Geometric tamely ramified local theta correspondence in terms of geometric Langlands functoriality

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GEOMETRIC TAMELY RAMIFIED LOCAL THETA CORRESPONDENCE IN TERMS OF GEOMETRIC LANGLANDS FUNCTORIALITY

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Abstract. In this Paper we are interested in the geometric local theta correspondence at the Iwahori level especially for dual reductive pairs of type II over a non Archimedean field F of characteristic $p \neq 2$ as well as the geometric Arthur-Langlands functoriality at the Iwahori level. We consider the geometric version of the invariants of the Weil representation of the Iwahori-Hecke algebras. We give a complete geometric description of the corresponding category for $(\mathbf{GL}_1, \mathbf{GL}_m)$. Additionally given two reductive connected groups G, H over F and a morphism $\check{G} \times \mathrm{SL}_2 \to \check{H}$ of Langlands dual groups, we suggest a bimodule over the affine extended Hecke algebras of H and G that should realize the local geometric Arthur-Langlands functoriality at the Iwahori level. We propose a conjecture describing the geometric Howe correspondence in terms of this bimodule and prove our conjecture for the pair $(\mathbf{GL}_1, \mathbf{GL}_m)$ for any m.

Keywords— Local theta correspondence, geometric Langlands program, Langlands functoriality, Hecke algebra, perverse sheaves, K-theory.

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Contents

1. Introduction

Our aim in this paper is to study the local theta correspondence known also as the Howe correspondence and Arthur-Langlands functoriality at the Iwahori level in the framework of geometric Langlands program. We develop this work in two directions. The first path consists in geometrizing the classical Howe correspondence at the Iwahori level by means of perverse sheaves and understanding the underlying geometry. The second path consists in studying the relation between this geometric Howe correspondence and the Arthur-Langlands funtoriality at the Iwahori level. Some of the constructions are done in all generality while some others are only established for the dual reductive pairs of type II.

The basic notions of the Howe correspondence from the classical point of view have been presented in [36]. Let $\mathbf{k} = \mathbb{F}_q$ of characteristic different from 2 and $F = \mathbf{k}((t))$ and $O = \mathbf{k}[[t]]$. All representations will be assumed to be smooth and will be defined over $\overline{\mathbb{Q}}_\ell$, where ℓ is a prime number different from p. Let (G, H) be a split dual reductive pair in some symplectic group $\mathrm{Sp}(W)$ over $\mathbf{k}$. Let $\widetilde{\mathrm{Sp}}(W)$ denote the metaplectic group which is the twofold topological covering of the symplectic group $\mathrm{Sp}(W)$. Denote by $(\mathcal{S}, \omega)$ the Weil (metaplectic) representation of $\widetilde{\mathrm{Sp}}(W)$, see [36]. We assume that the metaplectic cover admits a section over $G(F)$ and $H(F)$. In this case, the Howe correspondence becomes a correspondence between some class of representations of $G(F)$ and representations of $H(F)$.

It is of great interest to understand the relation between the Howe correspondence and Arthur-Langlands functoriality. Adams in [1] suggested conjectural relations with functoriality. Let $\check{G}$ (resp. $\check{H}$) denotes the Langlands dual group of G (resp. of H) over $\overline{\mathbb{Q}}_\ell$. Under some assumptions, it is expected that there is a morphism $\check{G} \times \mathrm{SL}_2 \to \check{H}$ such that if π is a smooth irreducible representation of $G(F)$ appearing in the Weil representation $\mathcal{S}$ and π' is the smooth irreducible representation of $H(F)$ the image of π under the Howe correspondence then the Arthur packet of π' is the image of the Arthur packet of π under the above map. For more details the reader can refer to [3], [22], [34], [35], [37]. It is well known that the Howe correspondence realizes the Langlands functoriality in some special cases. In the classical setting the reader may refer to [18], [22], [32], [37], and in the geometric setting one can refer to [23], [31].

It is also interesting to understand the geometry underlying the Howe correspondence and establish its analog in the framework of geometric Langlands program. This has been initiated by V.Lafforgue and Lysenko in [23]. The second author then studied the unramified case in [31] from global and local point of view for dual reductive pairs $(\mathrm{Sp}_{2n}, \mathrm{S}\mathbb{O}_{2m})$ and $(\mathbf{GL}_m, \mathbf{GL}_n)$. One of our motivations is to extend the results in [31] to the geometric setting of tamely ramified case (the Iwahori level).

Let B_G (resp. B_H) be a Borel subgroup of G (resp. of H) and let I_G (resp. I_H) be the corresponding Iwahori subgroup in $G(F)$ (resp. $H(F)$). At the Iwahori level we are interested in the class of tamely ramified representations. A irreducible smooth representation of (π, V) of $G(F)$ is called tamely ramified if the space of invariants under the Iwahori subgroup I_G is non-zero. The category of tamely ramified representations is the full subcategory of smooth representations of finite length consisting of those representations whose all irreducible subquotients are tamely ramified. Denote by $\mathcal{H}_{I_G}$ the Iwahori Hecke algebra of G. Consider the functor sending a tamely ramified representation (π, V) of $G(F)$ to the space of invariants V^{I_G}. According to [11, Theorem 4.10] this is an exact functor and an equivalence of categories between the category of tamely ramified smooth representations of $G(F)$ and the category of finite-dimensional $\mathcal{H}_{I_G}$-modules. In the classical setting, at the tamely ramified case, one strategy is to start by describing the space of invariants $\mathcal{S}^{I_G\times I_H}$ as a module over the tensor product $\mathcal{H}_{I_G}\otimes\mathcal{H}_{I_H}$ of Iwahori-Hecke algebras. So, our first step toward the geometrization is to define the geometric counterpoint of the invariants space $\mathcal{S}^{I_G\times I_H}$ and the Hecke actions of $\mathcal{H}_{I_G}$ and $\mathcal{H}_{I_H}$ on this space. This is done in §3, where we define the category of $I_G \times I_H$-equivariant perverse sheaves $P_{I_G\times I_H}(M(F))$ on some ind-scheme $M(F)$ as well as the derived category $D_{I_G\times I_H}(M(F))$. The construction of these categories uses some limit procedure. Moreover, we define two Hecke functors (4.1) geometrizing the bimodule structure of $\mathcal{S}^{I_G\times I_H}$ in §4, which is the action of the category $P_{I_G}(\mathcal{F}l_G)$ of I_G-equivariant perverse sheaves on the affine flag variety $\mathcal{F}l_G$ (the same for H) on $P_{I_G\times I_H}(M(F))$. These Hecke functors are actually defined at the level of the derived category $D_{I_G\times I_H}(M(F))$.

In the sequel (except in §8), we restrict ourselves to the case of dual reductive pairs of type II. More precisely, let $L_0 = \mathbf{k}^n$ and $U_0 = \mathbf{k}^m$ with $n \leq m$ and let $G = \mathbf{GL}(L_0)$ and $H = \mathbf{GL}(U_0)$. Denote by $\Pi(F)$ the space $U_0 \otimes L_0(F)$ and $\mathcal{S}(\Pi(F))$ the Schwartz space of locally constant functions with compact support on $\Pi(F)$. This Schwartz space realizes the restriction of the Weil representation to $G(F) \times H(F)$ [36]. According to Minguez [32] the Howe correspondence associate to any smooth irreducible representation π of $G(F)$ a unique irreducible smooth representation of $H(F)$ denoted by $\theta_{n,m}(\pi)$ such that $\pi \otimes \theta_{n,m}(\pi)$ is a quotient of the restriction of the Weil representation to $G(F) \times H(F)$. Moreover, he describes this correspondence explicitly in terms of Langlands parameters.

The next step toward the study of the Hecke functors on $P_{I_G\times I_H}(\Pi(F))$ is to consider the category $P_{H(O)\times I_G}(\Pi(F))$ of $H(O)\times I_G$-equivariant perverse sheaves on $\Pi(F)$. This category is acted on by the category $P_{H(O)}(Gr_H)$ of $H(O)$-equivariant perverse sheaves on the affine Grassmannian Gr_H and the category $P_{I_G}(\mathcal{F}l_G)$. In §5, we prove in Proposition 5.16 and Corollary 5.18 that at least at the level of Grothendieck groups, there is an isomorphism between $K(P_{H(O)\times I_G}(\Pi(F)))$ and $K(P_{I_G}(\mathcal{F}l_G))$ under the action of $K(P_{I_G}(\mathcal{F}l_G))$ and the action of $K(P_{H(O)}(Gr_H))$. At the unramified level, this isomorphism is actually verified at level of categories themselves [31] and one would hope the same would be true at the Iwahori level.

We continue our way toward understanding the geometric Weil representation $P_{I_G\times I_H}(\Pi(F))$ by describing Theorem 6.7 in §6 the simple objects of this category as the IC-sheaves of some $I_G\times I_H$-orbits on $\Pi(F)$. The precise formulation uses some combinatorics. Then in §7 we restrict ourselves to the case of the dual pairs $\mathbf{GL}_1$ and $\mathbf{GL}_m$ for all $m \geq 1$. In this setting, in a series of propositions we are able to give a complete geometric description in (7.10) of the module structure of $K(P_{I_G \times I_H}(\Pi(F)))$ under the action of the Hecke functors. More precisely we work with the category $DP_{I_G \times I_H}(\Pi(F))$ which takes in consideration the action of the multiplicative group $\mathbb{G}_m$ by cohomological shift -1. All our computations are at the level of perverse sheaves and the symmetry in this case comes from the action of the perverse sheaves in $P_{I_H}(\mathcal{F}l_H)$ associated with the elements of length zero in the affine extended Weyl group of H.

In §8 we start in a more general setting. Consider G and H two split reductive connected groups over $\mathbf{k}$ and a map $\check{G} \times \mathrm{SL}_2 \to \check{H}$ of dual Langlands groups over $\overline{\mathbb{Q}}_\ell$. To this data we attach a bimodule $K(\mathcal{X})$ over affine extended Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$. We propose Conjecture 8.7 stating that the bimodule $K(\mathcal{X})$ should realize the local geometric Arthur-Langlands functoriality at the Iwahori level for this map. This conjecture uses the essential Kazhdan-Lusztig-Ginzburg isomorphism between the $\check{G} \times \mathbb{G}_m$-equivariant Grothendieck group of the Steinberg variety of $\check{G}$ and the affine extended Hecke algebra $\mathbb{H}_G$. We describe some properties of this bimodule in this generality in §8.8.

In [15], the authors conjecture the existence of some category C_G over the stack of $\check{G}$-local systems over $D^* = \mathrm{Spec}(\mathbf{k}((t)))$ endowed with a "fiberwise" action of $G(F)$. Some conjectures about this category have been formulated in [15]. The construction of this category is more tractable at the Iwahori level, however a lot of improvements have been made in this area. Denote by $\mathcal{N}_{\check{G}}$ the nilpotent cone of $\check{G}$ and $\widetilde{\mathcal{N}}_{\check{G}}$ its Springer resolution. Denote by $C_{G,nilp}$ the category obtained from the base change $\mathcal{N}_{\check{G}}/\check{G} \to LS_{\check{G}}(D^*)$, where $LS_{\check{G}}(D^*)$ stands for the $\check{G}$-local systems on D^*. The authors conjecture [15, Formula 0.20] the following isomorphism

$$K(C_{G,nilp}^{I_G}) \widetilde{\longrightarrow} K(\widetilde{\mathcal{N}}_{\check{G}}/\check{G}), \tag{1.1}$$

where the left hand side is the Grothendieck group of the category of I_G-invariants in the category $C_{G,nilp}$ and the right hand side is the Grothendieck group of the category of coherent sheaves on the stack $\widetilde{\mathcal{N}}_{\check{G}}/\check{G}$. Moreover, this isomorphism should be compatible with the action of the affine extended Hecke algebra.

Consider a dual (split) reductive pair (G, H) over $\mathbf{k}$ with a map $\check{G} \times \mathrm{SL}_2 \to \check{H}$. Assume that the metaplectic cover splits over $G(F)$ and $H(F)$. In [23] the authors construct a category $\mathcal{W}$ called the Weil category equipped with an action of $(G \times H)(F)$. This is a geometrization of the Weil representation. Inspired from the series of conjectures presented in [15], Vincent Lafforgue has conjectured that there should exist the following equivalence of categories

$$\mathcal{W} \widetilde{\longrightarrow} C_G \times_{LS_{\check{H}}(D^*)} C_H \tag{1.2}$$

as categories equipped with an action of $(G \times H)(F)$.

By using Conjecture (1.1) and Conjecture (1.2), we propose the following conjecture at the level of Grothndieck groups linking the Howe correspondence and Arthur-Langlands functoriality at the Iwahori level. Consider the invariants of the category $D\mathcal{W}$ denoted by $(D\mathcal{W})^{I_G\times I_H}$. We conjecture in Conjecture 9.1 that the Grothendieck group of this category is isomorphic to $K(\mathcal{X})$ as a bimodule over the affine extended Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$. For this conjecture to make sense one should use the following facts. Denote by $DP_{I_G}(\mathcal{F}l_G)$ the category whose objects are direct sums of shifted I_G-equivariant perverse sheaves on $\mathcal{F}l_G$. This category is monoidal which takes in consideration the action of $\mathbb{G}_m$ by cohomological shift. The group $K(DP_{I_G}(\mathcal{F}l_G))\otimes\overline{\mathbb{Q}}_\ell$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$, and by Iwahori-Matsumoto [19] the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ is isomorphic to the affine extended Hecke algebra $\mathbb{H}_G$ after specialization and finally $\mathbb{H}_G$ is isomorphic to $\check{G}\times\mathbb{G}_m$-equivariant K-theory $K^{\check{G}\times\mathbb{G}_m}(Z_{\check{G}})$ of the Steinberg variety $Z_{\check{G}}$ of $\check{G}$. The usual Kazhdan-Lusztig-Ginzburg isomorphism is upgraded to the following isomorphism:

$$K(DP_{I_G}(\mathcal{F}l_G))\xrightarrow{\sim}K^{\check{G}\times\mathbb{G}_m}(Z_{\check{G}}).$$

Recently, Bezrukavnikov proved in [9] an equivalence of categories between $D_{I_G}(\mathcal{F}l_G)$ and the category of coherent sheaves on the Steinberg variety of $\check{G}$ linking two extreme parts of this chain of isomorphisms at the categorical level. By using these isomorphisms, both $K(\mathcal{X})$ and $K((D\mathcal{W})^{I_G\times I_H})$ are bimodules over the affine extended Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$.

We give a more precise definition and properties of the stack $\mathcal{X}$ and its K-theory while $G=\mathbf{GL}_n$ and $H=\mathbf{GL}_m$. The above conjecture in the case of linear groups states that there should exist an isomorphism of bimodules between $K(DP_{I_G\times I_H}(\Pi(F)))$ and $K(\mathcal{X})$ over the two affine extended Hecke algebras of G and H, see Conjecture 9.2. Finally in §9 we prove in Theorem 9.3 this conjecture for dual pairs $(\mathbf{GL}_1,\mathbf{GL}_m)$. This theorem expresses the Howe correspondence in terms of $K(\mathcal{X})$. The idea is that the explicit description of the Howe correspondence obtained by Minguez in [32] should be upgraded for a finer description of the bimodule in terms of the stack $\mathcal{X}$ attached to the map $\check{G}\times\mathrm{SL}_2\to\check{H}$. This opens an important perspective as the same description should also hold for other dual pairs. Especially it should be interesting to obtain a similar result for the dual pairs $(\mathrm{Sp}_{2\mathrm{n}},\mathrm{S}\mathbb{O}_{2m})$ and thus provide a conceptually new approach to the computation in [4]. Other important perspective is a hope that the whole derived category $D_{I_G\times I_H}(\Pi(F))$ could possibly be described in terms of the derived category of coherent sheaves over the stack $\mathcal{X}$ in the same esprite of the recent work in [9]. The main results of this paper have been announced in [14].

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2. Notation

Let $\mathbf{k}$ will be an algebraically closed field of characteristic $p > 2$ except in §8 and §9 where $\mathbf{k}$ is assumed to be finite. Let $F = \mathbf{k}((t))$ be the field of Laurent series with coefficients in $\mathbf{k}$ and $O = \mathbf{k}[[t]]$ its ring of integers. Let ℓ be a prime number different from p. We will denote by G a connected reductive group over $\mathbf{k}$ and by $G(F)$ the set of its F-points. Fix a maximal torus T and a Borel subgroup B of G containing T. Throughout the paper we denote by $\check{X}$ the lattice of characters of T and by X we denote the cocharacters lattice of T, see [12]. We denote by $\check{R}$ the set of roots and by R the set of coroots. Denote by $(\check{X}, \check{R}, X, R, \Delta)$ the root datum associated with (G, T, B), where Δ denotes the basis of simple roots. Denote by X^+ the set of dominant cocharacters of G. Denote by I_G the Iwahori subgroup of $G(F)$ associated with B. The reductive group $\check{G}$ associated with the dual root datum is called the Langlands dual group of G and will be considered over $\overline{\mathbb{Q}}_\ell$. All representations are assumed to be smooth and are considered over $\overline{\mathbb{Q}}_\ell$. Denote by $\mathrm{Rep}(\check{G})$ the category of smooth representations of $\check{G}$ over $\overline{\mathbb{Q}}_\ell$ and $\mathrm{R}(\check{G})$ the ring of smooth representations of $\check{G}$.

Denote by W_G the finite Weyl group of the root datum $(\check{X}, \check{R}, X, R, \Delta)$ and by $s_{\check{\alpha}}$ the simple reflection corresponding to the root $\check{\alpha}$. We denote by w_0 the longest element of the Weyl group W_G. In all our notation, if there is no ambiguity we will omit the subscript G. Denote by $\widetilde{W}_G$ the semi-direct product $W_G \ltimes X$, where W_G acts on X in a natural way. This is called the affine extended Weyl group. We will assume additionally that the root datum is irreducible and the unique highest root will be denoted by $\check{\alpha}_0$. Let $S_{aff} = \{s_\alpha | \alpha \in \Delta\} \cup \{s_0\}$, where $s_0 = t^{-\alpha_0} s_{\check{\alpha}_0}$. The subgroup W_{aff} of $\widetilde{W}_G$ generated by S_{aff} is the affine Weyl group associated with the root datum. Denote by ℓ the length function defined on the coxeter group W_{aff} which extends to a length function on $\widetilde{W}_G$. Let Q denote a subgroup of X generated by coroots. One has $W_{aff} \widetilde{\longrightarrow} W_G \ltimes Q$ and the subgroup W_{aff} is normal in $\widetilde{W}_G$ and admits a complementary subgroup $\Omega = \{w \in \widetilde{W}_G | \ell(w) = 0\}$, the elements of length zero. Moreover, we have $\widetilde{W}_G \widetilde{\longrightarrow} W_{aff} \rtimes \Omega$, which we will use as a description of $\widetilde{W}_G$.

For any scheme or stack S locally of finite type over $\mathbf{k}$, we denote by $D(S)$ the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves over S. Write $\mathbb{D} : D(S) \to D(S)$ for the Verdier duality functor. We denote by $P(S)$ the full subcategory of perverse sheaves in $D(S)$. We will also use a subcategory $DP(S)$ of $D(S)$ defined over any scheme or stack S. The objects of $DP(S)$ are the objects of $\bigoplus_{i\in\mathbb{Z}} P(S)[i]$, and for $K, K' \in P(S)$ and $i, j \in \mathbb{Z}$ the morphisms are:

$$Hom_{DP(S)}(K[i], K'[j]) = \begin{cases} Hom_{P(S)}(K, K') & \text{if } i = j; \\ 0 & \text{if } i \neq j. \end{cases}$$

Lt X be a scheme of finite type over $\mathbf{k}$ and let G be a connected algebraic group acting on X. We denote by $P_G(X)$ the full subcategory of $P(X)$ consisting of G-equivariant perverse sheaves. The derived category of G-equivariant $\overline{\mathbb{Q}}_\ell$-sheaves on X is denoted by $D_G(X)$. For Z a smooth d-dimensional irreducible locally closed subscheme of X

and $i : Z \to X$ the corresponding immersion, we define the intersection cohomology sheaf (IC-sheaf for short), IC(Z) as the perverse sheaf $i_{Z!*}(\overline{\mathbb{Q}}_\ell)[d]$.

We denote by Gr_G the affine Grassmannian defined as the $\mathbf{k}$-space quotient $G(F)/G(O)$. If G is the linear algebraic group $\mathbf{GL}_n$ over $\mathbf{k}$, the $\mathbf{k}$-points of Gr_G is naturally identified with the set of lattices in $\mathbf{k}((t))^n$, see [5]. The affine Grassmannian is an ind-scheme of ind-finite type. Given λ in X, the $G(O)$-orbit associated with $W_G.\lambda$ is $G(O).t^\lambda$ denoted by Gr_G^λ. Denote by $\overline{Gr_G^\lambda}$ the closure of the $G(O)$-orbit Gr_G^λ and the Cartan decomposition of $G(F)$

$$G(F) = \bigcup_{\lambda \in X^+} G(O)t^\lambda G(O).$$

For any λ and μ in X^+, $Gr_G^\mu \subset \overline{Gr_G^\lambda}$, if and only if $\lambda - \mu$ is a sum of positive coroots and

$$\overline{Gr^\lambda} = \bigsqcup_{\mu \leq \lambda} Gr^\mu.$$

For any λ in X^+, the dimension of Gr_G^λ is $\langle 2\check{\rho}, \lambda \rangle$, where $\check{\rho}$ is $\dfrac{1}{2} \sum_{\check{\alpha} \in \check{R}^+} \check{\alpha}$, the half sum of positive roots, see [7] and [33].

Denote by $\mathcal{F}l_G$ the affine flag variety for G defined as the quotient $\mathbf{k}$-space $G(F)/I_G$, which is an ind-scheme of ind-finite type as well, [17, Proposition 2.13]. The affine flag variety decomposes as a disjoint union

$$\mathcal{F}l_G = \bigcup_{w \in \widetilde{W}_G} I_G w I_G / I_G.$$

The closure of each Schubert cell $I_G w I_G / I_G$ is a union of Schubert cells and the closure relations are given by the Bruhat order:

$$\overline{I_G w I_G / I_G} = \bigcup_{w' \leq w} I_G w' I_G / I_G.$$

For any $w \in \widetilde{W}_G$ we will denote the Schubert cell $I_G w I_G / I_G$ by $\mathcal{F}l_G^w$. It is isomorphic to $\mathbb{A}^{\ell(w)}$.

Let R be a $\mathbf{k}$-algebra. A complete periodic flag of lattices inside $R((t))^n$ is a flag

$$L_{-1} \subset L_0 \subset L_1 \subset \dots$$

such that each L_i is a lattice in $R((t))^n$, each quotient L_{i+1}/L_i is a locally free R-module of rank one and $L_{n+k} = t^{-1}L_k$ for any k in $\mathbb{Z}$.

For $1 \leq i \leq n$, set

$$\Lambda_{i,R} = (\oplus_{j=1}^{i} t^{-1} R[[t]] e_j) \oplus (\oplus_{j=i+1}^{n} R[[t]] e_j).$$

For all i in $\mathbb{Z}$, we set $\Lambda_{i+n,R} = t^{-1}\Lambda_{i,R}$. This defines the standard complete lattice flag

$$\Lambda_{-1,R} \subset \Lambda_{0,R} \subset \Lambda_{1,R} \subset \dots$$

denoted by $\Lambda_{\bullet,R}$ in $R((t))^n$. Each point of $\mathbf{GL}_n(R((t)))$ gives rise to a flag of lattices inside $R((t))^n$ by applying it to the standard lattice flag. The Iwahori subgroup $I_G \subset$

$\mathbf{GL}_n(\mathbf{k}[[t]])$ is precisely the stabilizer of the standard lattice flag $\Lambda_{\bullet,\mathbf{k}}$. For any $\mathbf{k}$-algebra R, $\mathcal{F}l_{\mathbf{GL}_n}(R)$ is naturally in bijection with the set of complete periodic lattice flags in $R((t))^n$.

Let $Y = \varinjlim_i Y_i$ be an ind-scheme of ind-finite type, we define the category of perverse sheaves on it as $P(Y) = 2\varinjlim_i P(Y_i)$, where the functors $P(Y_i) \to P(Y_{i+1})$ are the direct image functors. This is again an abelian category since the functor of direct image under a closed embedding is exact. So, the notation of the category of perverse sheaves is well-defined for an ind-scheme of ind-finite type. Similarly one can define the derived category $D(Y) = 2\varinjlim D(Y_i)$ which is triangulated category due to the exactness property mentioned above. In the same way one may define the category of equivariant perverse sheaves on an ind-scheme for a nice action of an algebraic group on Y like the case of the affine Grassmannian and affine flag variety. We define the category of $G(O)$-equivariant (resp. I_G-equivariant) perverse sheaves on the affine Grassmannian Gr_G denoted by $P_{G(O)}(Gr_G)$ (resp. $P_{I_G}(Gr_G)$) and the category of I_G-equivariant perverse sheaves on the affine flag variety $\mathcal{F}l_G$ denoted by $P_{I_G}(\mathcal{F}l_G)$. The category $P_{G(O)}(Gr_G)$ is equipped with a geometric convolution product denoted by $\star$ which preserves perversity and makes $P_{G(O)}(Gr_G)$ into a symmetric monoidal category, see [33]. We define the extended geometric Satake equivalence in the following way:

$$DP_{G(O)}(Gr_G) \widetilde{\longrightarrow} \mathrm{Rep}(\check{G} \times \mathbb{G}_m),$$

for any perverse sheaf K in $P_{G(O)}(Gr_G)$ and any integer i, this functor sends $K[i]$ to $\mathrm{loc}(K) \otimes I^{\otimes -i}$, where I is the standard representation of $\mathbb{G}_m$ and $\mathrm{loc} : P_{G(O)}(Gr_G) \to \mathrm{Rep}(\check{G})$ is the satake equivalence.

One may define a geometric convolution product on $P_{I_G}(\mathcal{F}l_G)$ as well but this convolution product does not preserve perversity and the result is an element of the derived category $D_{I_G}(\mathcal{F}l_G)$, see [16] and [2].

Assume temporary that the ground field $\mathbf{k}$ is the finite field $\mathbb{F}_q$. We define the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ to be the space $C_c(I_G \backslash G(F)/I_G)$ of locally constant, I_G-bi-invariant compactly supported $\overline{\mathbb{Q}}_\ell$-valued functions on $G(F)$. We fix a Haar measure dx on $G(F)$ such that I_G be of measure 1 and endow $\mathcal{H}_{I_G}$ with the convolution product. There are two well-known presentations of this algebra by generators and relations. The first is due to Iwahori-Matsumoto [19] and the second is by Bernstein in [27] and [25]. We will use the second presentation. Moreover, we have the isomorphism $K(DP_{I_G}(\mathcal{F}l_G)) \otimes \overline{\mathbb{Q}}_\ell \widetilde{\longrightarrow} \mathcal{H}_{I_G}$.

3. Geometric Howe correspondence at the Iwahori level

Let M_0 be a finite-dimensional representation of G and let $M = M_0 \otimes_{\mathbf{k}} O$. The definitions of the derived category $D(M(F))$ of ℓ-adic sheaves on $M(F)$ and the category $P(M(F))$ of ℓ-adic perverse sheaves on $M(F)$ are given in [31]. We remind their definition briefly for the sake of the reader.

For any two integers $N, r \geq 0$ with $N+r > 0$, set $M_{N,r} = t^{-N}M/t^rM$. Given positive integers $N_1 \geq N_2$, $r_1 \geq r_2$, we have the following Cartesian diagram

$$\begin{tikzcd} M_{N_2,r_1} \arrow[r, hook, "i"] \arrow[d, "p"] & M_{N_1,r_1} \arrow[d, "p"] \\ M_{N_2,r_2} \arrow[r, hook, "i"] & M_{N_1,r_2} \end{tikzcd} \tag{3.1}$$

where i is the natural closed immersion and p is the projection.

Consider the following functor

$$\begin{aligned} D(M_{N,r_2}) &\longrightarrow D(M_{N,r_1}) \\ K &\longrightarrow p^*K[\dim rel(p)] \end{aligned} \tag{3.2}$$

According to [6, Proposition 4.2.5] the functor (3.2) is fully faithful and exact for the perverse t-structure. The functor i_* is fully faithful and exact for the perverse t-structure as well. This yields a commutative diagram of triangulated categories:

$$\begin{tikzcd} D(M_{N_2,r_1}) \arrow[r, hook, "i_*"] & D(M_{N_1,r_1}) \\ D(M_{N_2,r_2}) \arrow[r, hook, "i_*"] \arrow[u, "p^*[\dim]"] & D(M_{N_1,r_2}) \arrow[u, "p^*[\dim]"'] \end{tikzcd} \tag{3.3}$$

The derived category $D(M(F))$ is defined as the inductive 2-limit of derived categories $D(M_{N,r})$ as N, r go to infinity, see [20, Definition 4.2.1]. Similarly, $P(M(F))$ is defined as the inductive 2-limit of the categories $P(M_{N,r})$. The category $P(M(F))$ is a geometric analogue of the Schwartz space of locally constant functions with compact support on $M(F)$.

Assume $N + r > 0$. The subgroup $G(O)$ acts on $M_{N,r}$ via its finite dimensional quotient $G(O/t^{N+r}O)$. Denote by I_s the kernel of the map $G(O) \longrightarrow G(O/t^sO)$. The Iwahori subgroup I_G acts on $M_{N,r}$ via its finite-dimensional quotient I_G/I_{N+r}. For $s > 0$ denote by K_s the quotient I_G/I_s.

Let $r_1 \geq N+r > 0$, we have the projection $K_{r_1} \twoheadrightarrow K_{N+r}$. This leads to the following projection between stack quotients

$$q : K_{r_1}\backslash M_{N,r} \twoheadrightarrow K_{N+r}\backslash M_{N,r},$$

and gives rise to an equivalence of equivariant derived categories

$$D_{K_{N+r}}(M_{N,r}) \widetilde{\longrightarrow} D_{K_{r_1}}(M_{N,r}).$$

This equivalence is also exact for perverse t-structure. Denote by $D_{I_G}(M_{N,r})$ the derived category of K_{r_1}-equivariant ℓ-adic sheaves $D_{K_{r_1}}(M_{N,r})$ for any $r_1 \geq N + r$.

By taking the stack quotient of Diagram (3.1) by $K_{N_1+r_1}$, we obtain

$$\begin{tikzcd} D_{I_G}(M_{N_2,r_1}) \arrow[r, hook, "i_*"] & D_{I_G}(M_{N_1,r_1}) \\ D_{I_G}(M_{N_2,r_2}) \arrow[r, hook, "i_*"] \arrow[u, "p^*[dimrel]"] & D_{I_G}(M_{N_1,r_2}) \arrow[u, "p^*[dimrel]"'] \end{tikzcd} \tag{3.4}$$

where each arrow is fully faithful and exact for the perverse t-structure. Define $D_{I_G}(M(F))$ as the inductive 2-limit of $D_{I_G}(M_{N,r})$ as N, r go to infinity. Similarly we define the category $P_{I_G}(M(F))$. Since the Verdier duality $\mathbb{D}$ is compatible with the transition functors in both diagrams (3.3) and (3.4) we have the Verdier duality self-functors $\mathbb{D}$ on $D_{I_G}(M(F))$ and $D(M(F))$.

In order to define an action of the Hecke functors on $D_{I_G}(M(F))$, let us first define the equivariant derived category $D_{I_G}(M(F) \times \mathcal{F}l_G)$. Let $s_1, s_2 \geq 0$ and set

$$ {}_{s_1,s_2}G(F) = \{g \in G(F) | \ t^{s_1}M \subset gM \subset t^{-s_2}M\}. \tag{3.5}$$

Then ${}_{s_1,s_2}G(F) \subset G(F)$ is closed and stable under the left and right multiplication by $G(O)$. Further, ${}_{s_1,s_2}\mathcal{F}l_G = {}_{s_1,s_2}G(F)/I_G$ is closed in $\mathcal{F}l_G$. For $s_1' \geq s_1$ and $s_2' \geq s_2$, we have the closed embeddings ${}_{s_1,s_2}\mathcal{F}\ell_G \hookrightarrow {}_{s_1',s_2'}\mathcal{F}\ell_G$ and the union of ${}_{s_1,s_2}\mathcal{F}\ell_G$ is the affine flag variety $\mathcal{F}l_G$. The map sending g to g^{-1} yields an isomorphism between ${}_{s_1,s_2}G(F)$ and ${}_{s_2,s_1}G(F)$.

Assume M_0 is a faithful representation of G, then ${}_{s_1,s_2}\mathcal{F}\ell_G \subset \mathcal{F}\ell_G$ is a closed subscheme of finite type.

Lemma 3.6. *For any $s_1, s_2 \geq 0$, the action of $G(O)$ on ${}_{s_1,s_2}\mathcal{F}l_G$ factors through the quotient $G(O/t^{s_1+s_2+1}O)$.*

Proof. Choose a Borel B' in $GL(M_0)$ such that $B = G \cap B'$. Denote by

$$M \subset M_1 \subset \cdots \subset M_n = t^{-1}M$$

the full flag preserved by B'. The Iwahori subgroup I_G consists of the elements g of $G(F)$ preserving M together with the flag M_i above. Hence the map from $\mathcal{F}\ell_G$ to $\mathcal{F}\ell_{GL(M_0)}$ sending a point gI_G to the flag $(gM \subset gM_1 \subset \cdots \subset gM_n)$ is a closed immersion. Thus ${}_{s_1,s_2}\mathcal{F}l_G$ is realized as the closed subscheme in the scheme classifying a lattice M' such that $t^{s_1}M \subset M' \subset t^{-s_2}M$ together with the full flag

$$M' \subset M_1' \subset \cdots \subset M_n' = t^{-1}M'.$$

The action of $G(O)$ on the latter scheme factors through $G(O/t^{s_1+s_2+1}O)$. $\square$

The action of I_G on ${}_{s_1,s_2}\mathcal{F}l_G$ factors through $K_s = I_G/I_s$ for $s \geq s_1 + s_2 + 1$. Let $s \geq \max\{N + r, s_1 + s_2 + 1\}$, the group K_s acts on $M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G$ diagonally and the category $D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$ is well-defined. For $s' \geq s$ one has a canonical equivalence

$$D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) \widetilde{\longrightarrow} D_{K_{s'}}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G).$$

Define $D_{I_G}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$ as the category $D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$ for any $s \geq \max\{N + r, s_1 + s_2 + 1\}$.

Define the category $D_{I_G}(M(F) \times \mathcal{F}l_G)$ as the inductive 2-limit of $D_{I_G}(M_{N,r} \times_{s_1,s_2} \mathcal{F}l_G)$ as N, r, s_1, s_2 go to infinity. The subcategory $P_{I_G}(M(F) \times \mathcal{F}l_G) \subset D_{I_G}(M(F) \times \mathcal{F}l_G)$ of perverse sheaves is defined along the same lines.

4. Hecke functors at the Iwahori level

We use the same notation as in the previous section. Denote by $\check{\mu}$ in $\check{X}^+$ the character by which G acts on $\det(M_0)$. The connected components of the affine Grassmannian Gr_G are indexed by the algebraic fundamental group $\pi_1(G)$ of G, see [7]. For θ a cocharacter in $\pi_1(G)$, choose λ in X^+ whose image in $\pi_1(G)$ equals θ. Denote by Gr_G^θ the connected component of Gr_G containing Gr_G^λ. The affine flag manifold $\mathcal{F}l_G$ is a fibration over Gr_G with the typical fibre G/B. Hence the connected components of the affine flag variety $\mathcal{F}l_G$ are also indexed by $\pi_1(G)$. For θ in $\pi_1(G)$, denote by $\mathcal{F}l_G^\theta$ the preimage of Gr_G^θ in $\mathcal{F}l_G$. Set ${}_{s_1,s_2}\mathcal{F}l_G^\theta = \mathcal{F}l_G^\theta \cap {}_{s_1,s_2}\mathcal{F}l_G$.

Let us now define the Hecke functors (geometrization of the action of the Iwahori-Hecke algebra $\mathcal{H}_{I_G}$ on the invariants of the Weil representation $\mathcal{S}^{I_G}$) of $P_{I_G}(\mathcal{F}l_G)$ on $D_{I_G}(M(F))$, denoted by

$$\overleftarrow{H}_G : P_{I_G}(\mathcal{F}l_G) \times D_{I_G}(M(F)) \longrightarrow D_{I_G}(M(F)). \tag{4.1}$$

Consider the following isomorphism

$$\alpha : M(F) \times G(F) \longrightarrow M(F) \times G(F)$$
$$(v, g) \longrightarrow (g^{-1}v, g).$$

Let $(a, b) \in I_G \times I_G$ act on the source by $(a, b).(v, g) = (av, agb)$ and act on an element of the target (v', g') by $(a, b).(v', g') = (b^{-1}v', ag'b)$. The map α is $I_G \times I_G$-equivariant with respect to these two actions. Hence this yields a morphism of stacks

$$M(F) \times \mathcal{F}l_G \longrightarrow (M(F)/I_G) \times \mathcal{F}l_G,$$

and enables us to define the following morphism of stack quotients

$$act_q : I_G\backslash(M(F) \times \mathcal{F}l_G) \longrightarrow (M(F)/I_G) \times (I_G\backslash\mathcal{F}l_G),$$

where the action of I_G on $M(F) \times \mathcal{F}l_G$ is the diagonal one.

Lemma 4.2. *There exists an inverse image functor*

$$act_q^* : D_{I_G}(M(F)) \times D_{I_G}(\mathcal{F}l_G) \longrightarrow D_{I_G}(M(F) \times \mathcal{F}l_G)$$

which preserves perversity and is compatible with the Verdier duality in the following way: for any $\mathcal{K}$ in $D_{I_G}(M(F))$ and $\mathcal{F}$ in $D_{I_G}(\mathcal{F}l_G)$ we have

$$\mathbb{D}(act_q^*(\mathcal{K}, \mathcal{T})) \widetilde{\longrightarrow} act_q^*(\mathbb{D}(\mathcal{K}), \mathbb{D}(\mathcal{T})).$$

Proof. Given $N, r, s_1, s_2 \geq 0$ with $r \geq s_1$ and $s \geq \max\{N + r, s_1 + s_2 + 1\}$, one can define the following commutative diagram

$$
\begin{tikzcd}
 & M_{N,r} \times {}_{s_1,s_2}G(F) \arrow[r, "act"] \arrow[d, "q_G"'] & M_{N+s_1,r-s_1} \arrow[d, "q_M"] \\
M_{N,r} \arrow[d] & M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G \arrow[l, "pr_1"'] \arrow[r, "act_q"] \arrow[d] & K_s\backslash M_{N+s_1,r-s_1} \\
K_s\backslash M_{N,r} & K_s\backslash(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) \arrow[l, "pr"'] \arrow[r, "pr_2"] \arrow[ur, "act_{q,s}"] & K_s\backslash\,({}_{s_1,s_2}\mathcal{F}l_G)
\end{tikzcd}
$$

The action map *act* sends the couple (v, g) to $g^{-1}v$. The maps pr_1, pr_2 and pr are projections. The map q_G sends the couple (v, g) to (v, gI_G). All the vertical arrows are stack quotients for the action of the corresponding group. The group K_s acts diagonally on $M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G$ and the map act_q is equivariant with respect to this action. This enables us to define the following functor:

$$D_{I_G}(M_{N+s_1,r-s_1}) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \xrightarrow{temp} D_{I_G}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G)$$

sending $(\mathcal{K}, \mathcal{T})$ to

$$(act^*_{q,s}\mathcal{K}) \otimes pr_2^*\mathcal{T}[\dim(K_s) - c + s_1 \dim M_0]$$

where c equals $\langle\theta, \check{\mu}\rangle$ over ${}_{s_1,s_2}\mathcal{F}l^{\theta}_G$.

Consider $r_1 \geq r_2$ and $s \geq \max\{s_1 + s_2, N + r_1\}$. Then we have the diagram

$$
\begin{tikzcd}
K_s\backslash(M_{N,r_1} \times {}_{s_1,s_2}\mathcal{F}l_G) \arrow[r, "act_{q,s}"] \arrow[d] & K_s\backslash(M_{N+s_1,r_1-s_1}) \arrow[d] \\
K_s\backslash(M_{N,r_2} \times {}_{s_1,s_2}\mathcal{F}l_G) \arrow[r, "act_{q,s}"] & K_s\backslash(M_{N+s_1,r_2-s_1})
\end{tikzcd}
\tag{4.3}
$$

The functors *temp* and the transition functors in (4.3) are compatible. This gives rise to a functor

$$temp_{N,s_1,s_2} : D_{I_G}(M_{N+s_1}) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow D_{I_G}(M_N \times {}_{s_1,s_2}\mathcal{F}l_G),$$

where $M_N = t^{-N}M$.

Let $N_1 \geq N + s_2$ then $N \leq N_1 - s_2 \leq N_1 + s_1$ and one has the closed immersion $M_N \hookrightarrow M_{N_1+s_1}$. Thus we have

$$\begin{tikzcd} D_{I_G}(M_N) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \arrow[r, hook] & D_{I_G}(M_{N_1+s_1}) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \arrow[d, "temp_{N_1,s_1,s_2}"] \\ & D_{I_G}(M_{N_1} \times {}_{s_1,s_2}\mathcal{F}l_G) \arrow[d, hook] \\ & D_{I_G}(M(F) \times {}_{s_1,s_2}\mathcal{F}l_G) \end{tikzcd} \tag{4.4}$$

where the first inclusion is the extension by zero under the closed immersion $M_N \hookrightarrow M_{N_1+s_1}$. For any $\mathcal{K}$ in $D_{I_G}(M_N)$ and any $\mathcal{T}$ in $D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G)$, the image of $(\mathcal{K}, \mathcal{T})$ under the composition (4.4) does not depend on N_1. So we get a functor

$$temp_{s_1,s_2} : D_{I_G}(M_N) \times D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow D_{I_G}(M(F) \times {}_{s_1,s_2}\mathcal{F}l_G).$$

For any $s_1^{'} \geq s_1$, and $s_2^{'} \geq s_2$, we have the extension by zero functors

$$D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G) \hookrightarrow D_{I_G}({}_{s_1^{'},s_2^{'}}\mathcal{F}\ell_G),$$

which are compatible with our functor $temp_{s_1 s_2}$, so this yields the desired functor

$$act_q^* : D_{I_G}(M(F)) \times D_{I_G}(\mathcal{F}l_G) \overset{temp}{\longrightarrow} D_{I_G}(M(F) \times \mathcal{F}l_G).$$

One checks that $\mathbb{D}(act_q^*(\mathcal{K}, \mathcal{T})) \widetilde{\longrightarrow} act_q^*(\mathbb{D}(\mathcal{K}), \mathbb{D}(\mathcal{T}))$, and act_q^* preserves perversity. □

To define the convolution action, for any N, r, s_1, s_2 greater than zero satisfying the condition $s \geq \max\{N + r, s_1 + s_2 + 1\}$, consider the projection

$$pr : K_s \backslash (M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow K_s \backslash M_{N,r},$$

which gives us

$$pr_! : D_{K_s}(M_{N,r} \times {}_{s_1,s_2}\mathcal{F}l_G) \longrightarrow D_{K_s}(M_{N,r}).$$

These functors are compatible with the transition functors in (4.3) and yield a functor

$$pr_! : D_{I_G}(M(F) \times \mathcal{F}l_G) \longrightarrow D_{I_G}(M(F)).$$

For any $\mathcal{K}$ in $D_{I_G}(M(F))$ and $\mathcal{T}$ in $D_{I_G}(\mathcal{F}l_G)$, the Hecke operator $\overleftarrow{H}_G(\ ,\ )$ (4.1), is defined by

$$\overleftarrow{H}_G(\mathcal{T}, \mathcal{K}) = pr_!(act_q^*(\mathcal{K}, \mathcal{T})).$$

Moreover, this functor is compatible with the convolution product on $D_{I_G}(\mathcal{F}l_G)$. Namely, given $\mathcal{T}_1, \mathcal{T}_2$ in $D_{I_G}(\mathcal{F}l_G)$ and $\mathcal{K}$ in $D_{I_G}(M(F))$, one has naturally

$$\overleftarrow{H}_G(\mathcal{T}_1, \overleftarrow{H}_G(\mathcal{T}_2, \mathcal{K})) \widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{T}_1 \star \mathcal{T}_2, \mathcal{K}).$$

One may also consider the category $DP_{I_G}(\mathcal{F}l_G)$ and consider the Hecke functors in the form

$$\overleftarrow{H}_G : DP_{I_G}(\mathcal{F}l_G) \times D_{I_G}(M(F)) \longrightarrow D_{I_G}(M(F))$$

defined by $\overleftarrow{H}_G(\mathcal{T}[i], K) = \overleftarrow{H}_G(\mathcal{T}, K)[i]$ for $i \in \mathbb{Z}$ and $\mathcal{T} \in P_{I_G}(\mathcal{F}l_G)$.

Let $* : P_{I_G}(\mathcal{F}l_G) \tilde{\to} P_{I_G}(\mathcal{F}l_G)$ be the covariant equivalence of categories induced by the map $G(F) \to G(F)$, $g \mapsto g^{-1}$. We may define the right action $\overrightarrow{H}_G : D_{I_G}(\mathcal{F}l_G) \times D_{I_G}(M(F)) \to D_{I_G}(M(F))$ by $\overrightarrow{H}_G(\mathcal{T}, K) = \overleftarrow{H}_G(*\mathcal{T}, K)$.

Example. Let $R, r \geq 0$ and $t^r M \subset V \subset t^{-R}M$ be an intermediate lattice stable under I_G. Let $K \in P_{I_G}(M_{R,r})$ be a shifted local system on $V/t^r M \subset t^{-R}M/t^r M$. We are going to explain the above construction explicitly in this case. Let $\mathcal{T}$ be in $D_{I_G}({}_{s_1,s_2}\mathcal{F}l_G)$. Choose $r_1 \geq r + s_1$. If g is a point in ${}_{s_1,s_2}\mathcal{F}l_G$ then $t^{r_1}M \subset gV$. So we can define the scheme

$$(V/t^r M)\tilde{\times}_{s_1,s_2}\mathcal{F}l_G$$

as the scheme classifying pairs (gI_G, m) such that gI_G is an element of ${}_{s_1,s_2}\mathcal{F}l_G$ and m is in $(gV)/(t^{r_1}M)$. For a point (m, g) of this scheme we have $g^{-1}m$ in $V/t^r M$. Assuming $s \geq R + r$ we get the digram

$$M_{R+s_2,r_1} \xleftarrow{p} (V/t^r M)\tilde{\times}_{s_1,s_2}\mathcal{F}l_G \xrightarrow{act_{q,s}} K_s\backslash(V/t^r M),$$

where p is the map sending (gI_G, m) to m. For $gG(\mathcal{O}$ in Gr_G^θ, the virtual dimension of V/gV is $\langle\theta, \check{\mu}\rangle$. The space $(V/t^r M)\tilde{\times}_{s_1,s_2}\mathcal{F}l_G^\theta$ is locally trivial fibration over ${}_{s_1,s_2}\mathcal{F}l_G^\theta$ with fibre isomorphic to an affine space of dimension $\dim(V/t^{r_1}M) - \langle\theta, \check{\mu}\rangle$. Since K is a shifted local system, the tensor product $act_{q,s}^* K \otimes pr_2^*\mathcal{T}$ is a shifted perverse sheaf. Let $K\tilde{\boxtimes}\mathcal{T}$ be the perverse sheaf $act_{q,s}^* K \otimes pr_2^*\mathcal{T}[\dim]$. The shift $[\dim]$ in the definition depends on the dimension of the connected component and hence on $\check{\mu}$ as explained above and is such that the sheaf $act_{q,s}^* K \otimes pr_2^*\mathcal{T}[\dim]$ is perverse. Then

$$\overleftarrow{H}_G(\mathcal{T}, K) = p_!(K\tilde{\boxtimes}\mathcal{T}).$$

Compatibility. Assume temporary that $\mathbf{k} = \mathbb{F}_q$. Let us explain the relation between this geometrical convolution and classical convolution action in [32] and [36] at the level of functions. Given K in $D_{I_G}(M(F))$ we can associate with it the following function a_K in the Schwartz space $\mathcal{S}^{I_G}(M(F))$. If K is represented by the ind-pro-system $K_{N,r}$ in $D_{I_G}(M_{N,r})$ then for m in $t^{-N}M_0(\mathcal{O})$ one has

$$a_K(m) = Tr(Fr_{\overline{m}}, K_{N,r,\overline{m}})q^{\frac{rd}{2}},$$

where $d = \dim M_0$, the point $\overline{m}$ is the image of m in $M_{N,r}$, and $Fr_{\overline{m}}$ is the geometric Frobenius at $\overline{m}$. For large enough r, this is independent of r. The Hecke functors on $D_{I_G}(M(F))$ defined above geometrize the action of the Hecke algebras on $\mathcal{S}^{I_G}(M(F))$ corresponding to the following left action of $G(F)$ on $\mathcal{S}(M(F))$: for a point g in $G(F)$ and a function f in $\mathcal{S}(M(F))$ then

$$g.f(m) = |\det g|^{\frac{-1}{2}} f(g^{-1}m),$$

for any m in $M(F)$. To any $\mathcal{T}$ in $P_{I_G}(\mathcal{F}l_G)$ one can associate a function on $G(F)/I_G$ given by $a_{\mathcal{T}}(x) = Tr(Fr_x, \mathcal{T}_x)$ for x in $G(F)/I_G$. For $\mathcal{T}_i \in P_{I_G}(\mathcal{F}l_G)$ denote by f_i the corresponding function then we have

$$Tr(Fr_g, (\mathcal{T}_1 \star \mathcal{T}_2)_g) = \int_{x \in G(F)} f_1(x) f_2(x^{-1}g) dx,$$

where dx is the Haar measure on $G(F)$ such that I_G is of volume 1. Now if $\mathcal{F}$ is in $D_{I_G}(M(F))$, let $K = \overleftarrow{H}_G(\mathcal{T}, \mathcal{F})$ and denote by f the function associated to $\mathcal{F}$, then the function a_K associated to K is

$$a_K(m) = \int_{x \in G(F)} |\det x|^{-\frac{1}{2}} f(x^{-1}m) a_{\mathcal{T}}(x) dx,$$

for any m in $M(F)$.

In the following (except §**??**) we will restrict ourselves to the case of dual reductive pairs of type II. Let $L_0 = \mathbf{k}^n$ and $U_0 = \mathbf{k}^m$ with $n \leq m$ and let $G = \mathbf{GL}(L_0)$ and $H = \mathbf{GL}(U_0)$. We put $\Pi_0 = U_0 \otimes L_0$, $L = L_0(O)$, $U = U_0(O)$, and $\Pi = \Pi_0(O)$. For any O-module of finite rank M and any pair N, r of integers such that $N + r > 0$, we set $M_{N,r} = t^{-N}M/t^rM$. Let T_G (resp. T_H) be the maximal torus of diagonal matrices in G (resp. in H). Let B_G (resp. B_H) be the Borel subgroup of upper-triangular matrices in G (resp. H). Let I_G and I_H be the corresponding Iwahori subgroups. Denote by W_G the finite Weyl group of G, X_G (resp. X_H) the lattice of cocharacters of G (resp. H), and X_G^+ the lattice of dominant cocharacters of G. Let I_0 denote the constant perverse sheaf on Π. Using the previous construction we have the well-defined category of $I_G \times I_H$-equivariant perverse sheaves on $\Pi(F)$ inside the derived category $D_{I_G \times I_H}(\Pi(F))$ which is the geometrization of the invariants of the Schwartz space $\mathcal{S}(\Pi(F))^{I_G \times I_H}$, and two Hecke functors corresponding to the actions of $P_{I_G}(\mathcal{F}l_G)$ and $P_{I_H}(\mathcal{F}l_H)$ on $D_{I_G \times I_H}(\Pi(F))$:

$$\overleftarrow{H}_G : P_{I_G}(\mathcal{F}l_G) \times D_{I_G \times I_H}(\Pi(F)) \longrightarrow D_{I_G \times I_H}(\Pi(F))$$

and

$$\overleftarrow{H}_H : P_{I_H}(\mathcal{F}l_H) \times D_{I_G \times I_H}(\Pi(F)) \longrightarrow D_{I_G \times I_H}(\Pi(F)).$$

The ultimate goal is to understand these two functors $\overleftarrow{H}_G$ and $\overleftarrow{H}_H$. In this article we will first describe the simple objects of the category $P_{I_G \times I_H}(\Pi(F))$ and then we will restrict ourselves to the computation of $\overleftarrow{H}_G$ and $\overleftarrow{H}_H$ on the subcategory $P_{I_H \times I_G}(\Pi(F))$ of $D_{I_G \times I_H}(\Pi(F))$ in the case of $n = 1$ and $m \geq 1$.

5. Structure of the category $P_{H(O) \times I_G}(\Pi(F))$

The purpose here is to understand the module structure $P_{H(O) \times I_G}(\Pi(F))$ under the action of $P_{I_G}(\mathcal{F}l_G)$ and $P_{H(O)}(Gr_H)$. Let U^* denote the dual of U. A point v in $\Pi(F)$ may be seen as a O-linear map $v : U^* \to L(F)$. For v in $\Pi_{N,r}$, let $U_{v,r} = v(U^*) + t^rL$. Then $U_{v,r}$ is a O-module in $L(F)$. By identifying Gr_G with the ind-scheme of lattices in $L(F)$, we may view $U_{v,r}$ as a point of the affine Grassmannian Gr_G. The Iwahori subgroup I_G acts on the affine Grassmannian Gr_G as well. The

I_G-orbits are parametrized by cocharacters λ in X. Each orbit is an affine space. We have the decomposition

$$G(F) = \bigsqcup_{\lambda \in X} I_G t^\lambda G(O). \tag{5.1}$$

Let λ be in X^+, each $G(O)$-orbit Gr_G^λ decomposes into I_G-orbits which are parametrized by $W.\lambda$ and the orbit $I_G t^\lambda G(O)$ is open in Gr_G^λ. For any λ in X denote by O^λ for the I_G-orbit through $t^\lambda G(O)$ in Gr_G. Denote by $\overline{O^\lambda}$ its closure. The scheme $\overline{O^\lambda}$ is stratified by locally closed subschemes O^μ, where μ is in X. Remark that $O^\mu \subset \overline{O^\lambda}$ does not necessarily imply $\mu \leq \lambda$. Denote by $\mathcal{A}^\lambda$ the IC-sheaf of O^λ which is an object of $P_{I_G}(Gr_G)$.

Lemma 5.2. *The set of $H(O)$-orbits on $\Pi_{N,r}$ identifies with the scheme of lattices R such that $t^r L \subset R \subset t^{-N} L$ via the map sending v to $U_{v,r}$.*

Proof. Let M be a free O-module and M' be any O-module. If f_1 and f_2 are two surjections from M to M', then the kernel $\mathrm{Ker}(f_1)$ of the map f_1 is a free O-submodule of M and $\mathrm{Ext}^1(M, \mathrm{Ker}(f_1)) = 0$. Hence there is an O-linear endomorphism h of M such that $f_1 \circ h = f_2$. Let us now consider two elements v_1 and v_2 of $\Pi_{N,r}$ such that $U_{v_1,r} = U_{v_2,r}$. Adding to v_i a suitable element $t^r\Pi$, we may assume that both $v_i : U^* \to U_{v,r}$ are surjective for $i = 1, 2$. Then the previous argument implies that there exists h in $H(O)$ such that $v_1 \circ h = v_2$. Thus, for v_1 and v_2 in $\Pi_{N,r}$, the $H(O)$-orbits through v_1 and v_2 coincide if and only if $U_{v_1,r} = U_{v_2,r}$. It is straightforward to see that lattices R such that $t^r L \subset R \subset t^{-N} L$ are exactly of the form $U_{v,r}$ for some v in $\Pi_{N,r}$. □

Let $\check{\omega}_1 = (1, 0 \ldots, 0)$ be the highest weight of the standard representation of G and remind that w_0 is the longest element of the finite Weyl group W_G.

Lemma 5.3. *The $H(O) \times I_G$-orbits on $\Pi_{N,r}$ are parametrized by elements λ in X_G such that for any ν in $W_G.\lambda$*

$$\langle \nu, \check{\omega}_1 \rangle \leq r \qquad \text{and} \qquad \langle w_0(\nu), \check{\omega}_1 \rangle \leq N. \tag{5.4}$$

Denote each orbit indexed by λ as above by $\Pi_{\lambda,r}$. Each $\Pi_{\lambda,r}$ consists of points v such that $U_{v,r}$ lies in $I_G t^\lambda G(O)$.

Proof. Any lattice R satisfying $t^r L \subset R \subset t^{-N} L$ is of the form $U_{v,r}$ for some v in $\Pi_{N,r}$. Consider the lattice $U_{v,r}$ as a point in Gr_G. Then by Lemma 5.2 the $H(O) \times I_G$-orbits on $\Pi_{N,r}$ are exactly the locally closed subschemes $(\Pi_{\lambda,r})_{\lambda \in X_G}$ in $\Pi_{N,r}$ such that λ satisfies (5.4). □

The closure of the orbit $\Pi_{\lambda,r}$ will always be denoted in the sequel by $\overline{\Pi}_{\lambda,r}$. For a given cocharacter λ, the condition (5.4) is verified for large enough r. This defines a stratification of $\Pi_{N,r}$ by locally closed subschemes $\Pi_{\lambda,r}$.

For any λ in X_G, the perverse sheaves $\mathrm{IC}(\Pi_{\lambda,r})$ in $P_{H(O)\times I_G}(\Pi(F))$ are independent of the choice of r provided by the condition $\langle \nu, \check{\omega}_1 \rangle < r$ for any ν in $W_G\lambda$. Hence the resulting object of $P_{H(O)\times I_G}(\Pi(F))$ will be denoted by $\mathrm{IC}(\Pi_\lambda)$.

Proposition 5.5. *The irreducible objects of* $P_{H(O)\times I_G}(\Pi(F))$ *are in bijection with* X_G*. The irreducible object corresponding to a cocharacter* λ *in* X_G *is the intersection cohomology sheaf* $\mathrm{IC}(\Pi_\lambda)$.

Proposition 5.6. *For any* λ *in* X_G*, the complex* $\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0)$ *is canonically isomorphic to* $\mathrm{IC}(\Pi_\lambda)$.

Hence any irreducible object of the category $P_{H(O)\times I_G}(\Pi(F))$ is obtained by the action of $\mathcal{A}^\lambda$ on I_0 for some λ in X_G. The proof of this proposition will be presented in several steps. First remark that if λ is dominant then $\mathcal{A}^\lambda$ is $G(O)$-equivariant and in this case Proposition 5.6 results from [31, Proposition 5].

Let us give a description of the complex $\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0)$ by using Example. Choose two integers N, r satisfying $N + r > 0$ such that for any $\nu \in W_G.\lambda$, the condition (5.4) be satisfied. Let $\Pi_{0,r}\tilde{\times}\overline{O}^\lambda$ be the scheme classifying pairs $(v, gG(O))$, where $gG(O)$ belongs to $\overline{O}^\lambda$ and v is an O-linear map from U^* to gL/t^rL. Let

$$\pi : \Pi_{0,r}\tilde{\times}\overline{O}^\lambda \longrightarrow \Pi_{N,r} \tag{5.7}$$

be the map sending a pair $(v, gG(O))$ to the composition

$$U^* \xrightarrow{v} gL/t^rL \hookrightarrow t^{-N}L/t^rL.$$

This map is proper. The projection $p : \Pi_{0,r}\tilde{\times}\overline{O}^\lambda \to \overline{O}^\lambda$ is a vector bundle of rank $rnm - m\langle\lambda, \check{\omega}_n\rangle$, where $\check{\omega}_n = (1, \dots, 1)$. We obtain in this particular case an isomorphism

$$\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0) \widetilde{\longrightarrow} \pi_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda), \tag{5.8}$$

where the complex $\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda$ is normalized to be perverse, i.e.

$$\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda \widetilde{\longrightarrow} p^*\mathcal{A}^\lambda[\dim \mathrm{rel}(p)].$$

As mentioned before the category $P_{G(O)}(Gr_G)$ is equipped with a convolution product. Consider the following diagram

$$Gr_G \times Gr_G \xleftarrow{p} G(F) \times Gr_G \xrightarrow{q} G(F) \times_{G(O)} Gr_G \xrightarrow{m} Gr_G. \tag{5.9}$$

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two $G(O)$-equivariant perverse sheaves over Gr_G, the convolution product of these two perverse sheaves is by definition $\mathcal{F}_1 \star \mathcal{F}_2 = m_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2)$, where the sheaf $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2$ is perverse equipped with an isomorphism

$$p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \widetilde{\longrightarrow} q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2). \tag{5.10}$$

According to [16, Theorem 1] the category $P_{G(O)}(Gr_G)$ acts on $P_{I_G}(Gr_G)$ by convolution and this convolution product $\star$ preserves perversity. We want to use this result in order to give a dimension estimate for the objects of $P_{I_G}(Gr_G)$. Let $D = \mathrm{Spec}(\mathbf{k}[[t]])$ and $D^* = \mathrm{Spec}(\mathbf{k}((t)))$. Denote by E^0 the trivial G-torsor on D. The ind-scheme $G(F) \times_{G(O)} Gr_G$ classifies quadruplets $(E, E^1, \tilde{\beta}, \beta^1)$.

For μ in X_G^+, let $\mathcal{B}^\mu$ be the IC-sheaf associated with the $G(O)$-orbit $t^\mu G(O)$ in Gr_G. Then, for any cocharacter λ in X_G the convolution product $\mathcal{A}^\lambda \star \mathcal{B}^\mu$ is perverse. For any ν in X_G, and any point (E^1,β^1) in O^ν, let Y be the fibre of the map m over this point. The fibre Y identifies with the affine Grassmannian Gr_G. For η in X_G and δ in X_G^+, let $Y^{\eta,\delta}$ be the stratum of Y defined by the following two conditions:

(1) the G-torsor E is in I_G-position η with respect to the trivial G-torsor E^0.
(2) the G-torsor E^1 is in $G(O)$-position δ with respect to E.

Note that the restriction of $\mathcal{A}^\lambda \star \mathcal{B}^\mu$ to O^ν sites in usual degrees smaller than or equal to $-\dim O^\nu$, and the restriction of $\mathcal{A}^\lambda \tilde{\boxtimes} \mathcal{B}^\mu|_{Y^{\eta,\delta}}$ is the constant complex sitting in usual degrees smaller than or equal to $-\dim O^\eta - \dim Gr_G^\delta$.

Lemma 5.11. *For any η, ν in X_G and any δ in X_G^+ the following inequality holds:*

$$2\dim Y^{\eta,\delta} - \dim O^\eta - \dim Gr_G^\delta \leq -\dim O^\nu.$$

Proof. Let $\mathcal{B}^{\delta,!}$ (resp. $\mathcal{A}^{\eta,!}$) be the constant perverse sheaf on Gr_G^δ (resp.O^η) extended by zero (in the perverse sense) on Gr_G. The extension by zero functor is right exact for the perverse t-structure. Hence $\mathcal{B}^{\delta,!}$ (resp. $\mathcal{A}^{\eta,!}$) lies in non positive perverse degrees and so does the convolution product $\mathcal{A}^{\eta,!} \star \mathcal{B}^{\delta,!}$. The $*$-restriction of $\mathcal{A}^{\eta,!}\tilde{\boxtimes}\mathcal{B}^{\delta,!}$ to Y is the extension by zero from $Y^{\eta,\delta}$ to Y of the constant complex. Hence this complex lies in degrees $2\dim Y^{\eta,\delta} - \dim O^\eta - \dim Gr_G^\delta + \dim O^\nu$ and so we have the desired inequality. □

Proof of Proposition 5.6 : Let λ be in X_G and consider the complex $\pi_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda)$ appearing in (5.8). For ν in X_G, take a $H(O) \times I_G$-orbit $\Pi_{\nu,r}$ in $\Pi_{N,r}$. If v is in $\Pi_{\nu,r}$, let Y_v be the fibre of the map π over v defined in (5.7). The fibre Y_v is the scheme classifying $gG(O)$ in $\overline{O}^\lambda$ such that $U_{v,r}$ is a sublattice of gL. If v is in $\Pi_{\lambda,r}$ then Y_v is just a point and so the map π is an isomorphism over the open subscheme $\Pi_{\lambda,r}$. On one hand this implies directly that $\mathrm{IC}(\Pi_{\lambda,r})$ appears with multiplicity one in the complex of sheaves $\overleftarrow{H}_G(\mathcal{A}^\lambda, I_0)$. On the other hand this gives

$$\dim(\Pi_{\lambda,r}) = rnm - m\langle\lambda, \check{\omega}_n\rangle + \dim O^\lambda.$$

Let U be the open subscheme of $\Pi_{0,r}\tilde{\times}\overline{O}^\lambda$ consisting of pairs $(v, gG(O))$ such that $gG(O)$ lies in O^λ and $v : U^* \longrightarrow gL/t^rL$ is surjective. The image of U by π is contained in $\Pi_{\lambda,r}$. So, π induces a surjective proper map

$$\pi_\lambda : \Pi_{0,r}\tilde{\times}\overline{O}^\lambda \longrightarrow \overline{\Pi}_{\lambda,r}.$$

For v in $\Pi_{\nu,r}$, we stratify Y_v by locally closed subschemes Y_v^η indexed by cocharacters η in X_G. For any η, the stratum Y_v^η parametrizes elements $gG(O)$ in O^η. The $*$-restriction of $\overline{\mathbb{Q}}_\ell\tilde{\boxtimes}\mathcal{A}^\lambda$ to Y_v^η lives in usual degrees smaller than or equal to $-\dim O^\eta - rnm + m\langle\eta, \check{\omega}_n\rangle$ and the inequality is strict unless $\eta = \lambda$. We will show that

$$2\dim Y_v^\eta - \dim O^\eta - rnm + m\langle\eta, \check{\omega}_n\rangle \leq -\dim \Pi_{\nu,r} \tag{5.12}$$

and that the inequality is strict unless $\nu = \lambda$, this would imply our claim. Since we have $\dim(\Pi_{\nu,r}) = rnm - m\langle\nu, \check{\omega}_n\rangle + \dim O^\nu$, the inequality (5.12) becomes

$$2\dim Y_v^\eta \leq m\langle\nu - \eta, \check{\omega}_n\rangle + \dim O^\eta - \dim O^\nu. \tag{5.13}$$

Considering the map $\pi_\eta : \Pi_{0,r}\tilde{\times}\overline{O}^{\eta} \longrightarrow \overline{\Pi}_{\eta,r}$, we see that $\Pi_{\nu,r} \subset \overline{\Pi}_{\eta,r}$. A dominant cocharacter δ in X_G^+ is called *very positive* if

$$\delta = (b_1 \geq \cdots \geq b_n \geq 0).$$

It is natural to stratify Y_ν^η by locally closed subschemes $Y_\nu^{\eta,\delta}$, where δ runs through very positive cocharacters. For any such δ, the stratum $Y_\nu^{\eta,\delta}$ consists of elements $(v, gG(O))$ such that the lattice $U_{v,r}$ is in $G(O)$-position δ with respect to the lattice gL. For a point $(v, gG(O))$ of $Y_\nu^{\eta,\delta}$ the formula of virtual dimensions $\dim(L/gL) + \dim(gL/U_{v,r}) = \dim(L/U_{v,r})$ gives

$$\langle \delta + \eta - \nu, \check{\omega}_n \rangle = 0.$$

Finally the equation (5.13) is equivalent to

$$2 \dim Y_\nu^{\eta,\delta} \leq n\langle \delta, \check{\omega}_n \rangle + \dim O^\eta - \dim O^\nu.$$

By using Lemma 5.11 we are reduced to show that for any very positive δ, $\langle \delta, n\check{\omega}_n - 2\check{\rho}_G \rangle \geq 0$. To prove this inequality notice that

$$n\check{\omega}_n - 2\check{\rho}_G = (1, 3, 5, \ldots, 2n - 1).$$

Thus $n\check{\omega}_n - 2\check{\rho}_G$ is very positive and so for any very positive cocharacter δ we have $\langle \delta, n\check{\omega}_n - 2\check{\rho}_G \rangle \geq 0$. This proves the inequality (5.13). Moreover for any very positive δ this inequality is strict unless $\delta = 0$ which is the case if and only if $\nu = \eta$. This finishes the proof. □

Remind that according to the Satake isomorphism, $P_{H(O)}(Gr_H)$ is equivalent to the category $\mathrm{Rep}(\check{H})$ of representations of the Langlands dual group $\check{H}$ over $\overline{\mathbb{Q}}_\ell$. The module structure of $P_{H(O)\times G(O)}(\Pi(F))$ under the action of the category $P_{H(O)}(Gr_H)$ has been described in [31, § 5]. Let U_1 (resp. U_2) be the vector subspace of U_0, generated by the first n basis vectors (resp. by the last $m - n$ basis vectors) of U_0. Thus, $U_0 = U_1 \oplus U_2$. Let $P \subset H$ be the parabolic subgroup preserving U_1. Let $M \tilde{\to} \mathbf{GL}(U_1) \times \mathbf{GL}(U_2)$ the standard Levi factor in P and the map $\kappa : \check{G} \times \mathbb{G}_m \to \check{H}$ be the composition

$$\check{G} \times \mathbb{G}_m \xrightarrow{id \times 2\check{\rho}_{\mathbf{GL}(U_2)}} \check{G} \times \check{\mathbf{GL}}(U_2) = \check{M} \hookrightarrow \check{H}. \tag{5.14}$$

By using the extended Satake equivalence, write

$$\mathrm{gRes}^\kappa : P_{H(O)}(Gr_H) \longrightarrow DP_{G(O)}(Gr_G)$$

for the functor corresponding to the restriction $\mathrm{Rep}(\check{H}) \to \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$ with respect to κ.

Proposition 5.15. [31, Proposition 4] *The two functors*

$$P_{H(O)}(Gr_H) \to D_{H(O)\times G(O)}(\Pi(F))$$

given by

$$\mathcal{T} \to \overleftarrow{H}_H(\mathcal{T}, I_0) \qquad \textit{and} \qquad \mathcal{T} \to \overleftarrow{H}_G(\mathrm{gRes}^\kappa(\mathcal{T}), I_0)$$

are isomorphic.

Proposition 5.16. *For any λ in X_G and $\mathcal{T}$ in $P_{H(O)}(Gr_H)$ we have the following isomorphism*

$$\overleftarrow{H}_H(\mathcal{T}, \mathrm{IC}(\Pi_\lambda)) \widetilde{\rightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda \star \mathrm{gRes}^\kappa(\mathcal{T}), I_0).$$

Proof. Since $P_{I_G}(\mathcal{F}l_G)$ and $P_{H(O)}(Gr_H)$-actions on $P_{H(O)\times I_G}(\Pi(F))$ commute, we get from Proposition 5.6 and [31, Proposition 4]

$$\begin{aligned}
\overleftarrow{H}_H(\mathcal{T}, \mathrm{IC}(\Pi_\lambda)) &\widetilde{\longrightarrow} \overleftarrow{H}_H(\mathcal{T}, \overleftarrow{H}_G(\mathcal{A}^\lambda, I_0))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda, \overleftarrow{H}_H(\mathcal{T}, I_0))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda, \overleftarrow{H}_G(\mathrm{gRes}^\kappa(\mathcal{T}), I_0))\\
&\widetilde{\longrightarrow} \overleftarrow{H}_G(\mathcal{A}^\lambda \star \mathrm{gRes}^\kappa(\mathcal{T}), I_0).
\end{aligned}$$

□

From Proposition 5.6 it also follows that the functor

$$D_{I_G}(Gr_G) \to D_{H(O)\times I_G}(\Pi(F)) \tag{5.17}$$

given by $\mathcal{A} \mapsto \overleftarrow{H}_G(\mathcal{A}, I_0)$ is exact for the perverse t-structures. It is easy to see that neither of the categories $P_{I_G}(Gr_G)$ or $P_{H(O)\times I_G}(\Pi(F))$ is semi-simple. The functor (5.17) commutes with the actions of $P_{I_G}(\mathcal{F}l_G)$ by convolutions on the left. Let $P_{H(O)}(Gr_H)$ act on $D_{I_G}(Gr_G)$ via gRes^κ composed with the natural action of $D_{G(O)}(Gr_G)$ by convolutions on the right. According to Proposition 5.16, it is natural to expect that (5.17) commutes with the actions of $P_{H(O)}(Gr_H)$. From Proposition 5.6 and Proposition 5.15 one derives the following.

Corollary 5.18. *The functor* (5.17) *yields an isomorphism at the level of Grothendieck groups between $K(P_{I_G}(Gr_G))$ and $K(P_{H(O)\times I_G}(\Pi(F)))$ commuting with the above actions of $K(P_{H(O)}(Gr_H))$ and $K(P_{I_G}(\mathcal{F}l_G))$.*

6. Simple objects of $P_{I_G\times I_H}(\Pi(F))$

We use the same notation as in the previous section. Our goal is to describe the simple objects of $P_{I_H\times I_G}(\Pi(F))$. To do so we study the $I_H \times I_G$-orbits on $\Pi_{\lambda,r}$ defined in [§6, Lemma 5.3]. It turns out that it is not necessary to do the study for all cocharacters λ. Indeed if $\lambda = (a_1, \dots, a_n)$ we will restrict ourselves to the case where all a_i's are strictly smaller than r. This will be sufficient for our purpose.

Let

$$\mathrm{Stab}_\lambda = \{g \in I_G \mid g(t^\lambda L) = t^\lambda L\}$$

and

$$X^\lambda_{N,r} = \{v \in \Pi_{N,r} \mid U_{v,r} = t^\lambda L + t^r L\}.$$

Describing $I_H \times I_G$-orbits on $\Pi_{\lambda,r}$ is equivalent to describe $I_H \times \mathrm{Stab}_\lambda$-orbits on $X^\lambda_{N,r}$. Assume $n = m$.

Lemma 6.1. *The $I_H \times \mathrm{Stab}_\lambda$- orbits on $X^\lambda_{N,r}$ are in bijection with the finite Weyl group W_G.*

Proof. Let $\{e_1, e_2, \dots, e_n\}$ be the standard basis of the vector space L_0 such that the Borel subgroup B_G preserves the standard flag associated with the basis $(e_i)_{1\leq i\leq n}$. Let $(u_1^*, u_2^* \dots, u_m^*)$ be the standard basis of the dual space U_0^*. Let v be a point in $X_{N,r}^\lambda$ and consider the induced map

$$\bar{v} : U^*/tU^* \longrightarrow U_{v,r}/(tU_{v,r} + t^r L) = t^\lambda L/t^{\omega_n+\lambda}L, \tag{6.2}$$

where $\omega_n = (1, \dots, 1)$. The map $\bar{v}$ is an isomorphism and it may be considered as an element of $\mathrm{Aut}(t^\lambda L/t^{\omega_n+\lambda}L)$. Denote by

$$\dots \subset L_{-1} \subset L_0 \subset L_1 \subset \dots$$

the standard complete flag of lattices inside $L(F)$ preserved by the Iwahori group I_G. Then for any i in $\mathbb{Z}$ the images of $L_i \cap t^\lambda L$ in $t^\lambda L/t^{\omega_n+\lambda}L$ define a complete flag which is preserved by Stab_λ. Thus the image of Stab_λ in $\mathrm{Aut}(t^\lambda L/t^{\omega_n+\lambda}L)$ is a Borel subgroup of G but not necessary the standard one. Hence the $I_H \times \mathrm{Stab}_\lambda$-orbits on the set of isomorphisms (6.2) are parametrized by the finite Weyl group W_G. By Lemma 6.3 below each $I_H \times \mathrm{Stab}_\lambda$-orbit on $X_{N,r}^\lambda$ is the preimage of a $I_H \times \mathrm{Stab}_\lambda$-orbit on the scheme of isomorphisms (6.2). Finally one gets that $I_H \times \mathrm{Stab}_\lambda$-orbits on $X_{N,r}^\lambda$ are exactly indexed by W_G. □

Hence by this Lemma, any $I_H \times I_G$-orbit on $\Pi_{\lambda,r}$ is parametrized by W_G under the assumption that all a_i's are strictly smaller than r.

Lemma 6.3. *Let p, q be two integers such that $p \leq q$. Let B be a free $\mathcal{O}$-module of rank p and A be a free $\mathcal{O}$-module of rank q. Let $v_1, v_2 : A \to B$ be surjective $\mathcal{O}$-linear maps such that for $i = 1, 2$ the induced maps $\bar{v}_i : A/tA \to B/tB$ coincide. Then there is $h \in \mathbf{GL}(A)(\mathcal{O})$ with $h = 1 \bmod t$ such that $v_2 \circ h = v_1$.*

Proof. Let A_i be the kernel of v_i for $i = 1, 2$. These are free $\mathcal{O}$-modules of rank $q - p$. Choose a direct sum decomposition $A = A_i \oplus W_i$, where W_i is a free $\mathcal{O}$-module of rank p. Then there is a unique isomorphism $a : W_2 \longrightarrow W_1$ such that $W_2 \xrightarrow{a} W_1 \xrightarrow{v_1} A$ coincides with $W_2 \xrightarrow{v_2} A$. The images of $A_i \otimes_{\mathcal{O}} k$ in $A \otimes_{\mathcal{O}} k$ coincide, therefore there exists an isomorphism of $\mathcal{O}$-modules $b : A_2 \longrightarrow A_1$ such that $\bar{b} : A_2 \otimes_{\mathcal{O}} k \longrightarrow A_1 \otimes_{\mathcal{O}} k$ is identity. Then $a \oplus b$ is the desired map h. □

Lemma 6.4. *Any element w of $\widetilde{W}_G$ defines an $I_H \times I_G$-orbit on $\Pi_{N,r}$ for large enough r denoted by $\Pi_{N,r}^w$. More precisely if $w = t^\lambda \tau$ and $\lambda = (a_1, \dots, a_n)$, all values of r strictly bigger than a_i's are admissible.*

Proof. According to Lemma 6.1, for any cocharacter $\lambda = (a_1, \dots, a_n)$ in X_G such that for all i, a_i is strictly smaller than r the set of $I_H \times I_G$-orbits on $\Pi_{\lambda,r}$ is indexed by W_G. For any such λ and any τ in W_G, let $w = t^\lambda \tau$ be the associated element of $\widetilde{W}_G$. Then the $I_H \times I_G$-orbit passing through a point v of $\Pi_{N,r}$ is given by

$$v(u_i^*) = t^{a_{\tau_i}} e_{\tau(i)} \quad for \ \ i = 1, \dots, n.$$

For any $w = t^\lambda \tau$ in $\widetilde{W}_G$ we denote this orbit by $\Pi_{N,r}^w$. □

For any w in $\widetilde{W}_G$ denote by $\mathcal{I}^w$ the IC-sheaf of the $I_H \times I_G$-orbit $\Pi^w_{N,r}$ indexed by w, and by $\mathcal{I}^{w!}$ the constant perverse sheaf on $\Pi^w_{N,r}$ extended by zero to $\Pi_{N,r}$. As an object of $P_{I_H \times I_G}(\Pi(F))$ is independent of r, so that our notation is unambiguous. We underline that this notation is only introduced under the assumption $a_i < r$ for all i. Remind that $n = m$.

Proposition 6.5. *Any irreducible object of $P_{I_H \times I_G}(\Pi(F))$ is of the form $\mathcal{I}^w$ for some w in $\widetilde{W}_G$.*

Proof. An irreducible object of $P_{I_H \times I_G}(\Pi(F))$ is the IC-sheaf of an $I_H \times I_G$-orbit $\mathcal{Y}$ on $\Pi_{\lambda,r}$ for some integer r and for some cocharacter λ satisfying (5.4). In particular all a_i's are smaller than or equal to r. First we will show that we can restrict ourselves to the case where all a_i's are strictly less than r. Assume that $a_i = r$ for some i. For $s > r$ consider the projection $q : \Pi_{N,s} \to \Pi_{N,r}$. Then the $H(\mathcal{O}) \times I_G$-orbit $\Pi_{\lambda,s}$ is open in $q^{-1}(\Pi_{\lambda,r})$. The map $q : \Pi_{\lambda,s} \to \Pi_{\lambda,r}$ is not surjective but the sheaf $\mathrm{IC}(\mathcal{Y})$ is non-zero over the locus in $q^{-1}(\Pi_{\lambda,r})$ of maps $v : U^* \to t^{-N}L/t^rL$ whose geometric fibre of the image is of maximal dimension n. Hence the IC-sheaf of $\mathcal{Y}$ is a also an IC-sheaf of some $I_H \times I_G$-orbit on $\Pi_{\lambda,s}$. Thus we are reduced to the case where all a_i are strictly less than r. Remind that the geometric fibre of an $\mathcal{O}$-module $\mathcal{L}$ is $\mathcal{L} \otimes_{\mathbf{k}} \mathcal{O}$.

Next we are going to prove that each $I_H \times I_G$-equivariant local system on $\Pi^w_{N,r}$ is constant. The map $X^\lambda_{N,r} \to \mathrm{Isom}(U^*/tU^*, t^\lambda L/t^{\lambda+\omega_n}L)$ given by $v \to \bar{v}$ is an affine fibration. The group $\mathrm{Hom}(U^*, t^{\lambda+\omega_n}L/t^rL)$ acts freely and transitively on the fibres of this map. So we are reduced to show that any $B_G \times B_H$-equivariant local system on any $B_H \times B_G$-orbit on $U_0 \otimes L_0$ is constant. Indeed this is true because the stabilizer in B_G of a point in the double coset $B_G w B_G / B_G$ for any w in W_G is connected. $\square$

If λ is dominant then the image of Stab_λ in $\mathrm{Aut}(t^\lambda L/t^{\lambda+\omega_n}L)$ is the standard Borel subgroup of G. Thus when $w = t^\lambda$ with λ being dominant we have that $\Pi^w_{N,r}$ is an open subscheme of $\Pi_{\lambda,r}$ and $\mathcal{I}^w = \mathrm{IC}(\Pi_{\lambda,r})$.

Assume that $n \leq m$. In this case the map (6.2) is not an isomorphism but only a surjection. We may consider the $I_H \times \mathrm{Stab}_\lambda$-orbits on the set of surjections (6.2). Let $S_{n,m}$ be the set of pairs (s, I_s), where I_s is a subset of n elements of $\{1, \ldots, m\}$ and $s : I_s \longrightarrow \{1, \ldots, n\}$ is a bijection. Let $W_1 \subset W_2 \subset \cdots \subset W_m = U_0^*$ be a complete flag preserved by B_H. We denote by $\overline{W}_i$ the image of W_i under the map (6.2). Then $I_s = \{1 \leq i \leq m | \dim \overline{W}_i > \dim \overline{W}_{i-1}\}$.

Lemma 6.6. *Any element of $X_G \times S_{n,m}$ defines an $I_H \times I_G$-orbit on $\Pi_{N,r}$ for large enough r. More precisely if $w = (\lambda, s)$ and $\lambda = (a_1, \ldots, a_n)$, all values of r strictly bigger than a_i's are admissible.*

Proof. It is sufficient to describe the $I_H \times \mathrm{Stab}_\lambda$-orbits on the set $X^\lambda_{N,r}$. By Lemma 6.3 each $I_H \times \mathrm{Stab}_\lambda$-orbit on $X^\lambda_{N,r}$ is the preimage of a $I_H \times \mathrm{Stab}_\lambda$-orbit on the set of surjections (6.2). Let $\lambda = (a_1, \ldots, a_n)$. If all a_i's are strictly less than r, the $I_H \times \mathrm{Stab}_\lambda$-orbits on the set of surjections (6.2) are indexed by the set $S_{n,m}$. Let $w = (\lambda, s)$ be in

$X_G \times S_{n,m}$ then the $I_H \times I_G$-orbit passing through v a point of $\Pi_{N,r}$ is given by

$$\begin{cases} v(u_i^*) = t^{a_{si}} e_{si} & \text{for } i \in I_s; \\ v(u_i^*) = 0 & \text{for } i \notin I_s. \end{cases}$$

□

We denote this orbit by $\Pi^w_{N,r}$ and its closure by $\overline{\Pi}^w_{N,r}$. For any $w = (\lambda, s)$ in $X_G \times S_{n,m}$, where λ is in X_G and s is in $S_{n,m}$, denote by $\mathcal{I}^w$ the IC-sheaf of $\Pi^w_{N,r}$. The corresponding object of $D_{I_H \times I_G}(\Pi(F))$ is well-defined and independent of N, r.

Theorem 6.7. *Any irreducible object of $P_{I_H \times I_G}(\Pi(F))$ is of the form $\mathcal{I}^w$ for some w in $X_G \times S_{n,m}$.*

Proof. An irreducible object of $P_{I_H \times I_G}(\Pi(F))$ is the IC-sheaf of an $I_H \times I_G$-orbit $\mathcal{Y}$ on $\Pi_{\lambda,r}$ for some integer r and for some cocharacter $\lambda = (a_1, \dots, a_n)$ satisfying (5.4). As in the proof of Proposition 6.5 we may assume that all a_i's are strictly less than r. Consider a $I_H \times I_G$-orbit $\Pi^w_{N,r}$ on $\Pi_{N,r}$ passing through v as defined in Lemma 6.6. Let $St(v)$ be the stabilizer of v in $I_H \times I_G$. We are going to show that $St(v)$ is connected. This will imply that any $I_H \times I_G$-equivariant local system on $I_H \times I_G$-orbit $\Pi^w_{N,r}$ is constant.

The stabilizer $St(v)$ of v is a subgroup of $I_H \times \mathrm{Stab}_\lambda$. Let B_λ be the image of Stab_λ in $\mathrm{Aut}(t^\lambda L / t^{\lambda+\omega_n} L)$ then B_λ is a Borel subgroup of $\mathrm{Aut}(t^\lambda L / t^{\lambda+\omega_n} L)$. We define two groups $I_{0,\lambda}$ and $I_{0,H}$ by the exact sequences

$$1 \longrightarrow I_{0,\lambda} \longrightarrow \mathrm{Stab}_\lambda \longrightarrow B_\lambda \longrightarrow 1,$$

and

$$1 \longrightarrow I_{0,H} \longrightarrow I_H \longrightarrow B_H \longrightarrow 1.$$

Note that I_H is semi-direct product of $I_{0,H}$ and B_H. Let $St_0(v)$ be the stabilizer of v in $I_{0,H} \times I_{0,\lambda}$. By Lemma 6.3, the $I_{0,H} \times I_{0,\lambda}$-orbit through v on $X^\lambda_{N,r}$ is the affine space of surjections $f : U^* \longrightarrow t^\lambda L / t^r L$ such that $f = v \bmod t$. Thus $St_0(v)$ is connected. Let $\overline{v} : U^*/tU^* \longrightarrow t^\lambda L / t^{\lambda+\omega_n} L$ be the reduction of v mod t. The stabilizer $St(\overline{v})$ of $\overline{v}$ in $B_H \times B_\lambda$ is connected. By Lemma 6.3 the reduction map from $St(v)$ to $St(\overline{v})$ is surjective. Using the following exact sequence we obtain that $St(v)$ is connected.

$$1 \longrightarrow St_0(v) \longrightarrow St(v) \longrightarrow St(\overline{v}) \longrightarrow 1.$$

□

7. Study of Hecke functors, $n = 1$ and $m \geq 1$

We will assume that $n = 1$ and $m \geq 1$ in the entire section and we will give a complete description of $DP_{I_H \times I_G}(\Pi(F))$ under the actions of $P_{I_H}(\mathcal{F}l_H)$ et $P_{I_G}(\mathcal{F}l_G)$. We use the same notation as in previous section. Additionally for $1 \leq i \leq m$ we denote by ω_i the cocharacter of T_H equal to $(1, \dots, 1, 0, \dots, 0)$, where 1 appears i times. The Iwahori group I_H preserves $t^{-\omega_i} U$ and $t^{\omega_i} U^*$. Let Ω_H be the normal subgroup in the affine extended Weyl group $\widetilde{W}_H$ of elements of length zero. Note that $\omega_m = (1, \dots, 1)$ is in Ω_H.

For $1 \leq i \leq m$, let $U^i = t^{-\omega_i}U$. Define U^i for all $i \in \mathbb{Z}$ by the property that $U^{i+m} = t^{-\omega_m}U^i$ for all i. Thus,

$$\cdots \subset U^{-1} \subset U^0 \subset U^1 \subset \dots$$

is the standard flag preserved by I_H. For any integer k in $\mathbb{Z}$, we denote by IC^k the IC-sheaf of $U^k \otimes L$.

Proposition 7.1. *The irreducible objects of* $P_{I_H\times I_G}(\Pi(F))$ *are exactly the perverse sheaves* IC^k, $k \in \mathbb{Z}$.

Proof. The assertion follows from Theorem 6.7. □

We will denote by $\mathrm{IC}^{k,!}$ the constant perverse sheaf on $U^k \otimes L - U^{k-1} \otimes L$ extended by zero. This is a (non irreducible) perverse sheaf. In the Grothendieck group $K(DP_{I_H\times I_G}(\Pi(F)))$ of the category $P_{I_H\times I_G}(\Pi(F))$ we have

$$\mathrm{IC}^{k,!} = \mathrm{IC}^k + \mathrm{IC}^{k-1}(1/2).$$

At the level of functions, the objects $\mathrm{IC}^{k,!}$ for k in $\mathbb{Z}$ generate a subspace of codimension one in $\mathcal{S}(\Pi(F))^{I_G\times I_H}$. Denote by $I_0 = \mathrm{IC}^0$ the constant perverse sheaf on Π. Assume temporary that $\mathbf{k}$ is finite. For $w \in \widetilde{W}_G$, denote by j_w the inclusion of $\mathcal{F}l_G^w$ in $\mathcal{F}l_G$, and let $L_w = j_{w!*}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$, the IC-sheaf of $\mathcal{F}l_G^w$. We write $L_{w!} = j_{w!}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$ and $L_{w*} = j_{w*}\overline{\mathbb{Q}}_\ell[\ell(w)](\ell(w)/2)$ for the standard and costandard objects. As j_w is an affine map, both $L_{w!}$ and L_{w*} are perverse sheaves. They satisfy $\mathbb{D}(L_{w*}) = L_{w!}$, where $\mathbb{D}$ denotes the Verdier duality. Remark that in the notation of $L_{w!}$ and L_{w*} we wrote the Tate twists, when working over an algebraically closed field, we will forget the Tate twists. To $\mathcal{G}$ in $P_{I_G}(\mathcal{F}l_G)$ we attach a function $[\mathcal{G}] : G(F)/I_G \longrightarrow \overline{\mathbb{Q}}_\ell$ given by $[\mathcal{G}](x) = Tr(Fr_x, \mathcal{G}_x)$, for x a point in $G(F)/I_G$ and where Fr_x is the geometric Frobenius at x. The function $[\mathcal{G}]$ is an element of $\mathcal{H}_{I_G}$. In particular $[L_{w!}] = (-1)^{\ell(w)}q_w^{-1/2}T_w$ and $[L_{w*}] = (-1)^{\ell(w)}q_w^{1/2}T_{w^{-1}}^{-1}$, where $q_w = q^{\ell(w)}$. Here T_w denotes the characteristic function of the double coset I_GwI_G.

Let us describe $\overleftarrow{H}_H(\mathcal{A}^\lambda, I_0)$, for any cocharacter λ of H. Remind that $\mathcal{A}^\lambda$ is the IC-sheaf of the I_H-orbit O^λ through $t^\lambda H(O)$ in Gr_H. Let $\lambda = (a_1 \dots, a_m)$ and choose N, r such that $-N \leq a_i < r$ for all i. Let $\Pi_{0,r}\tilde{\times}\overline{O}^\lambda$ be the scheme classifying pairs $(v, hH(O))$, where $hH(O)$ is a point in $\overline{O}^\lambda$ and v is a O-linear map $L^* \to hU/t^rU$. Let

$$\pi : \Pi_{0,r}\tilde{\times}\overline{O}^\lambda \longrightarrow \Pi_{N,r}$$

be the map sending $(v, hH(O))$ to the composition $L^* \xrightarrow{v} hU/t^rU \longrightarrow t^{-N}U/t^rU$. By definition we have

$$\overleftarrow{H}_H(\mathcal{A}^\lambda, I_0) \widetilde{\longrightarrow} \pi_!(\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda),$$

where $\overline{\mathbb{Q}}_\ell \tilde{\boxtimes} \mathcal{A}^\lambda$ is normalized to be perverse. Denote by p_H the projection of $\mathcal{F}l_H \to Gr_H$. Note that for any $\mathcal{T}$ in $P_{I_H}(\mathcal{F}l_H)$ we have

$$\overleftarrow{H}_H(\mathcal{T}, I_0) \widetilde{\longrightarrow} \overleftarrow{H}_H(p_{H!}(\mathcal{T}), I_0).$$

For $1 \leq i < m$, let s_i be the simple reflection (permutation) $(i, i+1)$ in W_H.

Proposition 7.2. *For* $1 \le i < m$ *we have*

$$\overleftarrow{H}_H(L_{s_i}, I_0) \widetilde{\longrightarrow} I_0 \otimes \mathrm{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell) \widetilde{\longrightarrow} I_0 \otimes (\overline{\mathbb{Q}}_\ell[1] \oplus \overline{\mathbb{Q}}_\ell[-1]).$$

Similarly,

$$\overleftarrow{H}_H(L_{s_i!}, I_0) \widetilde{\longrightarrow} I_0[-1].$$

Proof. One has $p_{H!}(L_{s_i}) \widetilde{\longrightarrow} \mathrm{R}\Gamma(\mathbb{P}^1, \overline{\mathbb{Q}}_\ell)[1]$ and the assertion follows. □

Assume that $m > 1$ and let $s_m = t^\lambda \tau$, where $\lambda = (-1, 0, \dots, 0, 1)$ and $\tau = (1, m)$ is the reflection corresponding to the highest root. This is the unique affine simple reflection in $\widetilde{W}_H$.

Proposition 7.3. *If* $m > 1$, *we have the following canonical isomorphisms*

$$\overleftarrow{H}_H(L_{s_m}, I_0) \widetilde{\longrightarrow} \mathrm{IC}^1 \oplus \mathrm{IC}^{-1} \text{ and } \overleftarrow{H}_H(L_{s_m!}, I_0) \widetilde{\longrightarrow} \mathrm{IC}^{1,!} \oplus \mathrm{IC}^{-1}$$

Proof. The composition

$$\overline{\mathcal{F}l_H}^{s_m} \hookrightarrow \mathcal{F}l_H \xrightarrow{p_H} Gr_H$$

is a closed immersion and so $p_{H!}(L_{s_m}) \widetilde{\longrightarrow} \mathcal{A}^\lambda$. Thus we have

$$\overleftarrow{H}_H(L_{s_m}, I_0) \widetilde{\longrightarrow} \overleftarrow{H}_H(\mathcal{A}^\lambda, I_0).$$

In this case the scheme $\overline{O}^\lambda$ classifies lattices U' such that

$$\dots \subset U^{-1} \subset U' \subset U^1 \subset \dots$$

and $\dim(U'/U^{-1}) = 1$. Let $N = r = 1$, then the image of the projection

$$\pi : \Pi_{0,1} \tilde{\times} \overline{O}^\lambda \longrightarrow \Pi_{1,1}$$

is contained in $L \otimes (U^1/tU)$. Let v be a map from L^* to U^1/tU in the image of π. If v factors through U^{-1}/tU then the fibre of π over the point v is $\mathbb{P}^1$, otherwise it is a point. The first claim follows, the second is analogous. □

In a similar way one gets the following.

Proposition 7.4. *For for* $1 \le i \le m$, *we have*

$$\overleftarrow{H}_H(L_{s_i}, \mathrm{IC}^i) \widetilde{\longrightarrow} \mathrm{IC}^{i+1} \oplus \mathrm{IC}^{i-1} \text{ and } \overleftarrow{H}_H(L_{s_i!}, \mathrm{IC}^i) \widetilde{\longrightarrow} \mathrm{IC}^{i+1,!} \oplus \mathrm{IC}^{i-1}.$$

Proof. The proof follows from Lemma 7.2 and 7.3. □

The symmetry in our situation is due to the fact that Ω_H acts freely and transitively on the set of irreducible objects of $P_{I_G \times I_H}(\Pi(F))$.

For $1 \le i \le m$ there is a unique permutation σ_i in W_H such that $t^{-\omega_i}\sigma_i$ is of length zero. Indeed, σ_i is the permutation sending

$$(1, 2, \dots, m-i, m-i+1, \dots, m) \longrightarrow (i+1, i+2, \dots, m, 1, \dots, i).$$

For $1 \le i \le m$ we put $w_i = t^{-\omega_i}\sigma_i$. Extend this definition as follows, for any $i \in \mathbb{Z}$ let w_i in Ω_H be the unique element such that $w_i U^r = U^{r+i}$ for any r. For $1 \le i \le m-1$ we have $w_1 s_i w_1^{-1} = s_{i+1}$ and $w_1 s_m w_1^{-1} = s_1$. Thus, the affine Weyl group of H acts on the set $\{s_1, \dots, s_m\}$ by conjugation.

Proposition 7.5. *1) For any i and k in $\mathbb{Z}$ one has canonically*

$$\overleftarrow{H}_H(L_{w_i}, \mathrm{IC}^k) \xrightarrow{\sim} \mathrm{IC}^{k+i}.$$

2) For $1 \le i \le m$, $j \in \mathbb{Z}$ with $j \neq i \mod m$ one has

$$\overleftarrow{H}_H(L_{s_i}, \mathrm{IC}^j) \xrightarrow{\sim} \mathrm{IC}^j \otimes (\bar{\mathbb{Q}}_\ell[1] \oplus \bar{\mathbb{Q}}_\ell[-1]).$$

Propositions 7.4 and 7.5 describe completely the action of $P_{I_H}(\mathcal{F}l_H)$ on the simple objects IC^k, $k \in \mathbb{Z}$ and thus the module structure of $K(DP_{I_H \times I_G}(\Pi(F)))$ under the action of $K(P_{I_H}(\mathcal{F}l_H))$. Now we are going to define the action of the center of $P_{I_H}(\mathcal{F}l_H)$ on IC^k. Let us recall that there is a central functor constructed by Gaitsgory in [16] :

Theorem 7.6. [16, Theorem 1] *Let H be a connected reductive group. There exists a functor $\mathcal{Z} : P_{H(O)}(Gr_H) \longrightarrow P_{I_H}(\mathcal{F}l_H)$ verifying the following properties:*

1. *For $\mathcal{S} \in P_{H(O)}(Gr_H)$ and an arbitrary perverse sheaf $\mathcal{T}$ on $\mathcal{F}l_H$, the convolution product $\mathcal{T} \star \mathcal{Z}(\mathcal{S})$ is a perverse sheaf.*
2. *For $\mathcal{S} \in P_{H(O)}(Gr_H)$ and $\mathcal{T} \in P_{I_H}(\mathcal{F}l_H)$ there is a canonical isomorphism between $\mathcal{Z}(\mathcal{S}) \star \mathcal{T}$ and $\mathcal{T} \star \mathcal{Z}(\mathcal{S})$.*
3. *We have $\mathcal{Z}(\delta_{1_{Gr_H}}) = \delta_{\mathcal{F}l_H}$ and for any $\mathcal{S}^1$, $\mathcal{S}^2$ in $P_{H(O)}(Gr_H)$ there is a canonical isomorphism $\mathcal{Z}(\mathcal{S}^1) \star \mathcal{Z}(\mathcal{S}^2) \simeq \mathcal{Z}(\mathcal{S}^1 \star \mathcal{S}^2)$.*

Let $\sigma : \check{G} \times \mathbb{G}_m \longrightarrow \check{H}$ be given by (5.14). Denote by $\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$, the corresponding geometric restriction functor. For any $G(O)$-equivariant perverse sheaf $\mathcal{T}$ on Gr_H, $\mathcal{T}$ is naturally isomorphic to $p_{H!}(\mathcal{Z}(\mathcal{T}))$.

Denote by s the standard representation of $\mathbb{G}_m$ and by g the standard representation of $\check{G}$. The category $\mathrm{Rep}(\check{G} \times \mathbb{G}_m)$ acts on $DP_{I_G \times I_H}(\Pi(F))$ as follows:

$$\begin{cases} \overleftarrow{H}_G(s^j, \mathrm{IC}^k) \xrightarrow{\sim} \mathrm{IC}^k[j]. \\ \overleftarrow{H}_G(g^j, \mathrm{IC}^k) \xrightarrow{\sim} \mathrm{IC}^{k-mj}. \end{cases} \tag{7.7}$$

It follows that the representation ring $\mathrm{R}(\check{G} \times \mathbb{G}_m)$ acts on $K(DP_{I_G \times I_H}(\Pi(F)))$, which becomes in this way a free $\mathrm{R}(\check{G} \times \mathbb{G}_m)$-module of rank m with basis $\{\mathrm{IC}^0, \dots, \mathrm{IC}^{m-1}\}$.

Theorem 7.8. *The respective actions of the centre of $P_{I_H}(\mathcal{F}l_H)$ and the center of $P_{I_G}(\mathcal{F}l_G)$ on the category $DP_{I_G \times I_H}(\Pi(F))$ are compatible. More precisely, the center of $P_{I_H}(\mathcal{F}l_H)$ acts via the geometric restriction functor $\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_m)$ on the irreducible objects IC^k for any integer k.*

Proof. For any $\mathcal{S}$ in $P_{H(O)}(Gr_H)$, we have

$$\overleftarrow{H}_H(\mathcal{Z}(\mathcal{S}), \mathrm{IC}^0) \xrightarrow{\sim} \overleftarrow{H}_H(p_{H!}(\mathcal{Z}(\mathcal{S})), \mathrm{IC}^0) \xrightarrow{\sim} \overleftarrow{H}_H(\mathcal{S}, \mathrm{IC}^0) \xrightarrow{\sim} \overleftarrow{H}_G(\mathrm{Res}^\sigma(\mathcal{S}), \mathrm{IC}^0), \tag{7.9}$$

where the last isomorphism is [31, Proposition 5]. Remind that for any $k \in \mathbb{Z}$ we have $\overleftarrow{H}_H(L_{w_k}, \mathrm{IC}^0) \xrightarrow{\sim} \mathrm{IC}^k$ For $\mathcal{S}$ in $P_{H(O)}(Gr_H)$ we get by definition that $\mathcal{Z}(\mathcal{S})$ is central so

$$\overleftarrow{H}_H(\mathcal{Z}(\mathcal{S}), \mathrm{IC}^k) \xrightarrow{\sim} \overleftarrow{H}_H(L_{w_k}, \overleftarrow{H}_H(\mathcal{Z}(\mathcal{S}), \mathrm{IC}^0)) \xrightarrow{\sim} \overleftarrow{H}_G(\mathrm{Res}^\sigma(\mathcal{S}), \mathrm{IC}^k),$$

where the last isomorphism is from (7.9). The assertion follows. □

Assume that $\mathbf{k}$ is a finite field $\mathbb{F}_q$. Let us rewrite all useful formulas obtained above with taking in consideration the Tate twists which will be compared to the computation obtained in §9.

According to Propositions 7.2 and 7.3 and Proposition 7.5,

$$
\begin{cases}
\text{For } 1 \le i \le m : \overleftarrow{H}_H(L_{s_i}, \mathrm{IC}^i) \widetilde{\longrightarrow} \mathrm{IC}^{i+1} \oplus \mathrm{IC}^{i-1}. \\
\text{For } 1 \le i \le m : \overleftarrow{H}_H(L_{s_{i!}}, \mathrm{IC}^i) \widetilde{\longrightarrow} \mathrm{IC}^{i+1,!} \oplus \mathrm{IC}^{i-1}. \\
\text{If } j \neq i \bmod m : \overleftarrow{H}_H(L_{s_i}, \mathrm{IC}^j) \widetilde{\longrightarrow} \mathrm{IC}^j(\overline{\mathbb{Q}}_\ell[1](1/2) + \overline{\mathbb{Q}}_\ell[-1](-1/2)). \\
\text{If } j \neq i \bmod m : \overleftarrow{H}_H(L_{s_{i!}}, \mathrm{IC}^j) \widetilde{\longrightarrow} \mathrm{IC}^j[-1](-1/2). \\
\text{For any } i \text{ and } k \textit{ in } \mathbb{Z} : \overleftarrow{H}_H(L_{w_i}, \mathrm{IC}^k) \widetilde{\longrightarrow} \mathrm{IC}^{k+i}
\end{cases}
\tag{7.10}
$$

More generally, for $a < b$, denote by $\mathrm{IC}^{a,b,!}$ the sheaf $\overline{\mathbb{Q}}_\ell[b-a]$ defined on $(U^b/U^a) - \{0\}$ extended by zero to U^b/U^a. This is not perverse in general. In Grothendieck group $K(DP_{I_H \times I_G}(\Pi(F)))$ we have

$$
\mathrm{IC}^{a,b,!} = \mathrm{IC}^b - \mathrm{IC}^a[b-a].
$$

Let $\omega_i = (1, \dots, 1, 0, \dots, 0)$ where 1 appears i times and 0 appears $m-i$ times.

Proposition 7.11. *We have a canonical isomorphism in $K(DP_{I_H \times I_G}(\Pi(F)))$:*

$$
\overleftarrow{H}_H(L_{t^{\omega_i}!}, \mathrm{IC}^0) \widetilde{\longrightarrow} \mathrm{IC}^{i-m,0,!}[i - \langle \omega_i, 2\check{\rho}_H \rangle] + \mathrm{IC}^{-m}[m - i - \langle \omega_i, 2\check{\rho}_H \rangle].
$$

Proof. First remark that

$$
\overleftarrow{H}_H(L_{t^{\omega_i}}, \mathrm{IC}^0) \widetilde{\longrightarrow} \overleftarrow{H}_H(\mathcal{A}^{\omega_i !}, \mathrm{IC}^0).
$$

Let $N = r = 1$, the scheme O^{ω_i} classifies lattices $tU^0 \subset U' \subset U^0$ such that $\dim(U'/tU^0) = m-i$ and $(U'/tU^0) \cap (U^{m-i}/tU^0) = 0$. Therefore the orbit O^{ω_i} is an affine space of dimension $\ell(t^{\omega_i}) = \langle \omega_i, 2\check{\rho}_H \rangle = (m-i)i$. Let $\Pi_{0,1} \tilde{\times} O^{\omega_i}$ be the scheme classifying pairs (v, U'), where U' is in O^{ω_i} and v is a map from L^* to U'/tU^0. Consider the map

$$
\pi : \Pi_{0,1} \tilde{\times} O^{\omega_i} \longrightarrow \Pi_{0,1}
$$

sending (v, U') to v. Then we have

$$
\overleftarrow{H}_H(\mathcal{A}^{\omega_i !}, \mathrm{IC}^0) \widetilde{\longrightarrow} \pi_! \, \mathrm{IC}(\Pi_{0,1} \tilde{\times} O^{\omega_i})
$$

and the assertion follows from the remark above on the elements $\mathrm{IC}^{a,b,!}$. □

8. On the geometric local Langlands functoriality at the Iwahori level

For basic notions in equivariant K-theory, one can refer to [13, Chapter 5]. Some of the constructions we will use are recaled in the Appendix 10. Let us just recall the Kazhdan-Lusztig-Ginzburg isomorphism and fix some additional notation.

Let $\mathbf{k}$ be the finite filed $\mathbb{F}_q$. Let G be a connected reductive group over $\mathbf{k}$ and denote by $\check{G}$ its Langlands dual group over $\overline{\mathbb{Q}}_\ell$. Assume additionally that $[\check{G}, \check{G}]$ is simply connected. Let v be an indeterminate. Let (W, S) be the Coxeter group associated with the root datum defined on G, where W is the finite Weyl group and S the set of simple reflections. The finite Hecke algebra $\mathbb{H}_W$ is free $\mathbb{Z}[v^{-1}, v]$-algebra with basis $\{T_w, w \in W\}$ such that the following rules hold:

(1) $(T_s + 1)(T_s - v) = 0$ if $s \in S$ is a simple reflection.
(2) $T_y.T_w = T_{yw}$ if $\ell(yw) = \ell(y) + \ell(w)$.

These rules define completely the ring structure of $\mathbb{H}_W$ thus any algebra satisfying the properties of the proposition is isomorphic to the Hecke algebra $\mathbb{H}_W$. The group algebra $\mathbb{Z}[X]$ is isomorphic to $\mathrm{R}(\check{T})$, the representation ring of the dual torus to T. We will write e^λ for the element of $\mathrm{R}(\check{T})$ corresponding to the coweight λ in X. The affine extended Hecke algebra associated with G was introduced by Bernstein [8] (it first appeared in [26]) and is isomorphic to the so-called Iwahori-Hecke algebra of a split p-adic group with connected center. The latter was introduced in [19] and reflects the structure of $C_c(I_G\backslash G(F)/I_G)$ the space of locally constant compactly supported $\overline{\mathbb{Q}}_\ell$-valued functions on $G(F)$ which are bi-invariant under the action of I_G. The extended affine Hecke algebra $\mathbb{H}_G$ is a free $\mathbb{Z}[v, v^{-1}]$-module with basis $\{e^\lambda T_w | w \in W, \lambda \in X\}$, such that:

(1) The $\{T_w\}$ span a sub-algebra of $\mathbb{H}_G$ isomorphic to $\mathbb{H}_W$.
(2) The $\{e^\lambda\}$ span a $\mathbb{Z}[v, v^{-1}]$-sub-algebra of $\mathbb{H}_G$ isomorphic to $\mathrm{R}(\check{T})[v^{-1}, v]$.
(3) For any $s_\alpha \in S$ with $\langle \lambda, \check{\alpha}\rangle = 0$ we have $T_{s_\alpha} e^\lambda = e^\lambda T_{s_\alpha}$.
(4) For any $s_\alpha \in S$ with $\langle \lambda, \check{\alpha}\rangle = 1$ we have $T_{s_\alpha} e^{s_\alpha(\lambda)} T_{s_\alpha} = v e^\lambda$.

The properties (3),(4) together are equivalent to the following useful formula

$$T_{s_\alpha} e^{s_\alpha(\lambda)} - e^\lambda T_{s_\alpha} = (1 - v)\frac{e^\lambda - e^{s_\alpha(\lambda)}}{1 - e^{-\alpha}}, \tag{8.1}$$

where α is a simple coroot, s_α the corresponding simple reflection and $\lambda \in X$. The properties (1) and (2) give us two canonical embeddings of algebras

$$\mathrm{R}(\check{T})[v^{-1}, v] \hookrightarrow \mathbb{H}_G \qquad \textit{and} \qquad \mathbb{H}_W \hookrightarrow \mathbb{H}_G.$$

The multiplication in $\mathbb{H}_G$ gives rise to a $\mathbb{Z}[v^{-1}, v]$-module isomorphism

$$\mathbb{H}_G \simeq \mathrm{R}(\check{T})[v^{-1}, v] \otimes_{\mathbb{Z}[v^{-1}, v]} \mathbb{H}_W.$$

This is a v-analogue of the $\mathbb{Z}$-module isomorphism, [13, 7.1.14],

$$\mathbb{Z}[\widetilde{W}_G] \simeq \mathrm{R}(\check{T}) \otimes_{\mathbb{Z}} \mathbb{Z}[W_G].$$

Let $\check{\mathfrak{g}}$ be the Lie algebra of $\check{G}$, $\mathcal{B}_{\check{G}}$ be the variety of Borel subalgebras in $\check{\mathfrak{g}}$, and $\mathcal{N}_{\check{G}}$ be the nilpotent cone in $\check{\mathfrak{g}}$. The Springer resolution $\widetilde{\mathcal{N}}_{\check{G}}$ of $\mathcal{N}_{\check{G}}$ is given by

$$\widetilde{\mathcal{N}}_{\check{G}} = \{(x, \mathfrak{b}) \in \mathcal{N}_{\check{G}} \times \mathcal{B}_{\check{G}} | x \in \mathfrak{b}\}.$$

Let $\mu : \widetilde{\mathcal{N}}_{\check{G}} \to \mathcal{N}_{\check{G}}$ be the Springer map. Let s be the standard coordinate on $\mathbb{G}_m$. We let $\mathbb{G}_m$ act on $\check{\mathfrak{g}}$ by requiring that s sends an element x to $s^{-2}x$. We also define an action of $\check{G} \times \mathbb{G}_m$ on $\widetilde{\mathcal{N}}_{\check{G}}$ by the formula

$$(g, s).(x, b) = (s^{-2} g x g^{-1}, g \mathfrak{b} g^{-1}).$$

The map μ is $\check{G} \times \mathbb{G}_{\mathrm{m}}$-equivariant. The Steinberg variety is defined by

$$Z_{\check{G}} = Ntilde\check{G} \times_{\mathcal{N}_{\check{G}}} \widetilde{\mathcal{N}}_{\check{G}} = \{(x, , \mathfrak{b}, \mathfrak{b}') \in \mathcal{N}_{\check{G}} \times \mathcal{B}_{\check{G}} \times \mathcal{B}_{\check{G}} | x \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

The extended affine Hecke algebra $\mathbb{H}_G$ can be considered as $\mathbb{Z}[s, s^{-1}]$-algebra, where $v = s^2$. Viewing $\mathbb{Z}[s, s^{-1}]$ as the representation ring of $\mathbb{G}_m$, one has the following due to Kazhdan-Lusztig-Ginzburg:

Theorem 8.2. [13, Theorem 7.2.5] *There is a natural $\mathbb{Z}[s, s^{-1}]$-algebras isomorphism*

$$K^{\check{G} \times \mathbb{G}_{\mathrm{m}}}(Z_{\check{G}}) \xrightarrow{\sim} \mathbb{H}_G.$$

Now assume given two connected reductive groups G, H and a homomorphism $\check{G} \times \mathrm{SL}_2 \to \check{H}$, where $\check{G}$ (resp., $\check{H}$) denotes the Langlands dual group of G over $\overline{\mathbb{Q}}_\ell$ (resp., of H). We assume that the respective derived groups of $\check{G}$ and $\check{H}$ are simply connected. We suggest that there is a bimodule over the affine extended Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$ realizing the Arthur-Langlands functoriality at the Iwahori level for this homomorphism. We propose a definition of this explicit kernel at this level of generality given in Conjecture 8.7. It is based to a large extent on the Kazhdan-Lusztig-Ginzburg isomorphism in Theorem 8.2.

Let $\alpha : \mathbb{G}_m \to \mathrm{SL}_2$ be the standard maximal torus sending an element x to $\mathrm{diag}(x, x^{-1})$ and ξ (resp. η) be a homomorphism from SL_2 (resp. $\check{G}$) to $\check{H}$. We fix a maximal torus T_G (resp. T_H) in G (resp. H) and a Borel subgroup B_G (resp. B_H) in G (resp. H) containing T_G (resp. T_H). Assume we are given a morphism

$$\sigma : \check{G} \times \mathbb{G}_{\mathrm{m}} \longrightarrow \check{H}$$

as the composition

$$\check{G} \times \mathbb{G}_m \xrightarrow{\mathrm{id} \times \alpha} \check{G} \times \mathrm{SL}_2 \xrightarrow{\eta \times \xi} \check{H} \,. \tag{8.3}$$

For any element g in $\check{G}$ we will often denote its image $\eta(g)$ in $\check{H}$ by the same letter g as well as for the linearised morphisms between the corresponding Lie algebras. Denote by

$$\overline{\sigma} : \check{G} \times \mathbb{G}_{\mathrm{m}} \longrightarrow \check{H} \times \mathbb{G}_{\mathrm{m}}$$

the morphism whose first component is σ and whose second component is the second projection $pr_2 : \check{G} \times \mathbb{G}_{\mathrm{m}} \longrightarrow \mathbb{G}_{\mathrm{m}}$. The representation ring of $\check{G} \times \mathbb{G}_{\mathrm{m}}$ over $\overline{\mathbb{Q}}_\ell$ denoted by $\mathrm{R}(\check{G} \times \mathbb{G}_{\mathrm{m}})$ is isomorphic to $\mathrm{R}(\check{G})[s, s^{-1}]$. According to [31], the local Langlands functoriality at the unramified level sends the unramified representation with Langlands parameter γ in $\check{G}$ to the unramified representation with Langlands parameter $\sigma(\gamma, q^{1/2})$ of $\check{H}$. This is realized by the restriction homomorphism

$$\mathrm{Res}^\sigma : \mathrm{Rep}(\check{H}) \longrightarrow \mathrm{Rep}(\check{G} \times \mathbb{G}_{\mathrm{m}})$$

induced by σ.

On one hand, it is understood that the standard representation s of $\mathbb{G}_{\mathrm{m}}$ corresponds to the cohomological shift -1 in order to have the compatibility with [31]. On the other hand while specializing s, we should think of s as $q^{1/2}$ to makes things compatible with the theory of automorphic forms. The reader may refer to [29] for more details.

Let e denote the standard nilpotent element of $\mathrm{Lie}(SL_2)$

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

If $d\xi : \mathrm{Lie}(SL_2) \to \mathrm{Lie}(\check{H})$ is the linearised morphism associated to ξ, we denote $d\xi(e)$ by x.

Lemma 8.4. *The map f from $\mathcal{N}_{\check{G}}$ to $\mathcal{N}_{\check{H}}$ sending any element z in $\mathcal{N}_{\check{G}}$ to $z + x$ is a $\overline{\sigma}$-equivariant map. It defines a morphism of stack quotients*

$$\overline{f} : \mathcal{N}_{\check{G}}/(\check{G} \times \mathbb{G}_m) \longrightarrow \mathcal{N}_{\check{H}}/(\check{H} \times \mathbb{G}_m). \tag{8.5}$$

Proof. We have the following equality in $\mathrm{Lie}(\mathrm{SL}_2)$

$$ses^{-1} = s^2 e. \tag{8.6}$$

This implies that $s^{-2}\xi(s)x\xi(s)^{-1} = x$. For (g, s) in $\check{G} \times \mathbb{G}_m$, let $(h, s) = \overline{\sigma}(g, s) = (g\xi(s), s)$. Then for any z in $\mathcal{N}_{\check{G}}$

$$s^{-2}gzg^{-1} + x = s^{-2}h(z + x)h^{-1},$$

which implies that f is $\overline{\sigma}$-equivariant and the morphism of stack quotients $\overline{f}$ is well-defined. □

The Springer map $\widetilde{\mathcal{N}}_{\check{H}} \longrightarrow \mathcal{N}_{\check{H}}$ is $(\check{H} \times \mathbb{G}_m)$-equivariant. By using this and Lemma 8.4 we obtain the following diagram:

$$\begin{tikzcd}
\mathcal{X} = (\widetilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_m)) \times_{\mathcal{N}_{\check{H}}/(\check{H}\times\mathbb{G}_m)} (\widetilde{\mathcal{N}}_{\check{H}}/(\check{H} \times \mathbb{G}_m)) \arrow[r] \arrow[d] & \widetilde{\mathcal{N}}_{\check{G}}/(\check{G} \times \mathbb{G}_m) \arrow[d] \\
\widetilde{\mathcal{N}}_{\check{H}}/(\check{H} \times \mathbb{G}_m) \arrow[r] & \mathcal{N}_{\check{H}}/(\check{H} \times \mathbb{G}_m),
\end{tikzcd}$$

where the bottom horizontal map is induced from the Springer map for $\check{H}$ and the vertical right arrow is the composition of the $\check{G} \times \mathbb{G}_m$-equivariant Springer map for $\check{G}$ with the map $\overline{f}$ defined in Lemma 8.4. Note that in the left top corner of the diagram we took the fibre product in sense of stacks, see [24, §2.2.2], we denoted it by $\mathcal{X}$. The K-theory $K(\mathcal{X})$ of $\mathcal{X}$ is naturally a module over the associative algebras $K^{\check{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \widetilde{\mathcal{N}}_{\check{G}})$ and $K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}})$. The action is by convolution (see Appendix 10, § A, B, and C). Thanks to Theorem 8.2, these two algebras may be identified with the extended affine Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$ respectively. We may now state the conjecture :

Conjecture 8.7. *The bimodule over the affine extended Hecke algebras $K^{\check{G}\times\mathbb{G}_m}(Z_{\check{G}})$ and $K^{\check{H}\times\mathbb{G}_m}(Z_{\check{H}})$ realizing the local geometric Langlands functoriality at the Iwahori level for the map $\sigma : \check{G}\times\mathbb{G}_m \longrightarrow \check{H}$ identifies with $K(\mathcal{X})$.*

If $\check{G} = \check{H}$ then the map ξ is trivial, $\mathcal{X}$ equals $Z_{\check{G}}$ and $K(\mathcal{X})$ identifies with the extended affine Hecke algebra $\mathbb{H}_G$ for G. Thus $K(\mathcal{X})$ is naturally a free module of rank one over both algebras $\mathbb{H}_H$ and $\mathbb{H}_G$.

8.1. **Properties of the stack $\mathcal{X}$.** Consider the induced variety with respect to $\overline{\sigma}$ defined by

$$\mathcal{N}_{\check{G},\check{H}} = (\check{H}\times\mathbb{G}_m)\times_{\check{G}\times\mathbb{G}_m}\mathcal{N}_{\check{G}}.$$

Similarly we can define the induced space

$$\widetilde{\mathcal{N}}_{\check{G},\check{H}} = (\check{H}\times\mathbb{G}_m)\times_{\check{G}\times\mathbb{G}_m}\widetilde{\mathcal{N}}_{\check{G}}.$$

Proposition 8.8. *There exists a natural isomorphism of stacks*

$$\mathcal{X}\,\widetilde{\longrightarrow}\,(\widetilde{\mathcal{N}}_{\check{G},\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}})/(\check{H}\times\mathbb{G}_m),$$

and so an isomorphism of K-groups

$$K(\mathcal{X})\,\widetilde{\longrightarrow}\,K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G},\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}).$$

Proof. Since the map f defined in Lemma 8.4 is $\overline{\sigma}$-equivariant, it induces a $\check{H}\times\mathbb{G}_m$-equivariant map

$$f_1 : (\check{H}\times\mathbb{G}_m)\times_{\check{G}\times\mathbb{G}_m}\mathcal{N}_{\check{G}} \longrightarrow \mathcal{N}_{\check{H}}. \tag{8.9}$$

The map f_1 in (8.9) induces a map from $\widetilde{\mathcal{N}}_{\check{G},\check{H}}$ to $\mathcal{N}_{\check{H}}$ and we can consider the fibre product $\widetilde{\mathcal{N}}_{\check{G},\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$. Note that $\widetilde{\mathcal{N}}_{\check{G},\check{H}}/(\check{H}\times\mathbb{G}_m)$ is isomorphic to the stack quotient $\widetilde{\mathcal{N}}_{\check{G}}/(\check{G}\times\mathbb{G}_m)$, see Appendix 10 §$C$. It follows that $\mathcal{X}$ identifies with the stack quotient of $\widetilde{\mathcal{N}}_{\check{G},\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$ by the action of $\check{H}\times\mathbb{G}_m$ thanks to the following general fact : If $\phi : X \longrightarrow Z$ and $\psi : Y \longrightarrow Z$ are equivariant morphisms of G-schemes, then the fibre product $X/G\times_{Z/G} Y/G$ in the category of stacks identifies with the quotient stack $(X\times_Z Y)/G$. This yields the desired equivalence of categories. □

The module structure of $K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G},\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}})$ under the action of the two algebras $K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}})$ and $K^{\check{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{G}}}\widetilde{\mathcal{N}}_{\check{G}})$ is defined in Appendix, sections B and C.

If the map σ is the inclusion of $\check{G}$ in $\check{H}$, the natural map

$$\check{H}\times_{\check{G}}\mathcal{N}_{\check{G}} \to (\check{H}\times\mathbb{G}_m)\times_{\check{G}\times\mathbb{G}_m}\mathcal{N}_{\check{H},\check{G}} = \mathcal{N}_{\check{G},\check{H}}$$

is an isomorphism. We can identify $\mathcal{N}_{\check{G},\check{H}}$ with the variety of pairs

$$(h\check{G}\in\check{H}/\check{G}, v\in\mathcal{N}_{\check{H}})$$

satisfying $h^{-1}vh\in x+\mathcal{N}_{\check{G}}$ via the map sending any element of (h,z) of $\check{H}\times\mathcal{N}_{\check{G}}$ to $(h\check{G}, v = h(z+x)h^{-1})$. The latter map makes sense because $\check{G}$ centralizes x. Thus the map f_1 (8.9) becomes the projection sending any element $(h\check{G}, v)$ of $\mathcal{N}_{\check{G},\check{H}}$ to v. In this case the left $\check{H}\times\mathbb{G}_{\mathrm{m}}$-action on $\mathcal{N}_{\check{G},\check{H}}$ is such that for any (h_1, s) in $\check{H}\times\mathbb{G}_{\mathrm{m}}$ and any $(h\check{G}, v)$ in $\mathcal{N}_{\check{G},\check{H}}$ we have

$$(h_1, s).(h\check{G}, v) = (h_1 h\xi(s)^{-1}\check{G}, s^{-2}h_1 v h_1^{-1}).$$

Proposition 8.10. *There is a natural isomorphism*

$$K(\mathcal{X})\widetilde{\longrightarrow}K^{\check{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}),$$

and the $\mathrm{R}(\check{H}\times\mathbb{G}_{\mathrm{m}})$*-module structure on the right hand side is given by the functor* $\mathrm{Res}^{\overline{\sigma}}:\mathrm{R}(\check{H}\times\mathbb{G}_{\mathrm{m}})\to\mathrm{R}(\check{G}\times\mathbb{G}_{\mathrm{m}})$.

Proof. The scheme $\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$ classifies couples $((z,\mathfrak{b}_1),\mathfrak{b})$, where $(z,\mathfrak{b}_1)$ lies in $\widetilde{\mathcal{N}}_{\check{G}}$ and $\mathfrak{b}$ is Borel subalgebra in Lie(H) containing $z+x$. We define an action of $\check{G}\times\mathbb{G}_{\mathrm{m}}$ on $\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$ as follows: for any (g,s) in $\check{G}\times\mathbb{G}_{\mathrm{m}}$ and any $((z,\mathfrak{b}_1,\mathfrak{b})$ in $\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$

$$(g,s).((z,\mathfrak{b}_1),\mathfrak{b}) = (s^{-2}gzg^{-1}, g\mathfrak{b}_1 g^{-1}, g\xi(s)\mathfrak{b}\xi(s)^{-1}g^{-1}).$$

By Lemma $E.1$ in Appendix 10 we have an $\check{H}\times\mathbb{G}_{\mathrm{m}}$-equivariant isomorphism

$$(\check{H}\times\mathbb{G}_{\mathrm{m}})\times_{\check{G}\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}})\widetilde{\longrightarrow}\widetilde{\mathcal{N}}_{\check{G},\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}.$$

Combining this with Proposition 8.8 we get the desired isomorphism. □

In the following we will restrict ourselves to the case of $G=\mathbf{GL}_n$ and $H=\mathbf{GL}_m$ and we will describe a filtration and a grading on $K(\mathcal{X})$ in this special case. We will always use the same notation for $\mathbf{GL}_r$ and its Langlands dual over $\overline{\mathbb{Q}}_\ell$. In this setting we choose the morphism η to be the canonical inclusion of $\mathbf{GL}_n$ into $\mathbf{GL}_m$. The map σ is obtained by the composition

$$\mathbf{GL}_n\times\mathbb{G}_{\mathrm{m}}\to\mathbf{GL}_n\times\mathrm{SL}_2\xrightarrow{id\times\xi}\mathbf{GL}_n\times\mathbf{GL}_{m-n}\longrightarrow\mathbf{GL}_m,$$

where the last arrow is the inclusion of the standard Levi subgroup associated to the partition $(n, m-n)$ of m and ξ corresponds to the principal unipotent orbit as in [3]. Then the restriction of the map ξ to $\mathbb{G}_{\mathrm{m}}$ is the cocharacter $(0,\dots,0,m-n-1,m-n-3,\dots,1+n-m)$. Let $U_0=\mathbf{k}^m$ be the standard representation of $\mathbf{GL}_m$, and $\{u_1,\dots,u_m\}$ be the standard basis of U_0. The element $x=d\xi(e)$ is a nilpotent element of $\mathrm{Lie}(\mathbf{GL}_m)$ such that $x(u_i)=0$ for $1\le i\le n+1$ and that $x(u_{i+1})=u_i$ for $n+1\le i<m$. Let $G_2=\mathbf{GL}_{m-n}$ and B_2 be the unique Borel subgroup in G_2 such that x lies in $\mathrm{Lie}(B_2)$.

Let $Z_{G_2}(x)$ be the stabilizer of x in G_2. It acts naturally on $\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$: for any y in $Z_{G_2}(x)$ and any $(z,\mathfrak{b}_1,\mathfrak{b})$ in $\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$: we have

$$y.(z,\mathfrak{b}_1,\mathfrak{b}) = (z,\mathfrak{b}_1, y\mathfrak{b}y^{-1}).$$

For any s is in $\mathbb{G}_{\mathrm{m}}$ then the element $\xi(s)$ clearly normalizes $Z_{G_2}(x)$ and the semi-direct product $Z_{G_2}(x)\rtimes\mathbb{G}_{\mathrm{m}}$, is a subgroup of G_2. The group $Z_{G_2}(x)\rtimes\mathbb{G}_{\mathrm{m}}$ acts on $\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}}$ and this action commutes with the $\check{G}$-action.

Theorem 8.11. *There exists a $\check{G} \times \mathbb{G}_m$-invariant filtration*

$$\emptyset = F^0 \subset F^1 \subset \ldots \subset F^r = \widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}$$

such that for $0 \le i \le r$, each $K^{\check{G}\times\mathbb{G}_m}(F^i)$ is a submodule over both affine extended Hecke algebras $K^{\check{G}\times G_m}(\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \widetilde{\mathcal{N}}_{\check{G}})$ and $K^{\check{H}\times G_m}(\widetilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}})$. Moreover the spaces $K^{\check{G}\times\mathbb{G}_m}(F^i)$ for $0 \le i \le r$ define a filtration on $K(\mathcal{X})$.

Proof. For any $\check{G}$-orbit $\mathbb{O}$ on $\mathcal{N}_{\check{G}}$ we denote by $Y_{\mathbb{O}}$ the preimage of $\mathbb{O}$ in $\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}$ under the projection

$$\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}} \to \mathcal{N}_{\check{G}}$$

sending $(z, \mathfrak{b}_1, \mathfrak{b})$ to z. We refer the reader to [13, § 3.2] for details on nilpotent orbits and stratification of the nilpotent cone $\mathcal{N}_{\check{G}}$ into $\check{G}$-conjugacy classes and the stratification of the Steinberg variety of $\check{G}$. The orbits $Y_{\mathbb{O}}$ form a $\check{G}\times\mathbb{G}_m$-invariant stratification of $\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}$, which is also $Z_{G_2}(x)$-invariant. The $\check{G}$-orbit $\mathbb{O}$ is given by a partition $\theta = (n_1 \ge n_2 \ge \cdots \ge n_r \ge 1)$ of n. Let M_θ denote the standard Levi subgroup corresponding to this partition, Namely

$$M_\theta \widetilde{\longrightarrow} \mathbf{GL}_{n_1} \times \cdots \times \mathbf{GL}_{n_r}.$$

We denote by z_θ the standard upper triangular regular nilpotent element in $\mathrm{Lie}(M_\theta)$; z_θ lies in the orbit $\mathbb{O}$. Let Z_θ be the stabilizer of z_θ in $\check{G} \times \mathbb{G}_m$, Z_θ is connected.

Denote by $\mathcal{B}_{\check{G},\theta}$ the preimage of z_θ under the Springer map $\widetilde{\mathcal{N}}_{\check{G}} \to \mathcal{N}_{\check{G}}$. Let $\mathcal{B}_{\check{H},\theta}$ be the preimage of $z_\theta + x$ under the Springer map $\widetilde{\mathcal{N}}_{\check{H}} \to \mathcal{N}_{\check{H}}$. We have an isomorphism

$$(\check{G} \times \mathbb{G}_m) \times_{Z_\theta} (\mathcal{B}_{\check{G},\theta} \times \mathcal{B}_{\check{H},\theta}) \widetilde{\longrightarrow} Y_{\mathbb{O}}$$

sending $(g, s, \mathfrak{b}_1, \mathfrak{b})$ to $(s^{-2} g z_\theta g^{-1}, g\mathfrak{b}_1 g^{-1}, g\xi(s)\mathfrak{b}\xi(s)^{-1}g^{-1})$. Hence we have an isomorphism of groups

$$K^{\check{G}\times\mathbb{G}_m}(Y_{\mathbb{O}}) \widetilde{\longrightarrow} K^{Z_\theta}(\mathcal{B}_{\check{G},\theta} \times \mathcal{B}_{\check{H},\theta}). \tag{8.12}$$

According to [38] the scheme $\mathcal{B}_{\check{G},\theta}$ and $\mathcal{B}_{\check{H},\theta}$ respectively admit a finite paving by affine spaces stable under the action of Z_θ. Hence (8.12) is a free $\mathrm{R}(Z_\theta)$-module of finite type.

We enumerate the nilpotent orbits $\mathbb{O}_1, \mathbb{O}_2, \ldots, \mathbb{O}_r$ in $\mathcal{N}_{\check{G}}$ in such an order that

$$\dim(\mathbb{O}_1) \le \dim(\mathbb{O}_2) \le \cdots \le \dim(\mathbb{O}_r).$$

If $\overline{F}^j = \cup_{i \le j} \mathbb{O}_i$, then $\overline{F}^j$ is closed in $\mathcal{N}_{\check{G}}$ and we have a filtration

$$\emptyset = \overline{F}^0 \subset \overline{F}^1 \subset \cdots \subset \overline{F}^r = \mathcal{N}_{\check{G}}.$$

Let F^j be the preimage of $\overline{F}^j$ in $\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}$. We get a $\check{G} \times \mathbb{G}_m$-invariant filtration

$$\emptyset = F^0 \subset F^1 \subset \cdots \subset F^r = \widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}.$$

We can refine the filtration F^i in such way that the refined filtration be $\check{G} \times \mathbb{G}_m$-stable and the corresponding strata of the stack quotient of $(\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}})/(\check{G} \times \mathbb{G}_m)$ satisfy the assumptions of Lemma 8.13. Then by using this Lemma, we see that for each i the sequence

$$0 \longrightarrow K^{\check{G}\times\mathbb{G}_m}(F^{i-1}) \longrightarrow K^{\check{G}\times\mathbb{G}_m}(F^i) \longrightarrow K^{\check{G}\times\mathbb{G}_m}(Y_{\mathbb{O}_i}) \longrightarrow 0$$

is exact and $K^{\check{G}\times\mathbb{G}_m}(F^i)$, $0 \leq i \leq r$ define a filtration on $K(\mathcal{X})$. Moreover, for each i, $K^{\check{G}\times\mathbb{G}_m}(F^i)$ is a submodule over both extended affine Hecke algebras $K^{\check{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \widetilde{\mathcal{N}}_{\check{G}})$ and $K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}})$. □

Lemma 8.13 (Cellular fibration)**.** *Let us consider the following general situation: k is an algebraically closed field of arbitrary characteristic and $\mathcal{X}$ is a k-stack of finite type equipped with a filtration*

$$\emptyset = F^0 \subset F^1 \subset \cdots \subset F^r = \mathcal{X},$$

by closed substacks of $\mathcal{X}$. Assume that for $1 \leq i \leq r$ there exists an affine space E^i and a connected linear algebraic group P^i such that

$$F^i - F^{i-1} \xrightarrow{\sim} E^i/P^i,$$

where E^i/P^i is the stack quotient. Let U^i be the unipotent radical of P^i and $G^i = P^i/U^i$ be the reductive quotient. Then the natural sequence

$$0 \longrightarrow K(F^{i-1}) \longrightarrow K(F^i) \longrightarrow K(E^i/P^i) \longrightarrow 0$$

is exact and $K(F^i)$ is a free $\mathbb{Z}$-module.

Proof. Choose a section of the natural projection from P^i to P^i , it yields a map from E^i/G^i to E^i/P^i inducing an isomorphism

$$K(E^i/P^i) \xrightarrow{\sim} K(E^i/G^i) \xrightarrow{\sim} K^{G^i}(\mathrm{Spec}(k)) \xrightarrow{\sim} \mathrm{R}(G^i),$$

where $\mathrm{R}(G^i)$ denotes the ring of representations of G^i (which is a free $\mathbb{Z}$-module.) One has an exact sequence

$$K_1(E^i/P^i) \xrightarrow{\delta} K(F^{i-1}) \longrightarrow K(E^i/P^i) \longrightarrow 0.$$

Let us show that the map δ vanishes. By [13, 5.2.18], we have that

$$K_1^{P^i}(E^i) \xrightarrow{\sim} K_1^{G^i}(E^i)$$

and by Thom isomorphism [13, 5.4.17] we obtain that

$$K_1^{G^i}(E^i) \xrightarrow{\sim} K_1^{G^i}(\mathrm{Spec}(k)).$$

Now, by [39, Corollary 6.12], $K^{G^i}(\mathrm{Spec}(k))$ is isomorphic to $k^* \otimes_{\mathbb{Z}} S$, where S is a free abelian group generated by the irreducible representations of G^i. By induction on i we may assume that $K^i(F^{i-1})$ is a free $\mathbb{Z}$-module. To finish the proof note that for any free $\mathbb{Z}$-module S, one has $\mathrm{Hom}_{\mathbb{Z}}(k^*, S) = 0$. Indeed, Let ϕ be such a morphism and take y in the image of ϕ. Then there exists an element x in k^* such that $\phi(x) = y$. As k is algebraically closed, for any integer n , there exists t in k such that that $t^n = x$. This gives $n\phi(t) = y$ in S. As S is a free abelian group, the element y is dividable by a finite number of integers. Thus, $y = 0$ and ϕ vanishes.

□

9. Howe correspondence in terms of $K(\mathcal{X})$

Let $D\mathcal{W}$ be the category whose objects are direct sums of cohomologically shifted objects of $\mathcal{W}$ defined in [23]. This category is equipped with an action of $\mathbb{G}_m$ by chomological shifts. Consider the invariants of this category denoted by $(D\mathcal{W})^{I_G\times I_H}$. The group $K((D\mathcal{W})^{I_G\times I_H})$ is naturally a module over $K(DP_{I_G}(\mathcal{F}l_{I_G}))$. This K-group $K(DP_{I_H}(\mathcal{F}l_{I_H}))\otimes\overline{\mathbb{Q}}_\ell$ is isomorphic to the Iwahori-Hecke algebra $\mathcal{H}_{I_H}$. According to [19], the Iwahori-Hecke algebra $\mathcal{H}_{I_H}$ identifies with $\mathbb{H}_H\otimes_{\mathbb{Z}[s,s^{-1}]}\bar{\mathbb{Q}}_\ell$ for the map $\mathbb{Z}[s,s^{-1}]\to\bar{\mathbb{Q}}_\ell$ sending s to $q^{\frac{1}{2}}$. This isomorphism is naturally upgraded to the isomorphism

$$K(DP_{I_H}(\mathcal{F}l_H))\otimes\overline{\mathbb{Q}}_\ell\xrightarrow{\sim}\mathbb{H}_H\otimes_{\mathbb{Z}[s,s^{-1}]}\overline{\mathbb{Q}}_\ell$$

such that the multiplication by s in $\mathbb{H}_H$ corresponds to the cohomological shift by -1 in $K(DP_{I_H}(\mathcal{F}l_H))$. Hence under these isomorphisms and Kazhdan- Lusztig-Ginzburg isomorphism $K(\mathcal{X})$ and $K((D\mathcal{W})^{I_G\times I_H})$ are two bimodules over the affine extended Hecke algebras $\mathbb{H}_G$ and $\mathbb{H}_H$. We obtain the following conjecture:

Conjecture 9.1. *The bimodule $K(\mathcal{X})$ is isomorphic (after specializing s to $q^{1/2}$) to the Grothendieck group of the category $(D\mathcal{W})^{I_G\times I_H}$ under the action of the affine extended Hecke algebras $K^{\check{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G}}\times_{\mathcal{N}_{\check{G}}}\widetilde{\mathcal{N}}_{\check{G}})$ and $K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{H}}\times_{\mathcal{N}_{\check{H}}}\widetilde{\mathcal{N}}_{\check{H}})$*

Assume that $G=\mathbf{GL}_n$ and $H=\mathbf{GL}_m$, where $n\le m$. Then we obtain the following:

Conjecture 9.2. *The bimodule $K(\mathcal{X})$ over each extended affine Hecke algebra $\mathbb{H}_H$ and $\mathbb{H}_G$ is isomorphic (after specializing s to $q^{1/2}$) to the Grothendieck group of the category $DP_{I_H\times I_G}(\Pi(F))$.*

The principal result of this paper is the following theorem describing geometric Howe correspondence in terms of geometric Langlands functoriality for all dual reductive pairs $(\mathbf{GL}_1,\mathbf{GL}_m)$.

Theorem 9.3. *Conjecture* 9.2 *is true for* $(\mathbf{GL}_1,\mathbf{GL}_m)$ *for any m.*

9.1. **The proof of the conjecture.** The rest of the paper is devoted to the proof of Theorem 9.3. Let $n=1$ and $m\ge1$ and let $G=\mathbf{GL}_1$ and $H=\mathbf{GL}_m$, where we consider them as Langlands dual groups. The map $\check{G}\times\mathbb{G}_m\longrightarrow\check{H}$ is the composition

$$\check{G}\times\mathbb{G}_m\longrightarrow\check{G}\times\mathrm{SL}_2\longrightarrow\check{G}\times\mathbf{GL}_{m-1}\longrightarrow\check{H},$$

where the latter map is the inclusion of the standard Levi subgroup $\mathbf{GL}_1\times\mathbf{GL}_{m-1}$ in $\check{H}$ and $\xi:\mathrm{SL}_2\to\mathbf{GL}_{m-1}$ corresponds to the principal unipotent orbit. In particular the inclusion $\check{G}$ in $\check{H}$ is the coweight $(1,0,\dots,0)$ of the standard maximal torus of $\check{H}$. The restriction of ξ to the maximal torus $\mathbb{G}_m$ of SL_2 is the coweight $(0,m-2,m-4,\dots,2-m)$ of $\check{H}$. The element $x=d\xi(e)$ in $\mathcal{N}_{\check{H}}$ is the subregular nilpotent element given by $x(u_1)=x(u_2)=0$ and $x(u_{i+1})=u_i$ for all $2\le i<m$.

Proposition 9.4. *The bimodule $K(\mathcal{X})$ identifies with the Springer fibre $\mathcal{B}_{\check{H},x}$ of the Springer map $\widetilde{\mathcal{N}}_H\to\mathcal{N}_{\check{H}}$ over the point x*

Proof. In this case we have $\widetilde{\mathcal{N}}_{\check{G},\check{H}} = \check{H}/\check{G}$ in such way that the map $f_1 : \widetilde{\mathcal{N}}_{\check{G},\check{H}} = \check{H}/\check{G} \longrightarrow \mathcal{N}_{\check{H}}$ defined in (8.9) sends $h\check{G}$ to hxh^{-1}. The element s in $\mathbb{G}_m$ acts on the left hand side on $\widetilde{\mathcal{N}}_{\check{G},\check{H}}$ by sending the right coset $h\check{G}$ to $h\xi(s)^{-1}\check{G}$. The variety $\widetilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}$ identifies with the variety of pairs $(h\check{G}, \mathfrak{b})$ such that $\mathfrak{b}$ is a Borel subalgebra in $\check{H}$ and hxh^{-1} lies in $\mathfrak{b}$. Any element (h_1, s) in $\check{H} \times \mathbb{G}_m$ acts on $\widetilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}$ by the formula:

$$(h_1, s).(h\check{G}, \mathfrak{b}) = (h_1 h\xi(s)^{-1}\check{G}, h_1\mathfrak{b}h_1^{-1}).$$

Denote by $\mathcal{B}_{\check{H},x}$ the fibre of the Springer map $\widetilde{\mathcal{N}}_{\check{H}} \to \mathcal{N}_{\check{H}}$ over x. The map

$$\overline{\sigma} : \check{G} \times \mathbb{G}_m \longrightarrow \check{H} \times \mathbb{G}_m$$

sending (g, s) to $(g\xi(s), s)$ identifies $\check{G} \times \mathbb{G}_m$ with the stabilizer in $\check{H} \times \mathbb{G}_m$ of the right coset of the neutral element in $\check{H}/\check{G}$. Any element (g, s) of $\check{G} \times \mathbb{G}_m$ acts on the Springer fibre $\mathcal{B}_{\check{H},x}$ by

$$(g, s).\mathfrak{b}' = (g\xi(s)\mathfrak{b}'\xi(s)^{-1}g^{-1}).$$

This yields an isomorphism

$$K(\mathcal{X}) = K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{H},\check{G}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}}) \xrightarrow{\sim} K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x}).$$

□

To compute $K(\mathcal{X})$, we provide an explicit description of the Springer fibre $\mathcal{B}_{\check{H},x}$.

Lemma 9.5. *The Springer fibre $\mathcal{B}_{\check{H},x}$ is a configuration of projective lines $(V_i)_{1\le i\le m-1}$. For $1 \le i < j \le m-1$ the intersection $V_j \cap V_i$ is empty unless $j = i+1$. The fixed locus in $\mathcal{B}_{\check{H},x}$ under the action of $\check{G} \times \mathbb{G}_m$ consists of m points $p_1, p_2, \dots, p_{m-1}, p_m$, where p_1 and p_m are distinguished points on V_1 and V_m and for $2 \le i \le m-1$, the point p_i is the intersection of V_i with V_{i+1}.*

Proof. Denote by

$$F_1 \subset F_2 \subset \cdots \subset F_m = U_0$$

a complete flag on the standard representation U_0 of $\check{H}$ preserved by x. The vector space F_1 is a subspace of the vector space $\mathrm{Ker}(x) = \mathrm{Vect}(u_1, u_2)$. We have $\mathrm{Vect}(u_2) = \mathrm{Ker}(x) \cap \mathrm{Im}(x)$. If $F_1 \neq \mathrm{Vect}(u_2)$ then $F_2 = x^{-1}(F_1) = \mathrm{Vect}(u_1, u_2)$, $F_3 = x^{-1}(F_2) = \mathrm{Vect}(u_1, u_2, u_3), \dots$, and finally the space F_m is equal to $x^{-1}(F_{m-1}) = \mathrm{Vect}(u_1, u_2, \dots, u_m) = U_0$. So we may identify V_1 with the projective space of lines in $\mathrm{Vect}(u_1, u_2)$. The point p_2 is $F_1 = \mathrm{Vect}(u_2)$. If $F_1 = \mathrm{Vect}(u_2) \subset \mathrm{Im}(x)$ then $x^{-1}(F_1) = \mathrm{Vect}(u_1, u_2, u_3)$ and V_2 can be identified with the space of lines in $x^{-1}(F_1)/F_1$. Inside $\mathrm{Vect}(u_1, u_2, u_3)$ one has a distinguished subspace $\mathrm{Vect}(u_1, u_2, u_3) \cap \mathrm{Im}(x) = \mathrm{Vect}(u_2, u_3)$. If F_2 is different from this subspace then the whole flag F_i is uniquely defined. So the point p_3 of V_2 corresponds to $F_2 = \mathrm{Vect}(u_2, u_3)$. If now $F_1 = \mathrm{Vect}(u_2)$ and $F_2 = \mathrm{Vect}(u_2, u_3)$ then $x^{-1}(F_2) = \mathrm{Vect}(u_1, u_2, u_3, u_4)$ and D_3 is the space of lines in $x^{-1}(F_2)/F_2$. The point p_4 of V_3 corresponds to $F_3 = \mathrm{Vect}(u_2, u_3, u_4)$, and one can continue the construction till F_m. The points p_1 is the standard complete flag on U_0 and p_m is the flag $\mathrm{Vect}(u_2) \subset \mathrm{Vect}(u_2, u_3) \subset \cdots \subset \mathrm{Vect}(u_2, \dots, u_m) \subset \mathrm{Vect}(u_1, \dots, u_m)$. □

This result combined with the Cellular fibration Lemma in [13, § 5.5] implies the following

Proposition 9.6. *The K-group $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ is a free $\mathrm{R}(\check{G}\times\mathbb{G}_m)$-module of rank m. Moreover, the $\mathrm{R}(\check{H})$-module structure on $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ comes from $\mathrm{Res}^{\sigma} : \mathrm{R}(\check{H}) \longrightarrow \mathrm{R}(\check{G}\times\mathbb{G}_m)$.*

According to [13, Lemma 7.6.2] the assignment sending T_w to $s^{\ell(w)}$ for w in W_H, extends by linearity to an algebra homomorphism

$$\epsilon : \mathbb{H}_{W_H} \longrightarrow \mathbb{Z}[s, s^{-1}]$$

and it is known that the induced $\mathbb{H}_H$-module $\mathrm{Ind}_{\mathbb{H}_{W_H}}^{\mathbb{H}_H}\epsilon = \mathbb{H}_H \otimes_{\mathbb{H}_{W_H}} \epsilon$ is isomorphic to the polynomial representation [13, 7.6.8]. We have the following crucial set of isomorphisms of $\mathbb{Z}[s, s^{-1}]$-modules [13, Formula (7.6.5)]

$$K^{\check{H}\times\mathbb{G}_m}(T^*\mathcal{B}_{\check{H}}) \xrightarrow{\text{Thom}} K^{\check{H}\times\mathbb{G}_m}(\mathcal{B}_{\check{H}}) \xrightarrow{\alpha} \mathrm{R}(\check{T}_H)[s, s^{-1}] \xrightarrow{\beta} \mathrm{Ind}_{\mathbb{H}_{W_H}}^{\mathbb{H}_H}\epsilon,$$

where the first arrow is the Thom isomorphism [13, Theorem 5.4.16], the map α is the canonical isomorphism

$$K^{\check{H}\times\mathbb{G}_m}(\mathcal{B}_{\check{H}}) \xrightarrow{\sim} K^{\check{H}\times\mathbb{G}_m}(\check{H}/B_{\check{H}}) \xrightarrow{\sim} K^{B_{\check{H}}\times\mathbb{G}_m}(\mathrm{pt}) \tag{9.7}$$

$$\xrightarrow{\sim} \mathrm{R}(\check{T}_H\times\mathbb{G}_m) \xrightarrow{\sim} \mathrm{R}(\check{T}_H)[s, s^{-1}], \tag{9.8}$$

and the map β is given for any λ by $\beta(e^{\lambda}) = e^{-\lambda}$.

There is a natural action of $\mathbb{H}_H$ on $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ defined uniquely by the property that the inclusion of $\mathcal{B}_{\check{H},x}$ in $\mathcal{B}_{\check{H}}$ yields a $\mathrm{R}(\check{G}\times\mathbb{G}_m)\otimes_{\mathrm{R}(\check{H}\times\mathbb{G}_m)}\mathbb{H}_H$-equivariant surjection

$$\mathrm{R}(\check{G}\times\mathbb{G}_m) \otimes_{\mathrm{R}(\check{H}\times\mathbb{G}_m)} K^{\check{H}\times\mathbb{G}_m}(\mathcal{B}_{\check{H}}) \xrightarrow{\sim} K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H}}) \longrightarrow K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x}).$$

Consider the diagram

$$\begin{tikzcd} \mathbb{H}_H \arrow[r, "\gamma_1"] \arrow[d, "\gamma_2"] & K(DP_{I_H\times I_G}(\Pi(F))) \\ K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x}), & \end{tikzcd} \tag{9.9}$$

where γ_1 sends $\mathcal{T}$ to $\overleftarrow{H}_H(\mathcal{T}, I_0)$, and γ_2 sends $\mathcal{T}$ to the action of $\mathcal{T}$ on the structure sheaf O of $\mathcal{B}_{\check{H},x}$. Note that γ_1 and γ_2 are surjective. We are now going to construct a morphism

$$\mathfrak{J} : K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x}) \to K(DP_{I_H\times I_G}(\Pi(F))).$$

which will be induced by γ_1. One sees that γ_1 factors through the surjective morphism $\bar{\gamma}_1 : \mathbb{H}_H \otimes_{\mathbb{H}_{W_H}} \epsilon \to K(DP_{I_H\times I_G}(\Pi(F)))$ of $\mathbb{H}_H$-modules. Moreover, remark that If $m = 2$ then both $K(DP_{I_H\times I_G}(\Pi(F)))$ and $\mathbb{H}_H \otimes_{\mathbb{H}_{W_H}} \epsilon$ are free $R(\check{G}\times\mathbb{G}_m)$-modules of rank 2, and $\bar{\gamma}_1$ is an isomorphism. For proving Theorem 9.3 we are reduced to prove the following:

Proposition 9.10. *There is a unique isomorphism of $\mathbb{H}_H$-modules $\mathfrak{J}$ making diagram* (9.9) *commutative. The map $\mathfrak{J}$ commutes with the $\mathbb{H}_G$-actions.*

Note that if $n = m = 1$ then one has $I_H = H(O)$ and this proposition can be deduced from [31, Proposition 4]. We have seen in section 7 that the module $K(DP_{I_G\times I_H}(\Pi(F)))$ is free of rank m over $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathrm{m}})$. In the notation of this section, a basis of $K(DP_{I_G\times I_H}(\Pi(F)))$ is given by the elements IC^k for $0 \le k \le m-1$, and the action of $\mathrm{R}(\check{G}\times\mathbb{G}_{\mathrm{m}})$ is given on this basis in (7.10). Besides, according to Theorem 7.8, $\mathrm{R}(\check{H})$ acts via Res^{σ}. A part of these properties has been already proved for $K^{\check{G}\times\mathbb{G}_{\mathrm{m}}}(\mathcal{B}_{\check{H},x})$ in Proposition 9.6. In the sequel we will construct a basis of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ and we will identify the action of $\mathbb{H}_H$ on this basis and the basis IC^k. The morphism sending one basis to another will be induced by γ_1. Surprisingly the basis we will construct is not the canonical basis of Lusztig constructed in [30].

We will use the polynomial representation of the affine extended Hecke algebra $\mathbb{H}_H$ to describe the action of $\mathbb{H}_H$ on this new basis that we will construct. So let us first describe the representation of $\mathbb{H}_H$ in $\mathrm{R}(\check{T}_H)[s,s^{-1}]$. Consider the polynomial representation of the extended affine Hecke algebra $\mathbb{H}_H$ of H in $\mathrm{R}(\check{T}_H)[s,s^{-1}]$. For v in $\mathbb{H}_H$ and z in $\mathrm{R}(\check{T}_H)[s,s^{-1}]$ write $v * z$ for the action of v on z. The element e^{λ} denotes the element in $\mathrm{R}(\check{\mathrm{T}}_{\mathrm{H}})[s,s^{-1}]$ corresponding to λ; according to [13, Formula (7.6.1)], e^{λ} as an element of $\mathbb{H}_H$ acts on any element u of $\mathrm{R}(\check{T}_H)[s,s^{-1}]$ by

$$e^{\lambda} * u = e^{-\lambda}u, \tag{9.11}$$

and for any simple root α, the action of T_{s_α} on e^{λ} is given by the formula [13, Theorem 7.2.16]:

$$T_{s_\alpha} * e^{\lambda} = \frac{e^{\lambda} - e^{s_\alpha(\lambda)}}{e^{\alpha}-1} - s^2\frac{e^{\lambda} - e^{s_\alpha(\lambda)+\alpha}}{e^{\alpha}-1}. \tag{9.12}$$

This formula was discovered by Lusztig and was the starting point of the K-theoretic approach to Hecke algebras. The formulas (9.11) and (9.12) together completely determine the polynomial representation of $\mathbb{H}_H$. For λ dominant, the element e^{λ} corresponds in the Iwahori-Hecke algebra to the function $s^{-\ell(\lambda)}T_{t^{\lambda}}$, where $\ell(\lambda) = \langle\lambda, 2\check{\rho}_H\rangle$ and $T_{t^{\lambda}}$ is the characteristic function of the double coset $I_G t^{\lambda} I_G$. Denote by ω_i the coweight $(1,\dots,1,0,\dots,0)$, where 1 appears i times. For $1 \le i < m$, denote by $w_i = t^{\omega_i}\sigma_i$ the element of length zero. The element w_1 is the generator of the group Ω_H of length zero elements in $\widetilde{W}_H$; for any i in $\mathbb{Z}$, $w_i = w_1^i$. In the extended affine Hecke algebra $\mathbb{H}_H$ we have

$$T_{t^{\omega_i}}T_{w_i} = T_{\sigma_i}.$$

Further we have $\ell(t^{\omega_i}) = \ell(\sigma_i) = \langle\omega_i, 2\check{\rho}_{\check{H}}\rangle = i(m-i)$ and this gives

$$e^{\omega_i} = s^{i(i-m)}T_{t^{\omega_i}}. \tag{9.13}$$

In $\mathrm{R}(\check{T}_H)[s,s^{-1}]$, $T_{\sigma_i} * 1 = s^{2i(m-i)}$ and this yields

$$(s^{i(m-i)}e^{\omega_i}T_{w_i}) * 1 = s^{2i(m-i)},$$

and

$$T_{w_i} * 1 = s^{i(m-i)} e^{\omega_i}. \tag{9.14}$$

Till now we have described the action of the Wakimoto objects and the elements of length zero. We are going to compute the action of the simple reflections $s_i = (i, i+1)$ and the affine simple reflection $s_m = t^\lambda w_0$, where $\lambda = (-1, 0 \dots, 0, 1)$ and $w_0 = (1, m)$ is the longest element of the finite Weyl group of H. For $1 \le i \le m$ we have $T_{w_1} T_{s_i} T_w^{-1} = T_{s_{i+1}}$ and $T_{w_1} T_{s_m} T_{w_1}^{-1} = T_{s_1}$. For all integer j in $\mathbb{Z}$ set $s_j = s_{j+m}$ and rewrite the above formulas all together as

$$T_{w_1} T_{s_i} T_{w_1}^{-1} = T_{s_{i+1}}$$

Thus, for all i and j in $\mathbb{Z}$

$$T_{w_j} T_{s_i} T_{w_j}^{-1} = T_{s_{i+j}}.$$

For any cocharacter μ we have $w_i t^\mu w_i^{-1} = t^{\sigma_i(\mu)}$ and we get

$$T_{w_i} T_{t^\mu} T_{w_i}^{-1} = T_{t^{\sigma_i(\mu)}}.$$

Proposition 9.15. *In the polynomial representation the element T_{s_m} acts on 1 by $(s^2 - 1) + s^{2(m-1)} e^{\xi + \omega_1}$, where $\xi = (0, 0, \dots, 0, -1)$.*

Proof. Since $T_{s_m} = T_{w_1}^{-1} T_{s_1} T_{w_1}$, we get using (9.14) :

$$T_{s_m} * 1 = (T_{w_1}^{-1} T_{s_1}) * s^{m-1} e^{\omega_1}.$$

Let $\alpha_i = (0, \dots, 0, 1, -1, 0 \dots, 0)$ and $\mu_i = (0, \dots, 0, 1, 0 \dots, 0)$, where 1 appears on i^{th} place then

$$T_{s_1} * e^{\omega_1} = e^{\omega_1 - \alpha_1}.$$

Thus,

$$T_{s_m} * 1 = T_{w_1}^{-1} * s^{m-1} e^{\omega_1 - \alpha_1} \tag{9.16}$$

If $\xi = -\sigma_1^{-1} \omega_1 = (0, \dots, 0, -1)$, then ξ is a dominant character, and we have $T_{t^\xi} T_{w_1}^{-1} = T_{\sigma_1}^{-1}$. Thus

$$T_{w_1}^{-1} = s^{1-m} e^{-\xi} T_{\sigma_1^{-1}}.$$

Finally we have to compute

$$T_{s_m} * 1 = s^{1-m} e^{-\xi} T_{\sigma_1^{-1}} * s^{m-1} e^{\omega_1 - \alpha_1}.$$

On one hand the reduced decomposition of σ_1^{-1} is $s_{m-1} \dots s_2 s_1$ and it follows that $T_{\sigma_1^{-1}} = T_{s_{m-1}} \dots T_{s_2} T_{s_1}$. From (9.12) we get that $T_{s_1} * e^{\mu_2} = (s^2 - 1) e^{\mu_2} + s^2 e^{\omega_1}$. For $2 \le i \le m - 1$ we have $T_{s_i} * e^{\omega_1} = s^2 e^{\omega_1}$. We also have $T_{s_2} * e^{\mu_2} = e^{\mu_2 - \alpha_2} = e^{\mu_3}$ and more generally, for $1 \le i < m$, $T_{s_i} * e^{\mu_i} = e^{\mu_{i+1}}$. By induction we get

$$T_{\sigma_1^{-1}} * e^{\mu_2} = (s^2 - 1) e^{\mu_m} + s^{2(m-1)} e^{\omega_1}.$$

This implies that

$$T_{s_m} * 1 = (s^2 - 1) + s^{2(m-1)} e^{\xi + \omega_1} \tag{9.17}$$

□

In order to prove Proposition 9.10 we have to study the $\mathbb{H}_H$-module structure of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ and compare this action with the results obtained §(7.10). Now let us construct the desired basis of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$. Denote by L_λ the line bundle on $\mathcal{B}_{\check{H}}$ corresponding to coweight λ of H as in [13, §6.1.11]. The $\check{H}$-module $\mathrm{H}^0(\mathcal{B}_{\check{H}}, L_\lambda)$ vanishes unless $(a_1 \leq \cdots \leq a_m)$. Recall that the nilpotent subregular element x in $\mathrm{End}(U_0)$ is such that $x(u_1) = x(u_2) = 0$ and $x(u_i) = u_{i-1}$ for all $3 \leq i \leq m$. The natural morphism from $\mathrm{R}(\check{T}_H)[s, s^{-1}]$ to $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ sends an element e^λ to $L_{-\lambda}$. Besides, any element $\mathcal{L}$ in $K^{\check{H}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{H}})$ acts on $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ as the tensor product by $\mathcal{L}_{|\mathcal{B}_{\check{H},x}}$.

Let $\{u_1, \dots u_m\}$ be the canonical basis of U_0 and $\{u_1^*, \dots, u_m^*\}$ the corresponding dual basis. For $1 \leq i \leq m$ set

$$U_i = \mathrm{Vect}(u_1, \dots, u_i)$$

and for $1 \leq i \leq m-1$ set

$$U_i^{'} = \mathrm{Vect}(u_2, \dots, u_{i+1}),$$

with $U_0^{'}$ being equal to $\{0\}$. Note that for $0 \leq i \leq m-2$ the element x acts on $U_{i+2}/U_i^{'}$ by zero. For $1 \leq i < m$ let V_i be the projective line classifying flags

$$U_1^{'} \subset \dots U_{i-1}^{'} \subset W_i \subset U_{i+1} \subset \cdots \subset U_m,$$

where W_i is i-dimensional. The line V_i is isomorphic to $\mathbb{P}(\mathrm{Vect}(u_1, u_{i+1}))$ via the map sending a line l to the flag given by

$$U_1^{'} \subset \dots U_{i-1}^{'} \subset l \oplus U_{i-1}^{'} \subset U_{i+1} \subset \cdots \subset U_m.$$

Then we have $\mathcal{B}_{\check{H},x} = \cup_i V_i$, (see Lemma 9.5). Recall that there are m fixed points on $\mathcal{B}_{\check{H},x}$ under the action of $\check{G} \times \mathbb{G}_m$ corresponding to the following flags:

(1) $p_1 = U_1 \subset U_2 \subset \cdots \subset U_m$.

(2) For $2 \leq k \leq m-1$,

$$p_k = U_1^{'} \subset \dots U_{k-1}^{'} \subset U_k \subset \cdots \subset U_m.$$

(3) $p_m = U_1^{'} \subset U_2^{'} \subset \cdots \subset U_{m-1}^{'} \subset U_m$.

Note that for $2 \leq k \leq m-1$, the point p_k equals $V_{k-1} \cap V_k$.

Each line V_i is endowed with a tautological equivariant line bundle $\mathcal{O}_{V_i}(-1)$ which is an equivariant subbundle of $g\mathcal{O}_{V_i} \oplus s^{m-2i}\mathcal{O}_{V_i}$. Note that: for $1 \leq i \leq m-1$

$$\mathcal{O}_{V_i}(-p_i) = s^{2i-m}\mathcal{O}_{V_i}(-1) \quad \text{and} \quad \mathcal{O}_{V_i}(-p_{i+1}) = g^{-1}\mathcal{O}_{V_i}(-1).$$

Thanks to Lusztig [30, §4.7] the elements $\mathcal{O}_{p_1}, \mathcal{O}_{V_1}(-1), \dots, \mathcal{O}_{V_{m-1}}(-1)$ define a basis of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ over $\mathrm{R}(\check{G})[s, s^{-1}]$.

For $1 \leq i < m$, consider the line bundle L_{ω_i} on $\mathcal{B}_{\check{H},x}$ whose fibre at a point $F_1 \subset \cdots \subset F_m$ is $\det(F_i)$. Remind that $\det(U_i^{'}) \xrightarrow{\sim} s^{i(m-i-1)}$ as a $\check{G} \times \mathbb{G}_m$-representation. We also have $L_{\omega_m} = g\mathcal{O}$ in $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$.

Proposition 9.18. *The set of line bundles $\{\mathcal{O}, L_{-\omega_1}, \dots L_{-\omega_{m-1}}\}$ forms a basis of K-group $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ after specialization.*

Proof. For $1 \le k \le m-1$ and for any $\check{G} \times \mathbb{G}_m$-equivariant line bundle L on $\mathcal{B}_{\check{H},x}$, we have the following equality in $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$:

$$L = \sum_{j=1}^{k-1} L_{|V_j}(-p_{j+1}) + L_{|V_k} + \sum_{j=k+1}^{m-1} L_{|V_j}(-p_j)$$

We apply this formula to L_{ω_k}. Note that :

- If $j < k$, $L_{\omega_k|_{V_j}} = g s^{(k-1)(m-k)} \mathcal{O}_{V_j}$.
- If $j = k$, $L_{\omega_k|_{V_j}} = \mathcal{O}_{V_j}(-1)$.
- If $j > k$, $L_{\omega_k|_{V_j}} = s^{k(m-k-1)} \mathcal{O}_{V_j}$.

Hence we get:

$$L_{\omega_k} = \mathcal{O}_{V_k}(-1) + s^{(k-1)(m-k)} \Big[\sum_{j=1}^{k-1} \mathcal{O}_{V_j}(-1) + \sum_{j=k+1}^{m-1} s^{2(j-k)} \mathcal{O}_{V_j}(-1) \Big].$$

Lastly

$$\mathcal{O} = \mathcal{O}_{p_1} + \sum_{j=1}^{m-1} s^{2j-m} \mathcal{O}_{V_j}(-1).$$

Since $\mathcal{O}_{p_1}, \mathcal{O}_{V_1}(-1), \dots, \mathcal{O}_{V_{m-1}}(-1)$ is a basis of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$, the previous formulas imply that $\mathcal{O}, L_{\omega_1} \dots, L_{\omega_{m-1}}$ is a free family which becomes a basis after specializing s to $q^{1/2}$. If we apply the duality functor, we get the same result for the family $\mathcal{O}, L_{-\omega_1} \dots, L_{-\omega_{m-1}}$.

□

Consider the family $\{\mathcal{O}, s^{m-1} L_{-\omega_1}, s^{2(m-2)} L_{-\omega_2}, \dots, s^{m-1} L_{-\omega_{m-1}}\}$. Thanks to Proposition 9.18 this family is also a basis of $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$ after specialization. The map γ_1 factors through morphism $\mathfrak{I}$ sending this basis to $\{\mathrm{IC}^0, \dots, \mathrm{IC}^{m-1}\}$.

According to (9.14), we have

$$\gamma_2(T_{w_i}) = T_{w_i}(\mathcal{O}) = s^{i(m-i)} e^{\omega_i} = s^{i(m-i)} L_{-\omega_i}.$$

Hence the action of length zero elements on the basis is compatible with their action on $\{\mathrm{IC}^0, \dots, \mathrm{IC}^m\}$ in [§7, (7.10)].

Now we will compute the action of the affine simple reflection s_m. If λ be the cocharacter $(-1, 0 \dots, 0, 1)$, and consider the associated line bundle L_λ (resp. E) on $\mathcal{B}_{\check{H},x}$ whose fibre over a flag $F_1 \subset \cdots \subset F_m = U_m$ is $F_1^* \otimes F_m/F_{m-1}$(resp. F_m/F_{m-1}). The section u_m of the line bundle E yields an exact sequence

$$0 \longrightarrow s^{2-m}\mathcal{O} \longrightarrow E \longrightarrow (L_{m-1,m})_{p_m} \longrightarrow 0.$$

Note that $(E)_{p_m} = g\mathcal{O}_{p_m}$, and $(L_{-\omega_1})_{p_m} = s^{2-m}\mathcal{O}_{p_m}$. Tensoring by $L_{-\omega_1}$, we get the exact sequence on $\mathcal{B}_{\check{H},x}$

$$0 \longrightarrow s^{2-m} L_{-\omega_1} \longrightarrow L_\lambda \longrightarrow g s^{2-m} \mathcal{O}_{p_m} \longrightarrow 0.$$

Consider $u_1^* \wedge \cdots \wedge u_{m-1}^*$ as global section of $L_{-\omega_{m-1}}$ over $\mathcal{B}_{\check{H},x}$. It vanishes only at p_m and gives an exact sequence

$$0 \longrightarrow g^{-1}s^{2-m}O \longrightarrow L_{-\omega_{m-1}} \longrightarrow O_{p_m} \longrightarrow 0.$$

Finally we conclude that in $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$

$$L_\lambda = s^{2-m}L_{-\omega_1} + gs^{2-m}O_{p_m}, \quad \text{and} \quad gs^{2-m}O_{p_m} = gs^{2-m}L_{-\omega_{m-1}} - s^{4-2m}O.$$

Thus

$$L_\lambda = s^{2-m}L_{-\omega_1} + gs^{2-m}L_{-\omega_{m-1}} - s^{4-2m}O.$$

From Proposition 9.15 we obtain that

$$\gamma_2(T_{s_m}) = T_{s_m}(O) = (s^2-1)O + s^{2m-2}L_\lambda = -O + s^m L_{-\omega_1} + gs^m L_{-\omega_{m-1}}. \tag{9.19}$$

Finally, $s^{-1}T_{s_m}(O) + s^{-1}O$ corresponds to $\overleftarrow{H}_H(L_{s_m}, I_0)$, and the formula (9.19) is compatible with (7.10) by using the fact that L_{s_m} is isomorphic to $\bar{\mathbb{Q}}_\ell[1](\frac{1}{2})$ over $\overline{\mathcal{F}l}_H^{s_m}$. Moreover, for $1 \leq i < m$ one has $T_{s_i} * 1 = v$ in the polynomial representation, hence $T_{s_i}(O) = vO$ in $K^{\check{G}\times\mathbb{G}_m}(\mathcal{B}_{\check{H},x})$. The other relations are readily obtained by symmetry (the action of elements of length zero). This finishes the proof of Proposition 9.10 and so Conjecture 9.2.

10. Some generalities on convolution product

Let $\mathbf{k}$ be an algebraically closed field of characteristic zero. Let G be a linear algebraic group, and X be a scheme of finite type over $\mathbf{k}$. Denote by $\mathrm{R}(G)$ the ring of representation of G over $\overline{\mathbb{Q}}_\ell$.

Appendix A.

Let Y be a smooth G-variety and $\pi : Y \to X$ be a proper G-equivariant map then according to [13, 5.2.20] $K^G(Y \times_X Y)$ is an associative $\mathrm{R}(G)$-algebra. Moreover, $K^G(Y)$ is naturally a left module over $K^G(Y \times_X Y)$. Namely, for any L in $K^G(Y \times_X Y)$ and F in $K^G(Y)$, consider the restriction with supports of $L \boxtimes F$ an element of $K^G((Y \times_X Y) \times Y)$ with respect to the smooth closed embedding

$$\begin{array}{ccc} Y \times Y & \overset{\mathrm{id}\times\mathrm{diag}}{\rightarrow} & Y \times Y \times Y \\ \cup & & \cup \\ Y \times_X Y & \rightarrow & (Y \times_X Y) \times Y \end{array}$$

and denote the result by $L \otimes p_2^* F \in K^G(Y \times_X Y)$. Then we have $L * F = (p_1)_*(L \otimes p_2^* F) \in K^G(Y)$.

By using this $K^{\check{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G}})$ is a module over $K^{\check{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \widetilde{\mathcal{N}}_{\check{G}})$.

Appendix B.

Let Z be a smooth variety. Consider a G-equivariant morphism from Z to X. Then $K^G(Y \times_X Y)$ acts on $K^G(Z \times_X Y)$ by convolutions on the right. Additionally, this action is R(G)-linear. Namely, for any F in $K^G(Z \times_X Y)$ and any L in $K^G(Y \times_X Y)$, consider the element $p_{12}^* F \boxtimes p_{34}^* L$ in $K^G((Z \times_X Y) \times (Y \times_X Y))$. Let us apply the restriction with supports functor with respect to the smooth closed embedding $\mathrm{id} \times \mathrm{diag} \times \mathrm{id}$ in the following diagram to $p_{12}^* F \boxtimes p_{34}^* L$

$$\begin{array}{ccc} Z \times Y \times Y & \overset{\mathrm{id} \times \mathrm{diag} \times \mathrm{id}}{\rightarrow} & Z \times Y \times Y \times Y \\ \cup & & \cup \\ Z \times_X Y \times_X Y & \rightarrow & (Z \times_X Y) \times (Y \times_X Y) \end{array},$$

and denote the result by $p_{12}^* F \otimes p_{23}^* L$ in $K^G(Z \times_X Y \times_X Y)$. The projection $p_{13} : Z \times_X Y \times_X Y \rightarrow Z \times_X Y$ is proper, and we obtain the convolution product of F and L denoted by

$$F * L = (p_{13})_* (p_{12}^* F \otimes p_{23}^* L) \in K^G(Z \times_X Y).$$

By using this, $K^{\check{H} \times \mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}})$ appeared in §8 is a module over the algebra $K^{\check{H} \times \mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}})$.

Appendix C.

Let G be a closed subgroup of H and assume Y' to be smooth. Let $\pi' : Y' \rightarrow X'$ be a proper morphism of G-varieties. Assume that $X \xrightarrow{\sim} H \times_G X'$ and $Y \xrightarrow{\sim} H \times_G Y'$ as H-varieties, and that π is obtained from π' by induction in the sense of [13, 5.2.16]. Then we have the following isomorphism

$$Y \times_X Y \xrightarrow{\sim} H \times_G (Y' \times_{X'} Y')$$

as H-varieties. So, we have an isomorphism of stack quotients

$$(Y \times_X Y)/H \xrightarrow{\sim} (Y' \times_{X'} Y')/G,$$

and we get an isomorphism of algebras

$$K^H(Y \times_X Y) \xrightarrow{\sim} K^G(Y' \times_{X'} Y').$$

In a more general setting we have the following: let G and H be two algebraic groups, $\phi : G \rightarrow H$ be a morphism and X be a G-variety. The induced H-variety with $H \times_G X$ respect to ϕ is the stack quotient $(H \times X)/G$, where G acts on $H \times X$ by

$$g.(h, x) = (h\phi(g)^{-1}, g.x).$$

The group H acts on the stack $H \times_G X$ by $h'.(h, x) = (h'h, x)$, and $(H \times_G X)/H$ is isomorphic to the stack quotient X/G. There exists two functors "res" and "Ind" (restriction and induction) which are mutually inverse and give rise to the following isomorphisms of categories :

$$K^G(X) \leftrightarrows K^H(H \times_G X).$$

By using this construction we have the following equivalence of categories:

$$(\widetilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{G},\check{H}}} \widetilde{\mathcal{N}}_{\check{G},\check{H}})/\check{H} \xrightarrow{\sim} (\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \widetilde{\mathcal{N}}_{\check{G}})/\check{G}.$$

This allows us to define a left action by convolution of $K^{\check{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}}_{\check{G}} \times_{\mathcal{N}_{\check{G}}} \widetilde{\mathcal{N}}_{\check{G}})$ on $K^{\check{H}\times G_m}(\widetilde{\mathcal{N}}_{\check{G},\check{H}} \times_{\mathcal{N}_{\check{H}}} \widetilde{\mathcal{N}}_{\check{H}})$, see §8.

Appendix D.

Let Y be a smooth G-variety and $\pi : Y \to X$ a proper G-equivariant morphism. Let $X \to \bar{X}$ and $Z \to \bar{X}$ be G-equivariant morphisms of varieties. Assume Z to be smooth. Then $K^G(Y \times_X Y)$ acts on the left by convolution on $K^G(Y \times_{\bar{X}} Z)$. Indeed, for any F in $K^G(Y \times_{\bar{X}} Z)$ and L in $K^G(Y \times_X Y)$, consider $p_{12}^* L \boxtimes p_{34}^* F$ in $K^G((Y \times_X Y) \times (Y \times_{\bar{X}} Z))$. Apply the restriction with supports with respect to the smooth closed embedding $\mathrm{id} \times \mathrm{diag} \times \mathrm{id}$ in the following diagram to $p_{12}^* L \boxtimes p_{34}^* F$

$$\begin{array}{ccc} Y \times Y \times Z & \xrightarrow{\mathrm{id} \times \mathrm{diag} \times \mathrm{id}} & Y \times Y \times Y \times Z \\ \cup & & \cup \\ Y \times_X Y \times_{\bar{X}} Z & \to & (Y \times_X Y) \times (Y \times_{\bar{X}} Z). \end{array}$$

and denote the result by $p_{12}^* L \otimes p_{34}^* F$ in $K^G(Y \times_X Y \times_{\bar{X}} Z)$. The projection $p_{13} : Y \times_X Y \times_{\bar{X}} Z \to Y \times_{\bar{X}} Z$ is proper, and we obtain the convolution product of L and F denoted by

$$L * F = (p_{13})_*(p_{12}^* L \otimes p_{34}^* F) \in K^G(Y \times_{\bar{X}} Z).$$

Note actually that the essential thing we need is the fact that the structure sheaf $\mathcal{O}_Y$ of the diagonal $Y \subset Y \times Y$ admits a finite G-equivariant resolution by coherent locally free $\mathcal{O}_{Y\times Y}$-modules. Then restrict this resolution with respect to the flat projection $p_{23} : Y \times Y \times Y \times Z \to Y \times Y$. It seems that the smoothness of Z is not necessary. Moreover, assume Z to be smooth and $Z \to \bar{X}$ to be proper, then $K^G(Z \times_{\bar{X}} Z)$ acts on $K^G(Y \times_{\bar{X}} Z)$ by convolutions on the right, and the actions of $K^G(Z \times_{\bar{X}} Z)$ and of $K^G(Y \times_X Y)$ commute.

Let $\overline{x}$ be a G-fixed point in $\overline{X}$. Assume that the morphism $X \to \overline{X}$ factors through $X \to \overline{x} \to \overline{X}$. Let $Z_{\overline{x}}$ be the fibre of $Z \to \overline{X}$ over $\overline{x}$. Moreover, assume that $Z_{\overline{x}}$ is smooth and that it satisfies the conditions of Künneth of formula [13, Theorem 5.6.1]. Then, we have

$$K^G(Y \times_{\overline{X}} Z) \xrightarrow{\sim} K^G(Y \times Z_{\overline{x}}) \xrightarrow{\sim} K^G(Y) \otimes_{\mathrm{R}(G)} K^G(Z_{\overline{x}}). \tag{D.1}$$

Note that $K^G(Z_{\overline{x}})$ is naturally a $K^G(Z \times_{\overline{X}} Z)$-module and this is action is $\mathrm{R}(G)$-linear. The action of $K^G(Z \times_{\overline{X}} Z)$ on $K^G(Y \times_{\overline{X}} Z)$ is $\mathrm{R}(G)$-linear as well. One checks that the action of $K^G(Z \times_{\overline{X}} Z)$ on the right hand side of (D.1) comes by functoriality from the corresponding action on $K^G(Z_{\overline{x}})$.

Appendix E.

Let G be a closed algebraic subgroup of H, Y_1 be a G-scheme, and Y, $\tilde{Y}$ be two H-schemes. Consider the Cartesian diagram

$$\begin{tikzcd} Y_1 \times_Y \tilde{Y} \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ \tilde{Y} \arrow[r] & Y, \end{tikzcd}$$

where the map $\tilde{Y} \to Y$ is H-equivariant and the map $f : Y_1 \to Y$ is G-equivariant, the action of G on Y being induced by the action of defined H. The group G acts diagonally on the fibre product $Y_1 \times_Y \tilde{Y}$. This allows us to consider the induced space $H \times_G (Y_1 \times_Y \tilde{Y})$. On the other hand, we have a H-equivariant map $f_1 : H \times_G Y_1 \to Y$ given by $f_1(h, y_1) = hf(y_1)$. Consider the cartesian diagram

$$\begin{tikzcd} (H \times_G Y_1) \times_Y \tilde{Y} \arrow[r] \arrow[d] & H \times_G Y_1 \arrow[d, "f_1"] \\ \tilde{Y} \arrow[r] & Y, \end{tikzcd}$$

and let H act diagonally on the fibre product $(H \times_G Y_1) \times_Y \tilde{Y}$.

Lemma E.1. *There is a H-equivariant isomorphism of schemes*

$$H \times_G (Y_1 \times_Y \tilde{Y}) \widetilde{\longrightarrow} (H \times_G Y_1) \times_Y \tilde{Y}. \tag{E.2}$$

Proof. The isomorphism is furnished by the H-equivariant map

$$\begin{aligned} H \times_G (Y_1 \times_Y \tilde{Y}) &\to (H \times_G Y_1) \times_Y \tilde{Y} \\ (h, (y_1, u)) &\to ((h, y_1), hu) \end{aligned}$$

□

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