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Abstract

A graph G is *arbitrarily partitionable* (AP for short) if for every sequence $(n_1, ..., n_p)$ of positive integers summing up to $|V(G)|$ there exists a partition $(V_1, ..., V_p)$ of $V(G)$ such that $G[V_i]$ is a connected graph on n_i vertices for every $i \in \{1, ..., p\}$. We show that the Cartesian product $G \,\square\, K_l$ is AP whenever G is AP and K_l is a complete graph on $l \geqslant 1$ vertices.

1 Result

Please refer to [1] to understand our terminology and notation. We start with the following lemma.

Proposition 1:
Let $l \geqslant 1$ be a positive integer, and $\tau = (n_1, ..., n_p)$ be a sequence of positive integers such that $\|\tau\| \equiv 0 \bmod l$. If $p > l$, then τ can be divided into two non empty subsequences τ_1 and τ_2 such that $\|\tau_1\| \equiv 0 \bmod l$ and $\|\tau_2\| \equiv 0 \bmod l$.

Proof:
If τ contains an element n_i such that $n_i \equiv 0 \bmod l$, then if suffices to consider $\tau_1 = (n_i)$ and $\tau_2 = \tau - (n_i)$. Let us then consider that for every $i \in \{1, ..., p\}$, we have $n_i \not\equiv 0 \bmod l$. For every $x \in \{1, ..., p\}$, let $s_x = \sum_{i=1}^{x} n_i$ be the sum of the x first elements of τ. Because τ has more than l elements, there exist two values x_1 and x_2 such that $s_{x_1} \equiv s_{x_2} \bmod l$. Thus, $\tau_1 = \bigcup_{i=x_1+1}^{x_2} (n_i)$ and $\tau_2 = \tau - \tau_1$ satisfy our conditions ■

We will also be needing the following two connectivity theorems.

Theorem 1 ([4]):
If G and H are connected graphs, then the Cartesian product $G \,\Box\, H$ is $(k_1 + k_2)$-connected, where k_1 and k_2 are the connectivity of G and H, respectively.

Theorem 2 ([2, 3]):
If G is a k-connected graph, then G can be partitioned following every sequence of length at most k.

We now prove our main result.

Theorem 3:
The Cartesian product $G \,\Box\, K_l$ is AP whenever G is AP and $l \geqslant 1$.

Proof:
If $l = 1$, then $G \,\Box\, K_l$ is isomorphic to G and is AP by assumption. Let us thus consider that $l \geqslant 2$, and let consider any sequence $\tau = (n_1, ..., n_p)$ of positive integers summing up to $|V(G \,\Box\, K_l)|$. Since $G \,\Box\, K_l$ is l-connected by Theorem 1, we may also suppose that $|\tau| \geqslant l + 1$ since otherwise an obvious realization of τ in $G \,\Box\, K_l$ could be deduced thanks to Theorem 2.

By repeatedly applying Proposition 1, our sequence τ can be divided into $q \geqslant 2$ subsequences $\tau_1, ..., \tau_q$ such that $|\tau_i| \leqslant l$ and $\|\tau_i\| \equiv 0 \bmod l$ for every $i \in \{1, ..., q\}$. Let us put $\lambda_i = \frac{\|\tau_i\|}{l}$ for every $i \in \{1, ..., q\}$. These α_i's are integers, we have $\alpha_1 + ... + \alpha_q = |V(G)|$ and, because G is AP, there exists a realization $(V_1, ..., V_q)$ of $(\alpha_1, ..., \alpha_q)$ in G. Consider now U_i the extension of V_i in $G \,\Box\, K_l$ for every $i \in \{1, ..., q\}$ ($U_i = V_i^1 \cup ... \cup V_i^l$). Clearly, we have $|U_i| = \|\tau_i\|$ and, because $|\tau_i| \leqslant l$ and $(G \,\Box\, K_l)[U_i]$ is l-connected, there exists a realization R_i of τ_i in $(G \,\Box\, K_l)[U_i]$ for every $i \in \{1, ..., q\}$ according to Theorem 2. It follows that $\bigcup_{i=1}^{q} R_i$ is a realization of τ in $G \,\Box\, K_l$. ∎

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