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▶ To cite this version:

Sarah Ouadah. Uniform-in-bandwidth kernel estimation for censored data. 2012. hal-00766349

HAL Id: hal-00766349
https://hal.science/hal-00766349

Preprint submitted on 18 Dec 2012

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Uniform-in-bandwidth kernel estimation for censored data

Sarah Ouadah*

Laboratoire de Statistique Théorique et Appliquée
Université Pierre et Marie Curie–Paris 6
4 place Jussieu, 75252 Paris Cedex 05, France.

Draft version, September 27, 2012

Abstract

We present a sharp uniform-in-bandwidth functional limit law for the increments of the Kaplan-Meier empirical process based upon right-censored random data. We apply this result to obtain limit laws for nonparametric kernel estimators of local functionals of lifetime densities, which are uniform with respect to the choices of bandwidth and kernel. These are established in the framework of convergence in probability, and we allow the bandwidth to vary within the complete range for which the estimators are consistent. We provide explicit values for the asymptotic limiting constant for the sup-norm of the estimation random error.

AMS 2010 Classification: 62G15, 62G20, 62G30, 60F15, 60F17, 60N01, 60N02.

Keywords and phrases: Functional limit laws, nonparametric kernel density estimation, right censorship model, kernel lifetime density estimators, kernel failure rate estimators, Kaplan-Meier empirical process, uniform empirical process, convergence in probability.

1 Introduction and main results

1.1 An outline of our results

Let $X = X_1, X_2, \ldots$ be independent and identically distributed [iid] positive lifetimes jointly defined on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We assume that these random variables [rv] have common continuous distribution function [df] $F(\cdot) := \mathbb{P}(X \leq \cdot)$ and density $f(\cdot) := \frac{\partial F(\cdot)}{\partial \cdot}$, continuous and positive on $J := [A, B] \subseteq \mathbb{R}$. Denote by $C, C_1, C_2, \ldots$ iid positive censoring times independent of $X, X_1, X_2, \ldots$, with continuous df $G(\cdot) := \mathbb{P}(C \leq \cdot)$. Let $S_F := \sup\{x : F(x) < 1\}$ (resp. $S_G := \sup\{x : G(x) < 1\}$) be the upper endpoint of $F(\cdot)$ (resp. $G(\cdot)$), and fix $[A, B] \subseteq [0, \Theta]$, with $\Theta = \min(S_F, S_G) > 0$. In the right censorship model, the data set is given by the rv's $\{(T_i, \delta_i) : 1 \leq i \leq n\}$, where, for $i = 1, \ldots, n$,

$$\begin{cases} T_i = X_i \wedge C_i, \\ \delta_i = 1_{\{X_i \leq C_i\}}, \end{cases}$$

*E-mail: sarah.ouadah@upmc.fr

and 1_E denotes the indicator function of E. Our assumptions imply that T has df $H(\cdot) := \mathbb{P}(T \leq \cdot) = 1 - (1 - F(\cdot))(1 - G(\cdot))$. The nonparametric maximum likelihood estimators of $F(\cdot)$ and $G(\cdot)$ are the *product-limit* estimators introduced by Kaplan and Meier [19], and defined, for $x \in \mathbb{R}$, by (see, e.g., (1.1) and (1.2) in Deheuvels and Einmahl [9])

$$F_n(x) := 1 - \prod_{\substack{T_{i,n} \leq x \\ 1 \leq i \leq n}} \left\{ 1 - \frac{\delta_{i,n}}{n - i + 1} \right\} \tag{1}$$

and

$$G_n(x) := 1 - \prod_{\substack{T_{i,n} \leq x \\ 1 \leq i \leq n}} \left\{ 1 - \frac{1 - \delta_{i,n}}{n - i + 1} \right\}, \tag{2}$$

where, for all $n \geq 1$, $T_{1,n} < \ldots < T_{n,n}$ are the almost surely [a.s.] distinct order statistics of $T_1, \ldots, T_n$, and for each $i = 1, \ldots, n$, $\delta_{i,n}$ is the a.s. uniquely defined indicator δ_j for which $T_{i,n} = T_j$. The Kaplan-Meier empirical process $\{\alpha_n^{KM}(x) : x \in \mathbb{R}\}$ is given by

$$\alpha_n^{KM}(x) := n^{1/2}(F_n(x) - F(x)),$$

for $n \geq 1$ and $x \in \mathbb{R}$. For each *bandwidth* $h \geq 0$ and $t \in \mathbb{R}$, introduce the increment function

$$\xi_n^{KM}(h, t; s) := \alpha_n^{KM}(t + sh) - \alpha_n^{KM}(t), \quad s \in \mathbb{R}. \tag{3}$$

In §1.2 below, we present a limit law for the nonparametric kernel estimator of the lifetime density, which is uniform with respect to the choices of bandwidth and kernel (see, e.g., Theorem 1). This first result follows from a functional limit law for the increments of the Kaplan-Meier empirical process, which is stated in §1.3 (see, e.g., Theorem 2). Proofs of Theorems 1–2 are postponed until §2.1 and §2.2. We shall make an instrumental use of a functional limit law due to Deheuvels and Ouadah [12] which is described in §2.2.1. In §2.2.2–§2.2.5, we present some preliminaries needed in our proofs. In §3, we give some further applications of Theorem 1. We expose a generalization of Theorem 1 to kernel estimators of local functionals of lifetime densities in §3.1 (see, e.g., Theorem 3). As a consequence of this last result, we provide a limit law for the kernel failure rate estimator in §3.2 (see, e.g., Theorem 4). In §3.3, we construct uniform asymptotic certainty bands for these kernel estimators (see, e.g., Corollary 2).

1.2 Kernel lifetime density estimation

Consider the right censorship model of §1.1. Let $\mathcal{K}$ denote a collection of *kernels*, namely right-continuous functions $K(\cdot)$ on $\mathbb{R}$, of bounded variation and compact support on $\mathbb{R}$, such that $\int_{\mathbb{R}} K(t)dt = 1$ and there exists an $0 < M < \infty$ such that $\sup_{K \in \mathcal{K}} \int_{\mathbb{R}} |dK| \leq M$. The kernel estimator of $f(\cdot)$ (see, e.g., Watson and Leadbetter [27, 28], Tanner and Wong [25]) is defined, for each $K \in \mathcal{K}$, $h > 0$ and $x \in \mathbb{R}$, by

$$f_{n,K,h}(x) := \frac{1}{h} \int_{-\infty}^{\infty} K\left(\frac{x - t}{h}\right) dF_n(t), \tag{4}$$

where $F_n(\cdot)$ is as in (1). Fix a non degenerate interval $I := [C, D] \subset J$. Theorem 1 below, describing the *uniform in bandwidth and kernel* consistency of $f_{n,K,h}(\cdot)$, will be shown to follow from a functional limit law stated in Theorem 2 in the forthcoming §1.3.

Theorem 1 *Let $0 < a_n \leq b_n \leq 1$ be such that, as $n \to \infty$,*

$$b_n \to 0 \quad \textit{and} \quad \frac{na_n}{\log n} \to \infty. \tag{5}$$

Then, with $\mathcal{H}_n = [a_n, b_n]$, we have, as $n \to \infty$,

$$\sup_{h\in\mathcal{H}_n} \sup_{K\in\mathcal{K}} \Bigg| \left\{ \frac{nh}{2\log_+(1/h)} \right\}^{1/2} \sup_{x\in I} \Bigg(\pm \{ f_{n,K,h}(x) - \mathbb{E}(f_{n,K,h}(x)) \} \times \left\{ \frac{1-G(x)}{f(x)} \right\}^{1/2} \Bigg) - \left\{ \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2} \Bigg| = o_{\mathbb{P}}(1). \tag{6}$$

Remark 1 1°) *It easy to see that, under* (5)*, the limit law* (6) *holds with the formal replacement of $\pm \{ f_{n,K,h}(x) - \mathbb{E}(f_{n,K,h}(x)) \}$ by $|f_{n,K,h}(x) - \mathbb{E}(f_{n,K,h}(x))|$.*

2°) *Weighted versions of* (6)*, in the spirit of Theorem 3 in* §3.1*, may be obtained by the same arguments.*

3°) *Our theorem provides uniform asymptotic certainty bands for $\mathbb{E}(f_{n,K,h}(\cdot))$, in the spirit of that given in Deheuvels [5] (see, e.g., Corollary 2 in* §3.3*).*

We discuss below, the motivation and relevance of Theorem 1, with respect to the literature on functional estimation. *Uniform-in-bandwidth* results such as that given in Theorem 1, are motivated by the need of describing the limiting behavior of kernel estimators when their bandwidth is possibly random or data-dependent. Many elaborate schemes have been proposed in the statistical literature for constructing bandwidth sequences with asymptotically optimal properties (see, e.g., sections 2.4.1 and 2.4.2 in Deheuvels and Mason [11], and Berlinet and Devroye [1]). The use of bandwidths h of the form $h_n := Z_n n^{-1/5}$ where Z_n is a random sequence, stochastically bounded away from 0 and ∞, is often suggested. It turns out that Theorem 1 allows the description of the limiting behavior of the corresponding kernel lifetime density estimators. In the uncensored case, we refer to Einmahl and Mason [15], and Deheuvels and Ouadah [12], for discussions and references on this subject. We should mention that some authors (see, e.g., Epanechnikov [16], Marron and Nolan [21]) have introduced optimal choices of kernels (in a minimum variance sense) such as the *Epanechnikov* kernel, or *canonical kernels*, wich fall into $\mathcal{K}$, the general class of kernels we consider. To illustrate the sharpness of the conditions (5) implying (6), we set $\mathcal{H}_n = [h_n, h_n]$ in Theorem 1, and observe that, whenever $\{h_n : n \geq 1\}$ are constants fulfilling, as $n \to \infty$,

$$nh_n/\log n \to \infty, \text{ and } h_n \to 0, \tag{7}$$

and with a fixed kernel function $K \in \mathcal{K}$, then, as $n \to \infty$,

$$\left\{ \frac{nh_n}{2\log_+(1/h_n)} \right\}^{1/2} \sup_{x\in I} \Bigg(\pm \{ f_{n,K,h_n}(x) - \mathbb{E}(f_{n,K,h_n}(x)) \} \times \left\{ \frac{1-G(x)}{f(x)} \right\}^{1/2} \Bigg) \overset{\mathbb{P}}{\to} \left\{ \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2}. \tag{8}$$

Almost sure versions of (8) have been established, under various sets of assumptions, by Diehl and Stute [13] (for $c = \infty$), Deheuvels and Einmahl [8, 9], and Giné and Guillou [18]. We note that (8) and hence (6) do not hold almost surely for arbitrary choices of the continuous density $f(\cdot)$ on J, and bandwidth sequences $\{h_n : n \geq 1\}$ fulfilling (7). If we assume, in addition to (7), that

$$\log(1/h_n)/\operatorname{loglog} n \to c \in (0, \infty], \quad h_n \downarrow 0, \text{ and } nh_n \uparrow \infty, \tag{9}$$

then, setting $(c+1)/c := 1$ when $c = \infty$, by Theorem 1.1, pp. 1304-1305 in Deheuvels and Einmahl [9], we get, a.s.,

$$\begin{aligned}
&\limsup_{n\to\infty} \left\{ \frac{nh_n}{2\{\log_+(1/h_n) + \operatorname{loglog} n\}} \right\}^{1/2} \\
&\times \sup_{x\in I} \pm \{f_{n,h_n}(x) - \mathbb{E}(f_{n,h_n}(x))\} \left\{ \frac{1-G(x)}{f(x)} \right\}^{1/2} \\
&= \left(\frac{c+1}{c} \right)^{1/2} \left\{ \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
&\liminf_{n\to\infty} \left\{ \frac{nh_n}{2\{\log_+(1/h_n) + \operatorname{loglog} n\}} \right\}^{1/2} \\
&\times \sup_{x\in I} \pm \{f_{n,h_n}(x) - \mathbb{E}(f_{n,h_n}(x))\} \left\{ \frac{1-G(x)}{f(x)} \right\}^{1/2} \\
&= \left\{ \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2}.
\end{aligned}$$

This last result is known not to hold in general when the first condition in (9) is not fulfilled. Viallon [26] (see, e.g., Maillot and Viallon [20] [26]) has used the theory of *empirical processes indexed by functions* to obtain a uniform-in-bandwidth convergence theorem in the spirit of (6), without the condition of uniformity with respect to kernels. He showed that, for a specified $K \in \mathcal{K}$, we have a.s. as $n \to \infty$,

$$\begin{aligned}
\sup_{h\in\mathcal{H}_n} \left\{ \frac{nh}{2\log_+(1/h)} \right\}^{1/2} \sup_{x\in I} \Bigg(&\pm \{f_{n,K,h}(x) - \mathbb{E}(f_{n,K,h}(x))\} \\
&\times \left\{ \frac{1-G(x)}{f(x)} \right\}^{1/2} \Bigg) \xrightarrow{\mathbb{P}} \left\{ \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2}.
\end{aligned} \tag{10}$$

Here $\mathcal{H}_n = [h'_n, h''_n]$, and h'_n, h''_n are sequences of constants fulfilling (7)–(9) together with the additionnal condition $h''_n \leq [(B-D) \wedge (1 - H_1(D))]$ for each $n \geq 1$, and $H_1(\cdot)$ is as in (29) below. Independently of the conditions imposed on $\mathcal{H}_n$ in either Viallon [26] or (9), which are more strenuous than (5), we should point out that this last result is a much weaker statement than (6). Indeed, the asymptotic limiting constant in (10) relies on a specific $h \in \mathcal{H}_n$, whereas the limit law (6) provides the value of the asymptotic limiting constant for the sup-norm of the estimation random error, uniformly over $h \in \mathcal{H}_n$, and over $K \in \mathcal{K}$.

1.3 A functional limit law

In this section, we provide a uniform-in-bandwidth functional limit law for the increments of the Kaplan-Meier empirical process. As a consequence of this result, we obtain a uniform-in-bandwidth limit law for the modulus of continuity of this process (see Theorem

2 and Corollary 1 below). Throughout, we will denote by $\psi(\cdot)$ a specified continuous and positive function on J. Examples of such functions are provided in §3.1. We assume that ψ_n is a locally of bounded variation measurable estimator of ψ such that, as $n \to \infty$,

$$\sup_{x \in I} |\psi_n(x)/\psi(x) - 1| \to 0 \text{ in probability.} \tag{11}$$

Set $\log_+ s := \log(s \vee e)$ for $s \in \mathbb{R}$. Let $0 < a_n \leq b_n \leq 1$, $n = 1, 2, \ldots$ be positive constants, and fix $\mathcal{H}_n := [a_n, b_n]$. We are concerned with the limiting behavior, as $n \to \infty$, of the set of functions

$$\mathcal{F}_{n,I}^{KM}(h, \psi_n) := \left\{ \frac{\xi_n^{KM}(h, t; \cdot)}{\sqrt{2h \log_+(1/h)}} \times \left\{ \psi_n(t) \times \frac{1 - G(t)}{f(t)} \right\}^{1/2} : t \in I \right\}, \tag{12}$$

where $h > 0$ is restricted to vary in $\mathcal{H}_n$. Denote, by $(B[0,1],\mathcal{U})$ (resp. $(AC[0,1],\mathcal{U})$) the set of bounded (resp. absolutely continuous) functions on $[0,1]$, endowed with the uniform topology $\mathcal{U}$, induced by the sup-norm $\|f\| := \sup_{u \in [0,1]} |f(u)|$. For each $\epsilon > 0$ and $f \in B[0,1]$, set $\mathcal{N}_\epsilon(f) := \{g \in B[0,1] : \|f - g\| < \epsilon\}$, and, for each $A \subseteq B[0,1]$, set $A^\epsilon := \bigcup_{f \in A} \mathcal{N}_\epsilon(f)$, with the convention that $\bigcup_\emptyset(\cdot) := \emptyset$. Define the corresponding Hausdorff set-distance of $A, B \subseteq B[0,1]$, by

$$\Delta(A, B) = \begin{cases} \inf\left\{\theta > 0 : A \subseteq B^\theta \text{ et } B \subseteq A^\theta\right\} & \text{whenever such a } \theta \text{ exists,} \\ \infty & \text{otherwise.} \end{cases} \tag{13}$$

For each $f \in AC[0,1]$, denote by $\dot{f}(u) = \frac{d}{du} f(u)$ the Lebesgue derivative of f for $u \in [0,1]$. Consider the Hilbert norm defined on $B[0,1]$ by

$$|f|_{\mathbb{H}} := \begin{cases} \left\{\int_0^1 \dot{f}(u)^2 du\right\}^{1/2} & \text{when} \quad f(0) = 0 \text{ and } f \in AC[0,1], \\ \infty & \text{otherwise.} \end{cases}$$

For each $\lambda > 0$, set

$$\mathbb{S}_\lambda := \{f \in B[0,1] : |f|_{\mathbb{H}} \leq \lambda\} = \{\lambda^{1/2} f : f \in \mathbb{S}_1\}. \tag{14}$$

Notice that $\mathbb{S}_1 =: \mathbb{S}$ is the unit ball of the reproducing kernel Hilbert space of the usual Wiener process on $[0,1]$, shown by Strassen [24] to be the limit set in the functional law of the iterated logarithm for Wiener processes. Given these notations, our main result is the following uniform-in-bandwidth functional limit law.

Theorem 2 *Let $0 < a_n \leq b_n \leq 1$ be such that, as $n \to \infty$,*

$$b_n \to 0 \quad \textit{and} \quad \frac{n a_n}{\log n} \to \infty. \tag{15}$$

Then, with $\mathcal{H}_n = [a_n, b_n]$, we have, as $n \to \infty$,

$$\sup_{h \in \mathcal{H}_n} \Delta\left(\mathcal{F}_{n;I}^{KM}(h, \psi_n), \mathbb{S}_\Lambda\right) = o_{\mathbb{P}}(1), \tag{16}$$

where

$$\Lambda := \sup_{x \in I} \psi(x). \tag{17}$$

Remark 2 1°) *In the uncensored case, where $G \equiv 0$, $\psi \equiv 1$ and with X following the uniform distribution on* $[0,1]$*, Theorem 2 reduces to Theorem 1* (i) *of Deheuvels and Ouadah [12].*

2°) *Deheuvels and Einmahl [8, 9] established functional limit laws in the spirit of* (16)*, but without the uniformity in bandwidth. They considered the case $\mathcal{H}_n = [h_n, h_n]$.*

3°) *We shall mention that Viallon [26] obtained a uniform-in-bandwidth functional limit law in the spirit of* (16)*, but under more stringent conditions than* (15) *(see, e.g., the previous discussion in section* §1.2*).*

For the interval I and for any $h > 0$, consider the statistic

$$\Omega_{n;I}^{\pm KM}(h) = \sup_{t \in I} \pm\{\alpha_n^{KM}(t+h) - \alpha_n^{KM}(t)\}.$$

We obtain the following corollary of Theorem 2.

Corollary 1 *Let $\mathcal{H}_n = [a_n, b_n]$ be as in Theorem 2. Then, as $n \to \infty$, we have*

$$\sup_{h \in \mathcal{H}_n} \left| \frac{\Omega_{n;I}^{\pm KM}(h)}{\sqrt{2h \log_+(1/h)}} - \sup_{t \in I} \left\{ \frac{f(t)}{1 - G(t)} \right\}^{1/2} \right| = o_{\mathbb{P}}(1).$$

Remark 3 1°) *Deheuvels and Einmahl [8, 9] have given limit laws in the same spirit, but without the uniformity in bandwidth.*

Proof. The proof being similar to that of Corollary 1 in Deheuvels and Ouadah [12], is omitted.□

2 Proofs

2.1 Proof of Theorem 1

We provide below a proof of Theorem 1. We will need the following analytical result in the spirit of Lemma 1 in Deheuvels [6] (see, e.g., Lemma 1 in Deheuvels and Ouadah [12]). Let $\mathcal{M}$ denote a subset of $B[-T,T]$, such that $\mathbb{S}_\lambda \subseteq \mathcal{M} \subseteq B[-T,T]$, $\lambda > 0$, and let Γ denote a non-empty class of mappings $\Theta : \mathcal{M} \to \mathbb{R}$, continuous with respect to the uniform topology on $\mathcal{M}$. We assume that Γ has the following equicontinuity property. For each $\epsilon > 0$, there exists an $\eta(\epsilon) > 0$ such that, for each $\phi \in \mathcal{M}$ and $g \in \mathcal{S}_\lambda$, we have

$$\|\phi - g\| < \eta(\epsilon) \Rightarrow \sup_{\Theta \in \Gamma} |\Theta(\phi) - \Theta(g)| < \epsilon.$$

Fact 1 *For each $\varepsilon > 0$, there exists an $\eta(\varepsilon) > 0$ such that, for any $\mathcal{F} \subseteq \mathcal{M}$, we have*

$$\Delta(\mathcal{F}, \mathbb{S}_\lambda) < \eta(\varepsilon) \quad \Rightarrow \quad \sup_{\Theta \in \Gamma} \left| \sup_{g \in \mathcal{F}} \Theta(g) - \sup_{f \in \mathbb{S}_\lambda} \Theta(f) \right| < \varepsilon. \tag{18}$$

Proof of Theorem 1. We follow some of the arguments of the proof of Theorem 2 in Deheuvels and Ouadah [12]. We reduce the proof to the case where for some $0 < T < \infty$, $\widetilde{K}(u) := K(-u) = 0$ for all $|u| \geq T$ and $K \in \mathcal{K}$. We need only (see, e.g., (4.2.5)–(4.2.6) in

Deheuvels and Mason [10] and (1.22) in Deheuvels and Ouadah [12]) consider the limiting behavior of

$$
\begin{aligned}
&n^{1/2}h\left(f_{n,K,h}(x)-\mathbb{E}\left(f_{n,K,h}(x)\right)\right)\times\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2}\\
&=-\int_{-T}^{T}\left(\alpha_n^{KM}(x+hu)-\alpha_n^{KM}(x)\right)\times\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2}d\widetilde{K}(u),
\end{aligned}\tag{19}
$$

for $h\in\mathcal{H}_n$, $K\in\mathcal{K}$, and with x varying within $I=[C,D]$. Observe, via (19) and (3), that

$$
\begin{aligned}
&n^{1/2}h\left(f_{n,K,h}(x)-\mathbb{E}\left(f_{n,K,h}(x)\right)\right)\times\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2}\\
&=-\int_{-T}^{T}\xi_n^{KM}(h;x;u)\times\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2}d\widetilde{K}(u).
\end{aligned}
$$

It follows from Theorem 2 that, for each $\eta_0>0$, we have, as $n\to\infty$,

$$
\mathbb{P}\left(\sup_{h\in\mathcal{H}_n}\Delta\left(\left\{\frac{\xi_n^{KM}(h;x;\cdot\,)}{\sqrt{2h\log_+(1/(h))}}\times\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2}:x\in I\right\},\mathbb{S}_1\right)\geq\eta_0\right)\to 0.\tag{20}
$$

By Fact 1, taken with $\Theta(g):=\int_{-T}^{T}\mp g(u)d\widetilde{K}(u)$ and $\mathcal{M}$ being the set of all integrable functions $g(\cdot\,)$ on $[-T,T]$ with $g(0)=0$, for each $\varepsilon>0$, there exists an $\eta>0$ fulfilling (18). If we set $\eta_0=\eta$ in (20), we infer from (18) and (20) that, as $n\to\infty$,

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{h\in\mathcal{H}_n}\sup_{\widetilde{K}\in\mathcal{K}}\left|\sup_{x\in I}\left(\frac{\pm n^{1/2}h\left\{f_{n,\widetilde{K},h}(x)-\mathbb{E}\left(f_{n,\widetilde{K},h}(x)\right)\right\}}{\sqrt{2h\log_+(1/(h))}}\right.\right.\right.\\
&\left.\left.\left.\times\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2}\right)-\sup_{f\in\mathbb{S}_1}\int_{-T}^{T}\mp f(u)d\widetilde{K}(u)\right|\geq\varepsilon\right)\to 0.
\end{aligned}\tag{21}
$$

It is readily checked (see, e.g., (4.2.11) in Deheuvels and Mason [10]) that

$$
\sup_{f\in\mathbb{S}_1}\int_{-T}^{T}\mp f(u)d\widetilde{K}(u)=\left\{\int_{\mathbb{R}}K^2(u)du\right\}^{1/2}.
$$

Therefore, we obtain that, for all $\varepsilon>0$, as $n\to\infty$,

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{h\in\mathcal{H}_n}\sup_{K\in\mathcal{K}}\left|\left\{\frac{nh}{2\log_+(1/h)}\right\}^{1/2}\sup_{x\in I}\left(\pm\left\{f_{n,K,h}(x)-\mathbb{E}\left(f_{n,K,h}(x)\right)\right\}\right.\right.\right.\\
&\left.\left.\left.\times\left\{\frac{1-G(x)}{f(x)}\right\}^{1/2}\right)-\left\{\int_{\mathbb{R}}K^2(t)dt\right\}^{1/2}\right|\geq\varepsilon\right)\to 0.
\end{aligned}\tag{22}
$$

This completes the proof of Theorem 1.□

2.2 Proof of Theorem 2

2.2.1 A functional limit law in the uncensored case

We will make use of a functional limit law due to Deheuvels and Ouadah [12]. The following notation is needed for the statement of this result, stated in Fact 2 below. Let $U_1, U_2, \ldots$ be iid random variables with a uniform distribution on $(0,1)$. Denote by

$$\mathbb{U}_n(u) := n^{-1}\#\{U_i \leq u : 1 \leq i \leq n\} \text{ for } u \in \mathbb{R}, \tag{23}$$

the empirical df based upon the first $n \geq 1$ of these observations, with # denoting cardinality. Let,

$$\alpha_n(u) := n^{1/2}\left(\mathbb{U}_n(u) - u\right) \text{ for } u \in \mathbb{R}, \tag{24}$$

denote the uniform empirical process. For each choice of $h > 0$ and $t \in [0,1]$, consider, the increment function

$$\xi_n(h;t;u) := \alpha_n(t + hu) - \alpha_n(t) \text{ for } u \in \mathbb{R}, \tag{25}$$

together with the set of functions, defined, for $h > 0$, by

$$\mathcal{F}_{n,\mathcal{I},\gamma}(h) := \left\{ \frac{\xi_n(\gamma h;t;\cdot)}{\sqrt{2h\log_+(1/h)}} : t \in [0, 1-h] \cap \mathcal{I} \right\}, \tag{26}$$

where $\gamma > 0$ and $\mathcal{I} := [r,s] \subseteq [0,1]$ is a specified interval, with $r < s$. The functional limit law stated in Fact 2 below, is a version of Theorem 1 (i) in Deheuvels and Ouadah [12].

Fact 2 *Assume that $0 < a_n \leq b_n \leq 1$ are such that, as $n \to \infty$,*

$$b_n \to 0 \quad \text{and} \quad \frac{na_n}{\log n} \to \infty. \tag{27}$$

Then, with $\mathcal{H}_n = [a_n, b_n]$, for any $\gamma > 0$ and $\mathcal{I} = [u,v] \subseteq [0,1]$ with $u < v$, we have, as $n \to \infty$,

$$\sup_{h \in \mathcal{H}_n} \Delta\left(\mathcal{F}_{n,\mathcal{I},\gamma}(h), \mathbb{S}_\gamma\right) = o_{\mathbb{P}}(1). \tag{28}$$

2.2.2 Notation

In this section, we will adopt some basic notation taken from Deheuvels and Einmahl [9]. For any locally of bounded variation function $L(\cdot)$ (possibly discontinuous), we set $L(x-) := \lim_{t\uparrow x} L(t)$ and $L(x+) := \lim_{t\downarrow x} L(t)$. The distribution function of T, denoted for $x \in \mathbb{R}$, by $H(x) = H(x+)$, may be decomposed into

$$H(x) = 1 - (1 - F(x))(1 - G(x)) =: H_1(x) + H_0(x),$$

with

$$H_1(x) := \mathbb{P}\left(T \leq x \text{ and } \delta = 1\right) = \int_0^x (1 - G_-(t))dF(t) = H_1(x+), \tag{29}$$

and

$$H_0(x) := \mathbb{P}\left(T \leq x \text{ and } \delta = 0\right) = \int_0^x (1 - F_-(t))dG(t) = H_0(x+).$$

The empirical counterparts of $H(\cdot)$, $H_1(\cdot)$ and $H_0(\cdot)$ are defined, for $x \in \mathbb{R}$, by

$$H_n(x) := n^{-1}\#\{T_i \le x : 1 \le i \le n\} =: H_{n,1}(x) + H_{n,0}(x), \text{ with}$$

$$H_{n,1}(x) := n^{-1}\#\{\delta_i T_i \le x : 1 \le i \le n\}$$

and

$$H_{n,0}(x) = n^{-1}\#\{(1-\delta_i) T_i \le x : 1 \le i \le n\}.$$

Consider the empirical processes

$$\mathcal{H}_{n,j}(x) := n^{1/2}\left(H_{n,j}(x) - H_j(x)\right) \text{ for } j = 0, 1 \text{ and } x \in \mathbb{R}. \tag{30}$$

Introduce the empirical cumulated failure rate function defined by

$$\Lambda_n(x) = \int_0^x \frac{1}{1 - H_{n-}(u)} dH_{n,1}(u) = \Lambda_n(x+) \text{ for } x \ge 0.$$

The Kaplan-Meier estimators $F_n(\cdot)$ and $G_n(\cdot)$ defined in (1) and (2) can be rewritten into, for $x \in \mathbb{R}$ (see, e.g., p.295 in Shorack and Wellner [22]).

$$\begin{aligned} F_n(x) &:= 1 - \prod_{\substack{T_{i,n} \le x \\ 1 \le i \le n}} \left\{1 - \frac{\delta_{i,n}}{n - i + 1}\right\} \\ &= \int_0^x \left(1 - F_{n-}(u)\right) d\Lambda_n(u) \\ &= \int_0^x \frac{1}{1 - G_{n-}(u)} dH_{n,1}(u), \end{aligned}$$

and likewise

$$G_n(x) := \int_0^x \frac{1}{1 - F_{n-}(u)} dH_{n,0}(u).$$

2.2.3 Some useful facts

First, we decompose the Kaplan-Meier empirical process into (see, e.g., (4.18) in Deheuvels and Einmahl [9])

$$\begin{aligned} \alpha_n^{KM}(x) &= \int_0^x \frac{1}{1 - G_{n-}(u)} d\mathcal{H}_{n,1}(u) + \int_0^x \frac{\beta_{n-}(u)}{1 - G_{n-}(u)} dF(u) \\ &=: \alpha_n'(x) + \alpha_n''(x). \end{aligned} \tag{31}$$

Throughout, we will work on the probability space of Deheuvels and Einmahl [8], defined via the following fact.

Fact 3 *On a suitably enlarged probability space* $(\Omega, \mathcal{A}, \mathbb{P})$*, it is possible to define* $\{X_n : n \ge 1\}$ *and* $\{Y_n : n \ge 1\}$ *jointly with a sequence* $\{U_n : n \ge 1\}$ *of iid random variables with a uniform distribution on* $(0, 1)$*, such that the following properties hold. We have, almost surely,*

$$H_{n,1}(x) = \mathbb{U}_n\left(H_1(x)\right) \textit{ for } 0 < H_1(x) < p$$

and

$$H_{n,0}(x) = \mathbb{U}_n\left(H_0(x) + p\right) - \mathbb{U}_n(p) \textit{ for } 0 < H_0(x) < 1 - p,$$

where $p = \mathbb{P}(\delta = 1)$ *and* $\mathbb{U}_n(\cdot)$ *is as in* (23).

In the next fact, (32) follows from the Dvoretzky, Kiefer and Wolfowitz type inequality for the Kaplan-Meier estimator (see, e.g., Theorem 2 in Földes and Rejtő [17], and Theorem 1 in Bitouzé et al. [2]). Denote by $\{\beta_n^{KM}(t) : t \in \mathbb{R}\}$ the Kaplan-Meier empirical process for censoring times, defined, for $n \geq 1$ and $x \in \mathbb{R}$, by

$$\beta_n^{KM}(x) := n^{1/2}(G_n(x) - G(x)).$$

Fact 4 *For any specified* $0 \leq R < \Theta$, *we have, for all* $n \geq 1$,

$$\sup_{0 \leq t \leq R} \left|\beta_n^{KM}(t)\right| = O_{\mathbb{P}}(1). \tag{32}$$

2.2.4 Preliminaries lemmas

In this section, we provide three lemmas in the spirit of Lemmas 4.1–4.3 in Deheuvels and Einmahl [9]. The first lemma allows us to evaluate the modulus of continuity of $\alpha_n'(\cdot)$. The second lemma shows that the oscillations of $\alpha_n''(\cdot)$ can be neglected in forthcoming evaluations needed in our proofs. The third lemma provides an approximation of the increments $\xi_n^{KM}(h,t;s)$ for any $h \in \mathcal{H}_n$. We work throughout on the probability space of Fact 3. In view of (24) and (30), we set,

$$\begin{aligned} \omega_{n,1}(h) &:= \sup_{\substack{s,t \in I \\ |t-s| \leq h}} |\mathcal{H}_{n,1}(t) - \mathcal{H}_{n,1}(s)| \\ &= \sup_{\substack{s,t \in I \\ |t-s| \leq h}} |\alpha_n(H_1(t)) - \alpha_n(H_1(s))|, \quad h > 0 \end{aligned} \tag{33}$$

and

$$\omega_{n,1}^* := \sup_{h \in \mathcal{H}_n} \frac{\omega_{n,1}(h)}{\sqrt{2h \log_+(1/h)}}. \tag{34}$$

Now, in view of (31), for $s, t \in \mathbb{R}$, consider (see, e.g., (4.25) in Deheuvels and Einmahl [9])

$$\begin{aligned} A_{n,1}(s,t) &:= \alpha_n'(t) - \alpha_n'(s) - \frac{1}{1 - G_-(s)} \int_s^t d\mathcal{H}_{n,1}(u) \\ &= \left(\frac{1}{1 - G_{n-}(t)} - \frac{1}{1 - G_-(s)}\right) \{\mathcal{H}_{n,1}(t) - \mathcal{H}_{n,1}(s)\} \\ &\quad - \int_s^t \{\mathcal{H}_{n,1}(u) - \mathcal{H}_{n,1}(s)\} d\left\{\frac{1}{1 - G_{n-}(u)}\right\}. \end{aligned} \tag{35}$$

Lemma 1 *We have, as* $n \to \infty$,

$$\sup_{h \in \mathcal{H}_n} \sup_{\substack{s,t \in I \\ |t-s| \leq h}} \frac{|A_{n,1}(s,t)|}{\sqrt{2h \log_+(1/h)}} = \omega_{n,1}^* \times o_{\mathbb{P}}(1). \tag{36}$$

Proof. Making use of Fact 4, for all $n \geq 1$, we get

$$\sup_{t \in I} \left|\frac{1}{1 - G_{n-}(t)} - \frac{1}{1 - G_-(t)}\right| = O_{\mathbb{P}}(n^{-1/2}). \tag{37}$$

Since $G(\cdot)$ is continuous on J, we see that, as $n \to \infty$,

$$\sup_{h \in \mathcal{H}_n} \sup_{\substack{s,t \in I \\ |t-s| \leq h}} \left|\frac{1}{1 - G_-(t)} - \frac{1}{1 - G_-(s)}\right| = o_{\mathbb{P}}(1). \tag{38}$$

By combining the definition (34) of $\omega^*_{n,1}$ with the observations (37) and (38), as $n \to \infty$, we get the relations

$$\begin{aligned}
&\sup_{h\in\mathcal{H}_n} \sup_{\substack{s,t\in I\\|t-s|\le h}} \frac{1}{\sqrt{2h\log_+(1/h)}} \left| \left(\frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_-(s)}\right) \{\mathcal{H}_{n,1}(t) - \mathcal{H}_{n,1}(s)\} \right| \\
\le\ & \omega^*_{n,1} \times \left\{ \sup_{t\in I} \left| \frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_-(t)} \right| + \sup_{h\in\mathcal{H}_n} \sup_{\substack{s,t\in I\\|t-s|\le h}} \left| \frac{1}{1-G_-(t)} - \frac{1}{1-G_-(s)} \right| \right\} \\
=\ & \omega^*_{n,1} \times \left\{ O_{\mathbb{P}}(n^{-1/2}) + o_{\mathbb{P}}(1) \right\} \\
=\ & \omega^*_{n,1} \times o_{\mathbb{P}}(1).
\end{aligned}$$

Likewise, we observe that, as $n \to \infty$,

$$\begin{aligned}
&\sup_{h\in\mathcal{H}_n} \sup_{\substack{s,t\in I\\|t-s|\le h}} \frac{1}{\sqrt{2h\log_+(1/h)}} \left| \int_s^t \{\mathcal{H}_{n,1}(u) - \mathcal{H}_{n,1}(s)\} d\left\{\frac{1}{1-G_{n-}(u)}\right\} \right| \\
\le\ & \omega^*_{n,1} \times \sup_{h\in\mathcal{H}_n} \sup_{\substack{s,t\in I\\|t-s|\le h}} \left| \frac{1}{1-G_{n-}(t)} - \frac{1}{1-G_{n-}(s)} \right| \\
=\ & \omega^*_{n,1} \times \left\{ 2O_{\mathbb{P}}(n^{-1/2}) + o_{\mathbb{P}}(1) \right\} \\
=\ & \omega^*_{n,1} \times o_{\mathbb{P}}(1).
\end{aligned}$$

We combine the two above inequalities to conclude (36). □

Lemma 2 *Fix any $0 < R < \Theta$. Then, for all $n \ge 1$, we have uniformely over all $0 \le s \le t \le R$,*

$$\left|\alpha''_n(t) - \alpha''_n(s)\right| = \left| \int_s^t \frac{\beta^{KM}_{n-}(u)}{1-G_{n-}(u)} dF(u) \right| = O_{\mathbb{P}}(1) \times |t-s|. \tag{39}$$

Proof. Set $c(R) = \sup_{0\le u\le R} |f(u)|$. Making use of Fact 4 , we obtain for all $n \ge 1$,

$$\begin{aligned}
\left| \int_s^t \frac{\beta^{KM}_{n-}(u)}{1-G_{n-}(u)} dF(u) \right| &\le \frac{1}{1-G_{n-}(R)} \times \sup_{0\le u\le R} \left|\beta^{KM}_{n-}(u)\right| \times \{F(t) - F(s)\} \\
&\le \frac{c(R)}{1-G(R)} \times O_{\mathbb{P}}(1) \times |t-s| = O_{\mathbb{P}}(1) \times |t-s|. \square
\end{aligned}$$

For each $h \ge 0$ and $t \in \mathbb{R}$, set

$$\begin{aligned}
\xi^{KM}_{n,1}(h;t;s) &:= \frac{1}{1-G_-(t)} \{\mathcal{H}_{n,1}(t+hs) - \mathcal{H}_{n,1}(t)\} \\
&= \frac{1}{1-G_-(t)} \{\alpha_n(H_1(t+hs)) - \alpha_n(H_1(t))\}, \text{ for } s \in \mathbb{R}.
\end{aligned} \tag{40}$$

Lemma 3 *As $n \to \infty$, we have*

$$\begin{aligned}
&\sup_{h\in\mathcal{H}_n} \sup_{t\in I} \frac{1}{\sqrt{2h\log_+(1/h)}} \left\| \xi^{KM}_n(h;t;\cdot) - \xi^{KM}_{n,1}(h;t;\cdot) \right\| \\
&= \omega^*_{n,1} \times o_{\mathbb{P}}(1) + O_{\mathbb{P}}\left(\sqrt{\frac{b_n}{2\log_+(1/a_n)}} \right) =: A_n + B_n.
\end{aligned} \tag{41}$$

Proof. In view of the definitions (3), (31), (35) and (40), observe that

$$\begin{aligned}
&\sup_{h\in\mathcal{H}_n}\sup_{t\in I}\left\|\xi_n^{KM}(h;t;\cdot\,)-\xi_{n,1}^{KM}(h;t;\cdot\,)\right\| \\
&\quad\leq \sup_{h\in\mathcal{H}_n}\sup_{\substack{t\in I\\ s\in[0,1]}}\left|\alpha_n''(t+hs)-\alpha_n''(t)\right| \\
&\qquad+\sup_{h\in\mathcal{H}_n}\sup_{\substack{t\in I\\ s\in[0,1]}}\left|A_{n,1}(t,t+sh)\right|,
\end{aligned}$$

and combine (36) of Lemma 1 with (39) of Lemma 2. This completes our proof.□

2.2.5 Approximations and a functional limit law

The purpose of this section is to approximate (3) the increment function of the Kaplan-Meier empirical process, by a specified increment function of the uniform empirical process (see, e.g., (25), Lemmas 4–5), in view of an application of (50) a new functional limit law, we provide in Lemma 6. For $\mathcal{I}=[u,v]\subseteq[0,1]$ with $u<v$, consider the statistic

$$\omega_n^{\pm}(h,\mathcal{I}) \quad := \quad \sup_{\substack{s,t\in\mathcal{I}\\ |t-s|\leq h}} \pm\{\alpha_n(t)-\alpha_n(s)\}. \tag{42}$$

In view of Fact 2, the following fact hold (see, e.g., Corollary 1 in Deheuvels and Ouadah [12]).

Fact 5 *Let $\mathcal{H}_n=[a_n,b_n]$ and $\mathcal{I}$ be as in Fact 2. Then, as $n\to\infty$, for any $\gamma>0$, we have*

$$\sup_{h\in\mathcal{H}_n}\left|\frac{\omega_n^{\pm}(\gamma h,\mathcal{I})}{\sqrt{2h\log_+(1/h)}}-\gamma^{1/2}\right|=o_{\mathbb{P}}(1). \tag{43}$$

Lemma 4 *When $\mathcal{H}_n=[a_n,b_n]$ verifies the assumption (27) of Fact 2, we have, as $n\to\infty$,*

$$\sup_{h\in\mathcal{H}_n}\sup_{t\in I}\frac{1}{\sqrt{2h\log_+(1/h)}}\left\|\xi_n^{KM}(h;t;\cdot\,)-\xi_{n,1}^{KM}(h;t;\cdot\,)\right\|=o_{\mathbb{P}}(1). \tag{44}$$

Proof. Set

$$\kappa=\max_{u\in I}\varphi(u):=\max_{u\in I}f(u)(1-G(u))>0. \tag{45}$$

By (29) and the mean value theorem, we have uniformly over $s,t\in I$,

$$|H_1(t)-H_1(s)|\leq\kappa|t-s|. \tag{46}$$

This inequality, when combined with the definitions (33) and (42) implies that for all $h\in\mathcal{H}_n$,

$$\omega_{n,1}(h)\leq\omega_n^{\pm}(\kappa h,\mathcal{I}).$$

Whence, by (34) the definition of $\omega_{n,1}^{*}$ and (43) of Fact 5, for each $\varepsilon_0>0$, as $n\to\infty$

$$\mathbb{P}\left(\omega_{n,1}^{*}\geq\varepsilon_0+\kappa^{1/2}\right)\to 0.$$

By combining the previous observation with the fact that by condition (27),

$$B_n=O_{\mathbb{P}}\left(\sqrt{\frac{b_n}{2\log_+(1/a_n)}}\right)=o_{\mathbb{P}}\left(1\right),$$

we see that, for each $\varepsilon > 0$, as $n \to \infty$,

$$\mathbb{P}\left(|A_n + B_n| \geq \varepsilon\right) \to 0,$$

with A_n defined in (41). Hence, we infer from (41) that, as $n \to \infty$, (44) holds.□

Let $N \geq 1$ be an arbitrary fixed integer. For $1 \leq i \leq N$, set $t_{i,N} = C + (i-1)N^{-1}(D-C)$, where $[C, D] = I$, and recall the definitions (25) of the increment of the uniform empirical process and (45) of the function $\varphi(\cdot)$.

Lemma 5 *When $\mathcal{H}_n = [a_n, b_n]$ verifies the assumption* (27) *of Fact* 2*, for all N sufficiently large, we have, as $n \to \infty$,*

$$\sup_{h \in \mathcal{H}_n} \sup_{t \in I} \frac{1}{\sqrt{2h \log_+(1/h)}} \left\| \xi_{n,1}^{KM}(h; t; \cdot) - \frac{\xi_n(\varphi(t_{i,N})h; H_1(t); \cdot)}{1 - G(t)} \right\| = o_{\mathbb{P}}(1). \tag{47}$$

Proof. Set

$$\epsilon_N := \max_{1 \leq i \leq n} \left(\sup_{t_{i,N} \leq t \leq t_{i+1,N} + h} |f(t)(1 - G(t)) - f(t_{i,N})(1 - G(t_{i,N}))| \right).$$

Making use of the mean value theorem in combination with the above definition, we see that, for all $1 \leq i \leq N$, $t \in [t_{i,N}, t_{i+1,N}]$ and $s \in [0, 1]$, for all large n,

$$|H_1(t + sh) - H_1(t) - s\varphi(t_{i,N})h| \leq \epsilon_N h.$$

Therefore, in view of the definitions (25)–(40), we obtain the inequality

$$\sup_{h \in \mathcal{H}_n} \sup_{t \in I} \frac{1}{\sqrt{2h \log_+(1/h)}} \left\| \xi_{n,1}^{KM}(h; t; \cdot) - \frac{\xi_n(\varphi(t_{i,N})h; H_1(t); \cdot)}{1 - G(t)} \right\|$$
$$\leq \left\{ \frac{1}{1 - G(D)} \right\} \times \sup_{h \in \mathcal{H}_n} \frac{\omega_n^{\pm}(\epsilon_N h, \mathcal{I})}{\sqrt{2h \log_+(1/h)}}.$$

This, when combined with (43) of Fact 5 and the fact that, by choosing N large enough, $\epsilon_N > 0$ may be rendered as small as desired, implies (47).□

Now, let R denote a continuous and positive function on J and define

$$M^{1/2} := \sup_{t \in I} R(t) \left\{ \frac{f(t)}{1 - G(t)} \right\}^{1/2}. \tag{48}$$

The next lemma concerns the joint in $h \in \mathcal{H}_n$ limiting behavior, as $n \to \infty$, of the set of functions

$$\mathcal{G}_{n,I}(h, R) := \left\{ \frac{R(t)}{1 - G(t)} \times \frac{\xi_n(\varphi(t_{i,N})h; H_1(t); \cdot)}{\sqrt{2h \log_+(1/h)}} : t \in I \right\}. \tag{49}$$

Lemma 6 *When $\mathcal{H}_n = [a_n, b_n]$ verifies the assumption* (27) *of Fact* 2*, we have, as $n \to \infty$,*

$$\sup_{h \in \mathcal{H}_n} \Delta\left(\mathcal{G}_{n,I}(h, R), \mathbb{S}_M\right) = o_{\mathbb{P}}(1). \tag{50}$$

Proof. Fix any $\epsilon, \epsilon_0 > 0$ and consider $I = [C, D]$. In view of (13) the definition of the Haussdorf set-distance, we need only prove that, for each $\epsilon > 0$, as $n \to \infty$,

$$(i) \quad \mathbb{P}\left(\mathcal{G}_{n,I}(h, R) \subseteq \mathbb{S}_M^{\epsilon} : \ \forall h \in \mathcal{H}_n\right) \to 1, \tag{51}$$

and

$$(ii) \quad \mathbb{P}\left(\mathbb{S}_M \subseteq \mathcal{G}_{n,I}(h, R)^{\epsilon} : \ \forall h \in \mathcal{H}_n\right) \to 1. \tag{52}$$

Recall the set of functions (26). Since $\{H_1(t) : t \in I\} \subseteq [0, 1]$, we observe that, for all $h \in \mathcal{H}_n$,

$$\left\{ \frac{\xi_n(\varphi(t_{i,N})h; H_1(t); \cdot\,)}{\sqrt{2h \log_+(1/h)}} : t \in I \right\} \subseteq \mathcal{F}_{n,\mathcal{I},\phi(t_{i,N})}(h),$$

so that, as a consequence of (28) of Fact 2, for each $t \in I$ and for each $h \in \mathcal{H}_n$, there exists a function $g \in \mathbb{S}$ (see definition (14)), such that

$$\left\| \frac{\xi_n(\varphi(t_{i,N})h; H_1(t); \cdot\,)}{\sqrt{2h \log_+(1/h)}} - \varphi(t_{i,N})^{1/2} g(\cdot\,) \right\| < \epsilon_0 := \epsilon \times \frac{1 - G(t)}{R(t)},$$

which entails that, for each $t \in [t_{i,N}, t_{i+1,N}]$, $1 \leq i \leq N$ and for each $h \in \mathcal{H}_n$,

$$\left\| \frac{R(t)}{1 - G(t)} \frac{\xi_n(\varphi(t_{i,N})h; H_1(t); \cdot\,)}{\sqrt{2h \log_+(1/h)}} - \frac{R(t)}{1 - G(t)} \left\{ f(t_{i,N})(1 - G(t_{i,N}) \right\}^{1/2} g(\cdot\,) \right\| < \epsilon.$$

Therefore, by the above assertion and (48)–(49), the assertion (51)(i) holds. We now establish (51)(ii). Considering a function $g_i \in \mathbb{S}_{\varphi(t_{i,N})}$, $1 \leq i \leq N$ and each $h \in \mathcal{H}_n$, by (28), for all $\epsilon > 0$, there exists a $t_0 \in I$ such that

$$\left\| \frac{\xi_n(\varphi(t_{i,N})h; t_0; \cdot\,)}{\sqrt{2h \log_+(1/h)}} - g_i(\cdot\,) \right\| < \epsilon \times \frac{1 - G(t)}{R(t)},$$

with a fixed $t \in [t_{i,N}, t_{i+1,N}]$, $1 \leq i \leq N$. That implies, for each $h \in \mathcal{H}_n$,

$$\left\| \frac{R(t)}{1 - G(t)} \frac{\xi_n(\varphi(t_{i,N})h; t_0; \cdot\,)}{\sqrt{2h \log_+(1/h)}} - \frac{R(t)}{1 - G(t)} g_i(\cdot\,) \right\| < \epsilon.$$

Consider the function $g^*(\cdot\,) = \frac{R(t)}{1-G(t)} g_i(\cdot\,), t \in [t_{i,N}, t_{i+1,N}], 1 \leq i \leq N$. Since $g_i \in \mathbb{S}_{\varphi(t_{i,N})}$, we observe that $g* \in \mathbb{S}_M$. Then, we just choose $t_0 = H_1(t)$ to complete the proof of (51)(ii).□

2.2.6 Proof of Theorem 2

We have now in hand all the necessary ingredients for proving Theorem 2. We have the following relation

$$R(t) = \left\{ \psi(t) \times \frac{1 - G(t)}{f(t)} \right\}^{1/2} \Leftrightarrow \psi(t) = R(t)^2 \times \frac{f(t)}{1 - G(t)}.$$

Therefore, by (17) the definition of Λ and (48) the one of M, we see that

$$\Lambda = \sup_{t \in I} \psi(t) = M.$$

Considering the definitions (12)–(48) of the set of functions $\mathcal{F}_{n,I}^{KM}(h, \psi_n)$ and M, we shall combine the approximations (44) and (47) with the functional limit law (50). By (11), the theorem holds for $\psi(\cdot\,)$ replaced by $\psi_n(\cdot\,)$.□

3 Some applications

3.1 A generalization of Theorem 1

We provide below a more general setup of Theorem 1. Introduce the following examples of continuous and positive functions on J (see, e.g., (1.13) in Deheuvels and Einmahl [9]):

$$\begin{aligned} \psi^{(1)}(x) &= 1, \quad \psi^{(2)}(x) = \frac{1}{1-G(x)}, \\ \psi^{(3)}(x) &= f(x), \quad \psi^{(4)}(x) = \frac{f(x)}{1-G(x)}, \\ \psi^{(5)}(x) &= \frac{f(x)\varphi(x)}{(1-F(x))^2(1-G(x))}, \end{aligned} \tag{53}$$

where $\varphi(\cdot)$ is an auxiliary continuous and positive function on J. We shall obtain $\psi_n(\cdot)$ a locally of bounded variation measurable estimator of ψ by replacing in (53) the functions $f(\cdot)$, $F(\cdot)$ and $G(\cdot)$ by $f_{n,K,h}(\cdot)$, $F_n(\cdot)$ and $G_n(\cdot)$, respectively. Recall that $\psi_n(\cdot)$ is such that, as $n \to \infty$,

$$\sup_{x \in I} |\psi_n(x)/\psi(x) - 1| \to 0 \text{ in probability.}$$

The next theorem describes the uniform in bandwidth and kernel consistency of a series of kernel estimators of local functionals of lifetime densities.

Theorem 3 *Let $\mathcal{H}_n = [a_n, b_n]$ be as in Theorem 2. Then, as $n \to \infty$,*

$$\begin{aligned} \sup_{h \in \mathcal{H}_n} \sup_{K \in \mathcal{K}} \Bigg| \left\{ \frac{nh}{2\log_+(1/h)} \right\}^{1/2} \sup_{x \in I} \Bigg(& \pm \{ f_{n,K,h}(x) - \mathbb{E}(f_{n,K,h}(x)) \} \\ & \times \left\{ \psi_n(x) \times \frac{1-G(x)}{f(x)} \right\}^{1/2} \Bigg) - \sigma(\psi, K) \Bigg| = o_{\mathbb{P}}(1), \end{aligned} \tag{54}$$

with $\sigma(\psi, K) := \left\{ \sup_{x \in I} \psi(x) \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2}$.

Remark 4 1°) *The replacement of $F(\cdot)$ by $F_n(\cdot)$ in $\psi^{(5)}(\cdot)$, defined in (53), corresponds to an estimator of the hazard rate function $\lambda_{n,K,h}(\cdot)$ wich will be considered in §3.2. Our results also apply to this estimator.*

Proof of Theorem 3. The proof is essentially identical to the proof of Theorem 1, taking into account the function $\psi(\cdot)$ and its estimator $\psi_n(\cdot)$.□

3.2 Kernel failure rate estimation

Denote the failure rate function pertaining to $F(\cdot)$ by

$$\lambda(x) := \frac{f(x)}{1-F(x)} \text{ for } x \in \mathbb{R}. \tag{55}$$

We consider $\lambda_{n,K,h}(\cdot)$ the estimator of $\lambda(\cdot)$ defined, for $K \in \mathcal{K}$, $h > 0$ and $x \in \mathbb{R}$, by

$$\lambda_{n,K,h}(x) := \frac{f_{n,K,h}(x)}{1-F_n(x)}, \tag{56}$$

where $f_{n,K,h}(\cdot)$ is as in (4) and $F_n(\cdot)$ as in (1). The following theorem, describing the uniform in bandwidth and kernel consistency of $\lambda_{n,K,h}(\cdot)$, follows from Theorem 3.

Theorem 4 *Let $0 < a_n \leq b_n \leq 1$ be such that, as $n \to \infty$,*

$$b_n \to 0 \quad \textit{and} \quad \frac{na_n}{\log n} \to \infty.$$

Then, with $\mathcal{H}_n = [a_n, b_n]$, we have, as $n \to \infty$,

$$\sup_{h\in\mathcal{H}_n} \sup_{K\in\mathcal{K}} \left| \left\{ \frac{nh}{2\log_+(1/h)} \right\}^{1/2} \sup_{x\in I} \pm \left(\lambda_{n,K,h}(x) - \frac{\mathbb{E}\left(\lambda_{n,K,h}(x)\right)}{1-F(x)} \right.\right.$$
$$\left.\left. \times \left\{ \psi_n(x) \times \frac{1-H(x)}{\lambda(x)} \right\}^{1/2} \right) - \sigma(\psi, K) \right| = o_{\mathbb{P}}(1),$$

with $\sigma(\psi, K) := \left\{\sup_{x\in I} \psi(x) \int_{\mathbb{R}} K^2(t)dt\right\}^{1/2}$.

Remark 5 1°) *The uniform consistency of $\lambda_{n,K,h}(\cdot)$ over bounded intervals was investigated in Zhang [29], and Deheuvels and Einmahl [9].*

2°) *Our theorem can be used to construct uniform asymptotic certainty bands for $\lambda(\cdot)$, in the spirit of that given in Deheuvels [5] (see, e.g., §3.3).*

Proof of Theorem 4. We will make use of the next fact, which is a Dvoretzky, Kiefer and Wolfowitz type inequality for the Kaplan-Meier empirical process (see, e.g., Theorem 2 in Földes and Rejtő [17], and Theorem 1 in Bitouzé et al. [2]).

Fact 6 *For any specified $0 \leq R < \Theta$, we have, for all $n \geq 1$,*

$$\sup_{0\leq t\leq R} \left|\alpha_n^{KM}(t)\right| = O_{\mathbb{P}}(1). \tag{57}$$

Consider first the relation (22) in the proof of Theorem 3, in which we include $\psi_n(\cdot)$. For all $\varepsilon > 0$, as $n \to \infty$, we have

$$\mathbb{P}\left(\sup_{h\in\mathcal{H}_n} \sup_{K\in\mathcal{K}} \left| \left\{ \frac{nh}{2\log_+(1/h)} \right\}^{1/2} \sup_{x\in I} \left(\pm \left\{ f_{n,K,h}(x) - \mathbb{E}\left(f_{n,K,h}(x)\right) \right\} \right.\right.\right.$$
$$\left.\left.\left. \times \left\{ \psi_n(x) \times \frac{1-G(x)}{f(x)} \right\}^{1/2} \right) - \left\{ \sup_{x\in I} \psi(x) \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2} \right| \geq \varepsilon \right) \to 0.$$

We shall make the formal replacement of $\psi_n(\cdot)$ by

$$\left\{ \frac{1-F(\cdot)}{1-F_n(\cdot)} \right\}^2 \psi_n(\cdot).$$

Thus, by (55)–(56) the definitions of $\lambda(\cdot)$ and $\lambda_{n,K,h}(\cdot)$, and the relation $H \equiv 1-(1-F)(1-G)$, we obtain, as $n \to \infty$,

$$\mathbb{P}\left(\sup_{h\in\mathcal{H}_n} \sup_{K\in\mathcal{K}} \left| \left\{ \frac{nh}{2\log_+(1/h)} \right\}^{1/2} \sup_{x\in I} \left(\pm \left\{ \lambda_{n,K,h}(x) - \frac{\mathbb{E}\left(f_{n,K,h}(x)\right)}{1-F_n(x)} \right\} \right.\right.\right.$$
$$\left.\left.\left. \times \left\{ \psi_n(x) \times \frac{1-H(x)}{\lambda(x)} \right\}^{1/2} \right) - \left\{ \sup_{x\in I} \psi(x) \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2} \right| \geq \varepsilon \right) \to 0.$$

Observe that

$$\lambda_{n,K,h}(x) - \frac{\mathbb{E}\left(f_{n,K,h}(x)\right)}{1-F_n(x)} = \lambda_{n,K,h}(x) - \frac{\mathbb{E}\left(f_{n,K,h}(x)\right)\left(F_n(x)-F(x)\right)}{(1-F_n(x))(1-F(x))} - \frac{\mathbb{E}\left(f_{n,K,h}(x)\right)}{1-F(x)}.$$

We conclude by applying (57) to the second term in the right-hand side, combined with the fact that $h \in \mathcal{H}_n$ follows (5).□

3.3 Asymptotic certainty bands

We shall now show how Theorem 1 may be used to construct asymptotic certainty bands for $\mathbb{E}(f_{n,K,h}(\cdot))$ (see, e.g. p.232–233 in Deheuvels and Mason [11], Deheuvels and Derzko [7], Deheuvels [5]). Given $h \in \mathcal{H}_n$ fulfilling (5) a sequence of possibly data-dependent bandwidths and $K \in \mathcal{K}$, we consider positive possibly data-dependent functions of the form

$$L_{n,K,h}(x) := \left\{ \int_{\mathbb{R}} K^2(t)dt \right\}^{1/2} \times \left\{ \frac{2\log_+(1/h)}{nh} \right\}^{1/2} \times 1/\left\{ \frac{(1-G(x))}{f(x)} \right\}^{1/2}, \text{ for } x \in I.$$

It follows from Theorem 1, that for each choice of $h \in \mathcal{H}_n$ and $K \in \mathcal{K}$, as $n \to \infty$, we have

$$\sup_{x\in I} \pm \left\{ \frac{1}{L_{n,K,h}(x)} \right\} \{f_{n,K,h}(x) - \mathbb{E}(f_{n,K,h}(x))\} \xrightarrow{\mathbb{P}} 1. \tag{58}$$

Under (58), for each $0 < \varepsilon < 1$, we have, as $n \to \infty$,

$$\begin{aligned}&\mathbb{P}(\mathbb{E}f_{n,K,h}(x) \in [f_{n,K,h}(x) - (1+\varepsilon)L_{n,K,h}(x),\\ &\quad f_{n,K,h}(x) + (1+\varepsilon)L_{n,K,h}(x)], \text{ for all } x \in I, K \in \mathcal{K}, h \in \mathcal{H}_n) \to 1,\end{aligned}$$

$$\begin{aligned}&\mathbb{P}(\mathbb{E}f_{n,K,h}(x) \in [f_{n,K,h}(x) - (1-\varepsilon)L_{n,K,h}(x),\\ &\quad f_{n,K,h}(x) + (1-\varepsilon)L_{n,K,h}(x)], \text{ for all } x \in I, K \in \mathcal{K}, h \in \mathcal{H}_n) \to 0.\end{aligned}$$

Corollary 2 *When the two assertions above hold jointly for each $0 < \varepsilon < 1$, we obtain intervals which provide asymptotic certainty bands for $\mathbb{E}f_{n,K,h}(x)$ over $x \in I$ in the sense of Deheuvels and Mason [11].*

References

[1] Berlinet, A. and Devroye, L. A. (1994). Comparison of kernel density estimates. *Publ. Inst. Statist. Univ. Paris* **38**, 3–59.

[2] Bitouzé, D., Laurent, B. and Massart, P. (1999) A Dvoretzky-Kiefer-Wolfowitz type inequality for the Kaplan-Meier estimator. *Ann. Inst. H. Poincaré Probab. Statist.* **3**, 735–763.

[3] Bosq D. and Lecoutre J.P. (1987). Théorie de l'estimation fonctionnelle, Economica.

[4] Chung, K. (1949). An estimate concerning the Kolmogoroff limit distribution. *Trans. Amer. Math. Soc.* **67**, 36–50.

[5] Deheuvels, P. (2011). One bootstrap suffices to generate sharp uniform bounds in functional estimation. *Kybernetica*, **47**, 881–891.

[6] Deheuvels, P. (2012). Uniform-in-bandwidth functional limit laws for multivariate empirical processes. submitted.

[7] Deheuvels, P. and Derzko, G. (2008) Asymptotic certainty bands for kernel density estimators based upon a bootstrap resampling scheme. *Statistical models and methods for biomedical and technical systems.* 171–186, Stat. Ind. Technol., Birkhäuser Boston, Boston, MA.

[8] Deheuvels, P. and Einmahl, J. H. J. (1996) On the strong limiting behavior of local functionals of empirical processes based upon censored data. *Ann. Probab.* **24**, 504–525.

[9] Deheuvels, P. and Einmahl, J. H. J. (2000). Functional limit laws for the increments of Kaplan-Meier product-limit processes and applications. *Ann. Probab.* **28**, 1301–1335.

[10] Deheuvels, P. and Mason, D. M. (1992). Functional laws of the iterated logarithm for the increments of empirical and quantile processes. *Ann. Probab.* **20**, 1248–1287.

[11] Deheuvels, P. and Mason, D. M. (2004). General asymptotic confidence bands based on kernel-type function estimators. *Statist. Infer. Stoch. Processes.* **7**, 225–277.

[12] Deheuvels, P. and Ouadah, S. (2011). Uniform in bandwidth functionnal limit laws. *J. Theor. Probab.*, published online (DOI : 10.1007/s10959–011–0376–1).

[13] Diehl, S. and Stute, W. (1988). Kernel density and hazard function estimation in the presence of censoring. *J. Multivariate Anal.* **25**, 299–310.

[14] Dvoretzky, A., Kiefer, J. and Wolfowitz, J. (1956). Asymptotic minimax character of the sample distribution function and of the classical multinomial estimator. *Ann. Math. Statist.* **27**, 642–669.

[15] Einmahl, U. and Mason, D. M. (2005). Uniform in bandwidth consistency of kernel-type function estimators. *Ann. Statist.* **33**, 1380–1403.

[16] Epanechnikov, V.A. (1969). Non-parametric estimation of a multivariate probability density. *Theory of Probability and its Applications* **14**, 153–158.

[17] Földes, A. and Rejtő, L. (1981). A LIL type result for the product limit estimator. *Z. Wahrsch. Verw. Gebiete* **56**, 75–86.

[18] Giné, E. and Guillou, A. (2001). On consistency of kernel density estimators for randomly censored data: rates holding uniformly over adaptive intervals. *Ann. Inst. H. Poincaré Probab. Statist.* **37**, 503–522.

[19] Kaplan, E. L. and Meier, P. (1958). Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.* **53**, 457–481.

[20] Maillot, B. and Viallon, V. (2009). Uniform limit laws of the logarithm for nonparametric estimators of the regression function in presence of censored data. *Math. Methods Statist.* **18**, 159–184.

[21] Marron, J. S. and Nolan, D. (1988). Canonical kernels for density estimation. *Statist. Probab. Lett.* **7**, 195–199.

[22] Shorack, G. R. et Wellner, J. A. (1986) Empirical processes with applications to statistics. *Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics.* John Wiley and Sons.

[23] Silverman, B. W. (1986). Density estimation for statistics and data analysis. *Chapman and Hall*, London.

[24] Strassen, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeit. Verw. Gebiete.* **3**, 211–226.

[25] Tanner, M. A. and Wong, W.H. (1983). The estimation of the hazard function from randomly censored data by the kernel method. *Ann. Statist.* **11**, 989–993.

[26] Viallon, V. (2006). Processus empiriques, estimation non paramétrique et données censurées. *Doctoral Dissertation, Université Pierre et Marie Curie, Dec. 2, 2006.* Paris, France.

[27] Watson, G. S. and Leadbetter, M. R. (1964a). Hazard analysis. II. *Sankhya- Ser. A* **26**, 101–116

[28] Watson, G. S. and Leadbetter, M. R. (1964b). Hazard analysis. I. *Biometrika* **51**, 175–184.

[29] Zhang, B. (1996) A law of the iterated logarithm for kernel estimators of hazard functions under random censorship. *Scand. J. Statist.* **23**, 37–47.