Extension of Koiter's linear shell theory to materials exhibiting arbitrary symmetry

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Abstract: Koiter's linear shell theory applies to isotropic elastic materials and to anisotropic materials that exhibit reflection symmetry of the elastic properties with respect to the shell midsurface. To the extent that such shells are exceptional, classical linear shell theory is incomplete. This lacuna is addressed here through a systematic procedure, applicable equally to all kinds of material symmetry, which entails an expansion of the potential energy of the shell in powers of its thickness in a manner reminiscent of Koiter's work. The variables are the displacement field of the shell midsurface and certain director fields that arise in the course of the expansion procedure. The directors are constrained in accordance with necessary conditions arising in the exact three-dimensional theory, yielding the optimal expression for the potential energy among those of third order in the small thickness. For materials lacking reflection symmetry, it is found that the strain energy of the shell is sensitive to tangential gradients of strain in addition to the usual strain and bending strain of the conventional theory.

Keywords: shell theory, linear elasticity, strain gradients, functionally-graded materials

* *dedicated to J.G. Simmonds, Koiter medalist*

1. Introduction

Current research on the derivation of shell theory from three-dimensional elasticity emphasizes the method of gamma convergence [1], concerned with the variational problem that emerges in the small-thickness limit, or the method of asymptotic expansions applied to the weak forms of the three-dimensional equilibrium equations [2]. While these techniques have yielded rigorous derivations of membrane theory and pure-bending theory, neither has yielded a model in which membrane and bending effects are accommodated simultaneously in a single model. In particular, as remarked by Koiter [3], pure bending is rarely encountered in properly designed shell structures. Presumably this is due to Gauss' Theorema Egregium [2] linking the Gaussian curvature and the metric of the shell midsurface, requiring variations in the former (due to bending) to be accompanied by energetically costly variations in the latter (stretching), and thereby conferring significant potential stiffness which is available to be exploited by the designer. Indeed, it is a tribute to Koiter's early papers [3-6] that shell theory has advanced so little since their publication, despite the emergence of a voluminous literature that brings sophisticated modern methods to bear on the subject. On the contrary, it seems appropriate to con-

clude that the main success of the latter lies in its rigorous confirmation of Koiter's theory [2]. In the linear case, formal justification had already been accomplished through the application of asymptotic methods to the local three-dimensional equations [7], whereas rigorous justification was achieved via bounds derived from the potential and complementary potential energies [6].

An important exception occurs when tangential and transverse displacements decouple at leading order, as in the case of a flat plate exhibiting reflection symmetry of the material properties with respect to the midplane. In this circumstance the method of gamma convergence delivers a rigorous leading-order model for stretching and bending deformations [8]. This decoupling does not occur in curved shells, however. Nevertheless Koiter's [3] model for shells exhibiting reflection symmetry with respect to the midsurface, while not a limit model in the sense of gamma convergence, has been justified through careful asymptotic analysis of the three-dimensional theory for thin bodies (see [2] and references therein).

Our concern here is with the general case in which reflection symmetry need not obtain. We use the method discussed in [9-11], which delivers precisely the same model for stretching and bending as that obtained via the method of gamma convergence in circumstances when the latter yields definitive limit models [10]. This method has been used to generate optimal models for linearly elastic plates and laminates [10,11]. Here it is applied to derive a model for shells having arbitrary symmetry and which reduces to Koiter's upon specialization to reflection symmetry.

Standard bold face notation is used for vectors and tensors and indices to denote their components. Latin indices take values in $\{1,2,3\}$; Greek in $\{1,2\}$. The latter are associated with surface coordinates and associated vector and tensor components. A dot between bold symbols is used to denote the standard Euclidean inner product. Thus, if $\mathbf{A}_1$ and $\mathbf{A}_2$ are second-order tensors, then their inner product is $\mathbf{A}_1 \cdot \mathbf{A}_2 = tr(\mathbf{A}_1\mathbf{A}_2^t)$, where $tr(\cdot)$ is the trace and the superscript ^{t} is used to denote the transpose, and the associated norm is $|\mathbf{A}| = \sqrt{\mathbf{A}\cdot\mathbf{A}}$. The linear operators $Sym(\cdot)$ and $Skw(\cdot)$ deliver the symmetric and skew parts of their second-order tensor arguments, and the notation $\otimes$ identifies the standard tensor product of vectors. The symbols Div and D are used to denote the three-dimensional divergence and gradient operators, while div and ∇ are reserved for their two-dimensional counterparts. We use a combination of direct notation, commonplace in three-dimensional elasticity, and the indicial tensor notation preferred by shell theorists. Regarding the latter, useful background material together with a summary of the requisite differential geometry of surfaces may be found in [2,12]. In particular, commas followed by subscripts are used to denote partial derivatives with respect to surface coordinates, while semi-colons are used for covariant derivatives.

The three-dimensional equation of equilibrium without body force is

$$Div\tilde{\mathbf{P}} = \mathbf{0}, \tag{1}$$

where

$$\tilde{\mathbf{P}} = \mathbf{C}[\tilde{\mathbf{H}}]. \tag{2}$$

is the linear approximation to the Piola stress in the absence of residual stress, $\tilde{\mathbf{H}} = D\tilde{\mathbf{u}}$ is the displacement gradient, $\tilde{\mathbf{u}}(\mathbf{x})$ is the three-dimensional displacement field and $\mathbf{C}$ is the fourth-order tensor

of elastic moduli. The moduli possess the usual minor and major symmetries, the latter ensuring that

$$\tilde{\mathbf{P}} = U_{\tilde{\mathbf{H}}}, \tag{3}$$

where

$$U(\tilde{\mathbf{H}}) = \tfrac{1}{2}\tilde{\mathbf{H}} \cdot \mathbf{C}[\tilde{\mathbf{H}}] \tag{4}$$

is the strain energy.

We impose the usual requirement that the strain-energy function be positive definite on the linear space of symmetric tensors:

$$\mathbf{A} \cdot \mathbf{C}[\mathbf{A}] > 0 \quad \text{for all non-zero symmetric} \quad \mathbf{A}, \tag{5}$$

and the minor symmetries of $\mathbf{C}$ yield $\mathbf{A} \cdot \mathbf{C}[\mathbf{A}] = 0$ if and only if $\mathbf{A}$ is skew. This in turn yields the strong-ellipticity condition

$$\mathbf{v} \otimes \mathbf{w} \cdot \mathbf{C}[\mathbf{v} \otimes \mathbf{w}] > 0 \quad \text{for all} \quad \mathbf{v} \otimes \mathbf{w} \neq \mathbf{0}, \tag{6}$$

which is consistent with (5) as $\mathbf{v} \otimes \mathbf{w} = \mathbf{0}$ if and only if $Sym(\mathbf{v} \otimes \mathbf{w})$ vanishes.

Our approach is based on a development of the potential energy

$$\mathcal{E}[\tilde{\mathbf{u}}] = \int_R U dv - \int_{\partial R_t} \tilde{\mathbf{t}} \cdot \tilde{\mathbf{u}} da \tag{7}$$

of the three-dimensional shell in powers of its thickness, where

$$\tilde{\mathbf{t}} = \tilde{\mathbf{P}}\mathbf{n} \tag{8}$$

is assigned as a dead traction on a part ∂R_t of the boundary ∂R with exterior unit normal $\mathbf{n}$. Here $\partial R_t = \partial R \setminus \partial R_u$, where ∂R_u is the part where displacement is prescribed. Eqs. (1) and (8) are, of course, the Euler equation and natural boundary condition associated with $\mathcal{E}$. Specifically, we seek the optimal expression for the functional $E[\mathbf{u}]$ in the expansion

$$\mathcal{E} = E + o(h^3), \tag{9}$$

where h is the thickness of the shell and $\mathbf{u}$ is the displacement field of the surface Ω mid-way between the lateral surfaces. The associated Euler equations furnish an approximate system for the analysis of equilibria of the thin shell.

The present work is a continuation of [9], where a number of formulae required here are developed but which does not include a detailed study of the connection between shell theory and the three-dimensional theory. Sections 2 and 3 summarize those formulae needed to describe the geometry, kinematics and energetics of thin, three-dimensional elastic bodies. These are refined in Section 4, using necessary conditions in the three-dimensional theory, to obtain the optimal order-h^3 expression for the potential energy associated with a given mid-surface displacement field. There we also discuss the rigorous leading-order energies associated with membrane and pure-bending behavior, and, following Koiter [3,4], outline an approximate theory for combined stretching and bending that, remarkably, subsumes the exact membrane and pure-bending theories under a single model. The relevant strain measures are

shown to consist of the usual mid-surface strain, representing the linearized change of surface metric due to deformation; the bending strain representing the linearized change in curvature; and a third-order strain-gradient tensor arising in materials that lack reflection symmetry of the elastic properties with respect to the midsurface. The reduction to Koiter's theory is demonstrated for materials that possess reflection symmetry. The model is shown, in Section 5, to admit of a simple analog of the classical minimum-energy principle of conventional elasticity theory. Finally, in Section 6, the complete set of equilibrium equations is given in terms of certain dynamical variables that are constitutively connected to the mid-surface displacement field by the three-dimensional elastic properties.

2. Geometry and kinematics in 'shell space'

We seek equations of equilibrium for the shell involving as independent variables the coordinates θ^α that parametrize a curved base surface Ω. To this end we use the standard *normal-coordinate* parametrization of three dimensional space in the vicinity of the base surface [2,9,12]. Thus,

$$\mathbf{x}(\theta^\alpha, \varsigma) = \mathbf{r}(\theta^\alpha) + \varsigma \mathbf{n}(\theta^\alpha), \tag{10}$$

where $\mathbf{r}(\theta^\alpha)$ is the parametrization of Ω with unit-normal field $\mathbf{n}(\theta^\alpha)$ and ς is the coordinate in the direction perpendicular to Ω, the latter corresponding to $\varsigma = 0$. The lateral surfaces of the thin three-dimensional body are assumed to correspond to constant values of ς. These are separated by the distance h, the thickness of the shell. For simplicity we assume the thickness to be uniform. Further, h is assumed to be small against any other length scale at hand, such as a spanwise dimension or a local radius of curvature, and we choose the base surface to coincide with the midsurface.

The orientation of Ω is induced by the assumed right-handedness of the coordinate system $(\theta^\alpha, \varsigma)$; thus, $\mathbf{a}_1 \times \mathbf{a}_2 \cdot \mathbf{n} > 0$, where $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha} \equiv \partial \mathbf{r}/\partial \theta^\alpha$ span the tangent plane $T_{\Omega(p)}$ to Ω at the point p with coordinates θ^α. The *curvature* $\boldsymbol{\kappa}$ of the base surface is the symmetric linear map from $T_{\Omega(p)}$ to itself defined by the Weingarten equation

$$d\mathbf{n} = -\boldsymbol{\kappa} d\mathbf{r}, \tag{11}$$

where $d\mathbf{r} = \mathbf{a}_\alpha d\theta^\alpha$ and $d\mathbf{n} = \mathbf{n}_{,\alpha} d\theta^\alpha$. Accordingly,

$$d\mathbf{x} = d\mathbf{r} + \varsigma d\mathbf{n} + \mathbf{n} d\varsigma = \mathbf{G}(d\mathbf{r} + \mathbf{n} d\varsigma), \tag{12}$$

where

$$\mathbf{G} = \boldsymbol{\mu} + \mathbf{n} \otimes \mathbf{n}, \qquad \boldsymbol{\mu} = \mathbf{1} - \varsigma \boldsymbol{\kappa}, \tag{13}$$

and

$$\mathbf{1} = \mathbf{I} - \mathbf{n} \otimes \mathbf{n} = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha \tag{14}$$

is the projection onto - and the identity transformation for - $T_{\Omega(p)}$, on which $\{\mathbf{a}^\alpha\}$ is dual to $\{\mathbf{a}_\alpha\}$.

In Section 3 we require the volume measure induced by the coordinates. This is [9]

$$dv = \mu d\varsigma da, \tag{15}$$

where $da = \mathbf{n} \cdot d\mathbf{r}_1 \times d\mathbf{r}_2$ is the area measure on Ω, and

$$\mu = 1 - 2H\varsigma + \varsigma^2 K \tag{16}$$

is the (two-dimensional) determinant of $\boldsymbol{\mu}$ in which

$$H = \tfrac{1}{2} tr\boldsymbol{\kappa} \quad \text{and} \quad K = \det \boldsymbol{\kappa} \tag{17}$$

are the mean and Gaussian curvatures of Ω, respectively. This may be written $\mu = (1 - \varsigma\kappa_1)(1 - \varsigma\kappa_2)$, where κ_α - the *principal curvatures* - are the eigenvalues of $\boldsymbol{\kappa}$. The transformation from $(\theta^\alpha, \varsigma)$ to $\mathbf{x}$ is one-to-one and orientation preserving if and only if $\mu > 0$. Following common practice we refer to the region of space in which this condition holds as *shell space.* This is the region containing the base surface in which $|\varsigma| < \min\{r_1, r_2\}$, where $r_\alpha = |\kappa_\alpha|^{-1}$ are the principal radii of curvature. Thus, for example, (12) yields $d\varsigma(\mathbf{x}) = \mathbf{n} \cdot d\mathbf{x}$ and hence $\mathbf{n} = \nabla\varsigma$, implying that the normal to the base surface is also normal to the lateral surfaces of the shell. We make use of this fact in Section 4.

Let C^* be the line orthogonal to Ω and intersecting R at a point with surface coordinates θ^α. Let $\partial R_C = \partial\Omega \times C$, where C is the collection of such curves, be the ruled generating surface of the thin shell-like region R obtained by translating the points of $\partial\Omega$ along their associated lines C^*. Let s measure arclength on the curve $\partial\Omega$ with unit tangent $\boldsymbol{\tau}$ and rightward unit normal $\boldsymbol{\nu} = \boldsymbol{\tau} \times \mathbf{n}$. The oriented differential surface measure induced by the $(s, \varsigma)-$ parametrization of ∂R_C, required in Section 3, is [9] $\mathbf{N}da = \mathbf{G}^*\boldsymbol{\nu} ds d\varsigma$, where $\mathbf{G}^*$ is the cofactor of $\mathbf{G}$. From (13) it follows easily that $\mathbf{G}^* = \mu(\boldsymbol{\mu}^{-1} + \mathbf{n} \otimes \mathbf{n})$ and

$$\mathbf{N}da = \boldsymbol{\mu}^*\boldsymbol{\nu} d\varsigma ds, \tag{18}$$

where $\boldsymbol{\mu}^* = \mu\boldsymbol{\mu}^{-1}$ is the cofactor of $\boldsymbol{\mu}$. The Cayley-Hamilton formula

$$\boldsymbol{\kappa}^2 = 2H\boldsymbol{\kappa} - K\mathbf{1} \tag{19}$$

may be used to verify [12] that

$$\boldsymbol{\mu}^* = \mathbf{1} - \varsigma\boldsymbol{\kappa}^*, \tag{20}$$

where

$$\boldsymbol{\kappa}^* = 2H\mathbf{1} - \boldsymbol{\kappa} \tag{21}$$

is the cofactor of the curvature.

The representation

$$\boldsymbol{\kappa} = \kappa_\nu \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \kappa_{\boldsymbol{\tau}} \boldsymbol{\tau} \otimes \boldsymbol{\tau} + \tau(\boldsymbol{\nu} \otimes \boldsymbol{\tau} + \boldsymbol{\tau} \otimes \boldsymbol{\nu}), \tag{22}$$

where κ_ν, κ_τ and τ are the normal curvatures and twist of Ω on the $(\boldsymbol{\nu}, \boldsymbol{\tau})-$ axes, then yields $2H = \kappa_\nu + \kappa_{\boldsymbol{\tau}}$ and

$$\boldsymbol{\mu}^*\boldsymbol{\nu} = (1 - \varsigma\kappa_\tau)\boldsymbol{\nu} + (\varsigma\tau)\boldsymbol{\tau}, \tag{23}$$

which is needed in Section 3.

The model to be developed requires expressions for the three-dimensional displacement gradient and its through-thickness derivatives. Let $\tilde{\mathbf{u}}(\mathbf{x})$ be the displacement with gradient $\tilde{\mathbf{H}}(\mathbf{x}) = D\tilde{\mathbf{u}}$. We define

$$\hat{\mathbf{u}}(\theta^\alpha, \varsigma) = \tilde{\mathbf{u}}(\mathbf{r}(\theta^\alpha) + \varsigma\mathbf{n}(\theta^\alpha)) \quad \text{and} \quad \hat{\mathbf{H}}(\theta^\alpha, \varsigma) = \tilde{\mathbf{H}}(\mathbf{r}(\theta^\alpha) + \varsigma\mathbf{n}(\theta^\alpha)). \tag{24}$$

Then,

$$\hat{\mathbf{H}}(\boldsymbol{\mu} d\mathbf{r} + \mathbf{n} d\varsigma) = d\hat{\mathbf{u}} = \hat{\mathbf{u}}_{,\alpha} d\theta^\alpha + \hat{\mathbf{u}}' d\varsigma = (\nabla \hat{\mathbf{u}}) d\mathbf{r} + \hat{\mathbf{u}}' d\varsigma, \tag{25}$$

where $d\theta^\alpha = \mathbf{a}^\alpha \cdot d\mathbf{r}$, $\hat{\mathbf{u}}_{,\alpha} = \partial \hat{\mathbf{u}} / \partial \theta^\alpha$, $\hat{\mathbf{u}}' = \partial \hat{\mathbf{u}} / \partial \varsigma$ and

$$\nabla \hat{\mathbf{u}} = \hat{\mathbf{u}}_{,\alpha} \otimes \mathbf{a}^\alpha \tag{26}$$

is the surface displacement gradient. Thus,

$$\hat{\mathbf{H}} \mathbf{1} \boldsymbol{\mu} = \nabla \hat{\mathbf{u}} \quad \text{and} \quad \hat{\mathbf{H}} \mathbf{n} = \hat{\mathbf{u}}', \tag{27}$$

in which the orthogonal decomposition $\hat{\mathbf{H}} = \hat{\mathbf{H}} \mathbf{1} + \hat{\mathbf{H}} \mathbf{n} \otimes \mathbf{n}$ has been used. It follows that

$$\hat{\mathbf{H}} = (\nabla \hat{\mathbf{u}}) \boldsymbol{\mu}^{-1} + \hat{\mathbf{u}}' \otimes \mathbf{n}, \tag{28}$$

yielding

$$\hat{\mathbf{H}}' = (\nabla \hat{\mathbf{u}}') \boldsymbol{\mu}^{-1} + (\nabla \hat{\mathbf{u}}) (\boldsymbol{\mu}^{-1})' + \hat{\mathbf{u}}'' \otimes \mathbf{n} \quad \text{and} \quad \hat{\mathbf{H}}'' = (\nabla \hat{\mathbf{u}}'') \boldsymbol{\mu}^{-1} + 2 (\nabla \hat{\mathbf{u}}') (\boldsymbol{\mu}^{-1})' + (\nabla \hat{\mathbf{u}}) (\boldsymbol{\mu}^{-1})'' + \hat{\mathbf{u}}''' \otimes \mathbf{n}. \tag{29}$$

The associated mid-surface values are needed in Section 3. These are [9]

$$\hat{\mathbf{H}}_0 = \nabla \mathbf{u} + \mathbf{a} \otimes \mathbf{n}, \quad \hat{\mathbf{H}}_0' = \nabla \mathbf{a} + (\nabla \mathbf{u}) \boldsymbol{\kappa} + \mathbf{b} \otimes \mathbf{n} \tag{30}$$

and

$$\hat{\mathbf{H}}_0'' = \nabla \mathbf{b} + 2 (\nabla \mathbf{a}) \boldsymbol{\kappa} + 2 (\nabla \mathbf{u}) \boldsymbol{\kappa}^2 + \mathbf{c} \otimes \mathbf{n}, \tag{31}$$

where

$$\mathbf{u} = \hat{\mathbf{u}}_0, \quad \mathbf{a} = \hat{\mathbf{u}}_0', \quad \mathbf{b} = \hat{\mathbf{u}}_0'' \quad \text{and} \quad \mathbf{c} = \hat{\mathbf{u}}_0''' \tag{32}$$

are mutually *independent* functions of θ^α. The first of these is the mid-surface displacement field and the latter three are the *directors*. Together they furnish the coefficient vectors in the thickness-wise expansion

$$\hat{\mathbf{u}} = \mathbf{u} + \varsigma \mathbf{a} + \tfrac{1}{2} \varsigma^2 \mathbf{b} + \tfrac{1}{6} \varsigma^3 \mathbf{c} + ... \tag{33}$$

of the three-dimensional displacement.

It is known, for standard mixed boundary-value problems, that displacement fields satisfying (1), (2), (6) and (8) are of class C^∞ in the interior of the body [13]. Accordingly, the expansion (33) is valid without qualification and may be imposed without loss of generality in the course of constructing a potential energy functional for the shell, provided that its use is limited to the characterization of equilibria.

3. Thickness-wise expansion of the three-dimensional energy

The strain energy of the shell is

$$\int_R U dv = \int_\Omega W da, \tag{34}$$

where

$$W = \int_C \mu U d\varsigma \tag{35}$$

is the areal strain-energy density on Ω and $C = [-h/2, h/2]$. Leibniz' rule and Taylor's theorem yield [9]

$$W = h(1 + \tfrac{1}{12}h^2K)U_0 + \tfrac{1}{24}h^3(U_0'' - 4HU_0') + o(h^3), \tag{36}$$

where

$$U_0 = U(\hat{\mathbf{H}}_0), \quad U_0' = \mathbf{P}_0 \cdot \hat{\mathbf{H}}_0', \quad \text{and} \quad U_0'' = \mathbf{P}_0' \cdot \hat{\mathbf{H}}_0' + \mathbf{P}_0 \cdot \hat{\mathbf{H}}_0''. \tag{37}$$

We assume, as in [14], that the material properties do not vary through the thickness; i.e., that $\mathbf{C}' = \mathbf{0}$, yielding

$$U(\hat{\mathbf{H}}_0) = \tfrac{1}{2}\mathbf{P}_0 \cdot \hat{\mathbf{H}}_0, \quad \mathbf{P}_0 = \mathbf{C}[\hat{\mathbf{H}}_0] \quad \text{and} \quad \mathbf{P}_0' = \mathbf{C}[\hat{\mathbf{H}}_0']. \tag{38}$$

The subsequent development is eased considerably by using the decompositions (cf. (14))

$$\mathbf{P}_0 = \mathbf{P}_0\mathbf{1} + \mathbf{P}_0\mathbf{n} \otimes \mathbf{n} \quad \text{and} \quad \mathbf{P}_0' = \mathbf{P}_0'\mathbf{1} + \mathbf{P}_0'\mathbf{n} \otimes \mathbf{n} \tag{39}$$

in (37), leading to

$$\begin{aligned} U_0 &= \tfrac{1}{2}(\mathbf{P}_0\mathbf{1} \cdot \nabla\mathbf{u} + \mathbf{P}_0\mathbf{n} \cdot \mathbf{a}), \quad U_0' = \mathbf{P}_0\mathbf{1} \cdot [\nabla\mathbf{a} + (\nabla\mathbf{u})\boldsymbol{\kappa}] + \mathbf{P}_0\mathbf{n} \cdot \mathbf{b}, \quad \text{and} \\ U_0'' &= \mathbf{P}_0'\mathbf{1} \cdot [\nabla\mathbf{a} + (\nabla\mathbf{u})\boldsymbol{\kappa}] + \mathbf{P}_0'\mathbf{n} \cdot \mathbf{b} + \mathbf{P}_0\mathbf{1} \cdot [\nabla\mathbf{b} + 2(\nabla\mathbf{a})\boldsymbol{\kappa} + 2(\nabla\mathbf{u})\boldsymbol{\kappa}^2] + \mathbf{P}_0\mathbf{n} \cdot \mathbf{c}. \end{aligned} \tag{40}$$

Combining these with the Cayley-Hamilton formula (19) and the formula (21) for the cofactor, we derive from (36) the rather complicated preliminary expression

$$\begin{aligned} W &= \tfrac{1}{2}h(1 - \tfrac{1}{12}h^2K)\mathbf{P}_0\mathbf{1} \cdot \nabla\mathbf{u} + \tfrac{1}{2}h(1 + \tfrac{1}{12}h^2K)\mathbf{P}_0\mathbf{n} \cdot \mathbf{a} + \tfrac{1}{24}h^3\{\mathbf{P}_0'\mathbf{1} \cdot [\nabla\mathbf{a} + (\nabla\mathbf{u})\boldsymbol{\kappa}] + \mathbf{P}_0'\mathbf{n} \cdot \mathbf{b}\} \\ &\quad + \tfrac{1}{24}h^3[\mathbf{P}_0\mathbf{1} \cdot \nabla\mathbf{b} - 2\mathbf{P}_0\mathbf{1} \cdot \nabla\mathbf{a}(\boldsymbol{\kappa}^*) + \mathbf{P}_0\mathbf{n} \cdot (\mathbf{c} - 4H\mathbf{b})] + o(h^3). \end{aligned} \tag{41}$$

Assuming the lateral surfaces to be traction free, the load potential in (7) reduces to (cf. (18))

$$\int_{\partial R_t} \tilde{\mathbf{t}} \cdot \tilde{\mathbf{u}} da = \int_{\partial\Omega_t} (\int_C |\boldsymbol{\mu}^*\boldsymbol{\nu}|\, \tilde{\mathbf{t}} \cdot \tilde{\mathbf{u}} d\varsigma) ds, \tag{42}$$

where $\partial\Omega_t \times C = \partial R_C \cap \partial R_t$ is the part of the generating surface of the shell where tractions are assigned and $\boldsymbol{\mu}^*\boldsymbol{\nu}$ is given by (23). Leibniz' rule and Taylor's expansion furnish [9]

$$\int_{\partial R_t} \tilde{\mathbf{t}} \cdot \tilde{\mathbf{u}} da = \int_{\partial\Omega_t} (\mathbf{p}_u \cdot \mathbf{u} + \mathbf{p}_a \cdot \mathbf{a} + \mathbf{p}_b \cdot \mathbf{b}) ds + o(h^3), \tag{43}$$

where

$$\mathbf{p}_u = h(1 + \tfrac{1}{24}h^2\,\tau^2)\mathbf{t}_0 + \tfrac{1}{24}h^3(\mathbf{t}_0'' - 2\kappa_\tau\mathbf{t}_0'), \quad \mathbf{p}_a = \tfrac{1}{12}h^3(\mathbf{t}_0' - \kappa_\tau\mathbf{t}_0) \quad \text{and} \quad \mathbf{p}_b = \tfrac{1}{24}h^3\mathbf{t}_0. \tag{44}$$

These are connected to the deformation through $\tilde{\mathbf{t}} = \mathbf{P}(\boldsymbol{\mu}^*\boldsymbol{\nu}/\,|\boldsymbol{\mu}^*\boldsymbol{\nu}|)$ on ∂R_C (cf. (18)), yielding [9]

$$\mathbf{t}_0 = \mathbf{P}_0\mathbf{1}\boldsymbol{\nu}, \quad \mathbf{t}_0' - \kappa_\tau\mathbf{t}_0 = \mathbf{P}_0'\mathbf{1}\boldsymbol{\nu} + \tau\mathbf{P}_0\mathbf{1}\boldsymbol{\tau} - \kappa_\tau\mathbf{P}_0\mathbf{1}\boldsymbol{\nu}, \quad \mathbf{t}_0'' - 2\kappa_\tau\mathbf{t}_0' = \mathbf{P}_0''\mathbf{1}\boldsymbol{\nu} + 2(\tau\mathbf{P}_0'\mathbf{1}\boldsymbol{\tau} - \kappa_\tau\mathbf{P}_0'\mathbf{1}\boldsymbol{\nu}) - \tau^2\mathbf{P}_0\mathbf{1}\boldsymbol{\nu}. \tag{45}$$

If the lateral surfaces are loaded by small tractions of order h^3; i.e., if $\tilde{\mathbf{t}}^\pm = h^3\mathbf{t}^\pm + o(h^3)$ with $|\mathbf{t}^\pm|$ of order unity, then the total load potential is easily shown to be

$$\int_{\partial R_t} \tilde{\mathbf{t}} \cdot \tilde{\mathbf{u}} da = \int_{\partial\Omega_t} (\mathbf{p}_u \cdot \mathbf{u} + \mathbf{p}_a \cdot \mathbf{a} + \mathbf{p}_b \cdot \mathbf{b}) ds + \int_\Omega \mathbf{g} \cdot \mathbf{u} da + o(h^3), \quad \text{where} \quad \mathbf{g} = h^3(\mathbf{t}^+ + \mathbf{t}^-) \tag{46}$$

is the leading-order net traction on the lateral surfaces.

4. Construction of the optimal shell energy

4.1 Constraints derived from the exact theory

Our objective is an order-h^3 estimate of the potential energy that is as accurate as possible by the standard of three-dimensional elasticity theory. To this end we impose restrictions on the director fields $\mathbf{a}, \mathbf{b}$, etc. as required by that theory. For example, the exact expressions $\tilde{\mathbf{t}}^+ = \tilde{\mathbf{P}}^+\mathbf{n}$ and $\tilde{\mathbf{t}}^+ = -\tilde{\mathbf{P}}^-\mathbf{n}$ for the tractions at the lateral surfaces with exterior unit normals $\pm\mathbf{n}$ (cf. Section 2), together with Taylor expansions of $\tilde{\mathbf{P}}^\pm$, furnish

$$\tilde{\mathbf{t}}^+ + \tilde{\mathbf{t}}^- = h\mathbf{P}_0'\mathbf{n} + O(h^3) \quad \text{and} \quad \tilde{\mathbf{t}}^+ - \tilde{\mathbf{t}}^- = 2\mathbf{P}_0\mathbf{n} + O(h^2). \tag{47}$$

Accordingly, for traction-free lateral surfaces,

$$\mathbf{P}_0\mathbf{n} = \mathbf{0} \quad \text{and} \quad \mathbf{P}_0'\mathbf{n} = \mathbf{0}, \tag{48}$$

and these in turn ensure that the traction data are satisfied with an error of order h^2. Using (2) and (30), they are also seen to simplify (41) significantly while delivering $\mathbf{a} = \bar{\mathbf{a}}$ and $\mathbf{b} = \bar{\mathbf{b}}$, where

$$\mathbf{A}_n\bar{\mathbf{a}} = -(\mathbf{C}[\nabla\mathbf{u}])\mathbf{n} \quad \text{and} \quad \mathbf{A}_n\bar{\mathbf{b}} = -(\mathbf{C}[\nabla\bar{\mathbf{a}} + (\nabla\mathbf{u})\boldsymbol{\kappa}])\mathbf{n}, \tag{49}$$

in which $\mathbf{A}_n$ is the acoustic tensor defined by

$$\mathbf{A}_n\mathbf{w} = (\mathbf{C}[\mathbf{w}\otimes\mathbf{n}])\mathbf{n} \tag{50}$$

for any vector $\mathbf{w}$. That this is positive definite follows from (6); eqs. $(49)_{1,2}$ then yield unique directors $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ in terms of the gradients of the mid-surface position field. We then obtain $\mathbf{P}_0 = \bar{\mathbf{P}}_0\mathbf{1}$ and $\mathbf{P}_0' = \bar{\mathbf{P}}_0'\mathbf{1}$, where

$$\bar{\mathbf{P}}_0\mathbf{1} = (\mathbf{C}[\nabla\mathbf{u} + \bar{\mathbf{a}}\otimes\mathbf{n}])\mathbf{1} \quad \text{and} \quad \bar{\mathbf{P}}_0'\mathbf{1} = (\mathbf{C}[\nabla\bar{\mathbf{a}} + (\nabla\mathbf{u})\boldsymbol{\kappa} + \bar{\mathbf{b}}\otimes\mathbf{n}])\mathbf{1}. \tag{51}$$

Remark: The solutions $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ to $(49)_{1,2}$ minimize the quadratic forms $\mathbf{H}_0\cdot\mathbf{C}[\mathbf{H}_0]$ and $\mathbf{H}_0'\cdot\mathbf{C}[\mathbf{H}_0']$ with respect to $\mathbf{a}$ and $\mathbf{b}$, respectively, yielding $\bar{\mathbf{P}}_0\mathbf{1}\cdot\nabla\mathbf{u} = \bar{\mathbf{H}}_0\cdot\mathbf{C}[\bar{\mathbf{H}}_0]$ and $\bar{\mathbf{P}}_0'\mathbf{1}\cdot[\nabla\bar{\mathbf{a}} + (\nabla\mathbf{u})\boldsymbol{\kappa}] = \bar{\mathbf{H}}_0'\cdot\mathbf{C}[\bar{\mathbf{H}}_0']$. These results follow directly from a lemma proved in [11].

A further simplification may be achieved by imposing the exact equilibrium equation (1) in the form

$$(\tilde{\mathbf{P}}_{,i})\mathbf{g}^i = \mathbf{0}, \tag{52}$$

where $\mathbf{g}^i = \nabla\theta^i(\mathbf{x})$, $\{\theta^i\} = \{\theta^\alpha, \varsigma\}$ and $\mathbf{g}_i = \mathbf{x}_{,i}$, yielding (cf. (12)) $\mathbf{g}_\alpha = \boldsymbol{\mu}\mathbf{a}_\alpha$ and $\mathbf{g}_3 = \mathbf{n}$. This holds at all points of the body and on the base surface in particular; i.e.,

$$(\mathbf{P}_{0,\alpha})\mathbf{a}^\alpha + \mathbf{P}_0'\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Omega. \tag{53}$$

Using (14) and (39), this may be recast, using $\mathbf{a}^\alpha \cdot \mathbf{n}_{,\alpha} = -2H$, as

$$div(\mathbf{P}_0\mathbf{1}) + \mathbf{P}_0'\mathbf{n} - 2H\mathbf{P}_0\mathbf{n} = \mathbf{0}, \tag{54}$$

where $div(\mathbf{P}_0\mathbf{1}) = [(\mathbf{P}_0\mathbf{1})_{,\alpha}]\mathbf{a}^\alpha$.

A more useful form is obtained by using the decomposition $\mathbf{P}_0\mathbf{1} = \mathbf{P}_0^\alpha \otimes \mathbf{a}_\alpha$ with $\mathbf{P}_0^\alpha = \mathbf{P}_0\mathbf{a}^\alpha$. The Gauss equation

$$\mathbf{a}_{\alpha,\beta} = \Gamma^\lambda_{\alpha\beta}\mathbf{a}_\lambda + \kappa_{\alpha\beta}\mathbf{n}, \tag{55}$$

where $\Gamma^\lambda_{\alpha\beta}$ are the Christoffel symbols, then furnishes

$$div(\mathbf{P}_0\mathbf{1}) = \mathbf{P}^\alpha_{0;\alpha} = a^{-1/2}(a^{1/2}\mathbf{P}_0^\alpha)_{,\alpha}, \tag{56}$$

where $a = \det(a_{\alpha\beta})$, $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ is the (positive definite) metric on Ω induced by the parametrization $\mathbf{r}(\theta^\alpha)$ and we have used the identity $\Gamma^\beta_{\alpha\beta} = a^{-1/2}(a^{1/2})_{,\alpha}$. The Christoffel symbols are given in terms of the metric by the well-known formula [2,12]

$$\Gamma^\lambda_{\alpha\beta} = \mathbf{a}^\lambda \cdot \mathbf{a}_{\alpha,\beta} = \tfrac{1}{2}a^{\lambda\mu}(a_{\mu\alpha,\beta} + a_{\mu\beta,\alpha} - a_{\alpha\beta,\mu}), \tag{57}$$

where $a^{\lambda\mu}$ is the dual metric defined by $a^{\alpha\mu}a_{\mu\beta} = \delta^\alpha_\beta$ (the usual Kronecker delta) and $\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$.

For simply-connected surfaces we combine these formulae in an application of Stokes' theorem to the term involving $\mathbf{P}_0\mathbf{1} \cdot \nabla\mathbf{b}$ in (41). Thus,

$$\int_\Omega \psi^\alpha_{;\alpha} dv = \int_{\partial\Omega} \psi^\alpha \nu_\alpha ds, \tag{58}$$

where $\nu_\alpha = \boldsymbol{\nu} \cdot \mathbf{a}_\alpha$ and $\psi^\alpha = \mathbf{P}_0^\alpha \cdot \mathbf{b}$, and (54) yields

$$\int_\Omega \mathbf{P}_0\mathbf{1} \cdot \nabla\mathbf{b} da = \int_{\partial\Omega} \mathbf{P}_0\mathbf{1}\boldsymbol{\nu} \cdot \mathbf{b} ds + \int_\Omega \mathbf{b} \cdot (\mathbf{P}_0'\mathbf{n} - 2H\mathbf{P}_0\mathbf{n}) da. \tag{59}$$

Imposing (48) and using $\nabla\bar{\mathbf{a}}(\boldsymbol{\kappa}^*) = \bar{\mathbf{H}}_\mathbf{0}'(\boldsymbol{\kappa}^*) - K\nabla\mathbf{u}$, which follows from (19), (21) and $(30)_2$, we conclude that

$$\int_\Omega W da = \int_\Omega \bar{W} da + \tfrac{1}{24}h^3 \int_{\partial\Omega} \bar{\mathbf{P}}_0\mathbf{1}\boldsymbol{\nu} \cdot \bar{\mathbf{b}} ds + o(h^3), \tag{60}$$

where

$$\bar{W} = (1 + \tfrac{1}{12}h^2K)W_1 + W_2 + W_3 \tag{61}$$

with

$$W_1 = \tfrac{1}{2}h\bar{\mathbf{P}}_0\mathbf{1} \cdot \nabla\mathbf{u}, \quad W_2 = \tfrac{1}{24}h^3\bar{\mathbf{P}}_0'\mathbf{1} \cdot [\nabla\bar{\mathbf{a}} + (\nabla\mathbf{u})\boldsymbol{\kappa}] \quad \text{and} \quad W_3 = -\tfrac{1}{12}h^3\bar{\mathbf{P}}_0\mathbf{1} \cdot \bar{\mathbf{H}}_0'(\boldsymbol{\kappa}^*). \tag{62}$$

Invoking (46) and using $\mathbf{t}_0 = \mathbf{P}_0\mathbf{1}\boldsymbol{\nu}$ on $\partial\Omega_t$, we observe that the contribution from $\partial\Omega_t$ to the boundary integral balances the contribution $\mathbf{p}_b$ to the load potential (cf. $(44)_3$) in the expression (7) for the potential energy, leaving a residual integral of $\mathbf{P}_0\mathbf{1}\boldsymbol{\nu} \cdot \mathbf{b}$ over $\partial\Omega_u = \partial\Omega \setminus \partial\Omega_t$ which is fixed by the data, where $\partial\Omega_u \times C = \partial R_u \subset \partial R_C$ is the part of the generating surface where (three-dimensional) displacement is assigned. This follows from the fact that $\tilde{\mathbf{u}}$ is assigned on ∂R_u. Consequently, its (tangential) derivatives with respect to ς are also assigned. This then fixes the values of $\mathbf{u}, \mathbf{a}$ and $\mathbf{b}$ on $\partial\Omega_u$. Assuming the present model to be valid on the closure of Ω, the values of $\mathbf{a}$ and $\mathbf{b}$ obtained from

(49) must agree with those obtained from the assigned function $\tilde{\mathbf{u}}$ (cf. (33)), and this in turn imposes a restriction on admissible data. In addition, $(49)_1$ and the decomposition [15]

$$\nabla\mathbf{u} = \mathbf{u}_s \otimes \boldsymbol{\tau} + \mathbf{u}_\nu \otimes \boldsymbol{\nu}, \tag{63}$$

where $\mathbf{u}_s$ is the tangential (arclength) derivative on $\partial\Omega$, imply that the assignment of $\mathbf{u}$ and $\mathbf{a}$ on $\partial\Omega_u$ amounts to the assignment thereon of $\mathbf{u}$ and $\mathbf{u}_\nu$. This in turn fixes the values of $\nabla\mathbf{u}$, and hence those of $\mathbf{P}_0\mathbf{1}\boldsymbol{\nu}$, on $\partial\Omega_u$. With $\mathbf{b}$ fixed as discussed, the residual integral is then controlled by the data on $\partial\Omega_u$, contributing only a disposable constant to the total energy. Accordingly, $E = \bar{E} + o(h^3)$ in (9), where

$$\bar{E} = \int_\Omega (\bar{W} - \mathbf{g}\cdot\mathbf{u})da - \int_{\partial\Omega_t} (\mathbf{p}_u\cdot\mathbf{u} + \mathbf{p}_a\cdot\bar{\mathbf{a}})ds. \tag{64}$$

4.2 Membrane and pure-bending theories

(a) Membrane theory

Membrane theory is associated with the leading-order term in the potential energy. Thus, $\mathcal{E} = hE_M + O(h^3)$, where

$$E_M[\mathbf{u}] = \int_\Omega \tfrac{1}{2}\bar{\mathbf{P}}_0\mathbf{1}\cdot\nabla\mathbf{u}da - \int_{\partial\Omega_t}\mathbf{t}_0\cdot\mathbf{u}ds \tag{65}$$

is the membrane energy. Its analysis is facilitated by the standard decomposition

$$\mathbf{u} = u_\alpha\mathbf{a}^\alpha + w\mathbf{n} \tag{66}$$

of the displacement into tangential and normal components, yielding

$$\nabla\mathbf{u} = \mathbf{u}_{,\alpha}\otimes\mathbf{a}^\alpha = (u_{\lambda;\beta} - w\kappa_{\lambda\beta})\mathbf{a}^\lambda\otimes\mathbf{a}^\beta - \mathbf{n}\otimes\boldsymbol{\alpha}, \quad \text{where} \quad \boldsymbol{\alpha} = -(\nabla w + \boldsymbol{\kappa}\mathbf{u}) \tag{67}$$

and $u_{\lambda;\beta} = u_{\lambda,\beta} - u_\mu\Gamma^\mu_{\lambda\beta}$ is the usual covariant derivative.

Recalling that

$$\bar{\mathbf{P}}_0\mathbf{1}\cdot\nabla\mathbf{u} = \bar{\mathbf{H}}_0\cdot\mathbf{C}[\bar{\mathbf{H}}_0], \tag{68}$$

we see that the relevant strain is

$$Sym\bar{\mathbf{H}}_0 = Sym(\nabla\mathbf{u} + \bar{\mathbf{a}}\otimes\mathbf{n}), \tag{69}$$

where

$$Sym\nabla\mathbf{u} = \boldsymbol{\epsilon} - Sym(\boldsymbol{\alpha}\otimes\mathbf{n}) \tag{70}$$

and $\boldsymbol{\epsilon} = \epsilon_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta$, with

$$\epsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha;\beta} + u_{\beta;\alpha}) - w\kappa_{\alpha\beta}, \tag{71}$$

is the surface-strain measure used in the classical theory [2,6,15,16].,

To find $\bar{\mathbf{a}}$, we combine $(49)_1$, (67) and (70) with the definition (50) of $\mathbf{A}_n$, obtaining

$$\bar{\mathbf{a}} = -\mathbf{A}_n^{-1}(\mathbf{C}[\boldsymbol{\epsilon}])\mathbf{n} + \boldsymbol{\alpha}, \tag{72}$$

and hence

$$Sym\bar{\mathbf{H}}_0 = \boldsymbol{\epsilon} - Sym\{\mathbf{A}_n^{-1}(\mathbf{C}[\boldsymbol{\epsilon}])\mathbf{n}\otimes\mathbf{n}\}, \tag{73}$$

yielding the membrane strain-energy function (68) as a positive-definite quadratic form that vanishes if and only if $\boldsymbol{\epsilon}=\mathbf{0}$. In general the explicit expression in terms of the strain components is rather complicated and thus best left to specific applications. It is given in Section 4.5 for the special case of reflection symmetry with respect to the midsurface.

In view of the *Remark* following (51), the membrane energy given here is optimal for a given midsurface displacement field. Further, on division of the exact energy by h, followed by passage to the limit, it emerges as the rigorous leading-order energy of the thin shell, agreeing precisely with the result obtained by the method of gamma convergence.

(b) Pure bending

Pure bending is associated with deformations for which the membrane energy vanishes identically. In view of (68) and (73) this condition is satisfied if and only if $\boldsymbol{\epsilon}=\mathbf{0}$ on the closure of Ω, yielding $\mathcal{E}=h^3E_B+o(h^3)$, where

$$E_B=\int_\Omega\{\tfrac{1}{24}\bar{\mathbf{P}}_0'\mathbf{1}\cdot[\nabla\bar{\mathbf{a}}+(\nabla\mathbf{u})\boldsymbol{\kappa}]-\mathbf{g}\cdot\mathbf{u}\}da-\int_{\partial\Omega_t}[\tfrac{1}{24}(\mathbf{t}_0''-2\kappa_\tau\mathbf{t}_0')\cdot\mathbf{u}+\tfrac{1}{12}\mathbf{t}_0'\cdot\bar{\mathbf{a}}]ds \tag{74}$$

is the leading-order residual energy, incorporating the restriction $\bar{\mathbf{P}}_0=\mathbf{0}$ associated with null membrane energy. Here we use (71) to find

$$u_{\alpha;\beta}=w\kappa_{\alpha\beta}+\omega\varepsilon_{\alpha\beta}, \tag{75}$$

where $\varepsilon_{\alpha\beta}=a^{1/2}e_{\alpha\beta}$ are the covariant components of the permutation tensor ($e_{12}=-e_{21}=1$, $e_{11}=e_{22}=0$) and $\omega=\frac{1}{2}\varepsilon^{\alpha\beta}u_{\alpha;\beta}$, with $\varepsilon^{\alpha\beta}=a^{-1/2}e^{\alpha\beta}$ ($e^{\alpha\beta}=e_{\alpha\beta}$), is the rotation about the normal, yielding

$$\nabla\mathbf{u}=\omega\boldsymbol{\varepsilon}-\mathbf{n}\otimes\boldsymbol{\alpha}\quad\text{and}\quad(\nabla\mathbf{u})\boldsymbol{\kappa}=\omega\boldsymbol{\varepsilon}\boldsymbol{\kappa}-\mathbf{n}\otimes\boldsymbol{\kappa}\boldsymbol{\alpha}, \tag{76}$$

where $\boldsymbol{\varepsilon}=\varepsilon_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta$, which, when combined with (72), result in

$$\nabla\bar{\mathbf{a}}=\nabla\boldsymbol{\alpha}=\alpha_{\lambda;\beta}\mathbf{a}^\lambda\otimes\mathbf{a}^\beta+\mathbf{n}\otimes\boldsymbol{\kappa}\boldsymbol{\alpha} \tag{77}$$

and

$$\nabla\bar{\mathbf{a}}+(\nabla\mathbf{u})\boldsymbol{\kappa}=\mathbf{B},\quad\text{with}\quad\mathbf{B}=B_{\alpha\lambda}\mathbf{a}^\alpha\otimes\mathbf{a}^\lambda\quad\text{and}\quad B_{\alpha\lambda}=\alpha_{\alpha;\lambda}+\omega\varepsilon_{\alpha\beta}\kappa_\lambda^\beta. \tag{78}$$

Eq. $(67)_2$ and the skew symmetry of $\boldsymbol{\varepsilon}$ lead to

$$B_{\alpha\lambda}=\omega(\varepsilon_{\alpha\beta}\kappa_\lambda^\beta+\varepsilon_{\lambda\beta}\kappa_\alpha^\beta)-(w_{;\alpha\lambda}+\kappa_{\alpha;\lambda}^\beta u_\beta+w\kappa_\alpha^\beta\kappa_{\lambda\beta}) \tag{79}$$

and thus, with the Mainardi-Codazzi equations[1], to the conclusion that $\mathbf{B}$ is symmetric.

The constraint $(49)_2$ gives

$$\bar{\mathbf{b}}=-\mathbf{A}_n^{-1}(\mathbf{C}[\mathbf{B}])\mathbf{n} \tag{80}$$

and $(30)_2$ then furnishes (compare (73))

$$Sym\bar{\mathbf{H}}_0'=\mathbf{B}-Sym\{\mathbf{A}_n^{-1}(\mathbf{C}[\mathbf{B}])\mathbf{n}\otimes\mathbf{n}\}. \tag{81}$$

As observed in the previous subsection, these results imply that

$$\bar{\mathbf{P}}_0'\mathbf{1}\cdot[\nabla\bar{\mathbf{a}}+(\nabla\mathbf{u})\boldsymbol{\kappa}]=\bar{\mathbf{H}}_0'\cdot\mathbf{C}[\bar{\mathbf{H}}_0'], \tag{82}$$

yielding the strain-energy density as a positive-definite quadratic form that vanishes if and only $\mathbf{B}=\mathbf{0}$. Further, division of the exact energy by h^3 and passage to the limit yields E_B as the rigorous leading-order energy of the shell in the absence of surface strain. The *Remark* following (51) implies that precisely the same result is obtained by the method of gamma convergence.

We have not proved that $\mathbf{B}$ is a strain measure for pure bending, as the term is usually understood. This issue is revisited in Section 4.5, where it is shown that $\mathbf{B}$ is equivalent to a suitable measure of bending strain when the surface strain vanishes.

The condition $\boldsymbol{\epsilon}=\mathbf{0}$ associated with pure bending imposes a restriction on the deformation. To see this we use (67) and (75) with $d\mathbf{u}=\mathbf{u}_{,\alpha}d\theta^\alpha$ and $d\mathbf{r}=\mathbf{a}_\beta d\theta^\beta$ to conclude that [12]

$$d\mathbf{u}\cdot d\mathbf{r}=(u_{\lambda;\beta}-w\kappa_{\lambda\beta})d\theta^\lambda d\theta^\beta=\omega\varepsilon_{\lambda\beta}d\theta^\lambda d\theta^\beta=0, \tag{83}$$

which is equivalent to

$$d\mathbf{u}=\mathbf{v}\times d\mathbf{r},\quad\text{or}\quad\mathbf{u}_{,\alpha}=\mathbf{v}\times\mathbf{a}_\alpha, \tag{84}$$

for some vector field $\mathbf{v}$. Conversely, it follows from the formula

$$2\epsilon_{\alpha\beta}=\mathbf{u}_{,\alpha}\cdot\mathbf{a}_\beta+\mathbf{u}_{,\beta}\cdot\mathbf{a}_\alpha \tag{85}$$

that this restriction yields $\boldsymbol{\epsilon}=\mathbf{0}$ and so displacement fields satisfying (84) are equivalent to pure-bending deformations. These incorporate rigid-body deformations for which $\mathbf{v}$ is uniform on the closure of Ω. The integrability condition for $(84)_2$ is

$$\mathbf{0}=\varepsilon^{\alpha\beta}(\mathbf{v}\times\mathbf{a}_\alpha)_{;\beta}=\varepsilon^{\alpha\beta}\mathbf{v}_{,\beta}\times\mathbf{a}_\alpha, \tag{86}$$

where use has been made of Gauss' equation $\mathbf{a}_{\alpha;\beta}=\kappa_{\alpha\beta}\mathbf{n}$ and the symmetry of $\boldsymbol{\kappa}$. With this satisfied, path integration of $(84)_1$ furnishes the displacement field, modulo a constant vector, at any point of the shell. Accordingly, we view the general displacement field as being determined by solutions to (86), which, on use of the decomposition $\mathbf{v}=v^\alpha\mathbf{a}_\alpha+v\mathbf{n}$, is found to be equivalent to the three equations [12]

$$v^\alpha_{;\alpha}-2Hv=0\quad\text{and}\quad v_{,\beta}+\kappa^\mu_\beta v_\mu=0. \tag{87}$$

Solutions to this system may be used with $(76)_1$ and $(84)_2$ to establish the connections

$$\alpha_\lambda=\varepsilon_{\lambda\beta}v^\beta\quad\text{and}\quad v=-\omega, \tag{88}$$

where $\boldsymbol{\alpha}$ is defined by $(67)_2$. The field $\mathbf{v}$ thus determines the bending strain $\mathbf{B}$, and hence the bending energy, via $(78)_3$ and (82).

To explore the solvability of eqs. (87) we summarize the second pair in the form $\boldsymbol{\kappa}\mathbf{v}+\nabla v=\mathbf{0}$ and multiply by the cofactor $\boldsymbol{\kappa}^*$; using (19) and (21), this results in

$$K\mathbf{v}+\boldsymbol{\kappa}^*(\nabla v)=\mathbf{0}, \tag{89}$$

and, if K is non-zero, we obtain

$$v^\alpha = -K^{-1}\kappa^{*\alpha\beta}v_{,\beta}, \tag{90}$$

substitute into $(87)_1$, and arrive at a differential equation for v [12]:

$$(K^{-1}\kappa^{*\alpha\beta}v_{,\beta})_{;\alpha} + 2Hv = 0. \tag{91}$$

Boundary conditions on $\partial\Omega_u$ follow from those for $\mathbf{u}$ (hence $\mathbf{u}_s$) and $\mathbf{u}_\nu$. To derive them we combine (63) and $(76)_1$, obtaining

$$\mathbf{u}_s = (\nabla\mathbf{u})\boldsymbol{\tau} = \omega\boldsymbol{\nu} - (\boldsymbol{\alpha}\cdot\boldsymbol{\tau})\mathbf{n} \quad \text{and} \quad \mathbf{u}_\nu = (\nabla\mathbf{u})\boldsymbol{\nu} = -\omega\boldsymbol{\tau} - (\boldsymbol{\alpha}\cdot\boldsymbol{\nu})\mathbf{n}, \tag{92}$$

and thus infer that ω and $\boldsymbol{\alpha}$ are assigned on $\partial\Omega_u$. It then follows from (88) that

$$\mathbf{v} \quad \text{is assigned on } \partial\Omega_u. \tag{93}$$

A global restriction follows on multiplying (91) by v and integrating over the surface Ω, presumed to be simply connected. Thus,

$$\int_\Omega (K^{-1}\kappa^{*\alpha\beta}v_{,\alpha}v_{,\beta} - 2Hv^2)da = \int_{\partial\Omega} K^{-1}\kappa^{*\alpha\beta}v_{,\beta}v\nu_\alpha ds. \tag{94}$$

Suppose that $\partial\Omega_t$ is empty. Using v to denote the difference of two solutions to (91) and (93), we have $v = 0$ on $\partial\Omega$ and the right-hand side of (94) vanishes. If $\boldsymbol{\kappa}$ is positive (resp. negative) definite and H is non-positive (resp. non-negative), then (94) requires that v vanish in Ω and the solution is unique. The displacement field is then uniquely determined. The potential energy is trivial in this case and does not lead to equilibrium equations. The same conclusion follows if $\boldsymbol{\kappa}$ is positive (resp. negative) *semi*-definite and H is strictly negative (resp. positive). If the displacement $\mathbf{u}$ and its normal derivative vanish on $\partial\Omega$, then the associated field $\mathbf{v}$ vanishes there and the unique solution is $\mathbf{v} = \mathbf{0}$, implying, from (84), that the displacement field vanishes in Ω and thus that the surface is rigid. Further discussion of the general problem of rigidity may be found in [12,17,18]. Examples of surfaces not covered by this lemma include the hyperbolic paraboloids. For these it is conceivable that the surface is non-rigid, and indeed this is the case if $\partial\Omega_t$ is non-empty [18]. The specification of the displacement field must then be completed using equilibrium considerations (Section 6) in which $\boldsymbol{\epsilon} = \mathbf{0}$ is imposed as a constraint. This class of problems is the subject of *inextensional bending theory* ([16], Sect. 16.3).

In the alternative case, K vanishes and, if H is nonzero, the curvature is of the form $\kappa^{\alpha\beta} = \kappa m^\alpha m^\beta$ for some tangential unit vector field $\mathbf{m}$ and non-zero scalar field κ. Eq. $(87)_2$ yields $\mathbf{n}\times\mathbf{m}\cdot\nabla v = 0$, implying that v has no gradient along the trajectories orthogonal to $\mathbf{m}$, together with $\kappa\mathbf{m}\cdot\mathbf{v} = -\mathbf{m}\cdot\nabla v$, yielding one component of $\mathbf{v}$ in terms of v and the curvature. The other component is then controlled by $(87)_1$. Data for this problem are supplied by (93). A special class of solutions for which v vanishes identically (and thus too the rotation about the normal) corresponds to $v^\alpha = \varepsilon^{\alpha\beta}\varphi_{,\beta}$, where φ is a scalar field, while (87) reduces to $\mathbf{n}\times\mathbf{m}\cdot\nabla\varphi = 0$, implying that φ also has no gradient along the trajectories orthogonal to $\mathbf{m}$. Eqs. (88) yield the conclusion that the transverse displacement, w, has no gradient along these trajectories either, and that its gradient along $\mathbf{m}$ is $\mathbf{m}\cdot\nabla w = \mathbf{m}\cdot\nabla\varphi + \kappa\mathbf{m}\cdot\mathbf{u}$. For example, on a cylinder $\mathbf{m}$ is tangential to parallels of latitude, φ and w are functions only of the azimuthal angle and these determine the azimuthal displacement component.

Lastly, if H also vanishes then $\boldsymbol{\kappa}$ vanishes, Ω is a plane and $(87)_2$ and reduces to $\nabla\omega = \mathbf{0}$, implying that the field of rotations about the (fixed) normal is uniform. Eq. $(87)_1$ has the general solution $v^\alpha = \varepsilon^{\alpha\beta}\varphi_{,\beta}$ on simply-connected planes, with φ an arbitrary scalar field, and $(88)_1$ reduces to $w_{,\lambda} = \varepsilon_{\beta\lambda}\varepsilon^{\beta\mu}\varphi_{,\mu} = \varphi_{,\lambda}$, yielding $\varphi = w$ modulo a constant, while w is unrestricted and $(78)_3$ reduces to $B_{\alpha\beta} = -w_{;\alpha\beta}$.

4.3 Approximate model for combined bending and stretching

We have seen (cf. (68) and (82)) that the contributions W_1 and W_2 to the strain energy are expressible as positive-definite quadratic forms. However, the remaining term, W_3, which is effective if the surface strain is non-zero, is not sign definite. Its presence implies that equilibria generated by the model may not minimize the associated approximation to the potential energy. This stands in contrast to the situation in conventional shell theory [6,14,16], which does not contain indefinite terms. We note that application of Stokes' theorem does not ameliorate this situation, whether or not the Mainardi-Codazzi equations of surface theory are used[1]. Of course this term vanishes if $\boldsymbol{\kappa}^*$ vanishes and thus if $\boldsymbol{\kappa} = 2H\mathbf{1}$; the trace yields $H = 0$ and hence $\boldsymbol{\kappa} = \mathbf{0}$, implying, in this case, that the shell is a flat plate. The energy function (61) then reduces to that derived elsewhere by the same method [11].

The presence of the indefinite term is neither here nor there insofar as a consistent estimate of the energy is concerned. However, in the general case involving non-zero surface strain it poses an obstacle to the establishment of analogs of standard theorems in the three-dimensional theory such as the minimum-energy principle. A similar situation obtains in the primary literature [3,4,6,14] and has given rise to the practice of using order-of-magnitude estimates that preserve positive definiteness while delineating the range of applicability of the resulting equations. Of course well-posedness may be restored simply by using the full strain energy associated with the ansatz described by (33). However, the resulting model is too complicated to serve as a useful substitute for the three-dimensional theory.

To assess the importance of W_3 we derive an estimate for thin shells inspired by Koiter's work [4]. Our analysis is facilitated by using an inner product and norm on the linear space of symmetric tensors defined by

$$(\mathbf{A}, \mathbf{B}) = \mathbf{A} \cdot \mathbf{C}[\mathbf{B}] \quad \text{and} \quad \|\mathbf{A}\| = \sqrt{\mathbf{A} \cdot \mathbf{C}[\mathbf{A}]}, \tag{95}$$

respectively. Thus, from $(51)_1$ and $(62)_3$,

$$W_3 = -\tfrac{1}{12}h^3(Sym\bar{\mathbf{H}}_0, Sym(\bar{\mathbf{H}}_0'\boldsymbol{\kappa}^*)), \tag{96}$$

and

$$\left|(Sym\bar{\mathbf{H}}_0, Sym(\bar{\mathbf{H}}_0'\boldsymbol{\kappa}^*))\right| \leq \left\|Sym\bar{\mathbf{H}}_0\right\| \left\|Sym(\bar{\mathbf{H}}_0'\boldsymbol{\kappa}^*)\right\| \tag{97}$$

by the Cauchy-Schwartz inequality.

To estimate the second factor on the right-hand side we use the spectral decomposition

$$\boldsymbol{\kappa}^* = \kappa_v \mathbf{u} \otimes \mathbf{u} + \kappa_u \mathbf{v} \otimes \mathbf{v}, \tag{98}$$

where $\{\kappa_u, \kappa_v\}$ are the eigenvalues of $\boldsymbol{\kappa}$ and $\{\mathbf{u}, \mathbf{v}\} \in T_\Omega$ are the associated orthonormal eigenvectors. Imposing $\kappa_u \geq \kappa_v$ without loss of generality, we derive

$$\kappa_u = H + \sqrt{H^2 - K}, \quad \kappa_v = H - \sqrt{H^2 - K}, \tag{99}$$

substitute into (78), and obtain

$$\boldsymbol{\kappa}^* = H\mathbf{1} + \sqrt{H^2 - K}\boldsymbol{\Delta}, \quad \text{where} \quad \boldsymbol{\Delta} = \mathbf{v} \otimes \mathbf{v} - \mathbf{u} \otimes \mathbf{u} \tag{100}$$

and use has been made of $\mathbf{u} \otimes \mathbf{u} + \mathbf{v} \otimes \mathbf{v} = \mathbf{1}$. Accordingly,

$$Sym(\bar{\mathbf{H}}_0' \boldsymbol{\kappa}^*) = H Sym(\bar{\mathbf{H}}_0' \mathbf{1}) + \sqrt{H^2 - K} Sym(\bar{\mathbf{H}}_0' \boldsymbol{\Delta}), \tag{101}$$

and the triangle inequality yields

$$\left\| Sym(\bar{\mathbf{H}}_0' \boldsymbol{\kappa}^*) \right\| \leq |H| \left\| Sym(\bar{\mathbf{H}}_0' \mathbf{1}) \right\| + \sqrt{H^2 - K} \left\| Sym(\bar{\mathbf{H}}_0' \boldsymbol{\Delta}) \right\|. \tag{102}$$

Each of the norms on the right-hand side is bounded above by the sum of $\bar{\mathbf{H}}_0' \mathbf{u} \cdot \mathbf{A}_u(\bar{\mathbf{H}}_0' \mathbf{u})$ and $\bar{\mathbf{H}}_0' \mathbf{v} \cdot \mathbf{A}_v(\bar{\mathbf{H}}_0' \mathbf{v})$, where $\mathbf{A}_u$ and $\mathbf{A}_v$ are the (positive definite) acoustic tensors based on $\mathbf{u}, \mathbf{v}$ (cf. (50)), yielding

$$\left\| Sym(\bar{\mathbf{H}}_0' \boldsymbol{\kappa}^*) \right\| \leq 2\kappa \{ \bar{\mathbf{H}}_0' \mathbf{u} \cdot \mathbf{A}_u(\bar{\mathbf{H}}_0' \mathbf{u}) + \bar{\mathbf{H}}_0' \mathbf{v} \cdot \mathbf{A}_v(\bar{\mathbf{H}}_0' \mathbf{v}) \}, \tag{103}$$

where $\kappa = \max_\Omega \{ |\kappa_u|, |\kappa_v| \}$ and we have used $|H| + \sqrt{H^2 - K} = \frac{1}{2}(|\kappa_u + \kappa_v| + |\kappa_u - \kappa_v|)$.

Altogether, the order-h^3 strain-energy function $\bar{W}$ satisfies (cf. (61))

$$\left| \bar{W} - \check{W} \right| \leq \tfrac{h^2}{6}(h\kappa) \left\| Sym \bar{\mathbf{H}}_0 \right\| \{ \bar{\mathbf{H}}_0' \mathbf{u} \cdot \mathbf{A}_u(\bar{\mathbf{H}}_0' \mathbf{u}) + \bar{\mathbf{H}}_0' \mathbf{v} \cdot \mathbf{A}_v(\bar{\mathbf{H}}_0' \mathbf{v}) \} + \tfrac{1}{12}(h\kappa)^2 W_1, \tag{104}$$

where

$$\check{W} = W_1 + W_2 \tag{105}$$

and so $\bar{W} \to \check{W}$ as $h\kappa \to 0$. In a sufficiently thin shell the strain-energy function is thus well approximated by $W_1 + W_2$.

This result represents an improvement over estimates derived by Koiter [4], who used an argument similar to that given here to show that the term corresponding to our $H\mathbf{P}_0 \cdot \mathbf{H}_0'$ is negligible in comparison to $\check{W}$, but resorted to the equations of strain compatibility to assess the relative importance of a term corresponding to our $\mathbf{P}_0 \cdot \mathbf{H}_0''$ (cf. (36) and (37)), summarizing the results of the involved calculations in [4]. Here, by expressing the energy directly in terms of the displacement field, we are able to combine the terms in question into the single contribution W_3.

4.4 Strain measures in the approximate model and the reduced strain-energy function

The leading term W_1 in the strain-energy function has been determined in terms of the mid-surface displacement field in Section 4.2. To determine W_2 we require $\nabla \bar{\mathbf{a}} = \bar{\mathbf{a}}_{,\beta} \otimes \mathbf{a}^\beta$, where $\bar{\mathbf{a}}_{,\beta}$ follows from (72). However, it proves advantageous to instead proceed directly from $(49)_1$. Thus,

$$(\mathbf{A}_n \bar{\mathbf{a}})_{,\beta} = -\{ \mathbf{C}[(\nabla \mathbf{u})_{,\beta}] \} \mathbf{n} - (\mathbf{C}[\nabla \mathbf{u}]) \mathbf{n}_{,\beta} - (\mathbf{C}_{,\beta}[\nabla \mathbf{u}]) \mathbf{n}, \tag{106}$$

where (cf. (50))

$$(\mathbf{A}_n\bar{\mathbf{a}})_{,\beta} = \mathbf{A}_n\bar{\mathbf{a}}_{,\beta} + (\mathbf{C}[\bar{\mathbf{a}}\otimes\mathbf{n}_{,\beta}])\mathbf{n} + (\mathbf{C}[\bar{\mathbf{a}}\otimes\mathbf{n}])\mathbf{n}_{,\beta} + (\mathbf{C}_{,\beta}[\bar{\mathbf{a}}\otimes\mathbf{n}])\mathbf{n}, \tag{107}$$

and where

$$(\nabla\mathbf{u})_{,\beta} = (\mathbf{u}_{,\alpha}\otimes\mathbf{a}^\alpha)_{;\beta} = \mathbf{u}_{;\alpha\beta}\otimes\mathbf{a}^\alpha + [(\nabla\mathbf{u})\boldsymbol{\kappa}]\mathbf{a}_\beta\otimes\mathbf{n}, \tag{108}$$

with

$$\mathbf{u}_{;\alpha\beta} = \mathbf{u}_{,\alpha\beta} - \mathbf{u}_{,\lambda}\Gamma^\lambda_{\alpha\beta}, \tag{109}$$

in which use has been made of Gauss' equation in the form $\mathbf{a}^\alpha_{;\beta} = \kappa^\alpha_\beta\mathbf{n}$.

Before proceeding we introduce the tensors

$$\rho_{\alpha\beta} = b_{\alpha\beta} - \kappa_{\alpha\beta} \quad \text{and} \quad S^\lambda_{\alpha\beta} = \Gamma'^\lambda_{\alpha\beta} - \Gamma^\lambda_{\alpha\beta}, \tag{110}$$

where $b_{\alpha\beta}$ and $\Gamma'^\lambda_{\alpha\beta}$ respectively are the coefficients of the second fundamental form and the Christoffel symbols on the deformed surface induced by the $\{\theta^\alpha\}$, regarded as a system of convected coordinates. It is well known [15] that these are insensitive to superposed rigid-body motions and so may qualify as measures of strain, provided that the tensors induced by (110) are defined in terms of bases $\{\mathbf{a}_\alpha\}$ on the tangent planes of Ω. In connection with this claim we recall that the Christoffel symbols are not tensor components. However, the *differences* of the Christoffel symbols associated with a specified coordinate system, as defined in $(110)_2$, are in fact components of a tensor ([19], problem 3.25, and [20]).

We are interested here in the linearizations of (110) in terms of the displacement field. To this end we consider a one-parameter family of displacements that vanish identically at the reference configuration, corresponding to parameter value zero. The linearizations of the relevant variables are then given by their derivatives with respect to the parameter, evaluated at the reference configuration. These are also the Gateaux (or variational) derivatives. Using superposed dots to denote them, it may be shown that [2,20]

$$\dot{b}_{\alpha\beta} = \mathbf{n}\cdot\mathbf{u}_{;\alpha\beta}. \tag{111}$$

A formula for $\dot{\Gamma}'^\lambda_{\alpha\beta}$ may be derived by using (57) with $\dot{\mathbf{a}}_\alpha = \mathbf{u}_{,\alpha}$, yielding

$$\dot{\Gamma}'^\lambda_{\alpha\beta} = (\dot{a}^{\lambda\mu}\mathbf{a}_\mu + a^{\lambda\mu}\mathbf{u}_{,\mu})\cdot\mathbf{a}_{\alpha,\beta} + \mathbf{a}^\lambda\cdot\mathbf{u}_{,\alpha\beta}, \tag{112}$$

in which

$$\dot{a}^{\lambda\mu} = -a^{\lambda\nu}a^{\gamma\mu}\dot{a}_{\nu\gamma} \quad \text{and} \quad \dot{a}_{\alpha\beta} = \mathbf{u}_{,\alpha}\cdot\mathbf{a}_\beta + \mathbf{u}_{,\beta}\cdot\mathbf{a}_\alpha, \tag{113}$$

and we note in passing that $\dot{a}_{\alpha\beta} = 2\epsilon_{\alpha\beta}$, confirming, as is well known [14], that (71) furnishes the exact linearization of the mid-surface strain. Further algebra making use of (57), (67) and (109) leads to[2]

$$\dot{\Gamma}'^\lambda_{\alpha\beta} = \mathbf{a}^\lambda\cdot\mathbf{u}_{;\alpha\beta} - \alpha^\lambda\kappa_{\alpha\beta}. \tag{114}$$

Consequently,

$$\mathbf{u}_{;\alpha\beta} = S^\lambda_{\alpha\beta}\mathbf{a}_\lambda + \kappa_{\alpha\beta}\boldsymbol{\alpha} + \rho_{\alpha\beta}\mathbf{n} \tag{115}$$

in which, here and henceforth, we use the symbols $\rho_{\alpha\beta}$ and $S^\lambda_{\alpha\beta}$ to denote the *linearizations* of $(110)_{1,2}$ (i.e., $\rho_{\alpha\beta} = \dot{b}_{\alpha\beta}$ and $S^\lambda_{\alpha\beta} = \dot{\Gamma}'^\lambda_{\alpha\beta}$). The second of these may also be obtained by substituting the metric

$a_{\alpha\beta}+2\epsilon_{\alpha\beta}$ on the deformed surface into a formula like (57) for the Christoffel symbols $\Gamma'^{\lambda}_{\alpha\beta}$ and linearizing the resulting expression with respect to $\boldsymbol{\epsilon}$. This procedure leads to the exact linearization

$$S^{\lambda}_{\alpha\beta}=a^{\lambda\mu}S_{\mu\alpha\beta},\quad \text{where}\quad S_{\mu\alpha\beta}=\epsilon_{\mu\alpha;\beta}+\epsilon_{\mu\beta;\alpha}-\epsilon_{\alpha\beta;\mu}\quad\text{and}\quad \epsilon_{\alpha\beta;\mu}=\epsilon_{\alpha\beta,\mu}-\epsilon_{\beta\lambda}\Gamma^{\lambda}_{\alpha\mu}-\epsilon_{\alpha\lambda}\Gamma^{\lambda}_{\beta\mu} \tag{116}$$

is the usual covariant derivative, and thus to the interpretation of $S^{\lambda}_{\alpha\beta}$ as a tensorial measure of strain-gradient effects. Further, (67) and (111) may be used to derive

$$\rho_{\alpha\beta}=(\mathbf{n}\cdot\mathbf{u}_{,\alpha})_{;\beta}-\mathbf{u}_{,\alpha}\cdot\mathbf{n}_{,\beta}=-\alpha_{\alpha;\beta}+\kappa^{\mu}_{\beta}(u_{\mu;\alpha}-w\kappa_{\mu\alpha}), \tag{117}$$

which is symmetric in the subscripts by virtue of the Mainardi-Codazzi equations[1]. We note that this is precisely the measure of bending strain used in the classical theory [2,6,14].

With these formulae in hand, considerable algebra based on (72) and (106)-(109) leads to the complicated expression

$$\begin{aligned}\nabla\bar{\mathbf{a}}+(\nabla\mathbf{u})\boldsymbol{\kappa} \;=\;& -\boldsymbol{\rho}-S_{\lambda\alpha\beta}\mathbf{A}_n^{-1}(\mathbf{C}[\mathbf{a}^{\lambda}\otimes\mathbf{a}^{\alpha}])\mathbf{n}\otimes\mathbf{a}^{\beta}-\mathbf{A}_n^{-1}(\mathbf{C}_{,\beta}[\bar{\mathbf{H}}_0])\mathbf{n}\otimes\mathbf{a}^{\beta}\\ &+\mathbf{A}_n^{-1}(\mathbf{C}[\bar{\mathbf{H}}_0])\boldsymbol{\kappa}+\kappa_{\alpha\beta}\mathbf{A}_n^{-1}(\mathbf{C}[\mathbf{a}^{*}\otimes\mathbf{a}^{\alpha}])\mathbf{n}\otimes\mathbf{a}^{\beta},\end{aligned} \tag{118}$$

where $\boldsymbol{\rho}=\rho_{\alpha\beta}\mathbf{a}^{\alpha}\otimes\mathbf{a}^{\beta}$,

$$\mathbf{a}^{*}=-\mathbf{A}_n^{-1}(\mathbf{C}[\boldsymbol{\epsilon}])\mathbf{n} \tag{119}$$

and

$$(\mathbf{C}[\mathbf{a}^{\lambda}\otimes\mathbf{a}^{\alpha}])\mathbf{n}=C^{\mu 3\lambda\alpha}\mathbf{a}_{\mu}+C^{33\lambda\alpha}\mathbf{n}. \tag{120}$$

The expressions for $\bar{\mathbf{b}}$ and $\bar{\mathbf{H}}'_0$ follow from $(49)_2$ and $(30)_2$, and that for W_2 is then obtained from $(62)_2$. However, this may be simplified considerably by observing that the negligible term W_3 satisfies $|W_3|/W_2=O(\kappa\epsilon)$, where $\epsilon=|\boldsymbol{\epsilon}|$. This follows from (97) and (103) together with norm-equivalence on finite-dimensional spaces. Further, (118) involves the tangential derivatives of the moduli on the midsurface, which are calculated using Gauss' equation (55) in the form $\mathbf{a}_{\alpha;\beta}=\kappa_{\alpha\beta}\mathbf{n}$. Thus,

$$\mathbf{C}_{,\beta}=\mathbf{E}_{\beta}+O(\kappa)\quad\text{in which}\quad \mathbf{E}_{\beta}=C^{ijkl}_{;\beta}\bar{\mathbf{g}}_i\otimes\bar{\mathbf{g}}_j\otimes\bar{\mathbf{g}}_k\otimes\bar{\mathbf{g}}_l\quad\text{with}\quad \bar{\mathbf{g}}_{\alpha}=\mathbf{a}_{\alpha}\quad\text{and}\quad\bar{\mathbf{g}}_3=\mathbf{n}, \tag{121}$$

wherein C^{ijkl} is the projection of $\mathbf{C}$ onto $\bar{\mathbf{g}}^i\otimes\bar{\mathbf{g}}^j\otimes\bar{\mathbf{g}}^k\otimes\bar{\mathbf{g}}^l$ and thus pertains to the midsurface. This expression accounts for functional gradients along curves in the midsurface. We envisage applications in which the shell, while functionally graded, has no *intrinsic* functional gradient; that is, applications in which $\mathbf{E}_{\beta}$ vanishes. The intrinsic properties C^{ijkl} are then *covariant-constant* although the *tensor* of elastic moduli varies due to the curvature of the midsurface. In this case the properties, referred to the *fixed* basis $\{\mathbf{a}_{\alpha}\}_p$ spanning the tangent plane $T_{\Omega(p)}$ at a point p, are independent of position on that plane.

For consistency with our suppression of W_3 we neglect all terms of order $O(\kappa\epsilon)$ in (118). For the sake of simplicity we also consider the case of covariant-constant moduli, obtaining

$$\bar{\mathbf{H}}'_0=\bar{\mathbf{K}}_0+O(\kappa\epsilon), \tag{122}$$

where

$$\bar{\mathbf{K}}_0=-\boldsymbol{\rho}-S_{\lambda\alpha\beta}\mathbf{A}_n^{-1}(\mathbf{C}[\mathbf{a}^{\lambda}\otimes\mathbf{a}^{\alpha}])\mathbf{n}\otimes\mathbf{a}^{\beta}+\bar{\mathbf{b}}\otimes\mathbf{n} \tag{123}$$

and, to consistent order,

$$\bar{\mathbf{b}} = S_{\lambda\alpha\beta}\mathbf{A}_n^{-1}\{C^{\mu 3\lambda\alpha}(\mathbf{C}[\mathbf{A}_n^{-1}\mathbf{a}_\mu \otimes \mathbf{a}^\beta])\mathbf{n} + C^{33\lambda\alpha}(\mathbf{C}[\mathbf{A}_n^{-1}\mathbf{n} \otimes \mathbf{a}^\beta])\mathbf{n}\} + \mathbf{A}_n^{-1}(\mathbf{C}[\boldsymbol{\rho}])\mathbf{n}. \tag{124}$$

To summarize, the order-h^3 potential energy is given to consistent order by

$$E[\mathbf{u}] = \int_\Omega (W - \mathbf{g}\cdot\mathbf{u})da - \int_{\partial\Omega_t} (\mathbf{p}_u\cdot\mathbf{u} + \mathbf{p}_a\cdot\bar{\mathbf{a}})ds, \tag{125}$$

in which

$$W = \tfrac{1}{2}h\bar{\mathbf{H}}_0\cdot\mathbf{C}[\bar{\mathbf{H}}_0] + \tfrac{1}{24}h^3\bar{\mathbf{K}}_0\cdot\mathbf{C}[\bar{\mathbf{K}}_0]. \tag{126}$$

It is possible to estimate the expressions (44) and (45) for the loading terms in a manner consistent with the foregoing, but for reasons explained in Section 6 this does not result in any useful simplification of the theory. The explicit expression for W is rather complicated in the general case - all the more so for functionally-graded materials - and thus best left to specific applications. It is evidently determined by the strains, the bending strains (changes in curvature) and the strain gradients. We note that in the classical models [2,3,14,16], which are restricted to materials possessing reflection symmetry with respect to the midsurface, the strain energy depends on the strains and bending strains but does not involve strain gradients. We return to this point in Section 4.5 below, where the explicit strain-energy function for such materials is also recorded.

4.5 Reflection symmetry and Koiter's theory

For our present purposes the material is deemed to possess reflection symmetry with respect to the midsurface if and only if C^{ijkl} vanishes when an odd number of indices is equal to 3 ([14] and Section 5.4 of [16])[3]. For the sake of illustration we confine attention to covariant-constant elastic moduli. Eq. (50) furnishes the orthogonal decomposition

$$\mathbf{A}_n = C\mathbf{n}\otimes\mathbf{n} + C^{\alpha 3\beta 3}\mathbf{a}_\alpha\otimes\mathbf{a}_\beta, \quad \text{where} \quad C = C^{3333} \tag{127}$$

is an eigenvalue of $\mathbf{A}_n$ and hence positive. Accordingly, $\mathbf{A}_n^{-1}\mathbf{n} = C^{-1}\mathbf{n}$, whereas

$$(\mathbf{C}[\boldsymbol{\epsilon}])\mathbf{n}\otimes\mathbf{n} = C^{33\alpha\beta}\epsilon_{\alpha\beta}\mathbf{n}\otimes\mathbf{n}, \tag{128}$$

and (73) reduces to

$$Sym\bar{\mathbf{H}}_0 = \boldsymbol{\epsilon} - C^{-1}C^{33\alpha\beta}\epsilon_{\alpha\beta}\mathbf{n}\otimes\mathbf{n}. \tag{129}$$

Using $(48)_1$ we have $\mathbf{C}[\bar{\mathbf{H}}_0] = (\mathbf{C}[\bar{\mathbf{H}}_0])\mathbf{1}$ and thus conclude that

$$\mathbf{C}[\bar{\mathbf{H}}_0] = \mathcal{M}[\boldsymbol{\epsilon}], \tag{130}$$

where

$$\mathcal{M} = \mathcal{M}^{\alpha\beta\lambda\mu}\mathbf{a}_\alpha\otimes\mathbf{a}_\beta\otimes\mathbf{a}_\lambda\otimes\mathbf{a}_\mu, \quad \text{with} \quad \mathcal{M}^{\alpha\beta\lambda\mu} = C^{\alpha\beta\lambda\mu} - C^{-1}C^{\alpha\beta 33}C^{\lambda\mu 33}. \tag{131}$$

It then follows from (129) that

$$\bar{\mathbf{H}}_0\cdot\mathbf{C}[\bar{\mathbf{H}}_0] = \boldsymbol{\epsilon}\cdot\mathcal{M}[\boldsymbol{\epsilon}]. \tag{132}$$

In the same way we use (120), (123) and (124) to derive

$$\bar{\mathbf{K}}_0 = C^{-1}C^{33\lambda\alpha}S_{\lambda\alpha\beta}(\mathbf{a}^\beta \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{a}^\beta) - \boldsymbol{\rho} + \mathbf{A}_n^{-1}(\mathbf{C}[\boldsymbol{\rho}])\mathbf{n} \otimes \mathbf{n}. \tag{133}$$

Therefore,

$$Sym\bar{\mathbf{K}}_0 = -\boldsymbol{\rho} + C^{-1}C^{33\alpha\beta}\rho_{\alpha\beta}\mathbf{n} \otimes \mathbf{n}, \tag{134}$$

which combines with $\mathbf{C}[\bar{\mathbf{K}}_0] = (\mathbf{C}[\bar{\mathbf{K}}_0])\mathbf{1}$ (modulo negligible terms of order $O(\kappa\epsilon)$) to give

$$\bar{\mathbf{K}}_0 \cdot \mathbf{C}[\bar{\mathbf{K}}_0] = \boldsymbol{\rho} \cdot \mathcal{M}[\boldsymbol{\rho}]. \tag{135}$$

The strain energy (126) thus reduces precisely to Koiter's function [3,4]

$$W = \tfrac{1}{2}h\boldsymbol{\epsilon} \cdot \mathcal{M}[\boldsymbol{\epsilon}] + \tfrac{1}{24}h^3\boldsymbol{\rho} \cdot \mathcal{M}[\boldsymbol{\rho}], \tag{136}$$

consisting of pure-stretching and pure-bending energies, in which the first term is the exact membrane strain energy (cf. Section 4.2) and, of course, strain-gradient effects are absent. In fact the second term is the exact energy for pure bending (cf. Section 4.2). This follows from (81) and the consequent relation

$$\bar{\mathbf{H}}_0' \cdot \mathbf{C}[\bar{\mathbf{H}}_0'] = \mathbf{B} \cdot \mathcal{M}[\mathbf{B}], \tag{137}$$

together with (79), which yields $B_{\alpha\beta} = -\rho_{\alpha\beta}$ where $\rho_{\alpha\beta}$ is given by (117) in which (75) is imposed.

The result $\mathbf{B} = -\boldsymbol{\rho}$ is a kinematical identity for pure bending, and eqs. (81) and (122) imply that $\bar{\mathbf{H}}_0' \cdot \mathbf{C}[\bar{\mathbf{H}}_0'] = \bar{\mathbf{K}}_0 \cdot \mathbf{C}[\bar{\mathbf{K}}_0]$ is always true in pure bending regardless of the underlying material symmetry.

5. Theorem of minimum potential energy

Because the potential energy $E[\mathbf{u}]$ defined by (125) is a quadratic functional, it follows that

$$E[\acute{\mathbf{u}}] = E[\mathbf{u}] + \dot{E}[\mathbf{u}, \dot{\mathbf{u}}] + \tfrac{1}{2}\ddot{E}[\dot{\mathbf{u}}], \tag{138}$$

where $\mathbf{u}$ and $\acute{\mathbf{u}} = \mathbf{u} + \dot{\mathbf{u}}$ are kinematically possible mid-surface displacement fields ($\dot{\mathbf{u}} = \mathbf{0}$; hence $\dot{\mathbf{u}}_s = \mathbf{0}$, and $\dot{\mathbf{u}}_\nu = \mathbf{0}$ on $\partial\Omega_u$) and $\dot{E}$, $\ddot{E}$ are the induced first and second variations of the energy. Using (126), it is straightforward to show that

$$\ddot{E}[\dot{\mathbf{u}}] = \int_\Omega \varphi da, \tag{139}$$

where

$$\varphi = h(\bar{\mathbf{H}}_0)^{\cdot} \cdot \mathbf{C}[(\bar{\mathbf{H}}_\mathbf{0})^{\cdot}] + \tfrac{1}{12}h^3(\bar{\mathbf{K}}_0)^{\cdot} \cdot \mathbf{C}[(\bar{\mathbf{K}}_0)^{\cdot}], \tag{140}$$

and $(\bar{\mathbf{H}}_0)^{\cdot}$ and $(\bar{\mathbf{K}}_0)^{\cdot}$ are given by the linear expressions $(30)_1$ and (123) in which $\mathbf{u}$ is replaced by $\dot{\mathbf{u}}$.

By definition equilibria in this model are those displacements $\mathbf{u}$ that nullify the first variation $\dot{E}[\mathbf{u}, \cdot]$[4], yielding

$$E[\acute{\mathbf{u}}] - E[\mathbf{u}] = \tfrac{1}{2}\int_\Omega \varphi da. \tag{141}$$

Because φ is non-negative, the integral can vanish only if φ vanishes pointwise; and, because φ is the sum of two positive-definite quadratic forms, this in turn requires that

$$Sym(\bar{\mathbf{H}}_0)^{\cdot} = \mathbf{0} \quad \textit{and} \quad Sym(\bar{\mathbf{K}}_0)^{\cdot} = \mathbf{0} \quad \text{in} \quad \Omega. \tag{142}$$

The first of these conditions is seen, by virtue of (73), to require that $\dot{\boldsymbol{\epsilon}}$ vanish pointwise. Using (116), it follows that $\dot{S}_{\mu\alpha\beta}$ also vanishes pointwise, leaving (cf. (123), (124))

$$Sym(\bar{\mathbf{K}}_0)^{\cdot} = -\dot{\boldsymbol{\rho}} + Sym\{\mathbf{A}_n^{-1}(\mathbf{C}[\dot{\boldsymbol{\rho}}])\mathbf{n}\otimes\mathbf{n}\}, \tag{143}$$

and $(142)_2$ is then satisfied if and only if $\dot{\boldsymbol{\rho}}$ vanishes pointwise. We note that the local basis $\{\mathbf{a}_\alpha, \mathbf{n}\}$, the curvature and the metric involved in the formulae for the various kinematical quantities pertain to Ω and are therefore fixed when computing the associated variations.

The condition $\dot{\boldsymbol{\epsilon}} = \mathbf{0}$ implies, as in (84), that

$$\dot{\mathbf{u}}_{,\alpha} = \dot{\mathbf{v}}\times\mathbf{a}_\alpha \tag{144}$$

for some vector field $\dot{\mathbf{v}}$, yielding $\dot{\mathbf{u}}_{;\alpha\beta} = \dot{\mathbf{v}}_{,\beta}\times\mathbf{a}_\alpha + \kappa_{\alpha\beta}\dot{\mathbf{v}}\times\mathbf{n}$. Use of the linearization of $\dot{\boldsymbol{\rho}}$ (cf. (111)) results in

$$\dot{\rho}_{\alpha\beta} = \mathbf{n}\cdot\dot{\mathbf{u}}_{;\alpha\beta} = \varepsilon_{\lambda\alpha}\dot{\mathbf{v}}_{,\beta}\cdot\mathbf{a}^\lambda, \tag{145}$$

while the integrability condition for (144) is

$$\mathbf{0} = \varepsilon^{\alpha\beta}\dot{\mathbf{u}}_{;\alpha\beta} = \varepsilon^{\alpha\beta}\dot{\mathbf{v}}_{,\beta}\times\mathbf{a}_\alpha, \tag{146}$$

implying that $\dot{\boldsymbol{\rho}}$ is symmetric as required. The latter equation may be used to show that $\mathbf{n}\cdot\dot{\mathbf{v}}_{,\beta} = 0$ (cf. $(87)_2$, with $\mathbf{v}$ replaced by $\dot{\mathbf{v}}$) and therefore that $\dot{\mathbf{v}}_{,\beta} = (\mathbf{a}^\mu\cdot\dot{\mathbf{v}}_{,\beta})\mathbf{a}_\mu$. On the other hand, the condition $\dot{\rho}_{\alpha\beta} = 0$, when multiplied by $\varepsilon^{\mu\alpha}$, yields $\dot{\mathbf{v}}_{,\beta}\cdot\mathbf{a}^\mu = 0$ and therefore $\dot{\mathbf{v}}_{,\beta} = \mathbf{0}$. Thus $\dot{\mathbf{v}}$ is a constant vector and (144) delivers $\dot{\mathbf{u}} = \dot{\mathbf{v}}\times\mathbf{r} + \mathbf{c}$, where $\mathbf{c}$ is also constant, resulting in a rigid-body displacement field. This conclusion may be recognized as the linear version of the fundamental theorem of surface theory: a surface is determined, apart from translation and orientation, by its metric and curvature. The data on $\partial\Omega_u$ require $\dot{\mathbf{v}}\times\boldsymbol{\tau}$ and $\dot{\mathbf{v}}\times\boldsymbol{\nu}$ to vanish simultaneously, and this is possible only if $\dot{\mathbf{v}}$ vanishes, yielding $\dot{\mathbf{u}} = \mathbf{c}$ which also vanishes by virtue of the data.

We conclude that φ vanishes in Ω if and only if $\dot{\mathbf{u}}$ vanishes identically; that is, if and only if the configuration of the shell is fixed. For variations $\dot{\mathbf{u}}$ that do not vanish identically, φ is strictly positive in a set of non-zero area measure and its integral over Ω is then strictly positive. Eq. (141) leads finally to the conclusion that an equilibrium displacement field satisfies $E[\mathbf{u}+\dot{\mathbf{u}}] \geq E[\mathbf{u}]$ for all kinematically possible $\dot{\mathbf{u}}$ that do not vanish identically. The minimum is strict if $\partial\Omega_u$ is non-empty and non-strict in the alternative case as rigid-body displacements are then admissible. Further, this result is easily shown to remain valid if the restriction to covariant-constant moduli is relaxed.

We note that the same conclusion holds in the case of pure bending. This follows immediately on replacing $\bar{\mathbf{K}}_0$ by $\bar{\mathbf{H}}_0'$ in (126) and invoking (81) to conclude that $E[\mathbf{u}+\dot{\mathbf{u}}] \geq E[\mathbf{u}]$ if $\mathbf{u}$ is an equilibrium displacement field, reducing to an equality if and only if $\dot{\mathbf{B}}$ vanishes; and thus, with reference to the remarks following (137), if and only if $\dot{\boldsymbol{\rho}}(=-\dot{\mathbf{B}})$ vanishes.

The foregoing formalism also facilitates a standard uniqueness theorem for equilibria. This proceeds exactly as in the classical Kirchhoff theorem for the three-dimensional theory and is therefore omitted.

6. Euler equations and boundary data

We have seen that the strain-energy function (126) in the simplified theory for combined bending and stretching is determined by the strain measures $\epsilon_{\alpha\beta}$, $S^{\lambda}_{\alpha\beta}$ and $\rho_{\alpha\beta}$, which in turn are functions of the mid-surface displacement derivatives $\mathbf{u}_{,\alpha}$ and $\mathbf{u}_{;\alpha\beta}$; thus, $W = F(\mathbf{u}_{,\alpha}, \mathbf{u}_{;\alpha\beta})$ for some function F. Accordingly, its variational derivative is expressible in the form

$$\dot{F} = \mathbf{N}^{\alpha} \cdot \dot{\mathbf{u}}_{,\alpha} + \mathbf{M}^{\alpha\beta} \cdot \dot{\mathbf{u}}_{;\alpha\beta} \tag{147}$$

in which $\mathbf{M}^{\alpha\beta} = \mathbf{M}^{\beta\alpha}$.

The Euler equation for (125) may be summarized concisely as [9]

$$\mathbf{T}^{\alpha}_{;\alpha} + \mathbf{g} = \mathbf{0} \quad \text{in} \quad \Omega, \tag{148}$$

where

$$\mathbf{T}^{\alpha} = \mathbf{N}^{\alpha} - \mathbf{M}^{\beta\alpha}_{;\beta} \tag{149}$$

and

$$\mathbf{M}^{\beta\alpha}_{;\beta} = \mathbf{M}^{\beta\alpha}_{,\beta} + \mathbf{M}^{\beta\alpha}\Gamma^{\lambda}_{\lambda\beta} + \mathbf{M}^{\beta\lambda}\Gamma^{\alpha}_{\lambda\beta}. \tag{150}$$

It is customary to decompose the equilibrium equation into tangential and normal parts, although doing so rarely facilitates its analysis. Here we do so for the purpose of facilitating comparison with the classical theory, and to this end we write

$$\mathbf{N}^{\alpha} = N^{\beta\alpha}\mathbf{a}_{\beta} + N^{\alpha}\mathbf{n}, \quad \mathbf{M}^{\alpha\beta} = M^{\lambda\alpha\beta}\mathbf{a}_{\lambda} + M^{\alpha\beta}\mathbf{n}, \tag{151}$$

substitute into (149) and invoke the Gauss and Weingarten equations, obtaining

$$\mathbf{T}^{\alpha} = (N^{\lambda\alpha} + M^{\beta\alpha}\kappa^{\lambda}_{\beta} - M^{\lambda\beta\alpha}_{;\beta})\mathbf{a}_{\lambda} + (N^{\alpha} - M^{\beta\alpha}_{;\beta} - M^{\lambda\beta\alpha}\kappa_{\lambda\beta})\mathbf{n}, \tag{152}$$

where

$$M^{\lambda\beta\alpha}_{;\beta} = M^{\lambda\beta\alpha}_{,\beta} + M^{\lambda\beta\alpha}\Gamma^{\mu}_{\mu\beta} + M^{\lambda\beta\mu}\Gamma^{\alpha}_{\mu\beta} + M^{\mu\beta\alpha}\Gamma^{\lambda}_{\mu\beta}. \tag{153}$$

Projection of (148) onto $\mathbf{a}^{\mu}$ and $\mathbf{n}$ yields[5]

$$(N^{\mu\alpha} + M^{\beta\alpha}\kappa^{\mu}_{\beta} - M^{\mu\beta\alpha}_{;\beta})_{;\alpha} + (M^{\beta\alpha}_{;\beta} + M^{\lambda\beta\alpha}\kappa_{\lambda\beta} - N^{\alpha})\kappa^{\mu}_{\alpha} + g^{\mu} = 0 \tag{154}$$

and

$$(N^{\alpha} - M^{\beta\alpha}_{;\beta} - M^{\lambda\beta\alpha}\kappa_{\lambda\beta})_{;\alpha} + (N^{\beta\alpha} + M^{\lambda\alpha}\kappa^{\beta}_{\lambda} - M^{\beta\mu\alpha}_{;\mu})\kappa_{\beta\alpha} + g = 0, \tag{155}$$

respectively, where $g = \mathbf{g} \cdot \mathbf{n}$.

As for the boundary conditions, if $\partial\Omega$ is smooth[6] the residual virtual-work statement obtained from (125), with (148) satisfied, reduces to [9]

$$\int_{\partial\Omega} \{[\mathbf{T}^{\alpha}\nu_{\alpha} - (\mathbf{M}^{\alpha\beta}\nu_{\alpha}\tau_{\beta})_{s}] \cdot \dot{\mathbf{u}} + \mathbf{M}^{\alpha\beta}\nu_{\alpha}\nu_{\beta} \cdot \dot{\mathbf{u}}_{\nu}\} ds = \int_{\partial\Omega_t} [(\mathbf{p}_u \cdot \dot{\mathbf{u}} + \mathbf{p}_a \cdot (\bar{\mathbf{a}})^{\cdot}] ds, \tag{156}$$

where τ_{α} and ν_{α} respectively are the covariant components of the unit tangent $\boldsymbol{\tau}$ and unit normal $\boldsymbol{\nu}$ to the edge, superposed dots refer to variational (Gateaux) derivatives, and use has been made of (63) in the variational form

$$\dot{\mathbf{u}}_{,\alpha} = \tau_{\alpha}\dot{\mathbf{u}}_{s} + \nu_{\alpha}\dot{\mathbf{u}}_{\nu}. \tag{157}$$

From (72) it follows that the variation $(\bar{\mathbf{a}})^{\cdot}$ is a linear function of the $\dot{\mathbf{u}}_{,\alpha}$ and thus of $\dot{\mathbf{u}}_s$ and $\dot{\mathbf{u}}_\nu$. It therefore vanishes together with $\dot{\mathbf{u}}$ on $\partial\Omega_u$, permitting extension of the domain of integration on the right-hand side of (156) to $\partial\Omega$. Integration by parts on $\partial\Omega$ may then be used to express the virtual work of the edge loads in the form

$$\int_{\partial\Omega_t}[(\mathbf{p}_u\cdot\dot{\mathbf{u}}+\mathbf{p}_a\cdot(\bar{\mathbf{a}})^{\cdot}]ds=\int_{\partial\Omega_t}(\mathbf{f}\cdot\dot{\mathbf{u}}+\mathbf{c}\cdot\dot{\mathbf{u}}_\nu)ds, \tag{158}$$

wherein $\dot{\mathbf{u}}$ and $\dot{\mathbf{u}}_\nu$ may be specified independently on $\partial\Omega_t$. The procedure for effecting this reduction is described in detail in [11]. In practice $\mathbf{f}$ and $\mathbf{c}$ are assigned as data and the detailed function giving $\{\mathbf{f},\mathbf{c}\}$ in terms of $\{\mathbf{p}_u,\mathbf{p}_a\}$ is thus of secondary interest.

Equations (156) and (158) finally yield the natural boundary conditions

$$\mathbf{f}=\mathbf{T}^\alpha\nu_\alpha-(\mathbf{M}^{\alpha\beta}\nu_\alpha\tau_\beta)_s\quad\text{and}\quad\mathbf{c}=\mathbf{M}^{\alpha\beta}\nu_\alpha\nu_\beta\quad\text{on}\quad\partial\Omega_t, \tag{159}$$

in which $\mathbf{f}$ and $\mathbf{c}$ are the assigned force and couple per unit length of the boundary.

It remains is to express the variables $\mathbf{N}^\alpha$ and $\mathbf{M}^{\alpha\beta}$ in terms of the strain measures. The easiest way to proceed is to write the variational derivative of the strain energy W in the form (147) and then identify coefficients. For example, the variation of the coefficient of h in the strain-energy function is

$$\tfrac{1}{2}(\bar{\mathbf{H}}_0\cdot\mathbf{C}[\bar{\mathbf{H}}_0])^{\cdot}=\mathbf{C}[\bar{\mathbf{H}}_0]\cdot(\bar{\mathbf{H}}_0)^{\cdot}=(\mathbf{C}[\bar{\mathbf{H}}_0])\mathbf{1}\cdot\nabla\dot{\mathbf{u}}=(\mathbf{C}[\bar{\mathbf{H}}_0])\mathbf{1}\mathbf{a}^\alpha\cdot\dot{\mathbf{u}}_{,\alpha}, \tag{160}$$

In the same way, the variation of the coefficient of $h^3/12$ is

$$\tfrac{1}{2}(\bar{\mathbf{K}}_0\cdot\mathbf{C}[\bar{\mathbf{K}}_0])^{\cdot}=(\mathbf{C}[\bar{\mathbf{K}}_0])\mathbf{1}\mathbf{a}^\beta\cdot[\bar{\mathbf{a}}_{,\beta}+(\nabla\mathbf{u})\boldsymbol{\kappa}\mathbf{a}_\beta]^{\cdot}, \tag{161}$$

where, from (123),

$$[\bar{\mathbf{a}}_{,\beta}+(\nabla\mathbf{u})\boldsymbol{\kappa}\mathbf{a}_\beta]^{\cdot}=-\dot{S}_{\lambda\alpha\beta}\mathbf{A}_n^{-1}(\mathbf{C}[\mathbf{a}^\lambda\otimes\mathbf{a}^\alpha])\mathbf{n}-\dot{\rho}_{\alpha\beta}\mathbf{a}^\alpha \tag{162}$$

in the case of covariant-constant moduli.

We base the calculation of $\dot{S}_{\lambda\alpha\beta}$ and $\dot{\rho}_{\alpha\beta}$ on (67) and (115), obtaining

$$\dot{S}_{\lambda\alpha\beta}=\mathbf{a}_\lambda\cdot\dot{\mathbf{u}}_{;\alpha\beta}+\kappa_{\alpha\beta}\mathbf{n}\cdot\dot{\mathbf{u}}_{,\lambda}\quad\text{and}\quad\dot{\rho}_{\alpha\beta}=\mathbf{n}\cdot\dot{\mathbf{u}}_{;\alpha\beta}. \tag{163}$$

Using (147) and (151) we then have

$$\dot{W}=[N^{\beta\mu}\mathbf{a}_\beta+(N^\mu-M^{\mu\alpha\beta}\kappa_{\alpha\beta})\mathbf{n}]\cdot\dot{\mathbf{u}}_{,\mu}+M^{\alpha\beta}\dot{\rho}_{\alpha\beta}+M^{\lambda\alpha\beta}\dot{S}_{\lambda\alpha\beta}. \tag{164}$$

Taking account of the symmetry of $\dot{S}_{\lambda\alpha\beta}$ in the second pair of subscripts we thus derive

$$\begin{aligned}N^{\beta\mu}&=h\mathbf{a}^\beta\cdot\mathbf{C}[\bar{\mathbf{H}}_0])\mathbf{a}^\mu,\quad M^{\beta\mu}=-\tfrac{1}{12}h^3\mathbf{a}^\beta\cdot(\mathbf{C}[\bar{\mathbf{K}}_0])\mathbf{a}^\mu,\quad N^\mu=M^{\mu\alpha\beta}\kappa_{\alpha\beta}+h\mathbf{n}\cdot(\mathbf{C}[\bar{\mathbf{H}}_0])\mathbf{a}^\mu,\\ M^{\lambda\alpha\beta}&=-\tfrac{1}{24}h^3\{(C^{\mu3\lambda\alpha}\mathbf{A}_n^{-1}\mathbf{a}_\mu+C^{33\lambda\alpha}\mathbf{A}_n^{-1}\mathbf{n})\cdot(\mathbf{C}[\bar{\mathbf{K}}_0])\mathbf{a}^\beta...\},\end{aligned} \tag{165}$$

where the missing term is the same as that shown explicitly, but with the indices α and β interchanged. Moreover, using (67) and (72), with some effort we obtain

$$\mathbf{n}\cdot(\mathbf{C}[\bar{\mathbf{H}}_0])\mathbf{a}^\mu=-C^{\mu3\beta3}\mathbf{a}_\beta\cdot\mathbf{A}_n^{-1}(\mathbf{C}[\boldsymbol{\epsilon}])\mathbf{n}. \tag{166}$$

The minor symmetries of $\mathbf{C}$ imply that the expressions given for $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are symmetric. Further, the $M^{\lambda\alpha\beta}$ vanish identically for materials exhibiting reflection symmetry. In this case

$\mathbf{A}_n^{-1}(\mathbf{C}[\boldsymbol{\epsilon}])\mathbf{n} = C^{-1}C^{33\alpha\beta}\epsilon_{\alpha\beta}\mathbf{n}$, implying that the N^μ also vanish while (154) and (155) reduce precisely to the equilibrium equations of Koiter's theory ([2,6]).

Finally, we note that inextensional bending theory is covered by (154) and (155) on the understanding that $N^{\alpha\beta}$ is to be replaced by a symmetric matrix of Lagrange multipliers. These are needed to accommodate the constraints $\dot{\epsilon}_{\alpha\beta} = 0$ in the course of deriving the equilibrium equations from the stationarity of the energy. The use of such multipliers, which are determined solely by the equilibrium equations and boundary conditions, is implicit in the treatments of inextensional bending in [7] and [16].

Footnotes:

1. These are $\kappa^\lambda_{\alpha;\beta} = \kappa^\lambda_{\beta;\alpha}$ and are equivalent to the two equations $\kappa^{*\alpha\beta}_{;\beta} = 0$.

2. To interpret $\boldsymbol{\alpha}$ in kinematical terms we use (67) to obtain $\boldsymbol{\alpha} = -(\nabla\mathbf{u})^t\mathbf{n} = -\mathbf{a}^\alpha(\mathbf{n}\cdot\mathbf{u}_{,\alpha})$. But $\mathbf{n}\cdot\mathbf{u}_{,\alpha} = -\mathbf{a}_\alpha\cdot\dot{\mathbf{n}}$, where $\dot{\mathbf{n}}$ is the linearized change in the orientation of the midsurface. It then follows from $\dot{\mathbf{n}}\cdot\mathbf{n} = 0$ that $\boldsymbol{\alpha} = \dot{\mathbf{n}}$. This represents rotations about axes in the tangent plane to the midsurface. Rotation about the normal is described by the scalar field ω discussed in Section 4.2. Together, $\boldsymbol{\alpha}$ and ω comprise a rotation 3-vector that figures in Koiter's treatment of linear shell kinematics [6].

3. The characterization of reflection symmetry in [16] presumes the use of a basis $\{\mathbf{g}_i\}$ for which $\mathbf{g}_3\cdot\mathbf{g}_\alpha = 0$. This provision is satisfied in the present setting because $\mathbf{g}_3 = \mathbf{n}$ and $\mathbf{g}_\alpha = \boldsymbol{\mu}\mathbf{a}_\alpha \in T_\Omega$ (cf. (12)).

4. This of course is the virtual-work principle.

5. These correct eqs. (184) and (185) of [9].

6. Piecewise smooth edges may support *corner forces* at the points where $\boldsymbol{\tau}$ is discontinuous [14].

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