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RESEARCH
REPORT
N° 8154
December 2012
Project-Teams Geometrica and
Vegas

ISSN 0249-6399 ISRN INRIA/RR--8154--FR+ENG

The monotonicity of f-vectors of random polytopes

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Research Report n° 8154 — December 2012 — 10 pages

Abstract: Let K be a compact convex body in $\mathbb{R}^d$, let K_n be the convex hull of n points chosen uniformly and independently in K, and let $f_i(K_n)$ denote the number of i-dimensional faces of K_n.
We show that for planar convex sets, $E[f_0(K_n)]$ is increasing in n. In dimension $d \geqslant 3$ we prove that if $\lim_{n\to\infty} \frac{E[f_{d-1}(K_n)]}{An^c} = 1$ for some constants A and $c > 0$ then the function $n \mapsto E[f_{d-1}(K_n)]$ is increasing for n large enough. In particular, the number of facets of the convex hull of n random points distributed uniformly and independently in a smooth compact convex body is asymptotically increasing. Our proof relies on a *random sampling* argument.

Key-words: Computational geometry, Stochastic geometry, Convex hull, Complexity .

Part of this work is supported by: ANR blanc PRESAGE (ANR-11-BS02-003)

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Monotonie des f-vecteurs de polytopes aléatoires

Résumé : Soit K un domaine convexe et compact de $\mathbb{R}^d$, K_n l'enveloppe convexe de n points choisis uniformément et indépendamment dans K, et $f_i(K_n)$ le nombre de faces de K_n de dimension i.

Nous montrons que pour des convexes du plan, $E[f_0(K_n)]$ est croissant avec n. En dimension $d \geqslant 3$ nous montrons que si $\lim_{n\to\infty} \frac{E[f_{d-1}(K_n)]}{An^c} = 1$ pour des constantes A et $c > 0$ alors la fonction $n \mapsto E[f_{d-1}(K_n)]$ est croissante pour n suffisamment grand. En particulier, le nombre de facettes de l'enveloppe convexe de n points aléatoires distribués uniformément et indépendamment dans un convexe compact à bord lisse est asymptotiquement croissant. Notre démonstration utilise un argument d'échantillonnage aléatoire.

Mots-clés : Géométrie algorithmique, Géométrie stochastique, Enveloppe convexe, Complexité.

1 Introduction

What does a random polytope, that is, the convex hull of a finite set of random points in $\mathbb{R}^d$, look like? This question goes back to Sylvester's *four point problem*, which asked for the probability that four points chosen at random be in convex position. There are, of course, many ways to distribute points randomly, and as with Sylvester's problem, the choice of the distribution drastically influences the answer.

Random polytopes. In this paper, we consider a *random polytope* K_n obtained as the convex hull of n points distributed uniformly and independently in a *convex body* $K \subset \mathbb{R}^d$, i.e. a compact convex set with nonempty interior. This model arises naturally in applications areas such as computational geometry [7, 11], convex geometry and stochastic geometry [15] or functional analysis [6, 10]. A natural question is to understand the behavior of the ***f*-vector** of K_n, that is, $\boldsymbol{f}(K_n) = (f_0(K_n), \ldots, f_{d-1}(K_n))$ where $f_i(K_n)$ counts the number of i-dimensional faces, and the behaviour of the *volume* $V(K_n)$. Bounding $f_i(K_n)$ is related, for example, to the analysis of the computational complexity of algorithms in computational geometry.

Exact formulas for the expectation of $\boldsymbol{f}(K_n)$ and $V(K_n)$ are only known for convex polygons [2, Theorem 5]. In general dimensions, however, the asymptotic behavior, as n goes to infinity, is well understood; the general picture that emerges is a dichotomy between the case where K is a polytope when

$$\begin{aligned} E[f_i(K_n)] &= c_{d,i,K} \log^{d-1} n + o(\log^{d-1} n), \\ E[V(K) - V(K_n)] &= c_{d,K} n^{-1} \log^{d-1} n + o(n^{-1} \log^{d-1} n), \end{aligned} \tag{1}$$

and the case where K is a smooth convex body (i.e. with boundary of differentiability class C^2 and with positive Gaussian curvature) when

$$\begin{aligned} E[f_i(K_n)] &= c'_{d,i,K} n^{\frac{d-1}{d+1}} + o(n^{\frac{d-1}{d+1}}), \\ E[V(K) - V(K_n)] &= c_{d,K} n^{-\frac{2}{d+1}} + o(n^{-\frac{2}{d+1}}). \end{aligned} \tag{2}$$

(Here $c_{d,i,K}$ and $c'_{d,i,K}$ are constants depending only on i, d and certain geometric functionals of K.) The results for $E[V(K_n)]$ follow from the corresponding results for $f_0(K_n)$ via Efron's formula [8]. The literature devoted to such estimates is consequent and we refer to Reitzner [14] for a recent survey.

Monotonicity of f-vectors. In spite of numerous papers devoted to the study of random polytopes, the natural question whether these functionals are monotone remained open in general.

Concerning the monotonicity of $E[V(K_n)]$ with respect to n, the elementary inequality

$$E[V(K_n)] \leqslant E[V(K_{n+1})] \tag{3}$$

for any fixed K follows immediately from the monotonicity of the volume. Yet the monotonicity (at first sight similarly obvious) with respect to K, i.e. the inequality that $K \subset L$ implies

$$E[V(K_n)] \leqslant E[V(L_n)]$$

surprisingly turned out to be false in general. This problem was posed by Meckes [9], see also [14], and a counterexample for $n = d + 1$ was recently given by Rademacher [12].

A "tantalizing problem" in stochastic geometry, to quote Vu [16, Section 8], is whether $f_0(K_n)$ is monotone in n, that is, whether similar to Equation (3):

$$E[f_0(K_n)] \leqslant E[f_0(K_{n+1})].$$

This is not a trivial question as the convex hull of $K_n \cup \{x\}$ has fewer vertices than K_n if x lies outside K_n and sees more than two of its vertices. Some bounds are known for the expected number of points of K_n seen by a random point x chosen uniformly in $K \backslash K_n$ [16, Corollary 8.3] but they do not suffice to prove that $E[f_0(K_n)]$ is monotone.

It is known that $E[f_0(K_n)]$ is always bounded from below by an increasing function of n, namely $c(d)\log^{d-1} n$ where $c(d)$ depends only on the dimension: this follows, via Efron's formula [8], from a similar lower bound on the expected volume of K_n due to Bárány and Larman [1, Theorem 2]. While this is encouraging, it does not exclude the possibility of small oscillations preventing monotonicity. In fact, Bárány and Larman [1, Theorem 5] also showed that for any functions s and ℓ such that $\lim_{n\to\infty} s(n) = 0$ and $\lim_{n\to\infty} \ell(n) = \infty$ there exists[1] a compact convex domain $K \subset \mathbb{R}^d$ and a sequence $(n_i)_{i\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$:

$$E[f_0(K_{n_{2i}})] > s(n_{2i}) n_{2i}^{\frac{d-1}{d+1}} \quad \text{and} \quad E[f_0(K_{n_{2i+1}})] < \ell(n_{2i+1}) \log^{d-1}(n_{2i+1}).$$

When general convex sets are considered, there may thus be more to this monotonicity question than meets the eye.

Results. This paper present two contributions on the monotonicity of the $\boldsymbol{f}$-vector of K_n. First, we show that for any *planar* convex body K the expectation of $f_0(K_n)$ is an increasing function of n. This is based on an explicit representation of the expectation $E[f_0(K_n)]$.

Theorem 1. *Assume K is a planar convex body. For all integers n,*

$$E[f_i(K_{n+1})] > E[f_i(K_n)], \; i = 0, 1.$$

Second, we show that if K is a compact convex set in $\mathbb{R}^d$ with a C^2 boundary then the expectation of $f_{d-1}(K_n)$ is increasing for n large enough (where "large enough"depends on K); in particular, for smooth compact convex bodies K in $\mathbb{R}^d$ the expectation of $f_0(K_n)$ becomes monotone in n for n large enough.

Theorem 2. *Assume $K \subset \mathbb{R}^d$ is a smooth convex body. Then there is an integer n_K such that for all $n \geqslant n_K$*

$$E[f_i(K_{n+1})] > E[f_i(K_n)], \; i = d-2, d-1.$$

Our result is in fact more general and applies to convex hulls of points i.i.d. from any "sufficiently generic" distribution (see Section 3) and follows from a simple and elegant random sampling technique introduced by Clarkson [4] to analyze *non-random* geometric structures in discrete and computational geometry.

2 Monotonicity for convex domains in the plane

Assume K is a planar convex body of volume one. Given a unit vector $u \in \mathbb{S}^1$, each halfspace $\{x \in \mathbb{R}^2 : x \cdot u \leqslant p\}$ cuts off from K a set of volume $s \in [0,1]$. On the contrary, given $u \in \mathbb{S}^1, s \in (0,1)$ there is a unique line $\{x \in \mathbb{R}^2 : x \cdot u = p\}$ and a corresponding halfspace $\{x \in \mathbb{R}^2 : x \cdot u \leqslant p\}$ which cuts of from K a set of volume precisely s. Denote by $L(s,u)$ the square of the length of this unique chord.

[1] More precisely, *most* (in the Baire sense) compact convex sets exhibit this behavior.

Using this notation, Buchta and Reitzner [2] showed that the expectation of the number of points on the convex hull of n points chosen randomly uniformly in K can be computed with the following formula.

Lemma 3 (Buchta,Reitzner)**.**

$$E[f_0(K_n)] = \frac{1}{6}n(n-1)\int_{\mathbb{S}^1}\int_0^1 s^{n-2}L(s,u)\,dsdu$$

Here du denotes integration with respect to Lebesgue measure on $\mathbb{S}^1$. By a change of variables we obtain

$$E[f_0(K_n)] = \frac{1}{6}(n-1)\int_{\mathbb{S}^1}\int_0^1 t^{-\frac{1}{n}}L(t^{\frac{1}{n}},u)\,dtdu = \frac{1}{6}\int_{\mathbb{S}^1}\mathcal{I}_n(u)du$$

with $\mathcal{I}_n(u) = (n-1)\int_0^1 t^{-\frac{1}{n}}L(t^{\frac{1}{n}},u)\,dt$. Observe that $L(1,u) = 0$ for almost all $u \in \mathbb{S}^1$. Also observe that the partial derivative $\frac{\partial}{\partial s}L(s,u)$ exists for almost all (s,u). This is a consequence of the a.e. differentiability of a convex function. In the following we write $L(\cdot) = L(\cdot,u)$, $\frac{\partial}{\partial s}L(\cdot,u) = L'(\cdot)$. Integration by parts in t gives

$$\mathcal{I}_n(u) = -\int_0^1 L'(t^{\frac{1}{n}})\,dt. \tag{4}$$

Finally, the convexity of K induces the following lemma.

Lemma 4. *Given a value u, the derivative $s \mapsto L'(s)$ is a decreasing function.*

Proof. Fix $u \in \mathbb{S}^1$. We denote $l(p) = l(p,u)$ the length of the chord $K \cap \{x \in \mathbb{R}^2 : \; x \cdot u = p\}$. Moreover, $L(s(p)) = l(p)^2$ where $s(p)$ is the volume of the part of K on the left of the chord of length $l(p)$. This volume $s(p)$ is a monotone and hence injective function of p with inverse $p(s)$, and we have $\frac{d}{dp}s(p) = l(p)$.

First observe that because K is a convex body, the chord length $l(p)$ is concave as a function of p. Thus its derivative $\frac{d}{dp}l(p)$ is decreasing.

Hence

$$L'(s) = 2l(p(s))\frac{d}{ds}l(p(s)) = 2\frac{d}{ds}l(p(s))\;(\frac{d}{dp}s(p))|_{p=p(s)} = 2\frac{d}{dp}l(p)|_{p=p(s)}.$$

Since the derivative $\frac{d}{dp}l(p)$ is decreasing and $p(s)$ is monotone, we see that $\frac{d}{ds}L(s,u)$ is decreasing in s. □

This allows us to conclude that the expectancy of the number of points on the convex hull is increasing.

Proof of Theorem 1. According to Lemma 4, $L'(s)$ is decreasing. Since for all $t \in [0,1]$, $t^{\frac{1}{n}} < t^{\frac{1}{n+1}}$, this implies that $L'(t^{\frac{1}{n}}) \geqslant L'(t^{\frac{1}{n+1}})$. Combined with equation (4) we have

$$\mathcal{I}_n(u) \leqslant \mathcal{I}_{n+1}(u)$$

which proves $E[f_0(K_n)] \leqslant E[f_0(K_{n+1})]$. In the planar case, the number of edges is also an increasing function because $f_0(K_n) = f_1(K_n)$. □

3 Random sampling

We denote by $\sharp S$ the cardinality of a finite set S and we let $\mathbb{1}_X$ denote the characteristic function of event X: $\mathbb{1}_{p\in F}$ is 1 if $p \in F$ and 0 otherwise. Let S be a finite set of points in $\mathbb{R}^d$ and let $k \geqslant 0$ be an integer. A *k-set of S* is a subset $\{p_1, p_2, \ldots p_d\} \subseteq S$ that spans a hyperplane bounding an open halfspace that contains exactly k points from S; we say that the k-set *cuts off* these k points. In particular, 0-sets are facets of the convex hull of S. For any finite subset S of D we let $s_k(S)$ denote the number[2] of k-sets of S.

Generic distribution assumption. Let $\mathcal{P}$ denote a probability distribution on $\mathbb{R}^d$; we assume throughout this section that $\mathcal{P}$ is such that d points chosen independently from $\mathcal{P}$ are generically affinely independent.

Determining the order of magnitude of the maximum number of k-sets determined by a set of n points in $\mathbb{R}^d$ has been an important open problem in discrete and computational geometry over the last decades. In the case d=2, Clarkson [4] gave an elegant proof that $s_{\leqslant k}(S) = O(nk)$ for any set S of n points in the plane, where:

$$s_{\leqslant k}(S) = s_0(S) + s_1(S) + \ldots + s_k(S).$$

(See also Clarkson and Shor [5] and Chazelle [3, Appendix A.2].) Although the statement holds for any fixed point set, and not only in expectation, Clarkson's argument is probabilistic and will be our main ingredient. It goes as follows. Let R be a subset of S of size r, chosen randomly and uniformly among all such subsets. A i-set of S becomes a 0-set in R if R contains the two points defining the i-set but none of the i points it cuts off. This happens approximately with probability $p^2(1-p)^i$, where $p = \frac{r}{n}$ (see [5] for exact computations). It follows that:

$$E[s_0(R)] \geqslant \sum_{0\leqslant i\leqslant k} p^2(1-p)^i s_i(S) \geqslant p^2(1-p)^k s_{\leqslant k}(S).$$

Since $E[s_0(R)]$ cannot exceed $\sharp R = r$, we have $s_{\leqslant k}(S) \leqslant \frac{n}{p(1-p)^k}$ which, for $p = 1/k$, yields the announced bound $s_{\leqslant k}(S) = O(nk)$. A similar random sampling argument yields the following inequalities.

Lemma 5. *Let $s_k(n)$ denote the expected number of k-sets in a set of n points chosen independently from $\mathcal{P}$. We have*

$$s_0(n) \geqslant s_0(n-1) + \frac{d\ s_0(n) - s_1(n)}{n} \tag{5}$$

and for any integer $1 \leqslant r \leqslant n$

$$s_0(r) \geqslant \frac{\binom{n-d}{r-d}}{\binom{n}{r}} s_0(n) + \frac{\binom{n-d-1}{r-d}}{\binom{n}{r}} s_1(n). \tag{6}$$

Proof. Let S be a set of n points chosen, independently, from $\mathcal{P}$ and let $q \in S$. The 0-sets of S that are not 0-sets of $S\backslash\{q\}$ are precisely those defined by q. Conversely, the 0-sets of $S\backslash\{q\}$ that are not 0-sets of S are precisely those 1-sets of S that cut off q. We can thus write:

$$s_0(S) = s_0(S\backslash\{q\}) + \sum_{F \text{ facet of } CH(S)} \mathbb{1}_{q\in F} - \sharp\text{1-sets cutting off } q.$$

[2] If the hyperplane separates S in two subsets of k elements $(2k + d = n)$ the k-set is counted twice.

Note that the equality remains true in the degenerate cases where several points of S are identical or in a non generic position if we count the facets of the convex hull of S with multiplicities in the sum and in $s_0(n)$. Summing the previous identity over all points q of S, we obtain

$$ns_0(S) = \left(\sum_{q\in S} s_0(S\backslash\{q\})\right) + \left(\sum_{F \text{ facet of } CH(S)} \sum_{q\in S} \mathbb{1}_{q\in F}\right) - s_1(S),$$

and since a face of $CH(S)$ has at least d vertices,

$$ns_0(S) \geqslant \left(\sum_{q\in S} s_0(S\backslash\{q\})\right) + ds_0(S) - s_1(S).$$

This inequality is actually an equality if $d = 2$ but not if $d \geqslant 3$. Taking n points chosen randomly and independently from $\mathcal{P}$, then deleting one of these points, chosen randomly with equiprobability, is the same as taking $n-1$ chosen randomly and independently from $\mathcal{P}$. We can thus average over all choices of S and obtain

$$ns_0(n) \geqslant ns_0(n-1) + d\ s_0(n) - s_1(n),$$

which implies Inequality (5).

Now, let $r \leqslant n$ and let R be a random subset of S of size r, chosen uniformly among all such subsets. For $k \leqslant \frac{r}{2}$, a k-set of S is a 0-set of R if R contains that k-set and none of the k points it cuts off. For each fixed k-set A of S, there are therefore $\binom{n-d-k}{r-d}$ distinct choices of R in which A is a 0-set. Counting the expected number of 0-sets and 1-sets of S that remain/become a 0-set in R, and ignoring those 0-sets of R that were k-sets of S for $k \geqslant 2$, we obtain:

$$E[s_0(R)] \geqslant \frac{\binom{n-d}{r-d}}{\binom{n}{r}} s_0(S) + \frac{\binom{n-d-1}{r-d}}{\binom{n}{r}} s_1(S).$$

Recall that the expectation is taken over all choices of a subset R of r elements of the fixed point set S. We can now average over all choices of a set S of n points taken, independently, from $\mathcal{P}$, and obtain Inequality (6). $\square$

Letting $p = \frac{r-d}{n-d}$, Inequality (6) yields the, perhaps more intuitive, inequality:

$$s_0\,(r) \geqslant p^d s_0(n) + p^d(1-p)s_1(n).$$

We can now prove our main result.

Theorem 6. *Let Z_n denote the convex hull of n points chosen independently from $\mathcal{P}$. If $E[f_{d-1}(Z_n)] \approx An^c$ for some A, $c > 0$ then there exists an integer n_0 such that for any $n \geqslant n_0$ we have $E[f_{d-1}(Z_{n+1})] > E[f_{d-1}(Z_n)]$.*

Proof of Theorem 6. Let $\mathcal{P}$ be a probability distribution on $\mathbb{R}^d$ and let $s_k(n)$ denote the expected number of k-sets in a set of n points chosen independently from $\mathcal{P}$. Recall that s_0 counts the expected number of facets in the convex hull of n points chosen independently and uniformly from $\mathcal{P}$. By Inequality (5), for s_0 to be increasing it suffices that $s_1(n)$ be bounded from above by $ds_0(n)$.

Let $\psi(r,n) = \frac{s_0(r)}{s_0(n)}$. Substituting into Inequality (6),

$$\psi(r,n)s_0(n) \geqslant \frac{\binom{n-d}{r-d}}{\binom{n}{r}} s_0(n) + \frac{\binom{n-d-1}{r-d}}{\binom{n}{r}} s_1(n),$$

which rewrites as

$$s_1(n) \leqslant \frac{\binom{n-d}{r-d}}{\binom{n-d-1}{r-d}} \left(\psi(r,n) \frac{\binom{n}{r}}{\binom{n-d}{r-d}} - 1 \right) s_0(n) \tag{7}$$

We let $q = \frac{n-d}{r-d}$. Developing the binomial expressions:

$$\frac{\binom{n-d}{r-d}}{\binom{n-d-1}{r-d}} = \frac{n-d}{n-r} = \frac{q}{q-1} \qquad \text{and} \qquad \frac{\binom{n}{r}}{\binom{n-d}{r-d}} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdots \frac{n-d+1}{r-d+1}$$

And for $0 \leqslant k < d$ we have $\frac{n-k}{r-k} < \frac{n-d}{r-d} = q$. Thus:

$$s_1(n) \leqslant q \frac{\psi(r,n)\frac{n}{r} q^{d-1} - 1}{q-1} s_0(n)$$

Assume now that we know a function g such that $s_0(n) \approx g(n)$. Then for any $\frac{1}{2} > \epsilon > 0$, there is $N_\epsilon \in \mathbb{N}$ such that for all $n > r > N_\epsilon$ we have:

$$\psi(r,n) = \frac{s_0(r)}{s_0(n)} < \left(\frac{1+\epsilon}{1-\epsilon}\right) \frac{g(r)}{g(n)} < (1+4\epsilon) \frac{g(r)}{g(n)}$$

which gives:

$$s_1(n) \leqslant q \frac{\frac{g(r)/r}{g(n)/n} q^{d-1} - 1}{q-1} s_0(n) + 4\epsilon \left(\frac{g(r)/r}{g(n)/n}\right) \frac{q^d}{q-1} s_0(n). \tag{8}$$

In the case where $s_0(n) \approx An^c$, we have:

$$\frac{g(r)/r}{g(n)/n} = \left(\frac{n}{r}\right)^{1-c} < q^{1-c}$$

And plugging back in Equation (8), we get:

$$s_1(n) \leqslant q \frac{q^{d-c} - 1}{q-1} s_0(n) + 4\epsilon \frac{q^{d+1}}{q-1} s_0(n) \tag{9}$$

The expression $q\frac{q^{d-c}-1}{q-1}$ converges toward $d-c$ when q approaches 1. Thus, there exists ϵ_d such that for all $1 < q < 1 + \epsilon_d$:

$$q \frac{q^{d-c} - 1}{q-1} < d - \frac{c}{2}$$

The second term of Equation (9) is bounded for all $1 + \frac{\epsilon_d}{2} < q < 1 + \epsilon_d$ by:

$$4\epsilon \frac{q^{d+1}}{q-1} s_0(n) < 8\epsilon \frac{(1+\epsilon_d)^{d+1}}{\epsilon_d} s_0(n)$$

Finally, let $\epsilon = \frac{c\epsilon_d}{32(1+\epsilon_d)^{d+1}}$. For all r such that:

$$N_\epsilon < r < n \qquad \text{and} \qquad 1 + \frac{\epsilon_d}{2} < \frac{n-d}{r-d} < \epsilon_d \tag{10}$$

we have:

$$s_1(n) < (d - \frac{c}{4})s_0(n)$$

With Inequality (5) this implies that $s_0(n) > s_0(n-1)$.

It remains to check that for n large enough, there always exists r satisfying Condition (10). We can rewrite condition (10) as:

$$d + \frac{n-d}{1+\epsilon_d} < r < d + \frac{n-d}{1+\epsilon_d/2}$$

In particular, there exists an integer r satisfying this condition as soon as $\frac{n-d}{1+\epsilon_d/2} - \frac{n-d}{1+\epsilon_d} > 1$. Thus, as soon as $n > max(N_\epsilon, d + \frac{1}{\frac{1}{1+\epsilon_d/2} - \frac{1}{1+\epsilon_d}})$, Condition (10) is satisfied, which concludes the proof. □

Now, from equations (1) and (2) we can see that Theorem 6 holds for random polytopes K_n when K is smooth, but not when K is a polytope. This proves that for smooth K the expectation $E[f_{d-1}(K_n)]$ is asymptotically increasing, i.e. the first part of Theorem 2.

The genericity assumption on $\mathcal{P}$ implies that the convex hull Z_n of n points chosen independently from $\mathcal{P}$ is almost surely simplicial. Thus, in Z_n any $(d-1)$-face is almost surely incident to exactly d faces of dimension $d-2$ and $f_{d-2}(Z_n) = \frac{d}{2} f_{d-1}(Z_n)$. Theorem 6 therefore implies that $f_{d-2}(Z_n)$ is asymptotically increasing, i.e. the second part of Theorem 2.. For $d = 3$, with Euler's relation this further implies that $f_0(Z_n)$ is asymptotically increasing.

Acknowledgment.

This work was initiated during the 10^{th} McGill - INRIA Workshop on Computational Geometry at the Bellairs institute. The authors wish to thank all the participants for creating a pleasant and stimulating atmosphere, in particular Sariel Har-Peled and Raimund Seidel for useful discussions at the early stage of this work. The authors also thank Imre Bárány and Pierre Calka for helpful discussions.

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Publisher
Inria
Domaine de Voluceau - Rocquencourt
BP 105 - 78153 Le Chesnay Cedex
inria.fr

ISSN 0249-6399