (b) [1, Proposition 5.5]  $G' \leq F^\circ(G)$ . In particular  $G/F^\circ(G)$  and  $G/F(G)$  are divisible abelian groups.
- (c) [1, Corollary 5.6] If  $G$  is nontrivial, then  $F^\circ(G)$  is nontrivial. In particular  $G$  has an infinite abelian characteristic definable subgroup.
- (d) [1, Lemma 3.6] If  $G$  is nilpotent and  $H$  is an infinite normal subgroup of  $G$ , then  $H \cap Z(G)$  is infinite.

If  $H$  and  $G$  are two subgroups of a group with  $G$  normalizing  $H$ , then a  $G$ -minimal subgroup of  $H$  is an infinite  $G$ -invariant definable subgroup of  $H$ , which is minimal with respect to these properties (and where definability refers to the fixed underlying structure, as usual). If  $H$  is definable and satisfies the *dcc* on definable subgroups, then  $G$ -minimal subgroups of  $H$  always exist. As the definably connected component of a definable subgroup is a definably characteristic subgroup, we get also in this case that any  $G$ -minimal subgroup of  $H$  should be definably connected.

**Lemma 34.** *Let  $G$  be a definably connected solvable group definable in an o-minimal structure, and  $A$  a  $G$ -minimal subgroup of  $G$ . Then  $A \leq Z^\circ(F(G))$ , and  $C_G(a) = C_G(A)$  for every nontrivial element  $a$  in  $A$ .*

*Proof.* By Fact 33(c),  $A$  has an infinite characteristic abelian definable subgroup. Therefore the  $G$ -minimality of  $A$  forces  $A$  to be abelian. In particular,  $A \leq F(G)$ . Since  $A$  is normal in  $F(G)$ , Fact 33(d) and the  $G$ -minimality of  $A$  now force that  $A \leq Z(F(G))$ . Since  $A$  is definably connected, we have indeed  $A \leq Z^\circ(F(G))$ .

Now  $F(G) \leq C_G(A)$ , and  $G/C_G(A)$  is definably isomorphic to a quotient of  $G/F(G)$ . In particular  $G/C_G(A)$  is abelian by Fact 33(b). If  $A \leq Z(G)$ , then

clearly  $C_G(a) = C_G(A)$  ( $= G$ ) for every  $a$  in  $A$ , and thus we may assume  $G/C_G(A)$  infinite. Consider the semidirect product  $A \rtimes (G/C_G(A))$ . Since  $A$  is  $G$ -minimal,  $A$  is also  $G/C_G(A)$ -minimal. Now an o-minimal version of Zilber's Field Interpretation Theorem for groups of finite Morley rank [27, Theorem 2.6] applies directly to  $A \rtimes (G/C_G(A))$ . It says that there is an infinite interpretable field  $K$ , with  $A \simeq K_+$  and  $G/C_G(A)$  an infinite subgroup of  $K^\times$ , and such that the action of  $G/C_G(A)$  on  $A$  corresponds to scalar multiplication. In particular,  $G/C_G(A)$  acts *freely* (or *semiregularly* in another commonly used terminology) on  $A \setminus \{1\}$ . This means exactly that for any nontrivial element  $a$  in  $A$ ,  $C_G(a) \leq C_G(A)$ , i.e.,  $C_G(a) = C_G(A)$ .  $\square$

For definably connected groups definable in an o-minimal structure which are not solvable, our study of Cartan subgroups will make heavy use of the main theorem about groups definable in o-minimal structures. It can be summarized as follows, compiling several papers to which we will refer immediately after the statement. Recall that a group is *definably simple* if the only definable normal subgroups are the trivial and the full subgroup.

**Fact 35.** *Let  $G$  be a definably connected group definable in an o-minimal structure  $\mathcal{M}$ . Then*

$$G/R(G) = G_1 \times \cdots \times G_n$$

where each  $G_i$  is a definably simple infinite definable group. Furthermore, for each  $i$ , there is an  $\mathcal{M}$ -definable real closed field  $R_i$  such that  $G_i$  is  $\mathcal{M}$ -definably isomorphic to a semialgebraically connected semialgebraically simple linear semialgebraic group, definable in  $R_i$  over the subfield of real algebraic numbers of  $R_i$ .

Besides, for each  $i$ , either

- (a)  $\langle G_i, \cdot \rangle$  and  $\langle R_i(\sqrt{-1}), +, \cdot \rangle$  are bi-interpretable; in this case  $G_i$  is definably isomorphic in  $\langle G_i, \cdot \rangle$  to the  $R_i(\sqrt{-1})$ -rational points of a linear algebraic group, or
- (b)  $\langle G_i, \cdot \rangle$  and  $\langle R_i, +, \cdot \rangle$  are bi-interpretable; in this case  $G_i$  is definably isomorphic in  $\langle G_i, \cdot \rangle$  to the connected component of the  $R_i$ -rational points of an algebraic group without nontrivial normal algebraic subgroups defined over  $R_i$ .

The description of  $G/R(G)$  as direct product of definably simple definable groups can be found in [26, 4.1]. The second statement about definably simple groups is in [26, 4.1 & 4.4], with the remark concerning the parameters in the proof of [28, 5.1]. The final alternative for each factor, essentially between the complex case and the real case, is in [27, 1.1].

When applying Fact 35 in the sequel we will also use the following.

**Remark 36.** *Let  $\mathcal{M}$  be an o-minimal structure,  $R$  a real closed field definable in  $\mathcal{M}$ , and  $X$  an  $R$ -definable subset of some  $R^n$ . Then  $\dim_{\mathcal{M}}(X) = \dim_R(X)$ .*

*Proof.* By o-minimality,  $\mathcal{M}$  is a geometric structure [27, Definition 3.2]. Moreover, since  $\dim_{\mathcal{M}}(R) = 1$  by [30, Proposition 3.11] we deduce that  $R$  itself is  $R$ -minimal in the sense of [27, Definition 3.3]. Hence by [27, Lemma 3.5] we have that  $\dim_{\mathcal{M}}(X) = \dim_{\mathcal{M}}(R) \dim_R(X) = \dim_R(X)$ .  $\square$

We finish the present section with specific results about definably compact groups which might be used when such specific groups are involved in the sequel.

**Fact 37.** *Let  $G$  be a definably compact definably connected group definable in an o-minimal structure.*

- (a) [29, Corollary 5.4] *Either  $G$  is abelian or  $G/Z(G)$  is semisimple. In particular, if  $G$  is solvable, then it is abelian.*
- (b) [10, Proposition 1.2]  *$G$  is covered by a single conjugacy class of a definably connected definable abelian subgroup  $T$  such that  $\dim(T)$  is maximal among dimensions of abelian definable subgroups of  $G$ .*

For a variation on Fact 37(b), see also [2, Corollary 6.13]. With Fact 37 we can entirely clarify properties of Cartan subgroups in the specific case of definably compact groups definable in o-minimal structures, with a picture entirely similar to that in compact real Lie groups.

**Corollary 38.** *Let  $G$  be a definably compact definably connected group definable in an o-minimal structure. Then Cartan subgroups  $T$  of  $G$  exist and are abelian, definable, definably connected, and conjugate, and  $G = T^G$ .*

*Proof.* Let  $T$  be a definably connected abelian subgroup as in Fact 37(b). Since  $G = T^G$ ,  $T$  is in particular weakly generous, and thus of finite index in its normalizer by Corollary 17. Hence  $T$  is a Carter subgroup of  $G$ . Since  $G = T^G$  again, and  $t \in T \leq C^\circ(t)$  for every  $t \in T$ , we have the property that  $g \in C^\circ(g)$  for every  $g$  in  $G$ .

We now prove our statement by induction on  $\dim(G)$ . By Lemma 5,  $T \leq Q$  for some Cartan subgroup such that  $Q^\circ = T$ . This takes care of the existence of Cartan subgroups of  $G$ , and their definability follows from Lemma 5(a). We also have  $G = T^G$ . We now claim that  $T = Q$ . Otherwise,  $T = Q^\circ < Q$ , and we find by Fact 6 an element  $a$  in  $Q \setminus T$  centralizing  $T$ . Since  $a \in T^g$  for some  $g \in G$ , we have  $T$  and  $T^g$  in  $C^\circ(a)$ . Now the Carter subgroups  $T$  and  $T^g$  of  $C^\circ(a)$  are conjugate by an element of  $C^\circ(a)$ , obviously if  $C^\circ(a) = G$  and by induction otherwise. Since  $a \in T^g \leq C^\circ(a)$ , we get  $a \in T$ , a contradiction. Hence  $T = Q$  is a Cartan subgroup of  $G$ .

It remains just to show that Cartan subgroups of  $G$  are conjugate. Let  $Q_1$  be an arbitrary Cartan subgroup of  $G$ , and  $z$  a nontrivial element of  $Z(Q_1)$  (Fact 6 and Fact 33(d)). We also have  $z \in T^g$  for some  $g \in G$ , and thus  $Q_1, T^g \leq C(z)$ . If  $C^\circ(z) < G$ , the induction hypothesis applied in  $C^\circ(z)$  yields the conjugacy of  $Q_1^\circ$  and of  $T$ , giving also  $Q_1 = Q_1^\circ$  by maximal nilpotence of  $T$ . So we may assume  $z \in Z(G)$ . If  $Z(G)$  is finite, then  $G/Z(G)$  has a trivial center by Fact 2(b), and the previous argument applied in  $G/Z(G)$ , together with Lemma 8(a), yields the conjugacy of  $Q_1$  and  $T$ . Remains the case  $Z(G)$  infinite: then applying the induction hypothesis in  $G/Z(G)$ , and using Lemma 8(a), also gives the conjugacy of  $Q_1$  and  $T$ . This completes our proof.  $\square$

We have seen in the proof of Corollary 38 that the “maximal definable-tori”  $T$  of Fact 37(b) must be Cartan subgroups, and then the two types of subgroups coincide by the conjugacy of Cartan subgroups. We note that the conjugacy of the “maximal definable-tori”  $T$  as in Fact 37(b) was also shown in [10]. Besides, we note that the maximal nilpotence of a Cartan subgroup  $T$  of a group  $G$  always implies that  $C_G(T) = Z(T)$ . In particular, in Corollary 38,  $C(T) = T$  and the “Weyl group”  $W(G, T) := N(T)/C(T)$  acts faithfully on  $T$ .

Finally, we take this opportunity to mention, parenthetically, a refinement of Fact 37(a).

**Corollary 39.** *Let  $G$  be a definably compact definably connected group definable in an  $o$ -minimal structure. Then  $R(G) = Z(G)$ .*

*Proof.* By Fact 37(a) and [1, Lemma 3.13].  $\square$

## 8. THE DEFINABLY CONNECTED SOLVABLE CASE

In the present section we are going to prove the following theorem.

**Theorem 40.** *Let  $G$  be a definably connected solvable group definable in an  $o$ -minimal structure. Then Cartan subgroups of  $G$  exist and are conjugate, and they are definably connected and selfnormalizing. Moreover, they are largely generous in the following strong sense: for any Cartan subgroup  $Q$ , the (definable) set of elements of  $Q$  contained in a unique conjugate of  $Q$  is large in  $Q$  and largely generous in  $G$ .*

We first look at the minimal configuration for our analysis which can be thought of an abstract analysis of Borel subgroups of  $SL_2$  (over  $\mathbb{C}$  or  $\mathbb{R}$ ), first studied by Nesin in the case of groups of finite Morley rank [3, Lemma 9.14].

**Lemma 41.** *Let  $G$  be a definably connected solvable group definable in an  $o$ -minimal structure, with  $G'$  a  $G$ -minimal subgroup and  $Z(G)$  finite. Then  $G = G' \rtimes Q$  for some (abelian) selfnormalizing definably connected definable largely generous complement  $Q$ , and any two complements of  $G'$  are  $G'$ -conjugate. More precisely, we also have:*

- (a)  $F(G) = Z(G) \times G' = C_G(G')$ .
- (b) For any  $x$  in  $G \setminus F(G)$ ,  $xG' = x^{G'}$ ,  $G = G' \rtimes C(x)$ , and  $C(x)$  is the unique conjugate of  $C(x)$  containing  $x$ .

*Proof.* We elaborate on the proof given in [14, Theorem 3.14] in the finite Morley rank case. Since  $Z(G)$  is finite, the definably connected group  $G$  is not nilpotent by Fact 33(d), and in particular  $C_G(G') < G$ . By  $G$ -minimality of  $G'$  and Lemma 34,  $G' \leq Z^\circ(F(G))$  and  $C_G(a) = C_G(G')$  for every non-trivial element  $a$  of  $G'$ .

For any element  $x$  in  $G \setminus C_G(G')$ , we now show that  $Q := C_G(x)$  is a required complement of  $G'$ . Since  $x \notin C_G(G')$ ,  $C_{G'}(x) = 1$  and in particular  $\dim(x^G) \geq \dim(G')$ . On the other hand,  $x^G \subseteq xG'$  as  $G/G'$  is abelian, and it follows that  $\dim(x^G) = \dim(G')$ , or in other words that  $\dim(G/Q) = \dim(G')$ . Since  $Q \cap G' = 1$ , the definable subgroup  $G' \rtimes Q$  has maximal dimension in  $G$ , and since  $G$  is definably connected we get that  $G = G' \rtimes Q$ . Of course  $Q \simeq G/G'$  is abelian, and definably connected as  $G$  is. We also see that  $N_{G'}(Q) = C_{G'}(Q) = 1$ , since  $C_{G'}(x) = 1$ , and thus the definable subgroup  $Q = C_G(x)$  is selfnormalizing.

(a). The finite center  $Z(G)$  is necessarily in  $Q = C_G(x)$  in the previous paragraph, and in particular  $Z(G) \cap G' = 1$ . Since  $G = G' \rtimes Q$  and  $Q$  is abelian,  $C_Q(G') \leq Z(G)$ , and since  $G' \leq Z(F(G))$  one gets  $Z(G) \times G' \leq F(G) \leq C_G(G') \leq Z(G) \times G'$ , proving item (a).

(b). Let again  $x$  be any element in  $G \setminus F(G)$ . The map  $G' \rightarrow G' : u \mapsto [x, u]$  is a definable group homomorphism since  $G'$  is abelian, with trivial kernel as  $C_{G'}(x) = 1$ , and an isomorphism onto  $G'$  since the latter is definably connected. It follows that any element of the form  $xu'$ , for  $u' \in G'$ , has the form  $xu' = x[x, u] = x^u$  for some  $u \in G'$ , i.e.,  $xG' = x^{G'}$ .

Next, notice that any complement  $Q_1$  of  $G'$  is of the form  $Q_1 = C_G(x_1)$  for any  $x_1 \in Q_1 \setminus Z(G)$ . Indeed,  $x_1 \notin Z(G)$  and  $Q_1$  abelian imply  $x_1 \notin C_G(G')$ , and

as above  $C_G(x_1)$  is a definably connected complement of  $G'$  containing  $Q_1$ , and comparing the dimensions we get  $Q_1 = C_G(x_1)$ .

Moreover, if  $Q_1 = C_G(x_1)$  and  $Q_2 = C_G(x_2)$  are two complements of  $G'$ , we can always choose  $x_1$  and  $x_2$  in the same  $G'$ -coset; then they are  $G'$ -conjugate, as well as  $Q_1$  and  $Q_2$ . It is also now clear that, for any  $x \in G \setminus F(G)$ ,  $C_G(x)$  is the unique complement of  $G'$  containing  $x$ , proving item (b).

It is clear from item (b) that two complements of  $G'$  are  $G'$ -conjugate, and that such complements are largely generous in  $G$ .  $\square$

**Corollary 42.** *Let  $G$  be a group as in Lemma 41. Then:*

- (a) *If  $X$  is an infinite subgroup of a complement  $Q$  of  $G'$ , then  $N_G(X) = Q$  and  $N_G(X) \cap G' = 1$ .*
- (b) *If  $X$  is a nilpotent subgroup of  $G$  not contained in  $F(G)$ , then  $X$  is in an abelian complement of  $G'$ .*
- (c) *Complements of  $G'$  in  $G$  are both Carter and Cartan subgroups of  $G$ , and all are of this form.*

*Proof.* (a). We have  $Q \leq N_G(X)$ , and thus  $N_G(X) = N_{G'}(X) \rtimes Q$ . But  $[N_{G'}(X), X] \leq N_{G'}(X) \cap X = 1$  since  $Q \cap G' = 1$ . In view of Lemma 41, and since  $X$  is infinite, the only possibility is that  $N_{G'}(X) = 1$ . Hence  $N_G(X) = Q$ , which is disjoint from  $G'$ .

(b).  $X$  contains an element  $x$  outside of  $F(G) = C_G(G')$ . Replacing  $X$  by its definable hull  $H(X)$  and using Fact 3(b), we may assume without loss that  $X$  is definable. As in the proof of Lemma 41,  $X \cap G' = \{[x, u] \mid u \in X \cap G'\}$ , and the nilpotency of  $X$  forces that  $X \cap G' = 1$ . Hence  $X$  is abelian, and in the complement  $C(x)$  of  $G'$ .

(c). Complements of  $G'$  are selfnormalizing Carter subgroups by Lemma 41, and thus also Cartan subgroups by Lemma 5. Conversely, one sees easily that a Carter or a Cartan subgroup of  $G$  cannot be contained in  $F(G)$ , and then must be a complement of  $G'$  by item (b).  $\square$

Crucial in our proof of Theorem 40, the next point shows that any definably connected nonnilpotent solvable group has a quotient as in Lemma 41.

**Fact 43.** (Cf. [13, Proposition 3.5]) *Let  $G$  be a definably connected nonnilpotent solvable group definable in an o-minimal structure. Then  $G$  has a definably connected definable normal subgroup  $N$  such that  $(G/N)'$  is  $G/N$ -minimal and  $Z(G/N)$  is finite.*

*Proof.* The proof works formally exactly as in [13, Proposition 3.5] in the finite Morley rank case. All facts used there about groups of finite Morley rank have their formal analogs in Fact 33(a) and Lemma 34 in the o-minimal case. We also use the fact that lower central series and derived series of definably connected solvable groups definable in o-minimal structures are definable and definably connected, which follows from Fact 32 here.  $\square$

We now pass to the proof of the general Theorem 40. At this stage we could follow the analysis by *abnormal* subgroups of [6] in finite solvable groups, as developed in the case of infinite solvable groups of finite Morley rank in [13]. However we provide a more conceptual proof of Theorem 40, mixing the use of Fact 43 with our general genericity arguments, in particular of Section 6. We note that the proof of Theorem 40 we give here would work equally in the finite Morley rank case (in that case there is no elimination of imaginaries but the dimension is well defined on imaginaries),

providing a somewhat more conceptual proof of the analogous theorem in [13] in that case.

**Proof of Theorem 40.** We proceed by induction on  $\dim(G)$ . Clearly a minimal counterexample  $G$  has to be nonnilpotent, and then has a definably connected definable normal subgroup  $N$  as in Fact 43. In what follows we use the notation “ $\overline{\phantom{x}}$ ” to denote quotients by  $N$ . Notice that  $\overline{G}$  is necessarily infinite in Fact 43, and  $N$  is a subgroup of infinite index in  $G$ .

**Claim 44.**  *$G$  contains a definably connected and selfnormalizing Cartan subgroup  $Q$  which is largely generous in the following sense: the (definable) set of elements of  $Q$  contained in a unique conjugate of  $Q$  is large in  $Q$  and largely generous in  $G$ .*

*Proof.* Let  $H$  be a definable subgroup of  $G$  containing  $N$  such that  $\overline{H}$  is a selfnormalizing largely generous Carter subgroup of  $\overline{G}$  as in Lemma 41. Notice that  $H$  is definably connected since  $\overline{H}$  and  $N$  are. As  $\overline{G}$  is infinite,  $\dim(\overline{H}) < \dim(\overline{G})$ , and  $\dim(H) < \dim(G)$ . We can thus apply the induction hypothesis in  $H$ , and assume that  $H$  contains a definably connected and selfnormalizing Cartan subgroup  $Q$  with the strong large generosity property: the set of elements of  $Q$  contained in a unique  $H$ -conjugate of  $Q$  is large in  $Q$  and largely generous in  $H$ . We will show that  $Q$  is the required subgroup.

First note that  $Q$ , being definably connected, is a largely generous Carter subgroup of  $H$ . By Corollaries 26 and 27,  $Q$  must be a largely generous Carter subgroup of  $G$ . We now show that  $Q$  is selfnormalizing in  $G$ . Notice that  $Q$  has an infinite image in  $\overline{H}$ , since it is largely generous in  $H$  and  $N$  is normal and proper in  $H$ . If  $x \in N_G(Q)$ , then  $\overline{x} \in N_{\overline{G}}(\overline{Q}) = \overline{H}$  by Corollary 42(a), and since  $Q$  is selfnormalizing in  $H$  it follows that  $x \in N_H(Q) = Q$ . Thus  $Q$  is selfnormalizing in  $G$ . By Lemma 5,  $Q$  is also a Cartan subgroup of  $G$ .

It remains just to show the largeness issue. Let  $Q_0$  denote the set of elements of  $Q$  contained in a unique  $H$ -conjugate of  $Q$ . We know that  $Q_0$  is large in  $Q$  and that  $[Q_0]^H$  is large in  $H$ , so  $[Q_0]^G (= [[Q_0]^H]^G)$  is large in  $G$  by Proposition 24. This shows that  $Q$  is largely generous in  $G$ , and thus it remains only to show it is in the strong sense of our claim. For that purpose, one easily sees that it is enough to show that the subset  $X$  of elements of  $Q_0$  contained in a unique  $G$ -conjugate of  $Q$  is still large in  $Q_0$ , given the large partition of  $G$  as in Corollary 30 and Theorem 31 (see also Proposition 12). Since  $Q$  is largely generous in  $H$  and the preimage  $L$  in  $H$  of  $F(\overline{G})$  is normal and proper in  $H$ , we get that  $Q \not\leq L$ , and thus it suffices to show that  $Q_0 \setminus X$  is in  $L$ . Suppose towards a contradiction that an element  $x$  in  $Q_0$  and not in  $L$  is in  $Q^g$  for some  $g$  not in  $N_G(Q)$ . Looking at images in  $\overline{G}$  and since  $\overline{x} \in \overline{H} \setminus Z(\overline{G})$ , we then see with Lemma 41 that  $\overline{g} \in N_{\overline{G}}(\overline{H}) = \overline{H}$ , and thus  $g \in H$ . Then  $x \in Q \cap Q^g$  for some  $g \in H \setminus N_H(Q)$ , a contradiction since  $x$  is in a unique  $H$ -conjugate of  $Q$ . This completes our proof of Claim 44.  $\square$

**Claim 45.** *Carter subgroups of  $G$  are conjugate.*

*Proof.* There are indeed at this stage two quick ways to argue for the conjugacy of Carter subgroups, either by quotienting by a  $G$ -minimal subgroup of  $G$  as in [14, Proof of Theorem 3.11], or still looking at the quotient  $\overline{G}$ . Since we have already used  $\overline{G}$  for the existence of a largely generous Carter subgroup we keep on this second line of arguments.

Let  $Q_1$  be an arbitrary Carter subgroup of  $G$ . By Theorem 31, it suffices to prove that  $Q_1$  is a largely generous Carter subgroup of  $G$ . Let  $L$  be the preimage

of  $[\overline{G}]'$  in  $G$ ; notice that  $L$  is definably connected as  $[\overline{G}]'$  and  $N$  are. If  $Q_1 \leq L$ , then a Frattini Argument applied in  $L$ , using the induction hypothesis in  $L$ , gives  $G = L \cdot N_G(Q_1)$ , and since  $Q_1$  is a Carter subgroup this gives that  $L$  has finite index in  $G$ , a contradiction. Therefore  $Q_1 \not\leq L$ , and since  $\overline{Q_1}$  is definably connected we also get  $\overline{Q_1} \not\leq F(\overline{G})$  by Lemma 41(a). In particular, by Corollary 42(b),  $Q_1$  is contained in a definably connected definable subgroup  $H$  as in the proof of Claim 44. Since  $H < G$ , the induction hypothesis applies in  $H$ , and thus  $Q_1$  must be conjugate in  $H$  to a largely generous Carter subgroup  $Q$  of  $H$ . In particular, by the proof of Claim 44,  $Q_1$  is a largely generous Carter subgroup of  $G$ , as required.  $\square$

The Cartan subgroup  $Q$  provided by Claim 44 is also a Carter subgroup by definable connectedness and Lemma 5(a'). If  $Q_1$  is an arbitrary Cartan subgroup, then  $Q_1^\circ$  is a Carter subgroup by Lemma 5(a'), hence a conjugate of  $Q$  by Claim 45, and the maximal nilpotence of  $Q$  forces  $Q_1^\circ = Q_1$ . Hence Cartan subgroups are definably connected and conjugate. This completes the proof of Theorem 40.  $\square$

**Corollary 46.** *In a definably connected solvable group definable in an o-minimal structure, Cartan subgroups and Carter subgroups coincide.*

*Proof.* If  $Q$  is a Cartan subgroup, then it is definably connected by Theorem 40, and thus a Carter subgroup by Lemma 5(a'). If  $Q$  is a Carter subgroup, then  $Q$  is the definably connected component of a Cartan subgroup  $\tilde{Q}$  by Lemma 5, and thus  $Q = \tilde{Q}$  by Theorem 40.  $\square$

There are other aspects refining further the structure of definably connected solvable groups that we won't follow here, but which could be. It includes the already mentioned approach of Cartan/Carter subgroups as *minimal* abnormal subgroups [6, 13], as well as covering properties of nilpotent quotients by Cartan/Carter subgroups (see also [14, §4-5]), and also the peculiar theory of “generalized centralizers” of [13, §5.3]. We merely mention the most basic covering property, but before that a Frattini argument following Theorem 40.

**Corollary 47.** *Let  $G$  be a group definable in an o-minimal structure,  $N$  a definably connected definable normal solvable subgroup, and  $Q$  a Cartan/Carter subgroup of  $N$ . Then  $G = N_G(Q)N$ .*

*Proof.* By a standard Frattini argument, following the conjugacy in Theorem 40.  $\square$

**Lemma 48.** *Let  $G$  be a definably connected solvable group definable in an o-minimal structure,  $N$  a definable normal subgroup such that  $G/N$  is nilpotent, and  $Q$  a Cartan/Carter subgroup of  $G$ . Then  $G = QN$ .*

*Proof.* Suppose  $QN < G$ . Then  $QN/N$  is a definable subgroup of infinite index in the definably connected nilpotent group  $G/N$ . By Lemma 4, and since  $N_G(QN)$  is the preimage in  $G$  of  $N_{G/N}(QN/N)$ , we have thus  $QN$  of infinite index in  $K := N_G(QN)$ . But  $Q$  is a Cartan/Carter subgroup of the definably connected solvable group  $[QN]^\circ$ , normal in  $K$ , and thus  $K = N_K(Q)[QN]^\circ = N_K(Q)N^\circ$  by Corollary 47. Since  $Q$  is a Carter subgroup, we get that  $QN$  must have finite index in  $K$ , a contradiction.  $\square$

We note that Lemma 48 always applies with  $N = F^\circ(G)$ , in view of Fact 33(b), giving thus in particular  $G = QF^\circ(G)$  for any definably connected solvable group  $G$  and any Cartan/Carter subgroup  $Q$  of  $G$ .

## 9. ON LIE GROUPS

In this section we collect properties needed in the sequel concerning Cartan subgroups (in the sense of Chevalley as usual) of Lie groups. These are facts more or less known, but because of the different notions of a Cartan subgroup used in the literature we will be careful with references.

By a *Lie algebra* we mean a finite dimensional real Lie algebra. We are going to make use of the following concepts about Lie algebras: subalgebras, commutative, nilpotent, and semisimple Lie algebras [4, I.1.1, I.1.3, I.4.1 and I.6.1]. If  $\mathfrak{g}$  is a Lie algebra and  $x \in \mathfrak{g}$ , the linear map  $\text{ad}_x : \mathfrak{g} \rightarrow \mathfrak{g} : y \mapsto [x, y]$  is called the *adjoint map* of  $x$ . If  $\mathfrak{h}$  is a subalgebra of  $\mathfrak{g}$ , the *normalizer* of  $\mathfrak{h}$  in  $\mathfrak{g}$  is  $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} : \text{ad}_x(\mathfrak{h}) \subseteq \mathfrak{h}\}$  and the *centralizer* of  $\mathfrak{h}$  in  $\mathfrak{g}$  is  $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} : [\text{ad}_x]_{|\mathfrak{h}} = \text{id}_{|\mathfrak{h}}\}$ .

**Definition 49.** *Let  $\mathfrak{g}$  be a Lie algebra and  $\mathfrak{h}$  a subalgebra of  $\mathfrak{g}$ . We say that  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$  if  $\mathfrak{h}$  is nilpotent and selfnormalizing in  $\mathfrak{g}$ .*

The two following facts can be found in [33, Theorem 4.1.2] and [33, Theorem 4.1.5] respectively.

**Fact 50.** *Every Lie algebra has a Cartan subalgebra.*

**Fact 51.** *Let  $\mathfrak{g}$  be a semisimple Lie algebra and  $\mathfrak{h}$  a subalgebra of  $\mathfrak{g}$ . Then  $\mathfrak{h}$  is a Cartan subalgebra of  $\mathfrak{g}$  if and only if*

- (a)  $\mathfrak{h}$  is a maximal abelian subalgebra of  $\mathfrak{g}$ , and
- (b) For every  $x \in \mathfrak{h}$ ,  $\text{ad}_x$  is a semisimple endomorphism of  $\mathfrak{g}$ , i.e.,  $\text{ad}_x$  is diagonalizable over  $\mathbb{C}$ .

By a *Lie group* we mean a finite dimensional real Lie group  $G$ . The *connected component of the identity* is denoted by  $G^\circ$ . The Lie algebra of  $G$  is denoted by  $\mathfrak{L}(G)$ . A connected Lie group  $G$  is called a *semisimple Lie group* if  $\mathfrak{L}(G)$  is a semisimple Lie algebra (equivalently, every normal commutative connected immersed subgroup of  $G$  is trivial [4, Proposition III.9.8.26]). If  $g$  is an element of a Lie group  $G$ , then  $\text{Ad}(g) : \mathfrak{L}(G) \rightarrow \mathfrak{L}(G)$  denotes the differential at the identity of  $G$  of the map from  $G$  to  $G$  mapping  $h$  to  $ghg^{-1}$ , for each  $h \in G$ . If  $\mathfrak{g}$  is the Lie algebra of  $G$  and  $\mathfrak{h}$  a subalgebra of  $\mathfrak{g}$ , the *centralizer* of  $\mathfrak{h}$  in  $G$  is  $Z_G(\mathfrak{h}) := \{g \in G : \text{Ad}(g)(x) = x \text{ for every } x \in \mathfrak{h}\}$ .

**Fact 52.** *Let  $G$  be a connected semisimple Lie group with Lie algebra  $\mathfrak{g}$ , and let  $H$  be a subgroup of  $G$ . Then  $H$  is a Cartan subgroup of  $G$  if and only if  $H = Z_G(\mathfrak{h})$  for some Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ . Moreover, in this case,  $\mathfrak{h}$  is  $\mathfrak{L}(H)$ .*

*Proof.* As  $G$  is connected, [23, Theorem A.4] implies that  $H$  is a Cartan subgroup of  $G$  if and only if

- (C0)  $H$  is a closed subgroup of  $G$ ;
- (C1)  $\mathfrak{h} (= \mathfrak{L}(H))$  is a Cartan subalgebra of  $\mathfrak{g}$ , and
- (C2)  $H = C(\mathfrak{h})$ .

Here  $C(\mathfrak{h})$  is defined by a centralizer-like condition. To avoid introducing more notation, instead of properly defining  $C(\mathfrak{h})$ , we make use of [23, Lemma I.5], which states that  $C(\mathfrak{h}) = Z_G(\mathfrak{h})$  provided  $\mathfrak{h}$  is reductive in  $\mathfrak{g}$ , which is our case. Indeed,  $G$  is a semisimple Lie group, so  $\mathfrak{g}$  is a semisimple Lie algebra, hence  $\mathfrak{g}$  is reductive [4, Proposition I.6.4.5], and then by [23, Lemma I.4] every Cartan subalgebra of  $\mathfrak{g}$  is reductive in  $\mathfrak{g}$ ; in particular  $\mathfrak{h}$  is reductive in  $\mathfrak{g}$ .

For the converse, we observe that if  $H = Z_G(\mathfrak{h})$  for some Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , then  $H$  is closed in  $G$  and  $\mathfrak{L}(H) = \mathfrak{h}$ . Indeed,  $H$  is closed by definition of centralizers, and by [4, Proposition III.9.3.7],  $\mathfrak{L}(H) = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ . Now  $\mathfrak{h}$  is abelian by Fact 51, and hence  $\mathfrak{h} \subseteq \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ . Moreover, if  $x \in \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ , the subalgebra of  $\mathfrak{g}$  generated by  $x$  and  $\mathfrak{h}$  is abelian, so it must coincide with  $\mathfrak{h}$  by maximality of  $\mathfrak{h}$ , and  $x \in \mathfrak{h}$ ; hence  $\mathfrak{h} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$ . We then conclude as above, first applying Lemma I.5 and then Theorem A.4 from [23].  $\square$

**Fact 53.** *Let  $G$  be a connected semisimple centreless Lie group and  $H$  a subgroup of  $G$ . If  $H$  is a Cartan subgroup of  $G$ , then  $H$  is abelian.*

*Proof.* By Fact 52,  $H = Z_G(\mathfrak{h})$  with  $\mathfrak{h} = \mathfrak{L}(H)$  a Cartan subalgebra of  $\mathfrak{g}$ . By [16, Lemma 8, p. 556] we have that  $H/Z(G)$  is abelian (see also [34, Theorem 1.4.1.5], noting that since  $G$  is semisimple the general assumption (1.1.5) holds). Hence  $H$  is abelian.  $\square$

We note that the assumption  $Z(G) = 1$  is essential to get the Cartan subgroup abelian in Fact 53. For example  $\mathrm{SL}_3(\mathbb{R})$  has a simply-connected double covering with non-abelian Cartan subgroups [21, p.141], an example which can also occur in the context of our Theorem 62 below.

**Fact 54.** *Let  $G$  be a connected semisimple Lie group. Then:*

- (a) *There are only finitely many conjugacy classes of Cartan subgroups of  $G$ . All Cartan subgroups of  $G$  have the same dimension.*
- (b) *If  $H_1$  and  $H_2$  are two Cartan subgroups of  $G$  with  $H_1^\circ = H_2^\circ$ , then  $H_1 = H_2$ . In particular, if  $H_1^\circ$  and  $H_2^\circ$  are conjugate, then  $H_1$  and  $H_2$  are conjugate as well.*
- (c) *For any Cartan subgroup  $H$  of  $G$ , the set of elements of  $H$  contained in a unique conjugate of  $H$  is dense in  $H$ .*

*Proof.* (a). Let  $\mathfrak{g} = \mathfrak{L}(G)$ . Then  $\mathfrak{g}$  is semisimple and it has finitely many Cartan subalgebras, say  $\mathfrak{h}_1, \dots, \mathfrak{h}_s$ , such that any Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  is conjugate to one of them by an element of  $\mathrm{Ad}(G)$ , i.e.,  $\mathrm{Ad}(g)(\mathfrak{h}) = \mathfrak{h}_i$  for some  $i \in \{1, \dots, s\}$  and some  $g \in G$  (see [15, Corollary to Lemma 2] or [34, Corollary 1.3.1.11]).

Next, note that for every  $g$  in  $G$  and every (Cartan) subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , we have  $Z_G(\mathrm{Ad}(g)(\mathfrak{h})) = gZ_G(\mathfrak{h})g^{-1}$ . For,  $h \in Z_G(\mathrm{Ad}(g)(\mathfrak{h}))$  if and only if  $\mathrm{Ad}(h)\mathrm{Ad}(g)x = \mathrm{Ad}(g)x$  for every  $x \in \mathfrak{h}$ , and the latter is equivalent to  $g^{-1}hg \in Z_G(\mathfrak{h})$ . Therefore, conjugate Cartan subalgebras correspond to conjugate centralizers, and by Fact 52 to conjugate Cartan subgroups.

We prove the second part. By Fact 52 the Lie algebra of a Cartan subgroup is a Cartan subalgebra. By [33, Corollary 4.1.4] all Cartan subalgebras have the same dimension.

(b). It is clear since  $\mathfrak{L}(H_i) = \mathfrak{L}(H_i^\circ)$ , for  $i = 1, 2$ , and  $H_i = Z_G(\mathfrak{L}(H_i))$ . (Actually, to prove (b) we do not need  $G$  to be semisimple: just consider the  $C(\mathfrak{L}(H_i))$ 's of the proof of Fact 52, instead of the centralizers.)

(c). We essentially refer to [17]. Recall, by Fact 52 and its proof, that in the semisimple case our notion of a Cartan subgroup equals the one used in that paper and  $C(\mathfrak{h}) = Z_G(\mathfrak{h})$  for any Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g} := \mathfrak{L}(G)$ . Let  $\mathrm{Reg}(G)$  be the set of regular elements of  $G$ , as defined after Lemma 1.3 in [17]. We first show that each element  $g$  of  $\mathrm{Reg}(G)$  lies in a unique Cartan subgroup of  $G$ . Fix  $g \in \mathrm{Reg}(G)$ . By the proof of [17, Prop. 1.5] we have that  $\mathfrak{g}^1(\mathrm{Ad}(g)) := \{x \in \mathfrak{g} : (\exists n \in \mathbb{N})(\mathrm{Ad}(g) - 1)^n x = 0\}$  is a Cartan subalgebra of  $\mathfrak{g}$  and  $g$  belongs to the Cartan

subgroup  $Z_G(\mathfrak{g}^1(\text{Ad}(g)))$ . To show the uniqueness, let  $H$  be a Cartan subgroup of  $G$  containing  $g$ . By Fact 52,  $H = Z_G(\mathfrak{h})$  with  $\mathfrak{h} = \mathfrak{L}(H)$  a Cartan subalgebra of  $\mathfrak{g}$ . Since  $g \in Z_G(\mathfrak{h})$  we have that  $\mathfrak{h} \subseteq \mathfrak{g}^1(\text{Ad}(g))$  and hence  $\mathfrak{h} = \mathfrak{g}^1(\text{Ad}(g))$  by maximality of Cartan subalgebras. Therefore  $H = Z_G(\mathfrak{g}^1(\text{Ad}(g)))$ .

Finally, by [17, Proposition 1.6], the subset  $\text{Reg}(G) \cap H$  is dense in  $H$  for all Cartan subgroup  $H$  of  $G$ .  $\square$

For the following, we refer directly to [36, Proposition 5] and (the proof of) [36, Lemma 11] respectively.

**Fact 55.** *Let  $G$  be a connected Lie group. Then:*

- (a) *The union of all Cartan subgroups of  $G$  is dense in  $G$ .*
- (b) *For any Cartan subgroup  $H$  of  $G$ ,  $[H^\circ]^G$  contains an open subset.*

We finish this section with a remark which, as far as we know, does not seem to have been made before. We will show later that all Cartan and Carter subgroups of a group definable in an o-minimal structure are, as indicated by Fact 55(b), weakly generous in the sense of Definition 16(a). Our remark is essentially that the stronger notion of generosity of Definition 16(b) may be satisfied or not, depending of the Carter subgroups considered, and this phenomenon occurs even inside  $\text{SL}_2(\mathbb{R})$ . Recall that the Cartan subgroups of  $\text{SL}_2(\mathbb{R})$  are, up to conjugacy, the subgroup  $Q_1$  of diagonal matrices and  $Q_2 = \text{SO}_2(\mathbb{R})$ . Considering the characteristic polynomial, the two following equalities are easily checked:

$$Q_1^{\text{SL}_2(\mathbb{R})} = \{A \in \text{SL}_2(\mathbb{R}) : |\text{tr}(A)| > 2\} \cup \{I, -I\}$$

$$Q_2^{\text{SL}_2(\mathbb{R})} = \{A \in \text{SL}_2(\mathbb{R}) : |\text{tr}(A)| < 2\} \cup \{I, -I\}$$

**Remark 56.** *Let  $G = \text{SL}_2(\mathbb{R})$ . Then, according to Definition 16(b):*

- (a) *The Cartan subgroup  $Q_1$  of diagonal matrices is generous in  $G$ .*
- (b) *The Cartan subgroup  $Q_2 = \text{SO}_2(\mathbb{R})$  is not generous in  $G$ .*

*Proof.* (a). Fix  $a, b \in (0, \frac{1}{13})$  and consider the matrices  $A_1 = I$ ,

$$A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}, \text{ and } A_4 = \begin{pmatrix} 0 & -b^{-1} \\ b & 0 \end{pmatrix}.$$

We show that  $G = \cup_{i=1}^4 A_i Q_1^G$ . Suppose there exists

$$M = \begin{pmatrix} x & y \\ u & v \end{pmatrix} \in G$$

with  $M \notin \cup_{i=1}^4 A_i Q_1^G$ . Since  $M \notin A_1 Q_1^G \cup A_2 Q_1^G$ , we have  $x = \epsilon - v$  and  $y = u + \delta$  for some  $\epsilon, \delta \in [-2, 2]$ . Since  $M \notin A_3 Q_1^G$  we have that  $|ax + a^{-1}v| = |a(\epsilon - v) + a^{-1}v| \leq 2$ , so that  $v \in [\frac{-2a - a^2\epsilon}{1 - a^2}, \frac{2a - a^2\epsilon}{1 - a^2}]$ . Since  $\epsilon \in [-2, 2]$ , we deduce that  $v \in [\frac{-2a}{1 - a}, \frac{2a}{1 - a}]$ . Similarly, it follows from  $M \notin A_4 Q_1^G$  that  $u \in [\frac{-2b}{1 - b}, \frac{2b}{1 - b}]$ .

Finally, since  $a, b < \frac{1}{13}$  we have that  $|v|, |u| < \frac{1}{6}$  and  $|x|, |y| < 2 + \frac{1}{6} < 3$ . In particular,  $\det(M) = |xv - uy| \leq |x||v| + |u||y| < 1$ , a contradiction.

(b). We show that the family of matrices

$$M_x = \begin{pmatrix} x^2 & x - 1 \\ 1 & x^{-1} \end{pmatrix}$$

with  $x > 0$  cannot be covered by finitely many translates of  $Q_2^G$ . It suffices to prove that for a fixed matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$$

we have that  $\{x \in \mathbb{R}^{>0} : |\operatorname{tr}(A^{-1}M_x)| > 2\} \subseteq \{x \in \mathbb{R}^{>0} : M_x \notin AQ_2^G\}$  is not bounded. Since  $\operatorname{tr}(A^{-1}M_x) = x^2d - b - c(x-1) + ax^{-1}$  and  $x$  is positive, it follows that  $|\operatorname{tr}(A^{-1}M_x)| > 2$  if and only if one of the following two conditions holds:

$$(1) \quad dx^3 - cx^2 - (b - c + 2)x + a > 0$$

$$(2) \quad dx^3 - cx^2 - (b - c - 2)x + a < 0$$

It is easy to check that if  $d \neq 0$ , then either (1) or (2) is satisfied for large enough  $x$ . If  $d = 0$ , then  $c \neq 0$  (otherwise  $\det(A) = 0$ ) and again the same holds.  $\square$

In Remark 56, the generous Cartan subgroup is noncompact and the nongenerous one is compact. One can then wonder about the various possibilities for generosity depending on compactness. But considering  $Q_1 \times Q_2$  in  $\operatorname{SL}_2(\mathbb{R}) \times \operatorname{SL}_2(\mathbb{R})$  one gets from Remark 56 a nongenerous and noncompact Cartan subgroup. Besides, any compact group is typically covered by a single conjugacy class of compact Cartan subgroups by Corollary 38, and these compact Cartan subgroups are in particular generous.

## 10. FROM LIE GROUPS TO DEFINABLY SIMPLE GROUPS

We now return to the context of groups definable in o-minimal structures. In the present section we prove the following theorem, essentially transferring via Fact 35 the results of Section 9 on Lie groups to definably simple groups definable in an o-minimal structure.

**Theorem 57.** *Let  $G$  be a definably simple group definable in an o-minimal structure. Then  $G$  has definable Cartan subgroups and the following holds.*

- (1)  *$G$  has only finitely many conjugacy classes of Cartan subgroups.*
- (2) *If  $Q_1$  and  $Q_2$  are Cartan subgroups of  $G$  and  $Q_1^\circ = Q_2^\circ$ , then  $Q_1 = Q_2$ .*
- (3) *Cartan subgroups of  $G$  are abelian and have the same dimension.*
- (4) *If  $Q$  is a Cartan subgroup of  $G$ , then the set of elements of  $Q$  contained in a unique conjugate of  $Q$  is large in  $Q$ . In particular, if  $a \in Q$ , then the set of elements of  $aQ^\circ$  contained in a unique conjugate of  $aQ^\circ$  is large in  $aQ^\circ$ , and  $aQ^\circ$  is weakly generous in  $G$ .*
- (5) *The union of all Cartan subgroups of  $G$ , which is definable by (1), is large in  $G$ .*

Before passing to the proof of Theorem 57, we explain the ‘‘In particular’’ part of item (4). So let  $Q$  be a Cartan subgroup such that the set  $Q_0$  of elements of  $Q$  contained in a unique conjugate of  $Q$  is large in  $Q$ . Let  $[aQ^\circ]_0$  be the set of elements of  $aQ^\circ$  contained in a unique conjugate of  $aQ^\circ$ , for some  $a \in Q$ . We see easily that  $Q_0 \cap aQ^\circ \subseteq [aQ^\circ]_0$ , and since  $Q_0$  is large in  $Q$  we get that  $Q_0 \cap aQ^\circ$  is large in  $aQ^\circ$ , as well as  $[aQ^\circ]_0$ . Now, since  $Q^\circ \leq N(aQ^\circ) \leq N(Q^\circ)$  and  $\dim(Q^\circ) = \dim(N(Q^\circ))$ , we get that  $\dim([aQ^\circ]_0) = \dim(N(aQ^\circ))$ , and Corollary 17 gives the weak generosity of  $aQ^\circ$ .

We now embark on the proof of Theorem 57, bearing in mind that for item (4) we only need to prove the first statement. We first begin with some lemmas. By a

*system of representatives* we mean a system of representatives of conjugacy classes of a set of subgroups of a given group.

**Lemma 58.** *Let  $\mathcal{M}$  be an  $o$ -minimal expansion of an ordered group,  $A \subseteq M$  a set of parameters containing an element different from 0, and  $G$  a group definable in  $\mathcal{M}$  over  $A$ . Assume  $G$  has, for some  $s \in \mathbb{N}$ , at least  $s$  non-conjugate Carter subgroups. Then  $G$  has at least  $s$  non-conjugate Carter subgroups definable over  $A$ . In particular, if  $G$  has a finite number of Carter subgroups up to conjugacy, then in each conjugacy class there exists a Carter subgroup definable over  $A$ .*

*Proof.* The second part follows easily from the first one. Let  $Q_1, \dots, Q_s$  be non-conjugate Carter subgroups of  $G$ . We denote them by  $Q_1^{\bar{b}}, \dots, Q_s^{\bar{b}}$  to stress the fact that they are defined over the tuple  $\bar{b}$ . For each  $i = 1, \dots, s$ , let  $s_i = [N(Q_i) : Q_i]$  and  $r_i$  be the nilpotency class of  $Q_i$ . Consider the set  $\Xi$  of tuples  $\bar{c}$  satisfying the following conditions for each  $i$ .

- (1)  $Q_i^{\bar{c}}$  is a nilpotent subgroup of  $G$ , of nilpotency class  $r_i$ .
- (2)  $[N(Q_i^{\bar{c}}) : Q_i^{\bar{c}}] = s_i$ .
- (3) For any  $j = 1, \dots, s$  with  $j \neq i$ ,  $Q_i^{\bar{c}}$  and  $Q_j^{\bar{c}}$  are not conjugate.
- (4)  $Q_i^{\bar{c}}$  is definably connected.

The three first properties are clearly first-order definable. The fact that the fourth is also definable is well-known, and for completeness we sketch the proof (following Y. Peterzil). Let  $X \subseteq M^{n+m}$  be a definable set and for each  $d \in M^n$  denote by  $X_d$  the fiber of  $X$  over  $d$ . We have to show that the set  $\{d \in M^n : X_d \text{ is definably connected}\}$  is definable (here definable connectedness is in the topological sense, but by [30, Proposition 2.12] for a definable group the topological notion of definable connectedness coincides with the one generally in use here). By the cell decomposition [32, Thm. III.2.11],  $X$  is the union of definably connected definable sets  $C_1, \dots, C_k$  with the property that for each  $d \in M^n$  the fiber  $(C_i)_d$  is also definably connected. Finally, it suffices to note that for each  $d \in M^n$  the set  $X_d = \bigcup_{i=1}^k (C_i)_d$  is definably connected if and only if there is an ordering  $(C_{i_1})_d, \dots, (C_{i_k})_d$  such that  $\overline{((C_{i_1})_d \cup \dots \cup (C_{i_j})_d)} \cap (C_{i_{j+1}})_d \neq \emptyset$  or  $((C_{i_1})_d \cup \dots \cup (C_{i_j})_d) \cap \overline{(C_{i_{j+1}})_d} \neq \emptyset$ .

Now the set  $\Xi$  is definable, over  $A$  since  $G$  is, and it is non-empty since it contains  $\bar{b}$ . Since  $\mathcal{M}$  expands a group and  $A$  contains an element different from 0, the definable closure in  $\mathcal{M}$  of  $A$  is an elementary substructure of  $\mathcal{M}$ : the theory of  $\mathcal{M}$  expanded with a symbol for each element in  $A$  has definable Skolem functions [32, Chap. 6 §1(1.1-3)], and we may apply the Tarski-Vaught test (see also [22, §2.3]). Hence there exists a tuple  $\bar{c} \in \Xi$  with each coordinate in the definable closure of  $A$ . Now  $Q_1^{\bar{c}}, \dots, Q_s^{\bar{c}}$  are non-conjugate Carter subgroups of  $G$ , and each can be defined with parameters in  $A$ .  $\square$

**Corollary 59.** *Let  $\mathcal{M}$ ,  $A$ , and  $G$  be as in Lemma 58. Assume  $G$  has a finite number of Cartan subgroups up to conjugacy. Then, in each conjugacy class there exists a Cartan subgroup definable over  $A$ .*

*Proof.* By Lemma 5, a finite number of conjugacy classes of Cartan subgroups implies a finite number of conjugacy classes of Carter subgroups. Hence, by Lemma 58, there exists a finite system of representatives of Carter subgroups  $Q_1^\circ, \dots, Q_s^\circ$ , each defined over  $A$ . Now given any Cartan subgroup  $Q$ , we have up to conjugacy  $Q^\circ = Q_i^\circ$  for some  $i$  by Lemma 5( $a'$ ), and in particular  $Q \leq N(Q_i^\circ)$ . Since both

$N(Q_i^\circ)$  and the finite group  $N(Q_i^\circ)/Q_i^\circ$  are definable over  $A$ , we deduce that  $Q$  is definable over  $A$  up to conjugacy, as desired.  $\square$

We will also make use of the following elementary remark, actually valid in any context where Lemma 5 hold.

**Remark 60.** *Let  $G$  be a group definable in an o-minimal structure such that for every pair of Cartan subgroups  $Q_1$  and  $Q_2$ ,  $Q_1 = Q_2$  if and only if  $Q_1^\circ = Q_2^\circ$ . Then the cardinality of a system of representatives of Cartan subgroups of  $G$  equals the cardinality of a system of representatives of Carter subgroups of  $G$ .*

From now on we will use a standard notation from model theory, namely, if  $\mathcal{N}_1$  is a substructure of  $\mathcal{N}_2$  and  $X$  is definable in  $\mathcal{N}_1$  (respectively in  $\mathcal{N}_2$  with parameters in  $N_1$ ), then  $X(N_2)$  (resp.  $X(N_1)$ ) denotes the realization of  $X$  in  $\mathcal{N}_2$  (resp. in  $\mathcal{N}_1$ ).

**Corollary 61.** *Let  $\mathcal{M}$ ,  $A$ , and  $G$  be as in Lemma 58. Assume  $G$  satisfies properties (1-5) of Theorem 57. Then:*

- (a) *If  $\mathcal{N}$  is an elementary substructure of  $\mathcal{M}$  with  $A \subseteq N$ , then  $G(N)$  also satisfies properties (1-5).*
- (b) *If  $\mathcal{N}$  is an elementary extension of  $\mathcal{M}$ , then  $G(N)$  also satisfies properties (1-5).*

*Proof.* (a). Since  $G$  satisfies property (1), it follows from Corollary 59 that there is a finite system of representatives  $Q_1, \dots, Q_s$  of Cartan subgroups of  $G$  defined over  $A$ . Moreover, by Lemma 5 and property (2) of  $G$  it follows as in Remark 60 that  $Q_1^\circ, \dots, Q_s^\circ$  form a system of representatives of Carter subgroups of  $G$  (all defined over  $A$ ).

We claim that

- ( $\dagger$ )  $Q_1^\circ(N), \dots, Q_s^\circ(N)$  form a system of representatives of Carter subgroups of  $G(N)$ , and
- ( $\ddagger$ )  $Q_1(N), \dots, Q_s(N)$  form a system of representatives of Cartan subgroups of  $G(N)$ .

The claim ( $\dagger$ ) follows from the definition of a Carter subgroup. Indeed, for each  $i \in \{1, \dots, s\}$ , since  $Q_i^\circ$  is definably connected, nilpotent, and almost selfnormalizing,  $Q_i^\circ(N)$  satisfies the same properties, and is a Carter subgroup of  $G(N)$ . If  $Q^\circ$  is a Carter subgroup of  $G(N)$ , then as before  $Q^\circ(M)$  is a Carter subgroup of  $G$ , and is  $Q_i^\circ$  for some  $i$  up to conjugacy in  $G$ . Since  $\mathcal{N} \preceq \mathcal{M}$ ,  $Q^\circ = Q_i^\circ(N)$  up to conjugacy in  $G(N)$ . Similarly, the groups  $Q_i^\circ(N)$  cannot be conjugate because the groups  $Q_i^\circ$  are not, proving ( $\dagger$ ).

We now show ( $\ddagger$ ). We first observe: if  $R$  is a nilpotent definable subgroup of  $G(N)$  with  $[R^\circ]^g = Q_i^\circ(N)$  for some  $g \in G(N)$  and  $i \in \{1, \dots, s\}$ , then  $R^g \leq Q_i(N)$ . Indeed,  $Q_i^\circ = [R^\circ]^g(M) (= [R^\circ(M)]^g = [R(M)^\circ]^g)$ . Since  $[R(M)]^g (= [R^g(M)])$  is nilpotent and  $[R(M)^g]^\circ = [R(M)^\circ]^g = Q_i^\circ$  is a Carter subgroup, by Lemma 5(b)  $R(M)^g$  must be contained in a Cartan subgroup which must be  $Q_i$  by property (2) of  $G$ . Therefore  $R^g \leq Q_i(N)$ , as required. Now we deduce ( $\ddagger$ ) as follows. Each  $Q_i(N)$  is a Cartan subgroup: by Lemma 5 there is a Cartan subgroup  $Q$  with  $Q^\circ = Q_i(N)^\circ$  and by the observation above we have  $Q \leq Q_i(N)$ , and  $Q = Q_i(N)$  by maximal nilpotence of  $Q$ . It just remains to see that  $Q_1(N), \dots, Q_s(N)$  form a system of representatives. Let  $Q$  be a Cartan subgroup of  $G(N)$ . By Lemma 5(a')  $Q^\circ$  is a Carter subgroup and then by ( $\dagger$ ) there exist  $g \in G(N)$  and  $k \in \{1, \dots, s\}$  such

that  $[Q^\circ]^g = Q_k^\circ(N)$ . Hence  $Q^g \leq Q_k(N)$  because of the observation above, and  $Q^g = Q_k(N)$  by maximal nilpotence of  $Q$ . Finally, observe that  $Q_1(N), \dots, Q_s(N)$  cannot be conjugate in  $G(N)$ , since  $Q_1, \dots, Q_s$  are not in  $G$ , proving  $(\ddagger)$ .

We now deduce properties (1-5) for  $G(N)$  from  $(\dagger)$  and  $(\ddagger)$ . Property (1) is exactly  $(\ddagger)$ . For (2), let  $R_1$  and  $R_2$  be Cartan subgroups of  $G(N)$  such that  $R_1^\circ = R_2^\circ$ . By  $(\dagger)$ ,  $R_1^{\circ g} = R_2^{\circ g} = Q_i^\circ(N)$  for some  $g \in G(N)$  and some  $i$ , and by the observation in  $(\ddagger)$  above we get  $R_1^g, R_2^g \leq Q_i(N)$ , and an equality by maximal nilpotence. In particular  $R_1^g = R_2^g$ , and  $R_1 = R_2$ . Since the dimension in o-minimal structures is invariant under elementary substructures, and one considers only definable sets, properties (3-5) transfer readily from  $G$  to  $G(N)$ .

(b). Let  $Q_1, \dots, Q_s$  be a system of representatives of Cartan subgroups of  $G$ . By Lemma 5 and property (2) of  $G$  it follows, as in Remark 60, that  $Q_1^\circ, \dots, Q_s^\circ$  form a system of representatives of Carter subgroups. We first prove that  $Q_1^\circ(N), \dots, Q_s^\circ(N)$  is a system of representatives of Carter subgroups of  $G(N)$ . As in (a), we see that  $Q_1^\circ(N), \dots, Q_s^\circ(N)$  are (non-conjugate) Carter subgroups of  $G(N)$ . To see that they represent all the conjugacy classes, suppose there is a Carter subgroup  $Q^\circ$  of  $G(N)$  which is non-conjugate with  $Q_1^\circ(N), \dots, Q_s^\circ(N)$ . By Corollary 58 we can assume that  $Q^\circ$  is defined over  $M$ . Since  $Q^\circ(M)$  is clearly a Carter subgroup of  $G$ ,  $Q^\circ(M)^g = Q_i^\circ$  for some  $g \in G$  and some  $i$ . Therefore  $[Q^\circ]^g = Q_i^\circ(N)$ , a contradiction.

We next prove that  $Q_1(N), \dots, Q_s(N)$  is a system of representatives of Cartan subgroups of  $G(N)$ . As in (a), it suffices to observe: if  $R$  is a nilpotent definable subgroup of  $G(N)$  with  $[R^\circ]^g = Q_i^\circ(N)$  for some  $g \in G(N)$  and  $i \in \{1, \dots, s\}$ , then  $R^g \leq Q_i(N)$ . Indeed, since  $[R^\circ]^g = Q_i^\circ(N)$  and  $R^g \leq N_{G(N)}(Q_i^\circ(N))$ ,  $R^g$  is defined over  $M$ . Hence  $R^g(M)$  is a definable nilpotent subgroup of  $G$  such that  $R^g(M)^\circ = [R^g]^\circ(M) = Q_i^\circ$ . Then, by Lemma 5(b) and property (2) of  $G$ ,  $R^g(M) \leq Q_i$ . In particular  $R^g \leq Q_i(N)$ , as required.

Now we can transfer properties (1-5) from  $G$  to  $G(N)$  as in (a).  $\square$

**Proof of Theorem 57.** Let  $\mathcal{M}$  denote the ground o-minimal structure. By Fact 35, there is an  $\mathcal{M}$ -definable real closed field  $R$  (with no extra structure) such that  $G$  is  $\mathcal{M}$ -definably isomorphic to a semialgebraically connected semialgebraically simple semialgebraic group, definable in  $R$  over the real algebraic numbers  $\mathbb{R}_{alg}$ . By Remark 36, the dimensions of sets definable in  $R$ , computed in  $\mathcal{M}$  or  $R$ , are the same. Since  $\mathcal{M}$ -definable bijections preserve dimensions, all the conclusions of Theorem 57 would then be true if we prove them in this semialgebraic group definable in  $R$ . Therefore, replacing  $\mathcal{M}$  by  $R$ , we may suppose that  $\mathcal{M}$  is a pure real closed field, and that  $G = G(M)$  is a semialgebraically connected semialgebraically simple group defined over  $\mathbb{R}_{alg}$ .

By quantifier elimination  $\mathbb{R}_{alg} \preceq R$  and by Corollary 61(b) it suffices to show our statements for  $G(\mathbb{R}_{alg})$ . By quantifier elimination again,  $\mathbb{R}_{alg} \preceq \mathbb{R}$ , and by Corollary 61(a) it now suffices to prove our statements for  $G(\mathbb{R})$ .

Now, we observe that  $G(\mathbb{R})$  is a finite dimensional semisimple centerless connected real Lie group. By Facts 50 and 52 it has Cartan subgroups, necessarily definable as usual by Lemma 5(b). It remains just to notice that all items (1-5) are true in the connected real Lie group  $G(\mathbb{R})$  by Facts 53, 54, and 55(a). For item (4), we recall that it suffices to prove the first claim, as explained just after the statement of Theorem 57. It follows from Fact 54(c), noticing that a definable

subset has maximal dimension if and only if it has interior [30, Proposition 2.14], and thus is dense if and only if it is large.  $\square$

We note that the second claim in Theorem 57(4) could also have been shown using Fact 55(b).

### 11. THE SEMISIMPLE CASE

We now prove a version of Theorem 57 for definably connected semisimple groups definable in an o-minimal structure. Recall that a definably connected group  $G$  is semisimple if  $R(G) = Z(G)$  is finite; modulo that finite center,  $G$  is a direct product of finitely many definably simple groups by Fact 35.

**Theorem 62.** *Let  $G$  be a definably connected semisimple group definable in an o-minimal structure. Then  $G$  has definable Cartan subgroups and the following holds.*

- (1)  $G$  has only finitely many conjugacy classes of Cartan subgroups.
- (2) If  $Q_1$  and  $Q_2$  are Cartan subgroups of  $G$  and  $Q_1^\circ = Q_2^\circ$ , then  $Q_1 = Q_2$ .
- (3) If  $Q$  is a Cartan subgroup, then  $Z(G) \leq Q$ ,  $Q' \leq Z(G)$ , and  $Q^\circ \leq Z(Q)$ . Furthermore all Cartan subgroups have the same dimension.
- (4) If  $Q$  is a Cartan subgroup of  $G$  and  $a \in Q$ , then the set  $[aQ^\circ]_0$  of elements of  $aQ^\circ$  contained in a unique conjugate of  $aQ^\circ$  is large in  $aQ^\circ$ , and  $aQ^\circ$  is weakly generous in  $G$ . In addition, if  $a_1$  belongs to another Cartan subgroup  $Q_1$ , then either  $[aQ^\circ]_0 \cap a_1Q_1^\circ = \emptyset$  or  $aQ^\circ = a_1Q_1^\circ$ .
- (5) The union of all Cartan subgroups of  $G$ , which is definable by (1), is large in  $G$ . In fact, there are finitely many pairwise disjoint definable sets of the form  $[aQ^\circ]_0^G$  with  $Q$  a Cartan subgroup of  $G$  and  $a \in Q$ , each weakly generic and consisting of pairwise disjoint conjugates of  $[aQ^\circ]_0$ , whose union is large in  $G$ .

*Proof.* Assume first  $R(G) = Z(G) = 1$ . By Fact 35,  $G = G_1 \times \cdots \times G_n$  where each  $G_i$  is an infinite definably simple definable factor. Now by Corollary 10 Cartan subgroups  $Q$  of  $G$  are exactly of the form

$$Q = \tilde{Q}_1 \times \cdots \times \tilde{Q}_n$$

with  $\tilde{Q}_i$  is a Cartan subgroup of  $G_i$  for each  $i$ . In particular  $G$  has definable Cartan subgroups by Theorem 57. Since  $Q^\circ = \tilde{Q}_1^\circ \times \cdots \times \tilde{Q}_n^\circ$  and the dimension is additive, items (1-3) follow easily from Theorem 57(1-3). By additivity of the dimension, the first claim in item (4) also transfers readily from Theorem 57(4). If some element  $\alpha$  belongs to  $[aQ^\circ]_0 \cap a_1Q_1^\circ$ , for some Cartan subgroups  $Q$  and  $Q_1$  and some  $a \in Q$  and  $a_1 \in Q_1$ , then  $Q_1^\circ \leq C_G^\circ(\alpha) = Q^\circ$  by the commutativity of  $Q$  and  $Q_1$  and Lemma 18, and  $Q^\circ = Q_1^\circ$ . In particular  $aQ^\circ = \alpha Q^\circ = \alpha Q_1^\circ = a_1Q_1^\circ$ , proving item (4). For item (5), notice that if some  $[aQ^\circ]_0^G \cap [a_1Q_1^\circ]_0^G$  is non empty in item (4), then  $aQ^\circ = [a_1Q_1^\circ]^g$  for some  $g$  (conjugating in particular  $Q^\circ$  to  $Q_1^\circ$ ), so the finitely many weakly generic definable sets of the form  $[aQ^\circ]_0^G$  are pairwise disjoint and consist of a disjoint union of  $G$ -conjugates of  $[aQ^\circ]_0$ . By the largeness of  $[aQ^\circ]_0^G$  in  $[aQ^\circ]^G$  provided by Corollary 17 and the largeness of the union of all Cartan subgroups provided by Theorem 57(5), the union of all these sets  $[aQ^\circ]_0^G$  is large in  $G$ , proving item (5).

Assume now just  $R(G) = Z(G)$  finite, and let the notation “ $\bar{\quad}$ ” denote the quotients by  $Z(G)$ . By the centerless case, all the conclusions of Theorem 62 hold

in  $\overline{G}$ . By Lemma 8, Cartan subgroups of  $G$  contain  $Z(G)$  and are exactly the preimages in  $G$  of Cartan subgroups of  $G/Z(G)$ . In particular,  $G$  has definable Cartan subgroups, and we now check that they still satisfy (1-5).

(1) Since  $Z(G)$  is contained in each Cartan subgroup, item (1) transfers from the centerless case. (2) If  $Q_1$  and  $Q_2$  are two Cartan subgroups of  $G$  with  $Q_1^\circ = Q_2^\circ$ , then  $\overline{Q_i^\circ} = [\overline{Q_i}]^\circ$  and  $\overline{Q_1} = \overline{Q_2}$  by (2) in  $\overline{G}$ , giving  $Q_1 = Q_2$ . (3) By the centerless case  $\overline{Q}$  is abelian, and thus  $Q' \leq Z(G)$ . In particular  $[Q, Q^\circ]$  is in the finite center  $Z(G)$ , but since  $[Q, Q^\circ]$  is definable and definably connected by [1, Corollary 6.5] we get  $[Q, Q^\circ] = 1$ , proving the first claim of (3). Since the natural (and definable) projection from  $G$  onto  $\overline{G}$  has finite fibers one gets by axioms A2-3 of the dimension that  $\dim(\overline{Q}) = \dim(Q)$ , transferring also from  $\overline{G}$  to  $G$  the second claim of (3). (4) Let  $Q$  and  $Q_1$  be two Cartan subgroups,  $a \in Q$  and  $a_1 \in Q_1$ . If some element  $\alpha$  belongs to  $[aQ^\circ]_0 \cap a_1Q_1^\circ$ , one sees as in the centerless case, still using Lemma 18 but now the fact that  $Q^\circ \leq Z(Q)$  and  $Q_1^\circ \leq Z(Q_1)$ , that  $aQ^\circ = a_1Q_1^\circ$ . We now show that  $[aQ^\circ]_0$  is large in  $aQ^\circ$ . For that purpose, first notice that  $[aQ^\circ]_0$  is exactly the set of elements of  $aQ^\circ$  contained in finitely many conjugates of  $aQ^\circ$ : for, if  $\alpha$  is in  $aQ^\circ$  and in only finitely many of its conjugates, say  $(aQ^\circ)^{g_1}, \dots, (aQ^\circ)^{g_k}$ , then as above Lemma 18 yields  $Q^\circ = C^\circ(\alpha)$ , and  $aQ^\circ = (aQ^\circ)^{g_1} = \dots = (aQ^\circ)^{g_k}$ . For the largeness of  $[aQ^\circ]_0$  in  $aQ^\circ$ , it suffices as in item (3) to show that  $[aQ^\circ]_0$  contains the preimage of the set of elements  $\overline{\alpha}$  of  $\overline{aQ^\circ}$  contained in a unique  $\overline{G}$ -conjugate of  $\overline{aQ^\circ}$ . So assume towards a contradiction that there exists an element  $\alpha$  in  $aQ^\circ$ , in infinitely many  $G$ -conjugates of  $aQ^\circ$  but such that  $\overline{\alpha}$  is in a unique conjugate of  $\overline{aQ^\circ}$ . Now for  $g$  varying in infinitely many cosets of  $N(aQ^\circ)$ , and in particular in infinitely many cosets of  $N^\circ(aQ^\circ) = N^\circ(Q^\circ) = Q^\circ$ , we have  $aZ(G)Q^\circ = [aZ(G)Q^\circ]^g$ . But such elements  $g$  must normalize the subgroup  $Z(G)Q^\circ$ , and in particular  $[Z(G)Q^\circ]^\circ = Q^\circ$ , and hence cannot vary in infinitely many cosets of  $Q^\circ$ . This contradiction proves that  $[aQ^\circ]_0$  is large in  $aQ^\circ$ , and the weak generosity of  $aQ^\circ$  in  $G$  follows as usual with Corollary 17. (5) Using the projection from  $G$  to  $\overline{G}$ , the non weak genericity of the complement of the union of all Cartan subgroups passes from  $\overline{G}$  to  $G$ , and thus the union of all Cartan subgroups of  $G$  is large in  $G$ . Then all other claims of item (5) follow as in the case  $Z(G) = 1$ .  $\square$

In Theorem 62(3) Cartan subgroups need not be abelian outside of the centerless case, since the simply-connected double covering of  $\mathrm{SL}_3(\mathbb{R})$  with non-abelian Cartan subgroups mentioned after Fact 53 is definable in  $\mathbb{R}$ . The following question then arises naturally.

**Question 63.** *Let  $G$  be a definably connected semisimple group definable in an  $\mathcal{O}$ -minimal structure, and  $Q$  a Cartan subgroup of  $G$ . When is it the case that  $Q$  is abelian? That  $Q = Q^\circ Z(G)$ ?*

For Carter subgroups, one gets the following corollary of Theorem 62.

**Corollary 64.** *Let  $G$  be a definably connected semisimple group definable in an  $\mathcal{O}$ -minimal structure. Then  $G$  has finitely many conjugacy classes of Carter subgroups. Each Carter subgroup  $Q^\circ$  is abelian and weakly generous in the following strong sense: the set of elements of  $Q^\circ$  contained in a unique conjugate of  $Q^\circ$  is large in  $Q^\circ$  and weakly generous in  $G$ .*

*Proof.* We know by Lemma 5 that Carter subgroups are exactly the definably connected components  $Q^\circ$  of Cartan subgroups  $Q$  of  $G$ . In particular item (3) of

Theorem 62 shows that  $Q^\circ \leq Z(Q)$ , and  $Q^\circ$  is abelian. The other conclusions follow immediately from items (1) and (4) in Theorem 62.  $\square$

Before moving to more general situations, we make a few additional remarks about the semisimple case. We first mention a general result on control of fusion, reminiscent from [8, Corollary 2.12] in the finite Morley rank case.

**Lemma 65** (Control of fusion). *Let  $G$  be a group definable in an o-minimal structure,  $Q$  a Cartan subgroup of  $G$ ,  $X$  and  $Y$  two  $G$ -conjugate subsets of  $C(Q^\circ)$  such that  $C^\circ(Y)$  has a single conjugacy class of Carter subgroups. Then  $Y = X^g$  for some  $g$  in  $N(Q^\circ)$ .*

*Proof.* Let  $g$  in  $G$  be such that  $Y = X^g$ . Then  $C^\circ(Y) = C^\circ(X)^g$  contains both  $Q^\circ$  and  $Q^{\circ g}$ , so our assumption forces that  $[Q^\circ]^{g\gamma} = Q^\circ$  for some  $\gamma$  in  $C^\circ(Y)$ . Now  $g\gamma$  normalizes  $Q^\circ$  and  $X^{g\gamma} = Y^\gamma = Y$ .  $\square$

**Lemma 66.** *Let  $G$  be a definably connected semisimple group  $G$  definable in an o-minimal structure and  $Q$  a Cartan subgroup of  $G$ . Then  $Q = F(N_G(Q^\circ))$ .*

*Proof.* Any definable nilpotent subgroup containing the Carter subgroup  $Q^\circ$  is a finite extension of it by Lemma 4, and hence is in  $N_G(Q^\circ)$ . By Theorem 62(2), there is a unique maximal one. This proves that  $Q \trianglelefteq N_G(Q^\circ)$ . Hence  $Q \leq F(N_G(Q^\circ))$ , and in fact there is equality by maximal nilpotence of  $Q$ .  $\square$

With Lemma 65, we can rephrase the last part of Theorem 62(4).

**Corollary 67.** *Let  $G$  be a definably connected semisimple group definable in an o-minimal structure and  $Q$  a Cartan subgroup of  $G$ . If  $a_1$  and  $a_2$  are two  $G$ -conjugate elements of  $Q$  such that  $a_i \in [a_i Q^\circ]_0$  as in Theorem 62(4) for  $i = 1$  and  $2$ , then  $a_1 Q^\circ$  and  $a_2 Q^\circ$  are  $N(Q)$ -conjugate.*

*Proof.* By Theorem 62(3),  $a_i \in C(Q^\circ)$  for each  $i$ , and by Lemma 18  $Q^\circ = C^\circ(a_1) = C^\circ(a_2)$ . Lemma 65 implies then that  $a_2 = a_1^g$  for some  $g$  in  $N(Q^\circ)$ . But since  $Q \trianglelefteq N_G(Q^\circ)$  by Lemma 66,  $g \in N_G(Q)$ .  $\square$

As just seen in Corollary 67, if  $Q$  is a Cartan subgroup of a definably connected semisimple group  $G$  definable in an o-minimal structure, then

$$N_G(Q) = N_G(Q^\circ).$$

Now the finite group  $W(G, Q) := N_G(Q)/Q = N_G(Q^\circ)/Q$  can naturally be called the *Weyl group relative to  $Q$* , or, equivalently, *relative to  $Q^\circ$* . If  $G$  is definably simple, then one has the two alternatives at the end of Fact 35. In the first case  $G$  is essentially a simple algebraic group over an algebraically closed field (of characteristic 0). It is well known in this case that there is only one conjugacy of Cartan subgroups, the maximal (algebraic and split) tori which are also Carter subgroups (by divisibility). Then there is only one relative Weyl group, and their classification is provided by the classification of the simple algebraic groups. In the second alternative at the end of Fact 35, the group is essentially a simple real Lie group, and again the Weyl groups relative to the various Cartan subgroups, corresponding to the various split or non-split tori, are classified in this case. For a general definably connected semisimple ambient group  $G$ , the structure of the Weyl groups is inherited from that of the definably simple factors of  $G/R(G)$ , as we will see in Section 13.

Theorem 62(5) equips any definably connected semisimple group with some kind of a partition into finitely many canonical “generic types”. We finish this section by counting them precisely.

**Remark 68.** *The number  $n(G)$  of weakly generic definable sets of the form  $[aQ^\circ]_0^G$  as in Theorem 62(5) is clearly bounded by the sum  $\sum_{Q \in \mathcal{Q}} |Q/Q^\circ|$  where  $\mathcal{Q}$  is a system of representatives of the set of Cartan subgroups of  $G$ . But it might happen in Theorem 62(4) that two distinct sets of the form  $aQ^\circ$  and  $a'Q^\circ$ , for  $a$  and  $a'$  in a common Cartan subgroup  $Q$ , are conjugate by the action of the Weyl group  $W(G, Q) = N_G(Q)/Q$ . If one denotes by  $\sim_Q$  the equivalence relation on  $Q/Q^\circ$  by the action of  $W(G, Q)$  naturally induced by conjugation on  $Q/Q^\circ$ , then one sees indeed with Corollary 67 that*

$$n(G) = \sum_{Q \in \mathcal{Q}} |[Q/Q^\circ]_{\sim_Q}|.$$

## 12. THE GENERAL CASE

We now analyze the general case of a group definable in an o-minimal structure. As far as possible, we will restrict ourselves to definably connected groups only when necessary. We start by lifting Carter subgroups.

**Lemma 69.** *Let  $G$  be a group definable in an o-minimal structure, and  $N$  a definable normal subgroup of  $G$  such that  $N^\circ$  is solvable. Then Carter subgroups of  $G/N$  are exactly of the form  $QN/N$  for  $Q$  a Carter subgroup of  $G$ .*

*Proof.* We may use the notation “ $\overline{\phantom{x}}$ ” to denote the quotients by  $N$ . Let  $Q$  be a Carter subgroup of  $G$ . Then  $Q$  is also a Carter subgroup of the definable subgroup  $QN$ . The preimage in  $G$  of  $N_{\overline{G}}(\overline{Q})$  normalizes  $[QN]^\circ = QN^\circ$ , and thus is contained in  $N_G(Q)N$  by Corollary 47. Hence  $\overline{Q}$ , which is definable and definably connected, must have finite index in its normalizer in  $\overline{G}$ , and is thus a Carter subgroup of  $\overline{G}$ . Conversely, let  $X/N$  be a Carter subgroup of  $\overline{G}$  for some subgroup  $X$  of  $G$  containing  $N$ . Since  $X/N$  is definable,  $X$  must be definable. By Theorem 40,  $X^\circ$  has a Carter subgroup  $Q$ , and of course  $Q$  must also be a Carter subgroup of  $X$ . Since  $X = X^\circ N$  and  $X^\circ = Q(X^\circ \cap N)$  by Lemma 48, we get that  $X = QN$ . Since  $QN/N$  is a Carter subgroup  $\overline{G}$ , we get that  $QN$  has finite index in  $N_G(QN)$ . Since  $N_G(Q) \leq N_G(QN)$  and  $Q$  has finite index in  $N_{QN}(Q)$ , we get that  $Q$  has finite index in  $N_G(Q)$ . Hence  $X = QN$  for a Carter subgroup  $Q$  of  $G$ .  $\square$

The following special case of Lemma 69 with  $N = R^\circ(G)$  is of major interest, and for the rest of the paper one should bear in mind that

$$R^\circ(G) = R^\circ(G^\circ).$$

**Corollary 70.** *Let  $G$  be a group definable in an o-minimal structure. Then Carter subgroups of  $G/R^\circ(G)$  are exactly of the form  $QR^\circ(G)/R^\circ(G)$  for  $Q$  a Carter subgroup of  $G$ .*

At this stage, we can prove our general Theorem 1 giving the existence, the definability, and the finiteness of the set of conjugacy classes of Cartan subgroups in an arbitrary group definable in an o-minimal structure.

**Proof of Theorem 1.** Let  $G$  be our arbitrary group definable in an arbitrary o-minimal structure. The quotient  $G^\circ/R^\circ(G)$  is semisimple by Fact 2, and has Carter subgroups by Theorem 62. Hence  $G^\circ$  has Carter subgroups by Corollary 70. This takes care of the existence of Carter subgroups of  $G^\circ$ , and of course of  $G$  as well. Now  $G$  has Cartan subgroups by Lemma 5. Their definability is automatic as usual in view of Lemma 5( $a'$ ). To prove that Cartan subgroups fall into only finitely many conjugacy classes, it suffices by Lemma 5( $a'$ ) to prove it for

Carter subgroups. We may then assume  $G$  definably connected. Now groups of the form  $QR^\circ(G)/R^\circ(G)$ , for  $Q$  a Carter subgroup of  $G$ , are Carter subgroups of the semisimple quotient  $G/R^\circ(G)$ . By Theorem 62(1), there are only finitely many  $G/R^\circ(G)$ -conjugacy classes of groups of the form  $QR^\circ(G)/R^\circ(G)$ , and thus only finitely many  $G$ -conjugacy classes of groups of the form  $QR^\circ(G)$ . Replacing  $G$  by such a  $QR^\circ(G)$ , we may thus assume  $G$  definably connected and solvable. But now in  $G$  there is only one conjugacy class of Carter subgroups by Theorem 40. This completes our proof of Theorem 1.  $\square$

We mention the following form of a Frattini Argument as a consequence of Theorem 1.

**Corollary 71.** *Let  $G$  be a definably connected group definable in an  $o$ -minimal structure and  $N$  a definable normal subgroup of  $G$ . Then  $G = N_G^\circ(Q)N^\circ$  for any Cartan subgroup  $Q$  of  $N$ .*

*Proof.* Clearly, for any element  $g$  of  $G$ ,  $Q^g$  is a Cartan subgroup of  $N$ . On the other hand, the set  $\mathcal{Q}$  of conjugacy classes of Cartan subgroups of  $N$  is finite by Theorem 1, and the action of  $G$  on  $N$  by conjugation naturally induces a definable action on the finite set  $\mathcal{Q}$ . Since  $G$  is definably connected, Fact 2(a) shows that this action must be trivial. Hence, for any  $g$  in  $G$ ,  $Q^g$  is indeed in the same  $N$ -conjugacy class as  $Q$ , i.e.,  $Q^g = Q^h$  for some  $h \in N$ ; in particular  $g = gh^{-1}h \in N_G(Q)N$ . Hence  $G = N_G(Q)N$ , and in fact  $G = N_G^\circ(Q)N^\circ$  by definable connectedness.  $\square$

We shall now inspect case by case what survives of Theorem 62(2-5) in the general case. We first consider Theorem 62(2).

**Theorem 72.** *Let  $G$  be a definably connected group definable in an  $o$ -minimal structure and  $Q$  a Cartan subgroup of  $G$ . Then there is a unique (definable) subgroup  $K_Q$  of  $G$  containing  $R^\circ(G)$  such that  $K_Q/R^\circ(G)$  is the unique Cartan subgroup of  $G/R^\circ(G)$  containing  $Q^\circ R^\circ(G)/R^\circ(G)$ . Moreover,  $QR(G) \leq K_Q$  and*

$$Q = F(N_{K_Q}(Q^\circ)) = C_{K_Q}(Q^\circ)Q^\circ = C_G(Q^\circ)Q^\circ.$$

*Proof.* By Corollary 70, the group  $Q^\circ R^\circ(G)/R^\circ(G)$  is a Carter subgroup of the semisimple quotient  $G/R^\circ(G)$ . By Theorem 62(2), it is contained in a unique Cartan subgroup, of the form  $K/R^\circ(G)$  for some subgroup  $K$  containing  $R^\circ(G)$  and necessarily definable by Lemma 5(a'). We will show that  $K_Q = K$  satisfies all our claims. Since  $QR^\circ(G)/R^\circ(G)$  is nilpotent and contains the Carter subgroup  $Q^\circ R^\circ(G)/R^\circ(G)$ , we have  $QR^\circ(G) \leq K$ . Since  $R(G)/R^\circ(G)$  is the center of  $G/R^\circ(G)$ , it is contained in  $K/R^\circ(G)$  by Lemma 8(a), and thus  $R(G) \leq K$ . Hence,  $QR(G) \leq K$ .

To prove our last equalities, we first show that  $F(N_K(Q^\circ)) = C_K(Q^\circ)Q^\circ$ . Since  $Q^\circ = F^\circ(N_K(Q^\circ))$  by Lemma 4, the inclusion from left to right follows from Fact 6. For the reverse inclusion, notice that  $C_K(Q^\circ)Q^\circ$  is normal in  $N_K(Q^\circ)$ . Since Cartan subgroups of  $G/R^\circ(G)$  are nilpotent in two steps by Theorem 62(3), the second term of the descending central series of  $C_K(Q^\circ)Q^\circ$  is in  $R^\circ(G)$ , and thus in  $Q^\circ$  because  $Q^\circ$  is selfnormalizing in  $Q^\circ R^\circ(G)$  by Theorem 40. By keeping taking descending central series and using the nilpotency of  $Q^\circ$ , we then see that  $C_K(Q^\circ)Q^\circ$  is nilpotent, and thus in  $F(N_K(Q^\circ))$  by normality in  $N_K(Q^\circ)$ .

Since  $C_G(Q^\circ) \leq K$ , clearly by considering its image modulo  $R^\circ(G)$ , our last equality is true. Finally,  $Q = C_Q(Q^\circ)Q^\circ$  by Fact 6, and thus  $Q \leq C_K(Q^\circ)Q^\circ = F(N_K(Q^\circ))$ . Now the maximal nilpotence of  $Q$  forces  $Q = F(N_K(Q^\circ))$ , and our proof is complete.  $\square$

With Theorem 72 one readily gets the analog of Theorem 62(2). Of course definable connectedness is a necessary assumption here, since a finite group may have several Cartan subgroups.

**Corollary 73.** *Let  $G$  be a definably connected group definable in an  $o$ -minimal structure,  $Q_1$  and  $Q_2$  two Cartan subgroups. If  $Q_1^\circ = Q_2^\circ$ , then  $Q_1 = Q_2$ .*

We also get that  $QR^\circ(G)$  is normal in  $K_Q$ , and actually has a quite stronger uniqueness property in  $K_Q$ .

**Corollary 74.** *Same assumptions and notation as in Theorem 72. Then  $[K_Q]^\circ = Q^\circ R^\circ(G)$  and  $QR^\circ(G)$  is invariant under any automorphism of  $K_Q$  leaving  $[K_Q]^\circ$  invariant.*

*Proof.* The first equality comes from Lemma 69.

Let  $\sigma$  be an arbitrary automorphism of  $K_Q$  leaving  $[K_Q]^\circ$  invariant. Since  $Q^\circ$  is a Cartan subgroup of  $[K_Q]^\circ$  by Corollary 46, its image by  $\sigma$  is also a Cartan subgroup of  $[K_Q]^\circ$ , and with Theorem 40 one gets  $[Q^\circ]^\sigma = [Q^\circ]^k$  for some  $k$  in  $[K_Q]^\circ$ . Since  $QR^\circ(G)$  is normalized by  $k$ , we can thus assume that  $\sigma$  leaves  $Q^\circ$  invariant. But now  $\sigma$  leaves  $F(N_{K_Q}(Q^\circ))$  invariant. Hence by Theorem 72  $Q$  is left invariant by  $\sigma$ , and thus  $\sigma$  leaves  $Q[K_Q]^\circ = QR^\circ(G)$  invariant.  $\square$

The main question we are facing with at this stage is the following.

**Question 75.** *Is it the case, in Theorem 72, that  $K_Q = QR^\circ(G)$ ?*

Question 75 has a priori stronger forms, which are indeed equivalent as the following lemma shows.

**Lemma 76.** *Under the assumptions and notation of Theorem 72, the following are equivalent:*

- (a)  $K_Q = PR^\circ(G)$  for some Cartan subgroup  $P$  of  $G$
- (b)  $K_Q = PR^\circ(G)$  for any Cartan subgroup  $P$  of  $K_Q$ .

*Proof.* Assume  $K_Q = P_1R^\circ(G)$  for some Cartan subgroup  $P_1$  of  $G$ , and suppose  $P_2$  is a Cartan subgroup of  $K_Q$ . Then  $P_1^\circ$  and  $P_2^\circ$  are Carter subgroups of  $[K_Q]^\circ$  by Lemma 5(a'). Since they are  $[K_Q]^\circ$ -conjugate by Theorem 40, we may assume  $P_1^\circ = P_2^\circ$  up to conjugacy. Now applying Theorem 72 with the Cartan subgroup  $P_1$ , or just Corollary 73, we see that  $P_1 = P_2$  up to conjugacy, and thus  $K_Q = P_2R^\circ(G)$ .

Conversely, suppose  $K_Q = PR^\circ(G)$  for any Cartan subgroup  $P$  of  $K_Q$ . This applies in particular to the Cartan subgroup  $Q$  of  $G$ .  $\square$

By the usual Frattini Argument following the conjugacy of Cartan/Carter subgroups in  $[K_Q]^\circ$ , we have that  $K_Q = \hat{Q}R^\circ(G)$  where

$$\hat{Q} = N_{K_Q}(Q^\circ).$$

The subgroup  $\hat{Q}$  is solvable and nilpotent-by-finite, and with the selfnormalization property of  $Q^\circ$  in the definably connected solvable group  $Q^\circ R^\circ(G)$  one sees easily that  $\hat{Q}/Q \simeq K_Q/(QR^\circ(G))$ . Hence Question 75 is equivalent to proving that the finite quotient  $\hat{Q}/Q$  is trivial.

Retaining all the notation introduced so far, Theorem 62(3) takes the following form for a general definably connected group.

**Theorem 77.** *Same assumptions and notation as in Theorem 72. Then  $[K_Q]' \leq R(G)$ , and  $[\hat{Q}, [\hat{Q}']] \leq Q^\circ \cap R^\circ(G)$  where  $\hat{Q} = N_{K_Q}(Q^\circ)$ .*

*Proof.* By Theorem 62(3),  $[K_Q]' \leq R(G)$  and  $[K_Q, [K_Q]'] \leq R^\circ(G)$ . The second inclusion shows in particular that  $[\hat{Q}, [\hat{Q}']] \leq R^\circ(G)$ , and since  $Q^\circ$  is selfnormalizing in  $Q^\circ R^\circ(G)$  by Theorem 40, we get inclusion in  $Q^\circ$  as well.  $\square$

We now consider Theorem 62(4) and give its most general form in the general case (working in particular without any assumption of definable connectedness of the ambient group).

**Theorem 78.** *Let  $G$  be a group definable in an o-minimal structure,  $Q$  a Cartan subgroup of  $G$  and  $a \in Q$ . Then  $aQ^\circ$  is weakly generous in  $G$ . In fact, the set of elements of  $aQ^\circ$  contained in a unique conjugate of  $aQ^\circ$  is large in  $aQ^\circ$ . Furthermore, if  $G$  is definably connected, then the set of elements of  $Q$  contained in a unique conjugate of  $Q$  is large in  $Q$ .*

*Proof.* We first prove that the set of elements of  $Q^\circ$  contained in a unique  $G$ -conjugate of  $Q^\circ$  is large in  $Q^\circ$ . For that purpose, it suffices by Corollary 30 to show that the set of elements of  $Q^\circ$  contained in only finitely many  $G$ -conjugates of  $Q^\circ$  is large in  $Q^\circ$ . Assume towards a contradiction that the set  $Q_\infty$  of elements of  $Q^\circ$  contained in infinitely many  $G$ -conjugates of  $Q^\circ$  is weakly generic in  $Q^\circ$ . By Theorem 40, we may restrict  $Q_\infty$  to the subset of elements contained in a unique  $Q^\circ R^\circ(G)$ -conjugate of  $Q^\circ$ , and still have a weakly generic subset of  $Q^\circ$ . Now  $Q_\infty$  must have a weakly generic image in  $Q^\circ$  modulo  $R^\circ(G)$ . By Theorem 62(4), we must then find an element  $x \in Q_\infty$  which, modulo  $R^\circ(G)$ , is in a unique conjugate of  $Q^\circ$ . Then we have infinitely many Carter subgroups of  $Q^\circ R^\circ(G)$  passing through  $x$ , a contradiction since they are all  $Q^\circ R^\circ(G)$ -conjugate by Theorem 40.

We now consider the full Cartan subgroup  $Q$ , and an arbitrary element  $a$  in  $Q$ . For the weak generosity of  $aQ^\circ$  in  $G$ , it suffices to use our general Corollary 21. Indeed, by Corollary 17, it suffices to show the stronger property that the set of elements of  $aQ^\circ$  in a unique conjugate of  $aQ^\circ$  is large in  $aQ^\circ$ . Assume towards a contradiction that the set  $X$  of elements of  $aQ^\circ$  in at least two distinct conjugates of  $aQ^\circ$  is weakly generic in  $aQ^\circ$ . If  $n$  is the order of  $a$  modulo  $Q^\circ$ , then the set of  $n$ -th powers of elements of  $X$  would be weakly generic in  $Q^\circ$  by Corollary 23. Hence by the preceding paragraph one would find an element  $x$  in  $X$  such that  $x^n$  is in a unique conjugate of  $Q^\circ$ . This is a contradiction as usual since  $xQ^\circ$  must then be the unique conjugate of  $aQ^\circ$  containing  $x$ .

We now prove our last claim about  $Q$  when  $G$  is definably connected. Assume towards a contradiction that the set  $X$  of elements in  $Q$  and in at least two distinct conjugates of  $Q$  is weakly generic in  $Q$ . Then it should meet one of the cosets  $aQ^\circ$  of  $Q^\circ$  in  $Q$  in a weakly generic subset, say  $X'$ . By Corollary 23 again, one finds an element  $x$  in  $X'$  such that  $x^{|Q/Q^\circ|}$  is in a unique conjugate of  $Q^\circ$ . Now all the conjugates of  $Q$  passing through  $x$  should have the same definably connected component, and thus are  $N_G(Q^\circ)$ -conjugate. Then they are all equal by Corollary 73, a contradiction.  $\square$

In case Question 75 fails, we unfortunately found no way of proving Theorem 78 for  $a$  in  $\hat{Q} \setminus Q$ . Besides, our method for proving the weak generosity of  $aQ^\circ$  in  $G$  does not seem to be appropriate for attacking the following more refined question.

**Question 79.** *Let  $G$ ,  $Q$ , and  $a$  be as in Theorem 78, with  $G$  definably connected and such that, modulo  $R^\circ(G)$ ,  $a$  is in a unique conjugate of  $aQ^\circ$ .*

- (a) *Is it the case that  $[aQ^\circ]^{R^\circ(G)}$  is large in  $aQ^\circ R^\circ(G)$ ?*
- (b) *Same question, with  $a$  in  $\hat{Q}$  instead of  $a$  in  $Q$ ?*

By Theorem 78, the union of Cartan subgroups of a group definable in an o-minimal structure must be weakly generic, but the much stronger statement of Theorem 62(5) now becomes a definite question.

**Question 80.** *Let  $G$  be a definably connected group definable in an o-minimal structure. Is it the case that the union of its Cartan subgroups forms a large subset?*

We now prove that Question 80 can be seen on top of both Questions 75 and 79.

**Proposition 81.** *Let  $G$  be a definably connected group definable in an o-minimal structure whose Cartan subgroups form a large subset. Then*

- (a) *Cartan subgroups of  $G/R^\circ(G)$  are exactly of the form  $QR^\circ(G)/R^\circ(G)$  with  $Q$  a Cartan subgroup of  $G$ .*
- (b) *For every Cartan subgroup  $Q$  and  $a$  in  $Q$  such that, modulo  $R^\circ(G)$ ,  $a$  is in a unique conjugate of  $aQ^\circ$ ,  $[aQ^\circ]^{R^\circ(G)}$  is large in  $aQ^\circ R^\circ(G)$ .*

*Proof.* (a). Assume towards a contradiction that for some Cartan subgroup  $Q$ , and with the previously used notation, we have  $QR^\circ(G) < K_Q$ . Let  $\overline{B}$  be the large subset of  $(K_Q/R^\circ(G)) \setminus (QR^\circ(G)/R^\circ(G))$  then provided by Theorem 62(4), and  $B$  its pull back in  $G$ . By additivity of the dimension,  $B^G$  must be weakly generic in  $G$ . Now the largeness of the set of Cartan subgroups forces the existence of an element  $g$  in  $B \cap P$  for some Cartan subgroup  $P$  of  $G$ . Let  $\overline{g}$  denote the image of  $g$  in  $G/R^\circ(G)$ . We have  $g \in K_Q \setminus QR^\circ(G)$ , and  $C^\circ(\overline{g}) = Q^\circ R^\circ(G)/R^\circ(G)$  by considering the structure of Cartan subgroups in the semisimple quotient  $G/R^\circ(G)$  and the uniqueness property of  $\overline{g}$ . By Lemma 69 the group  $P^\circ$ , modulo  $R^\circ(G)$ , is a Carter subgroup of  $G/R^\circ(G)$ . Now  $P$ , modulo  $R^\circ(G)$ , is included in a Cartan subgroup of  $G/R^\circ(G)$ , and its definably connected component centralizes  $\overline{g}$  by Theorem 62(3). We then get  $P^\circ R^\circ(G)/R^\circ(G) \leq C^\circ(\overline{g}) = Q^\circ R^\circ(G)/R^\circ(G)$ , and actually equality since the first group is a Carter subgroup. Hence  $P^\circ R^\circ(G) = Q^\circ R^\circ(G)$  and Theorem 72 yields  $P \leq K_Q$ . Since  $Q^\circ$  and  $P^\circ$  are conjugate in  $Q^\circ R^\circ(G)$  by Theorem 40, we may also assume without loss that  $P^\circ = Q^\circ$ . But then  $P = Q$  by Corollary 73, a contradiction since  $g \notin QR^\circ(G)$ .

(b). Let  $A$  be the pull back in  $G$  of the large set of  $G/R^\circ(G)$  provided in Theorem 62(5), and

$$A = A_1 \sqcup \cdots \sqcup A_{n(G)}$$

the pull back in  $G$  of the corresponding partition of that large set equally provided in Theorem 62(5). Here  $n(G)$  is the number of “generic types” of  $G/R^\circ(G)$  computed with precision in Remark 68. By additivity of the dimension,  $A$  is large in  $G$  and each  $A_i$  is weakly generic. Our claim is that for  $Q$  a Cartan subgroup of  $G$  and  $a \in Q \cap A_i$  for some  $i$ , the set  $[aQ^\circ]^{R^\circ(G)}$  is large in  $aQ^\circ R^\circ(G)$ . Since  $Q^\circ$  normalizes the coset  $aQ^\circ$ , this is equivalent to showing that  $[aQ^\circ]^{Q^\circ R^\circ(G)}$  is large in  $aQ^\circ R^\circ(G)$ . But by Theorem 62(4-5) applied in  $G/R^\circ(G)$ , one can see that the largeness of the set of Cartan subgroups of  $G$  and the additivity of the dimension forces  $[aQ^\circ]^{Q^\circ R^\circ(G)}$  to be large in  $aQ^\circ R^\circ(G)$ .  $\square$

For instance, if  $G$  is a definably connected real Lie group definable in an o-minimal expansion of  $\mathbb{R}$ , then its Cartan subgroups form a large subset by Fact 55(a) and the fact that density implies largeness for definable sets (as seen in the proof of Theorem 57). Hence, by Proposition 81, such a  $G$  can produce a counterexample to neither Question 75 nor Question 79. Attacking Question 80 in general would seem to rely on an abstract version of Fact 55(a), but with a priori no

known abstract analog of regular elements (as in the proof of Fact 54(c)) it seems difficult to find any spark plug.

13. FINAL REMARKS

We begin this final section with additional comments on Question 75 in special cases. If  $G$  is a definably connected group definable in an o-minimal structure, then by Fact 35 we have

$$G/R(G) = G_1/R(G) \times \cdots \times G_n/R(G)$$

for some definable subgroups  $G_i$  containing  $R(G)$  and such that  $G_i/R(G)$  is definably simple. For each  $i$ ,  $G_i/R(G)$  is definably connected, and thus  $G_i = G_i^\circ R(G)$ . From the decomposition  $G = G_1 \cdots G_n$  we get  $G = G_1^\circ \cdots G_n^\circ R(G)$ . By definable connectedness of  $G$  we also get a decomposition

$$(*) \quad G = G_1^\circ \cdots G_n^\circ$$

where each  $G_i^\circ$  is definably connected, contains  $R^\circ(G)$ , and  $G_i^\circ/R^\circ(G)$  is finite-by-(definably simple), as  $R(G_i^\circ) = G_i^\circ \cap R(G)$  and  $G_i^\circ/R(G_i^\circ)$  is definably isomorphic to  $G_i/R(G)$ . We may analyze certain factors  $G_i^\circ$  individually with the following.

**Fact 82.** *Let  $\mathcal{M}$  be an o-minimal structure and  $G$  a definably connected group definable in  $\mathcal{M}$  with  $R(G) = Z(G)$  finite and  $G/R(G)$  definably simple.*

- (a) *If  $G/R(G)$  is stable as in the first case of Fact 35, then  $G$  is (definably isomorphic in  $\mathcal{M}$  to) an algebraic group over an algebraically closed field.*
- (b) *If  $G/R(G)$  is definably compact, then  $G$  is definably compact as well.*

*Proof.* As  $G$  is definably connected and semisimple, there is an  $\mathcal{M}$ -definable real closed field  $R$  such that  $G$  is definably isomorphic in  $\mathcal{M}$  to a semialgebraic group over the field of real algebraic numbers  $R_{alg} \subseteq R$ , by [18, 4.4(ii)] or [11]. In case (a) our claim follows from [18, 6.3] and thus we only have to consider case (b). Assume towards a contradiction that  $\alpha : (0, 1) \rightarrow G$  is a continuous definable curve not converging in  $G$ . Since  $G/Z(G)$  is definably compact, the composition of  $\alpha$  with the projection  $p : G \rightarrow G/Z(G)$  converges to a point  $x \in G/Z(G)$ . By [12, Prop.2.11],  $p$  is a definable covering map. In particular, there exists a definable open neighbourhood  $U$  of  $x$  in  $G/Z(G)$  such that each definable connected component of  $p^{-1}(U)$  is definably homeomorphic to  $U$  via  $p$ . Since  $\alpha$  does not converge to any point of  $p^{-1}(x) = \{y_1, \dots, y_s\}$ , by o-minimality there exist definable open neighbourhoods  $V_i \subseteq p^{-1}(U)$  of  $y_i$  and  $\delta \in (0, 1)$  such that  $\alpha(t) \notin V_1 \cup \dots \cup V_s$  for  $t \in (\delta, 1)$ . Hence  $p \circ \alpha(t)$  does not lie in the open neighbourhood  $p(V_1) \cap \dots \cap p(V_s)$  of  $x$  for  $t \in (\delta, 1)$ , which is a contradiction.  $\square$

**Corollary 83.** *If  $G$  is as in Fact 82, case (a) or (b), then it has a single conjugacy class of Cartan subgroups, which are divisible and definably connected.*

*Proof.* It is well known that in a connected reductive algebraic group over an algebraically closed field, Cartan subgroups are the selfcentralizing maximal algebraic tori, and are conjugate. They are isomorphic to a direct product of finitely many copies of the multiplicative group of the ground field (where the number of copies is the Lie rank of the group seen as a pure algebraic group). In particular they are divisible, and thus with no proper subgroup of finite index. In the definably compact case we refer to Corollary 38, getting the divisibility from the definable connectedness in this case.  $\square$

Consider the decomposition (\*) of a definably connected group  $G$  as above, and let  $I = \{1, \dots, n\}$ . Let  $I_1$  be the subset of elements  $i \in I$  such that  $G_i^\circ/R^\circ(G_i^\circ)$  is stable (as a pure group) or definably compact. Notice that, by Fact 82, it suffices to require the definably simple group  $G_i^\circ/R^\circ(G_i^\circ)$  to be stable (as a pure group) or definably compact. Let  $I_2$  be the subset of elements  $i \in I$  such that Cartan subgroups of  $G_i^\circ/R^\circ(G_i^\circ)$  are definably connected. Finally, let  $I_3$  be the subset of elements  $i \in I$  such that in  $G_i^\circ$  Question 75 has a positive answer for any Cartan subgroup. Corollary 83 shows that  $I_1 \subseteq I_2$  and Lemma 69 shows that  $I_2 \subseteq I_3$ . Hence

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq I$$

and the inclusion  $I_1 \subseteq I_3$  reads informally as the fact that the definably simple factors of  $G/R(G)$  which are algebraic or compact cannot produce any counterexample to the lifting problem of Question 75. More precisely, we have the following statement.

**Remark 84.** *If  $I_2 = I$ , then  $G$  cannot produce any counterexample to the lifting problem of Question 75.*

*Proof.* First one can check that, modulo  $R^\circ(G)$ , the decomposition (\*) of  $G$  becomes a central product:

$$G/R^\circ(G) = G_1^\circ/R^\circ(G) * \dots * G_n^\circ/R^\circ(G).$$

Indeed, if  $i \neq j$ , then  $[G_i^\circ, G_j^\circ] \leq R(G)$ , and  $R(G)$  is finite modulo  $R^\circ(G)$ . Hence any element in  $G_i^\circ/R^\circ(G)$  has a centralizer of finite index in the other factor  $G_j^\circ/R^\circ(G)$ , which must then be the full factor  $G_j^\circ/R^\circ(G)$  by definable connectedness. Therefore the factors  $G_i^\circ/R^\circ(G)$  pairwise commute, as claimed. Now Lemma 9 gives that Cartan subgroups of  $G/R^\circ(G)$  are exactly of the form  $Q_1/R^\circ(G) * \dots * Q_n/R^\circ(G)$  with, for each  $i$ ,  $Q_i/R^\circ(G)$  a Cartan subgroup of  $G_i^\circ/R^\circ(G)$ .

Assuming now that  $I_2 = I$  we get that, for each  $i$ , each Cartan subgroup  $Q_i/R^\circ(G)$  of  $G_i^\circ/R^\circ(G)$  is definably connected. We then see that Cartan subgroups of  $G/R^\circ(G)$  must be definably connected as well. Now Lemma 69 implies that Question 75 is positively satisfied for every Cartan subgroup of  $G$  (and that such Cartan subgroups of  $G$  are all definably connected and Carter subgroups by Corollary 46).  $\square$

The decomposition (\*) of a definably connected group  $G$  as above is also convenient for describing the various relative Weyl groups. If  $Q$  is a Cartan subgroup of  $G$ , then we still have that  $N_G(Q^\circ) = N_G(Q)$  by Corollary 73. If Question 75 is positively satisfied for  $Q$ , then retaining the notation of Section 12 and using the notation “ $\overline{\quad}$ ” for quotients modulo  $R^\circ(G)$  we get, as after Lemma 76, that

$$W(\overline{G}, \overline{K_Q}) \simeq N_G(Q)/Q.$$

We also see, with Theorem 72 or just Lemma 8(a), that  $\overline{R(G)}$  does not contribute to the Weyl group  $W(\overline{G}, \overline{K_Q})$ . Hence the latter is isomorphic to the direct product of the Weyl groups in  $G_i/R(G)$  relative to the factors of  $QR(G)/R(G)$  in its decomposition along the decomposition  $G_1/R(G) \times \dots \times G_n/R(G)$  of  $G/R(G)$  (Corollary 10). Since the group  $N_G(Q)/Q$  is isomorphic to  $W(\overline{G}, \overline{K_Q})$ , it has the same isomorphism type and may be called the *Weyl group relative to  $Q$* .

Without assuming the exact lifting of Question 75 for the Cartan subgroup  $Q$  we only get, with Corollary 74 and as after Lemma 76, that

$$W(\overline{G}, \overline{K_Q}) \simeq (N_G(Q)/Q)/(\hat{Q}/Q).$$

In this case the Weyl group  $W(\overline{G}, \overline{K_Q})$  has the same description as above, but  $N_G(Q)/Q$  just has a quotient isomorphic to  $W(\overline{G}, \overline{K_Q})$ .

We finish on a more model-theoretic note.

**Proposition 85.** *Let  $\mathcal{M}$  be an o-minimal structure,  $A \subseteq M$  a set of parameters such that  $\text{dcl}_{\mathcal{M}}(A) \preceq \mathcal{M}$ , and  $G$  a group definable in  $\mathcal{M}$  over  $A$ . Then  $G$  has a finite system of representatives of Cartan (resp. Carter) subgroups, both definable over  $A$ .*

Of course, having now Theorem 1 at hand, the proof of Proposition 85 is the same as in Lemma 58 and Corollary 59. As seen in the proof of Lemma 58, when  $\mathcal{M}$  expands an ordered abelian group, examples of  $A$  such that  $\text{dcl}_{\mathcal{M}}(A) \preceq \mathcal{M}$  include any  $A$  not contained in  $\{0\}$ .

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