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Abstract. We give a short argument why the tensor product norm on $K \otimes_k L$ is multiplicative when k is an algebraically closed valued field and K and L are valued extensions (valued in $\mathbf{R}$). When the valuation on k is non trivial we use the fact that $ACVF$, the theory of algebraically closed (non trivially) valued fields, has quantifier elimination.

It is a classical fact that any two extensions K and L of an algebraically closed field k the ring $K \otimes_k L$ is an integral domain (and this characterises algebraically closed fields). When K and L (and therefore k) are valued in $(\mathbf{R}^{\geq 0}, \cdot)$, the tensor product carries a natural norm; by analogy with the non valued case, if k is also algebraically closed, we would expect this norm to be "prime", i.e., multiplicative, extending to a valuation of the fraction field. This is indeed proved by Jérôme Poineau [Poi, Corollaire 3.14], with both the result and the proof stated in the language of Berkovich spaces, making them fairly obscure to those not familiar with this formalism (such as the author, who is indebted to Amaury Thuillier for having pointed to and explained Poineau's result). Here we propose a more direct proof, using quantifier elimination for the theory of algebraically closed valued fields.

Definition 1. A *valued field* is a pair $k = (k, O^k) = (k, O)$ where k is a field and the *valuation ring* $O \subseteq k$ is a sub-ring such that $k = O \cup \big(O \setminus \{0\}\big)^{-1}$. We call $(\Gamma, \cdot) = k^\times/O^\times$ the *value group*, let $|\cdot|\colon k^\times \to \Gamma$ denote the quotient map, and add a formal symbol $0 = |0|$. We order $\Gamma \cup \{0\}$ by $|a| \leq |b| \iff a \in bO$. By a *standard valued field* we mean one together with an embedding of Γ in $(\mathbf{R}^{>0}, \cdot)$.

An *embedding* of valued fields must respect the valuation ring (in both directions), and therefore induces an embedding of the value groups. An embedding of standard valued fields is also required to respect the embeddings of the value groups in the reals.

We refer the reader to any standard textbook on model theory, e.g., Poizat [Poi85] for a general discussion of elementary extensions and ultra-powers.

Fact 2. *If $\mathcal{M}$ is any structure, in the sense of first order logic (e.g., a valued field, or a pair of a valued field and a sub-field) and U is an ultra-filter then the ultra-power $\mathcal{M}^U$ is an elementary extension of k. Conversely, every elementary extension $\mathcal{N} \succeq \mathcal{M}$ embeds over $\mathcal{M}$ in some ultra-power of $\mathcal{M}$.*

When K/k is a field extension, let $\langle \ldots \rangle_k$ denote the span in K viewed as a k-vector space.

Lemma 3. *Let K/k be an extension of valued fields.*

(i) *Let U be an ultra-filter, $X \subseteq K$, $y \in K$, and r rational such that $|y| \leq r|y'|$ for all $y' \in y + \langle X \rangle_k$. Then the embeddings $k \subseteq k^U \subseteq K^U$ and $k \subseteq K \subseteq K^U$ commute, and $|y| \leq r|y'|$ for all $y' \in y + \langle X \rangle_{k^U}$ as well.*

(ii) *If k is algebraically closed and non trivially valued, then there exists an ultra-filter U and an embedding $\iota\colon K \to k^U$ which is the identity on k.*

(iii) *If K/k in the previous item is moreover an extension of standard valued fields then the embedding $\Gamma^k \subseteq \mathbf{R}^{>0}$ induces $\Gamma^{k^U} \subseteq (\mathbf{R}^U)^{>0}$, and $|a| = \operatorname{st} |\iota a|$ for all $a \in K^\times$, where* st *denotes the standard part map (so in particular, $|\iota a| \in \mathbf{R}^U$ lies in the convex hull of $\mathbf{R}^+$).*

Proof. For the first assertion, we use the fact that the pair $(K, k)^U = (K^U, k^U)$ is an elementary extension of (K, k). For the second assertion, we may assume that K is algebraically closed. Since the theory of algebraically closed, non trivially valued fields (commonly denoted $ACVF$) has quantifier elimination (see [HHM06] – all we need in fact is that it is model complete), we have $K \succeq k$, so K embeds in an ultra-power of k. For the last assertion, this happens automatically for any $a \in k^\times$, $|a| \neq 1$, which in turn implies the same for all $a \in K$. $\blacksquare_3$

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Definition 4 ([BGR84, 2.1.7])**.** Let k be a standard valued field, K and L two standard extensions. For $z \in K \otimes_k L$ we define

$$\|z\| = \inf_{z=\bar{x}^t \otimes \bar{y}} \max_i |x_i||y_i|,$$

where $\bar{x}^t \otimes \bar{y} = \sum_i x_i \otimes y_i$.

Lemma 5. *Let k be a valued field, $K \supseteq k$ an extension, and let $\bar{x} \in K^m$ and $\bar{r} \in \mathbf{Q}^m$ be such that $0 < |x_i| \leq r_i|x|$ for all $x \in x_i + \langle x_{<i}\rangle_k$.*

(i) *For every $\bar{a} \in k^m$:*

$$|\bar{a}^t \cdot \bar{x}| \prod r_i \geq \max_i |a_i||x_i|.$$

(ii) *Assume that K and L are standard valued field extensions of k. Then for every $\bar{y} \in L^m$:*

$$\|\bar{x}^t \otimes \bar{y}\| \prod r_i \geq \max_i |x_i||y_i|.$$

Proof. For the first assertion, let $s_k = \prod_{i<k} r_i$, let $t = \max_i |a_i x_i|$, and let k be maximal such that $t \leq s_k|a_k x_k|$. Then $s_{k+1}|\sum_{i\leq k} a_i x_i| \geq s_k|a_k x_k| \geq t$ and $s_{k+1}|a_i x_i| \leq s_i|a_i x_i| < t$ for $i > k$, so $s_m|\bar{a}^t \cdot \bar{x}| \geq s_{k+1}|\bar{a}^t \cdot \bar{x}| \geq t$.

For the second, let us show that if $\bar{x}^t \otimes \bar{y} = \bar{x}'^t \otimes \bar{y}'$ then $s_m \max |x'_i||y'_i| \geq \max_i |x_i||y_i|$. Applying identities of the form $y'_i = \sum_{i\neq j} c_j y'_j$ with $c_j \in k$ and $|y'_i| \geq |c_j y'_j|$ for $j \neq i$ and developing, $\max |x'_i||y'_i|$ can only decrease, reducing to the case where $\bar{y}'$ are independent. Then $\bar{x}' \subseteq \langle \bar{x}\rangle_k$ (otherwise, one can construct k-linear functionals $\lambda \in \langle \bar{x}, \bar{x}'\rangle_k^*$, $\mu \in \langle \bar{y}, \bar{y}'\rangle_k^*$ such that $(\lambda \otimes \mu)(z) = 1$ but λ vanishes on $\langle \bar{x}\rangle_k$, which is absurd), i.e., $x'_i = \sum_{j<m} a_{ij} x_j$. Since the hypothesis implies that $\bar{x}$ are also independent, $y_j = \sum_{i<n} a_{ij} y'_i$. Thus

$$s_m \max_i |x'_i||y'_i| \geq \max_{i,j} |a_{ij}||x_j||y'_i| \geq \max_j |x_j||y_j|,$$

as desired. $\blacksquare_5$

Theorem 6. *Let k be a standard algebraically closed valued field, let $K, L \supseteq k$ be standard valued field extensions, and let $A = K \otimes_k L$. Then the norm $\|\cdot\|$ is multiplicative on A and non zero on $A \smallsetminus \{0\}$, extending to valuation on $F = \mathrm{Frac}(A)$ making it an extension of both K and L.*

Proof. It is clear that $\|\cdot\|$ is ultra-metric on A, and by the above it is non zero on $A \smallsetminus \{0\}$, so it is enough to show that $\|\cdot\|$ is multiplicative.

Assume first that the valuation on k is not trivial, and let $\iota\colon L \hookrightarrow k^U$ be an embedding as per Lemma 3(ii). Since K is a sub-field of K^U, this gives us a natural map $\iota\colon A \to K^U$ such that $\|z\| \geq \mathrm{st}\,|\iota z|$ for all $z \in A$. For the converse inequality, let $r > 1$ be rational and write $z = \bar{x}^t \otimes \bar{y}$, say of length m, where $|x_i| \leq r|x|$ for all $x \in x_i + \langle x_{<i}\rangle_k$, and by Lemma 3(i), also for all $x \in x_i + \langle x_{<i}\rangle_{k^U} \subseteq K^U$, so

$$r^m\, \mathrm{st}\,|\iota z| \geq \mathrm{st} \max_i |x_i \iota y_i| = \max_i |x_i||y_i| \geq \|z\|.$$

Since $r > 1$ was arbitrary, $\mathrm{st}\,|\iota z| = \|z\|$, and since $|\cdot|$ is multiplicative in K^U, it is in A as well. Notice that a similar argument works if k and at least one of K or L is trivially valued, using quantifier elimination in ACF.

When k is trivially valued (and neither K or L is) we need a different argument. Call $z \in A$ (α, β)*-pure* if it can be written as $\bar{x}^t \otimes \bar{y}$ with $|x_i| = \alpha$ and $|y_i| = \beta = \|z\|/\alpha$ for all i. When $z, z' \in A$ are pure, we can multiply them by elements of K and L to reduce to the case where both are $(1,1)$-pure, which case $\|zz'\| = \|z\|\|z'\|$ holds since the tensor product of the residue fields is an integral domain.

Since k is trivial, on any finite-dimensional $V \subset K$ there are only $\dim V + 1$ possible valuations. Therefore any $z = \bar{x}^t \otimes \bar{y}$ can be re-written as $\bar{u}^t \otimes \bar{v}$, where $u_i \in x_i + \langle x_{<i}\rangle$ has least value (possibly zero), in which case $\|z\| = \max_i |u_i||v_i|$ by Lemma 5(ii). Thus every z can be written as $z_0 + \bar{x}^t \otimes \bar{y}$, where z_0 is (α, β)-pure for some α, β, and $\big(|x_i||y_i|, |x_i|\big) < (\alpha\beta, \alpha)$ in lexicographic order for all i. Conversely, by applying the same procedure, every z which can be written in this manner must have norm $\alpha\beta$. This observation together with the case of product of two pure elements yields the desired result. $\blacksquare_6$

Remark 7. In our definitions we only considered non Archimedean valued fields. More generally, an *absolute value* on a field k is a map $|\cdot|\colon k \to \mathbf{R}^{\geq 0}$, satisfying $|ab| = |a||b|$, $|a + b| \leq |a| + |b|$, $|0| = 0$ and $|1| = 1$. It is a standard fact (e.g., Artin [Art67]) that such a valuation is either *Archimedean*, i.e., $|a| = |\iota^k a|^\alpha$, where $\alpha = \log_2 |2| \in (0, 1]$ and $\iota^k\colon k \to \mathbf{C}$ is uniquely determined up to complex conjugation, or is a standard valuation as defined above. In particular, if K/k is an extension of valued fields in this sense, then one is Archimedean if and only if the other is, in which case we may choose ι^K so that that

$\iota^k \subseteq \iota^K$, and if k is algebraically closed (or merely such that the image of ι^k is not contained in $\mathbf{R}$), this determines ι^K. When K and L are two extensions of an algebraically closed Archimedean valued field k, we can define on $K \otimes_k L$:

$$\|\bar{x}^t \otimes \bar{y}\| = |\iota^K \bar{x}^t \cdot \iota^L \bar{y}|.$$

This is clearly multiplicative, but may have non trivial kernel (e.g., when $K = L \neq k$), inducing an Archimedean absolute value on $\operatorname{Frac}(A/\ker\|\cdot\|)$.

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