On the Proper Form of the Amplitude Modulation Equations for Resonant Systems

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Abstract. The complex amplitude modulation equations of a discrete dynamical system are derived under general conditions of simultaneous internal and external resonances. Alternative forms of the real amplitude and phase equations are critically discussed. First, the most popular polar form is considered. Its properties, known in literature for a multitude of specific problems, are here proven for the general case. Moreover, the drawbacks encountered in the stability analysis of incomplete motions (i.e. motions containing some zero amplitudes) are discussed as a consequence of the fact the equations are not in standard normal form. Second, a so-called Cartesian rotating form is introduced, which makes it possible to evaluate periodic solutions and analyze their stability, even if they are incomplete. Although the rotating form calls for the enlargement of the space and is not amenable to analysis of transient motions, it systematically justifies the change of variables sometimes used in literature to avoid the problems of the polar form. Third, a mixed polar-Cartesian form is presented. Starting from the hypothesis that there exists a suitable number of amplitudes which do not vanish in any motion, it is proved that the mixed form leads to standard form equations with the same dimension as the polar form. However, if such principal amplitudes do not exist, more than one standard form equation should be sought. Finally, some illustrative examples of the theory are presented.

Keywords: Perturbation methods, resonant systems, stability, classes of motion, standard form.

1. Introduction

Much attention has been paid in the literature to the analysis of dynamic interaction phenomena occurring in weakly nonlinear systems under internal and/or external resonance conditions. The Multiple Scale Method (MSM) is the most popular analytical tool employed to investigate such behavior [1]. The method leads to N evolution equations in the unknown complex amplitudes $A_n = 1/2 a_n \exp(i\theta_n)$, N being the number of eigenmodes involved in the resonance conditions. These equations are generally non-autonomous and are called Amplitude Modulation Equations (AME). Under the hypothesis that the remaining eigenmodes decay, the AME describe the asymptotic dynamics in the $2N$-dimensional (a_n, θ_n)-space, which therefore represents the center manifold of the dynamical system.

The analysis of the system's behavior usually starts with the search for periodic motions. However, since complex amplitudes are involved in the equations of motion rather than state variables, periodic motions *are not* the fixed points of the AME. In fact, while the real amplitudes a_n remain constant during a periodic motion, the phases θ_n vary linearly in time, as a result of the tuning of the nearly-resonant linear frequencies. To make the mathematical problem easier, the $2N$ amplitude-phase equations are combined and a (generally) reduced system of $M = N + R$ equations is obtained, R being the number of the linearly independent resonance conditions (usually $R < N$, except in certain special forced cases in which $R = N$) [1]. This step requires the introduction of R phase-differences γ_r, which remain constant during a periodic motion. Therefore, the fixed points of the (a_n, γ_r)-Reduced Amplitude Modulation Equations (RAME) actually *are* the periodic motions of the original system. The choice of the phases γ_r is naturally suggested by the procedure itself, and is often justified in literature by invoking the need to render the AME autonomous [1]. The number M of the RAME is called the *codimension* of the problem [2], since, in the system's parameter space, it is the codimension of the manifold on which N eigenvalues simultaneously cross the imaginary axis and, in addition, their imaginary parts satisfy R resonance constraints.

The possibility of reducing the AME to its codimension has the following remarkable meaning. Although the asymptotic motion develops in a $2N$-dimensional space, its *essential* aspects (e.g., the existence of periodic motions and their stability) can be described in a smaller M-dimensional space, where the evolution equations are autonomous. The remaining $2N–M$ dimensions govern *complementary* aspects of the motion (e.g., the frequency corrections), not affecting the qualitative character of the solution [3].

Nevertheless, the $a\gamma$-form of the RAME still entails some computational difficulty. The equations appear in a *non-standard* form, since the γ-equations admit some amplitudes a_h as factors which cannot be eliminated if a_h identically vanish in some classes of motion. In such cases some γ-equations are identically satisfied and the relevant γ's remain undetermined. This circumstance is an obvious consequence of having expressed the complex amplitudes A_n in polar form, which in fact leaves the phase of zero undetermined. From a geometrical point of view such fixed points are *not-isolated points*, lying on a manifold in the (a_n, γ_r) space. When the stability of these points is analyzed, the *standard method fails*, since some coefficients of the time-derivative $\delta\gamma_r'$ vanish in the variational equation, so that more difficult *ad hoc* methods must be employed.

To overcome this drawback, it is customary in the literature to return to the complex AME and perform the following steps: first, the amplitudes A_n are expressed in Cartesian (rather than polar) form; then, they are multiplied by *suitable time-exponential factors*, so to render the equation autonomous; finally, the variational equation is built up [4–9]. In these new variables the non-isolated fixed points of the $a\gamma$-form equations reduce to isolated points, so that the standard variational equation procedure works well.

Although the procedure described leads to correct results, it is unsatisfactory for the following reasons: (a) The use of different equations to analyze first steady-state solutions and then their stability is somewhat tedious from a computational point of view. (b) It appears suspicious that the stability analysis of some classes of motion should call for the enlargement of the space, from M to $2N$, whereas the stability of some other classes can be analyzed in the smaller space. (c) The role of the time-exponential factors used in the complex variational equation has not been clarified and, above all, no general rules to determine them are available. On the other hand, it would be desirable to use suitable variables that allow the RAME to be put in standard form, in order to accomplish the whole analysis by the same equations. Moreover, it is worth noting that commercial software (e.g. AUTO) requires the equations are put in standard form.

This paper contributes to clarify the matter along the following lines. (1) It is shown that the search for the fixed points representative of the periodic motions and the analysis of their stability can be performed through a unique $2N$-dimensional equation. This is obtained from the AME by expressing the complex amplitudes in Cartesian components on bases rotating with unknown angular velocities. It is explained that the exponential time factors used in the literature to obtain the variational equation represent the rotations of the bases; a rule to determine them is then given. (2) The problem of existence of a standard form for the RAME is then addressed. It is shown that, by using rotating bases and a mixed polar-Cartesian representation of the complex amplitudes, such a *standard form* does exist, although under restrictive conditions. Thus, a procedure followed in the literature by referring to specific problems [10, 11] is here generalized.

2. The Polar Form of Amplitude Modulation Equations

The asymptotic motion of a discrete or spatially continuous weakly nonlinear dynamical system in internal and external resonance conditions, according to the MSM [1], reads

$$\mathbf{x}(t) = \sum_{n=1}^{N} \varepsilon A_n(t_1, t_2, \ldots)\mathbf{u}_n\, \mathrm{e}^{i\omega_n t_0} + O(\varepsilon^2, \varepsilon^h A_0\, \mathrm{e}^{i\omega_0 t_0}) + \text{c.c.}\,, \tag{1}$$

where $\mathbf{x}$ is the state vector, t the time, $t_k = \varepsilon^k t$ $(k = 0, 1, \ldots)$ are time-scales, $\varepsilon \ll 1$ a perturbation parameter, $\mathbf{u}_n$ and ω_n eigenmodes and eigenfrequencies of the linearized system, respectively, and $A_n(t_1, t_2, \ldots)$ $(n = 1, 2, \ldots, N)$ are slow varying complex amplitudes. Moreover A_0 and ω_0 are the amplitude and the frequency of the *hard* $(h = 1)$ or *soft* $(h = 2, 3, \ldots)$ excitations, respectively [1].

The N linear frequencies ω_n and the frequency excitation ω_0 satisfy the following resonance conditions

$$\sum_{n=0}^{N} k_{sn}\omega_n = \varepsilon^{K_s}\sigma_s, \quad s = 1, 2, \ldots, S, \tag{2}$$

where $k_{sn} \in \mathbb{Z}$ are (small) integer numbers and $\sigma_s = O(1)$ are detuning parameters. The integer $K_s := \sum_{n=0}^{N} |k_{sn}| - 1$ is called the *order* of the s-th resonance [12]; this in fact appears for the first time in the perturbation process in the K_s-th perturbation equation. The remaining $n = N+1,\ N+2, \ldots,$ modes, not involved in the S resonances (2), are assumed to decay in time, so that they do not contribute to the asymptotic motion (1).

Generally, not all the S resonances (2) are independent; let us assume that the first $R := \operatorname{rank}[k_{sn}]$ of them are independent, the remaining $Q := S - R$ conditions being linear combinations of the former (i.e. $k_{qn} = \sum_{r=1}^{R} c_{qr}k_{rn}$, $\sigma_q = \sum_{r=1}^{R} c_{qr}\sigma_r$, $q = R+1, R+2, \ldots, S$).

The evolution of the amplitudes is governed by the AME, as furnished by the MSM [1]. The equations are obtained by zeroing in the ε^2-and higher order perturbation equations the nonlinear (and possibly the forcing) terms causing resonance, and by successively combining them according to the *reconstitution* procedure [2, 13]. The nonlinearities of order K produce, at any step of the perturbation procedure, linear combinations of K products, whose factors are $A_n \exp(i\omega_n t)$ and $\overline{A}_n \exp(-i\omega_n t)$; they read: $\left(\prod_{n\in\mathcal{N}^\pm} A_n^{l_n}\right) \exp\left[i \sum_{n\in\mathcal{N}^+}(l_n - l_{-n})\omega_n\right] t$, where $\mathcal{N}^+ = \{0, 1, \ldots, N\}$, $\mathcal{N}^\pm = \{\pm 0, \pm 1, \ldots, \pm N\}$, $A_{-n} := \overline{A}_n$, $l_n \in \mathbb{N}$ and $\sum_{n\in\mathcal{N}^\pm} l_n = K$. Now, if $k_{sm} \neq 0$, Equation (2) can be solved with respect to ω_m, to furnish

$$\omega_m = \sum_{n=0}^{N} k_{smn}\omega_n \pm \varepsilon^{K_s}\sigma_s, \quad k_{smn} := \mp k_{sn} + \delta_{mn}, \quad s = 1, 2, \ldots, S, \tag{3}$$

where the upper sign must be taken if $k_{sm} > 0$ and the lower sign if $k_{sm} < 0$. Therefore, nonlinearities would cause resonance on the m-th mode if $l_n - l_{-n} = k_{smn}$ $\forall n \in \mathcal{N}^+$; these terms therefore enter the AME. Moreover, terms as $A_m(A_n\overline{A}_n)^k$, $A_m(A_i\overline{A}_i)^{k_i}(A_j\overline{A}_j)^{k_j}, \ldots,$ $(n = 0, 1, 2, \ldots, N;\ \ (k, k_i, k_j) = 0, 1, \ldots)$, produced by odd nonlinearities, although *not associated with any resonance*, must be removed from the right hand member of the perturbation equations, in order to allow solvability. Therefore, the AME read

$$A'_m = \mathcal{L}_m\left(A_m(A_n\overline{A}_n)^k, \ldots, \prod_{n\in\mathcal{N}^\pm} A_n^{l_{smn}}\,\mathrm{e}^{\mp i\sigma_s t}, \ldots\right), \quad m = 1, 2, \ldots, N, \tag{4}$$

in which the exponents l_{smn} are (generally not unique) solutions of

$$l_{smn} - l_{sm,-n} = k_{smn}, \qquad \sum_{n\in\mathcal{N}^\pm} l_{smn} = K, \quad l_{smn} \in \mathbb{N}, \quad K \in [K_s, K_{\max}]. \tag{5}$$

In Equation (4), the prime denotes differentiation with respect to the reconstituted true time t, the parameter ε has been reabsorbed and $\mathcal{L}_m$ is a complex linear operator with constant coefficients. Moreover $K_{\max}$ is a prefixed maximum order of the resonance accounted for, and one (or more) terms are present for each K and for each of the S resonance conditions in which ω_m is involved (i.e. for which $k_{sm} \neq 0$).

The complex AME can be put in real form by adopting a polar representation for the amplitudes

$$A_0 = \frac{1}{2}a_0\,\mathrm{e}^{i\vartheta_0}, \quad A_n(t) = \frac{1}{2}a_n(t)\,\mathrm{e}^{i\vartheta_n(t)}, \quad n = 1, 2, \ldots, N, \tag{6}$$

where $a_n(t)$ are real amplitudes, $\vartheta_n(t)$ are phases and a_0 and ϑ_0 are the constant amplitude and phase of the excitation, respectively. By substituting Equation (6) in Equation (4), accounting for Equation (3_2) and separating real and imaginary parts, it follows that

$$\begin{aligned} a'_m &= \Re_m\left(a_m a_n^{2k}, \ldots, \prod_{n\in\mathcal{N}^\pm} a_n^{l_{smn}}\exp(\mp i\gamma_s), \ldots\right), \\ a_m\vartheta'_m &= \Im_m\left(a_m a_n^{2k}, \ldots, \prod_{n\in\mathcal{N}^\pm} a_n^{l_{smn}}\exp(\mp i\gamma_s), \ldots\right), \quad m = 1, 2, \ldots, N, \end{aligned} \tag{7}$$

where $\Re_m(\cdot) = \mathrm{Re}[\mathcal{L}_m(\cdot)]$ and $\Im_m(\cdot) = \mathrm{Im}[\mathcal{L}_m(\cdot)]$ (with the numerical coefficient 1/2 absorbed), $a_{-n} = a_n$ and

$$\gamma_s := \sum_{n\in\mathcal{N}^+} k_{sn}\vartheta_n + \sigma_s t, \quad s = 1, 2, \ldots, S. \tag{8}$$

In Equations (7) S new functions γ_s appear. However, due to the linear dependence of the resonance conditions (2), only the first R of them are independent of the remaining ones; i.e. $\gamma_s = \{\gamma_r, \gamma_q\}$ with

$$\begin{aligned} \gamma_r &:= \sum_{n=0}^{N} k_{rn}\vartheta_n + \sigma_r t, \quad r = 1, 2, \ldots, R, \\ \gamma_q &:= \sum_{r=1}^{R} c_{qr}\gamma_r, \quad q = R+1, R+2, \ldots, S, \end{aligned} \tag{9}$$

where the constant coefficients c_{qr} are known.

A differentiation of Equations (9_1) and substitution of Equations (7_2) leads to

$$\prod_{j\in J_r}^{N} a_j\gamma_r' = \sum_{n\in\mathcal{N}^+} k_{rm}\left(\prod_{\substack{j\in J_r\\ j\neq m}}^{N} a_j\right)\Im_m\left(a_m a_n^{2k},\ldots,\prod_{n\in\mathcal{N}^\pm} a_n^{l_{smn}}\exp(\mp i\gamma_s),\ldots\right)+\sigma_r,$$

$$J_r=\{j|k_{rj}\neq 0\},\quad r=1,2,\ldots,R \tag{10}$$

since $\vartheta_0'=0$.

The N real-amplitude Equations (7_1) and the R phase-combination Equations (10) (with γ_q expressed as functions of γ_r via Equation (9_2)) constitute a differential system of $M := N+R$ autonomous equations in the M (a_m, γ_r)-unknowns. Since M is generally less than $2N$, except for particular cases of forced systems in which $M=2N$, these equations will be referred to as Reduced Amplitude Modulation Equations (RAME). The integer M is called the *codimension* of the problem [2].

RAME have a remarkable property: *their fixed points $a_m'=0$, $\gamma_r'=0$ are periodic solutions* for the original dynamical system, i.e. $\mathbf{x}(t+T)=\mathbf{x}(t)$ with T the generally unknown period. In fact, Equations (7_2) and $a_m=\text{const}$, $\gamma_r=\text{const}$ entail

$$\vartheta_n=\nu_n t+\varphi_n,\quad n=0,1,2,\ldots,N, \tag{11}$$

where ν_n are *frequency corrections*, φ_n initial phases and the dummy equation $\vartheta_0=\nu_0 t+\varphi_0$ with $\nu_0=0$, has been appended. By substituting Equations (11) in Equation (9_1), since this is satisfied for any t, it follows that

$$\sum_{n\in\mathcal{N}^+} k_{rn}\nu_n+\sigma_r=0,\qquad \sum_{n\in\mathcal{N}^+} k_{rn}\varphi_n=\gamma_r,\quad r=1,2,\ldots,R. \tag{12}$$

Both Equations (12) have meaningful consequences. In fact, a combination of Equations (2) (with ε reabsorbed) and Equation (12_1) leads to

$$\sum_{n\in\mathcal{N}^+} k_{rn}\Omega_n=0,\quad r=1,2,\ldots,R, \tag{13}$$

where $\Omega_n := \omega_n+\nu_n$ ($n=0,1,2,\ldots,N$) are *nonlinear frequencies*. Equations (13) show that nonlinearities adjust the frequencies in such a way that the resonance conditions among the nearly-resonant linear frequencies are satisfied with no detunings by the nonlinear frequencies. Since these are in rational ratios, the motion (1)

$$\mathbf{x}(t)=\sum_{n\in\mathcal{N}^+}\frac{1}{2}a_n\mathbf{u}_n\,\mathrm{e}^{i(\Omega_n t+\varphi_n)}+\text{c.c.}+\text{higher-order terms} \tag{14}$$

is periodic.

A consequence of Equation (12_2) is the following: since γ_r assume a finite number of known values in $[0,2\pi)$, Equation (12_2) is a system of R linearly independent algebraic equations in the N unknown initial phases φ_n. Consequently, $L := N-R$ phases remain arbitrary and therefore *there exist ∞^L periodic motions which differ from each other in the values of the initial phases.* However, higher degrees of arbitrariness can exist in particular cases, in which the amplitudes are also involved, in addition to the phases. A very important class of such systems is represented by the (forced or unforced) conservative systems, for which the linear operators $\mathcal{L}_m$ appearing in the AME (4) are purely imaginary. Consequently the phase-combinations γ_r appear only as arguments of sinus in Equations (7_1); moreover, terms independent of γ_r are present only in Equations (7_2), together with cosines terms. Thus, by requiring $\sin\gamma_r = 0$ ($r = 1, 2, \ldots, R$), N equations are satisfied by $R \leq N$ unknowns, and therefore L amplitudes remain arbitrary. In conclusion, *for conservative systems, periodic motions constitute families of* $2(N-R)$ *parameters*, half of which are initial phases and the remaining half amplitudes.

It is worth noting that, due to the fact that phases ϑ_n vary linearly in time, the complex amplitudes A_n do not remain constant during a periodic motion; therefore, a periodic motion *is not* a fixed point for the complex AME (4) or for its polar representation (7).

Although the reduction of the AME to its codimension simplifies the mathematical problem, it still entails a drawback, due to the fact that the AME *are not in standard form*. Let us assume that a set of H amplitudes a_h (e.g. $h = 1, 2, \ldots, H$) exists in which if $a_h = 0\ \forall t$ the arguments of $\Re_h$ and $\Im_h$ ($h = 1, 2, \ldots, H$) identically vanish. Then, from Equations (7_2) and (10) it follows that all the phases ϑ_h ($h = 1, 2, \ldots, H$) and the phase-differences γ_r involving the phases ϑ_h *remain undetermined*. Therefore, such periodic motions are represented in the M-dimensional space (a_n, γ_r) by manifolds of non-isolated points. When their local stability is analyzed, the standard method based on the variational equation fails, since the coefficients of $\delta\gamma_r'$ vanish.

The circumstance illustrated is not a special case; on the contrary it is often met in practice. To overcome the drawback it is customary to analyze the stability of such (incomplete) classes of motion by taking the variation of the complex AME (4) and introducing a suitable change of variable in order to render the variational equations [4–9] autonomous. As will be shown in Section 3, such a procedure is a special case of a general method illustrated later.

3. The Rotating Form of Amplitude Modulation Equations

It has been observed in the previous section, that the complex amplitudes A_n ($n = 1, 2, \ldots, N$), do not remain constant during a periodic motion. They are represented on the complex plane by vectors rotating with constant (and unknown) angular velocities ν_n. However, such vectors appear as fixed vectors to N observers, each being solid with a base rotating with angular velocity ν_n. This circumstance suggests the introduction of the following change of variable

$$A_n = B_n(t)\,\mathrm{e}^{i\nu_n t}, \quad n = 0, 1, \ldots, N \tag{15}$$

in order to obtain a form of AME that admits the periodic motions as fixed points. By substituting Equation (15) in the AME (4), and using Equation (3_2), it follows that

$$B_m' + i\nu_m B_m = \mathcal{L}_m\Big(B_m(B_n\overline{B}_n)^k, \ldots, \prod_{n\in\mathcal{N}^\pm} B_n^{l_{smn}}, \ldots\Big), \quad m = 1, 2, \ldots, N \tag{16}$$

provided that the auxiliary parameters ν_n satisfy Equation (12_1), i.e.

$$\sum_{n\in\mathcal{N}^+} k_{rn}\nu_n + \sigma_r = 0, \quad r = 1, 2, \ldots, R \tag{17}$$

once linear dependence on the remaining resonance conditions ($q = R+1, R+2, \ldots, S$) has been accounted for.

Equations (16) will be referred to as the *rotating (bases) form* of the AME. Due to the choice (17) of parameters ν_n, the fixed points $B'_m = 0$ $(m = 1, 2, \ldots, N)$ of Equation (16) describe periodic motions.

It is worth noting that Equations (16) are autonomous. However, not all the changes of variables which lead to autonomous equations admit fixed points with this property. For example, if the detunings σ_r were all zero, the AME (4) would be autonomous, but the previous property would not hold.

The rotating base AME have a powerful property: they *naturally lead to real equations expressed in standard form*. In fact, by using the Cartesian representation

$$B_n = \frac{1}{2}(u_n + i v_n) \tag{18}$$

instead of the polar representation, Equations (16) read

$$\begin{aligned}
u'_m - \nu_m v_m &= \Re_m \left((u_m + i v_m)(u_n + i v_n)^{2k}, \ldots, \prod_{n \in \mathcal{N}^{\pm}} (u_n + i v_n)^{l_{smn}}, \ldots \right), \\
v'_m + \nu_m u_m &= \Im_m \left((u_m + i v_m)(u_n + i v_n)^{2k}, \ldots, \prod_{n \in \mathcal{N}^{\pm}} (u_n + i v_n)^{l_{smn}}, \ldots \right), \\
& m = 1, 2, \ldots, N.
\end{aligned} \tag{19}$$

They are a system of $2N$ equations in the $2N$ unknowns $(u_n, \ v_n)$ in which N unknown parameters ν_n appear. However, since they must satisfy $R \leq N$ resonance conditions (17), $L = N - R$ of them remain arbitrary. If Equations (19) must be numerically integrated for given initial conditions, the arbitrary ν_n's can be set equal to zero. In this way, however, periodic motions are not represented by $u_m = \text{const}$, $v_m = \text{const}$ and no advantages are gained by the change of variable (15). In contrast, if periodic solutions must be sought, the steady version of Equations (19) must be solved together with Equations (17). It is easy to prove that L initial phases $\varphi_n = \arctan(v_n/u_n)$ remain undetermined according to the results drawn by the polar form of the AME, while the parameters ν_n assume determined values (see Appendix A for an example). As an exception, if the system admits incomplete solutions, in which $u_h = v_h = 0$ for some h's, the associated parameters ν_h are left undetermined by Equations (19) since the relevant equations identically vanish. In some particular cases (especially for small N–R, see [8]) they can be evaluated by Equations (17); since they represent the frequency corrections of modes of zero amplitudes, they do not have a physical meaning. In contrast, in more general cases, Equations (17) are not sufficient to evaluate all the ν_h, so that some of them remain undetermined. This circumstance has some consequences on stability, as will be explained soon.

The stability of the steady solutions is analyzed by performing the variation of Equations (19). For complete solutions, no problem arises, since all the ν_h's have been determined. In contrast, for incomplete solutions, some arbitrariness still exists. However, it is easy to prove *the undetermined parameters ν's do not affect the (orbital) stability* of the steady solution. In fact, the perturbations δA and δB are still related by Equation (15), i.e. $\delta A_n = \delta B_n \exp(i \nu_n t)$. Since the δB_n are governed by constant coefficient equations, they vary in time as $\sum_{m=1}^{2N} c_{mn} \exp(\lambda_m t)$, where λ_m are the eigenvalues and c_{mn} linear combinations of the components of the eigenvectors of the variational equation. Consequently, $\delta A_n =$

$\sum_{m=1}^{2N} c_{mn} \exp[(\lambda_m + i\nu_n)t]$. Since the c_{mn} cannot vanish simultaneously, if $\mathrm{Re}(\lambda_m)$ depended on the arbitrary ν_n's, δA_n would not be unique for any given initial conditions. Therefore the arbitrariness can only affect the imaginary part of the eigenvalues λ_m. In Appendix B an example of the procedure is given which also illustrates the mechanism leading δA_n to be independent of the arbitrariness.

The procedure illustrated suggests the following comments. (a) When the stability of incomplete solutions is analyzed, the variational equation of Equations (19) does not present the pathology of the variational equation of the RAME. This property is a consequence of the fact that Equations (19), unlike the RAME, are expressed in standard form. (b) The variational equation of Equations (19) is precisely the equation used in the literature to investigate incomplete periodic solutions. As has been explained, it is usually employed after the RAME have been used to determine the periodic solutions. The present derivation has shown that the change of variable to be used to obtain a variational equation with constant coefficients is given by Equations (15). If some amplitude A_h is zero, the associated ν_h factors must be determined by Equations (17); if these are unable to determine all of them, the remaining ν_h can be put equal to zero.

The rotating form of the AME therefore represents a general tool for the unitary analysis of steady solutions as well as their stability, both for complete or incomplete motions.

4. The Standard Form of Amplitude Modulation Equations

Although the procedure previously described, leading to the rotating form of the AME, gives insight into the problem and allows some computational difficulties to be overcome, it is not completely satisfactory. According to this procedure, the problem is governed by $2N$ equations (Equations (19)) depending on L free parameters, in comparison with the $M = N + R$ equations of the reduced form (Equations (7_1) and (10)). Moreover, because of the presence of the undetermined parameters, they are not amenable to direct numerical integration. The reasons of the enlargement of the problem stand on the following two circumstances: (a) to obtain the rotating form of the AME, additional parameters ν_n have been introduced through the change of variable (15); (b) unlike the RAME, where the evaluation of the phase-modulation follows that of the amplitudes and phase-combinations, in the rotating form of the AME it is not possible to split the analysis into two steps, but all the quantities must be determined together.

Here a new form of AME, that maintains the dimensions of the RAME is sought. The basic idea is to use a mixed representation. In order to try to separate some phase equations from the others, certain complex amplitudes must be expressed in polar form; however, in order to try to obtain standard form equations, Cartesian components in rotating bases must be used for the remaining complex amplitudes.

To achieve this, the following change of variable, more general than Equation (15), is introduced

$$A_n = B_n(t)\,\mathrm{e}^{i\alpha_n(t)}, \quad n = 0, 1, \ldots, N, \tag{20}$$

where $\alpha_n(t)$ are unknown functions of time (except for the dummy $\alpha_0 \equiv 0$). Equations (20) reduce to Equation (15) when $\alpha_n(t) = \nu_n t + \text{const}$; also make it possible to express some amplitudes in polar form by letting $\alpha_n(t) \equiv \theta_n(t)$ and $B_n = 1/2a_n$. A substitution of Equations (20) in the AME (4) leads to

$$B'_m + i\alpha'_m B_m = \mathcal{L}_m\left(B_m(B_n\overline{B}_n)^k, \ldots, \prod_{n\in\mathcal{N}^\pm} B_n^{l_{smn}}, \ldots\right), \quad m = 1, 2, \ldots, N \tag{21}$$

if the functions $\alpha_n(t)$ are chosen in such a way that

$$\sum_{n\in\mathcal{N}^+} k_{rn}\alpha_n + \sigma_r t = 0, \quad r = 1, 2, \ldots, R. \tag{22}$$

Other solutions, obtained by equating the left side of Equation (22) to $2k\pi$ with k integer, are also possible; however, since they affect the unknowns α_r only by a constant, they are unessential. The similarity among Equations (21) and (22), and Equations (16) and (17), should be noted.

If $R < N$, Equations (22) are not sufficient to determine all the functions α_n, and $L = N - R$ of them (e.g. the first L) remain undetermined. In order to avoid indeterminacies, they are taken as equal to the phases θ_n of the associated amplitudes. By solving Equations (22) for the remaining α_n's, it follows that

$$\begin{aligned} \alpha_p &= \theta_p, \quad p = 1, 2, \ldots, L, \\ \alpha_q &= \sum_{p=1}^{L} c_{qp}\theta_p + d_{qr}\sigma_r t, \quad q = L+1, L+2, \ldots, N, \end{aligned} \tag{23}$$

where c_{qp} and d_{qr} are constant coefficients. According to this choice, the amplitudes B_p are assumed to be real (since they are associated with unknown exponents) while the amplitudes B_q are assumed to be complex (since they are associated with 'known' exponents), i.e.

$$\begin{aligned} B_p &= \frac{1}{2}a_p, \quad p = 1, 2, \ldots, L, \\ B_q &= \frac{1}{2}(u_q + iv_q), \quad q = L+1, L+2, \ldots, N. \end{aligned} \tag{24}$$

The phases θ_p and the associated real amplitudes a_p will be referred to as *principal phases* and *principal amplitudes*, respectively. From Equations (20), (23), and (24) it follows that

$$\begin{aligned} A_p &= \frac{1}{2}a_p\, \mathrm{e}^{i\theta_p}, \quad p = 1, 2, \ldots, L, \\ A_q &= \frac{1}{2}(u_q + iv_q)\, \mathrm{e}^{i\alpha_q}, \quad q = L+1, L+2, \ldots, N. \end{aligned} \tag{25}$$

In conclusion, the unknowns of the problem are still $2N$, as in the original problem: the L principal phases θ_p, the L principal amplitudes a_p, and the $2(N-L) = 2R$ components u_q and v_q of the complex amplitudes B_q, measured in bases each rotating with a time law given from Equation (23_2). With Equations (24), Equations (21) read

$$a'_p = \Re_p\left(a_p a_m^{2k}, a_p(u_n + iv_n)^{2k}, \ldots, \prod_{m=0}^{L} a_m^{l_{spm}} \prod_{n=L+1}^{N} (u_n + iv_n)^{l_{spn}}, \ldots\right),$$

$$a_p\theta_p' = \Im_p\left(a_p a_m^{2k}, a_p(u_n+iv_n)^{2k}, \ldots, \prod_{m=0}^{L} a_m^{l_{spm}} \prod_{n=L+1}^{N} (u_n+iv_n)^{l_{spn}}, \ldots\right),$$

$$u_q' - \alpha_q' v_q = \Re_q\Bigg((u_q+iv_q)a_m^{2k}, (u_q+iv_q)(u_n+iv_n)^{2k},$$

$$\ldots, \prod_{m=0}^{L} a_m^{l_{sqm}} \prod_{n=L+1}^{N} (u_n+iv_n)^{l_{sqn}}, \ldots\Bigg),$$

$$v_q' + \alpha_q' v_q = \Im_q\Bigg((u_q+iv_q)a_m^{2k}, (u_q+iv_q)(u_n+iv_n)^{2k},$$

$$\ldots, \prod_{m=0}^{L} a_m^{l_{sqm}} \prod_{n=L+1}^{N} (u_n+iv_n)^{l_{sqn}}, \ldots\Bigg), \tag{26}$$

in which, due to Equations (23_2)

$$\alpha_q' = \sum_{p=1}^{L} c_{qp}\theta_p' + d_{qr}\sigma_r. \tag{27}$$

Equations (26) will be referred to as the *mixed form of the* AME. Unlike the rotating form, it does not contain arbitrary quantities. To obtain periodic motions, $a_p' = u_p' = v_p' = 0$ must be enforced, together with $\theta_p' = \nu_p =$ const. Hence, by solving $2N$ algebraic equations, the $2N$ unknowns $(a_p, \nu_p, u_{q,} v_q)$ are evaluated. By substituting $\theta_p = \nu_p t + \varphi_p$ in Equations (23_2), where the φ_p's are L arbitrary initial phases, $\alpha_q = \nu_q t + \varphi_q$ is drawn, with the frequency corrections ν_q univocally determined and the initial phases φ_q depending on φ_p.

Similarly to the original AME, and due to the presence of the principal phases, periodic motions are not fixed points for the mixed form of the AME. To remove this drawback it is necessary to eliminate these phases from the equations. However, this operation is not always possible, as will be explained. Let us introduce the following fundamental hypothesis: *the L principal amplitudes a_p do not vanish in any motions*. In this case Equations (26_2) can be divided by $a_p \neq 0$ and put in the standard form $\theta_p' = \theta_p'(a_m, u_n, v_n)$. By using these equations in Equations (27), $\alpha_q' = \alpha_q'(a_m, u_n, v_n)$ follows. Therefore Equations (26_1), (26_3) and (26_4) become a standard form system $\mathbf{y}' = \mathbf{f}(\mathbf{y}, t)$ of $M = N + R$ equations in the M unknowns $\mathbf{y} = (a_p, u_q, v_q)^T$. This will be referred to as *the standard form of the* AME. Once the unknowns have been determined, from Equations (26_2) the evolution of the principal phases θ_p is first drawn and, from Equation (23_2), that of the phases α_q is finally obtained.

The standard form of the AME has the same peculiarities as the reduced form, namely: (a) it has the smallest dimension M (equal to the codimension of the problem) and, (b) it admits periodic motions as fixed points $a_p' = u_p' = v_p' = 0$. In addition, it suffers no problems when the stability of incomplete classes of motion (i.e. $u_h = v_h = 0$ for some h's) is analyzed. Appendices C and D illustrate the application of the procedure.

The standard form of AME is the most suitable for a study of the evolution of the amplitudes. However, to obtain it, it is necessary to select L principal amplitudes that do not vanish in any class of motion. If, in contrast, some of the principal amplitudes vanish in particular classes, the standard form is unable to give correct information about those motions. In these circumstances it would be possible to build more than one standard form, each valid for some classes of motion. However, such a procedure could be inconvenient from a computational point of view.

The standard form, when it exists, also has the following pathology. Although $a_p \neq 0$ by hypothesis, it can become small in some motions. In these cases, small denominators entailing numerical problems appear in the standard form, since in Equation (26_2) θ_p' is affected by a small coefficient. The problem can be overcome by adopting a *master*- and *slave*-amplitude representation, in which the non-principal (slave) amplitudes admit the principal (master) amplitude as factors. A general treatment of the problem will not be developed here, but the basic idea is outlined by an example in the Appendix C.

5. Conclusions

Alternative forms of Amplitude Modulation Equations (AME) governing the asymptotic dynamics of multiresonant systems have been discussed. The AME have been derived (to whitin their coefficients) under general simultaneous internal and external resonance conditions. By using polar, Cartesian and mixed representation for the complex amplitudes, different sets of real equations have been obtained. Their effectiveness in analyzing periodic motions and their stability has been described, above all when incomplete solutions (i.e. solutions in which some amplitudes identically vanish) have been studied. Finally, some illustrative examples have been worked out. The following conclusions are drawn:

1. The polar form (the most popular one) is also the most significant, due to the physical meaning of the variables, i.e. real amplitudes and phase-combinations. However, due to its non-standard normal form, it does not permit the stability of incomplete solutions to be analyzed.
2. In the rotating form the complex amplitudes are expressed through Cartesian components in suitably rotating frames. The equations are well-suited to find periodic solutions and are in standard normal form, so that no problems arise in analyzing stability. However, they are not amenable to direct numerical integrations. Moreover, they call for an enlargement of the state-space with respect to the polar form, since they simultaneously describe the evolution of both amplitudes and phases, rather than of phase-combinations. Therefore, their importance is chiefly conceptual, since they systematically explain the use of changes in variables which have been used in the literature to overcome the problems of the polar form.
3. The mixed (i.e. polar and Cartesian) form, leads to standard normal form equations suitable to analyze the stability of incomplete solutions. However, such a standard form does not always exist, since it is conditional on the finding of a suitable number of (principal) amplitudes which are different from zero in any motion. If this is not the case, more than one standard normal form should be sought, each one valid for some classes of motion.
4. The possible occurrence of small denominators in the standard normal form is highlighted. A strategy to overcome the problem is only sketched here; it could be the subject for future work.

The previous findings have illustrated the expedience of creating a tool that is able to ascertain in advance (i.e. before solving any forms of the AME) whether or not the system admits non-vanishing principal amplitudes, in order to proceed or not with the construction of the standard form. This subject is addressed in a companion paper, in which a method answering the problem is developed [14].

Appendix A: An Example of AME in the Rotating Form

As an example, a system is considered under $R = 2$ simultaneous internal combination and primary external resonance conditions, namely: $\omega_3 = \omega_1 + \omega_2 + \varepsilon\sigma_1,\ \omega_0 = \omega_3 + \varepsilon\sigma_2$. The relevant complex AME, up to order-two are [1]

$$
\begin{aligned}
&2i(A_1' + \mu_1 A_1) - c_1 A_3 \bar{A}_2\, \mathrm{e}^{i\sigma_1 t} = 0,\\
&2i(A_2' + \mu_2 A_2) - c_2 A_3 \bar{A}_1\, \mathrm{e}^{i\sigma_1 t} = 0,\\
&2i(A_3' + \mu_3 A_3) - c_3 A_2 A_1\, \mathrm{e}^{-i\sigma_1 t} - A_0\, \mathrm{e}^{i\sigma_2 t} = 0,
\end{aligned}
\tag{28}
$$

since other non-independent conditions like $\omega_0 = \omega_1 + \omega_2 + \varepsilon(\sigma_1 + \sigma_2)$ would lead to terms $A_0\bar{A}_1,\ A_0\bar{A}_2$ which are of higher order, being the excitation of soft type ($A_0 \ll A_i,\quad i = 1, 2, 3$). In Equations (28), the damping coefficients μ_n are allowed to assume negative values (self-excitation). By using $A_n = B_n\, \mathrm{e}^{i\nu_n t}$ and requiring, in accordance with Equations (17)

$$
\begin{aligned}
&\nu_3 - \nu_1 - \nu_2 + \sigma_1 = 0,\\
&-\nu_3 + \sigma_2 = 0,
\end{aligned}
\tag{29}
$$

the complex AME transform into

$$
\begin{aligned}
&2i(B_1' + i\nu_1 B_1 + \mu_1 B_1) - c_1 B_3 \bar{B}_2 = 0,\\
&2i(B_2' + i\nu_2 B_2 + \mu_2 B_2) - c_2 B_3 \bar{B}_1 = 0,\\
&2i(B_3' + i\nu_3 B_3 + \mu_3 B_3) - c_3 B_1 B_2 - A_0 = 0.
\end{aligned}
\tag{30}
$$

By expressing the rotating amplitudes in Cartesian components, $B_n = (u_n + i v_n)/2$, Equations (30) lead to

$$
\begin{aligned}
\dot{u}_1 &= \nu_1 v_1 - \mu_1 u_1 + c_1(u_2 v_3 - u_3 v_2)/4,\\
\dot{v}_1 &= -\nu_1 u_1 - \mu_1 u_1 - c_1(u_2 u_3 + v_3 v_2)/4,\\
\dot{u}_2 &= \nu_2 v_2 - \mu_2 u_2 + c_2(u_1 v_3 - u_3 v_1)/4,\\
\dot{v}_2 &= -\nu_2 u_2 - \mu_2 u_2 - c_2(u_1 u_3 + v_3 v_1)/4,\\
\dot{u}_3 &= \nu_3 v_3 - \mu_3 u_3 + c_3(u_1 v_2 + u_2 v_1)/4,\\
\dot{v}_3 &= -\nu_3 u_3 - \mu_3 u_3 - c_3(u_1 u_2 - v_1 v_2)/4.
\end{aligned}
\tag{31}
$$

Equations (31) are a system of equations in the six unknowns (u_n, v_n), depending on one of the ν_m-parameters ($m = 1, 2$), left arbitrary by the algebraic Equation (29_1). It can be checked, however, that such a parameter must assume a specified value in order for the steady version of Equations (31) to admit a solution, while a v_m/u_m ratio ($m = 1, 2$) remains undetermined.

To prove this, it is easier to work directly on Equations (30). By requiring $B_n' = 0$, solving and ignoring the trivial solution, it follows that

$$
\begin{aligned}
B_1 &= -\frac{ic_1\bar{B}_2B_3}{2(\mu_1 + i\nu_1)}, \\
B_2\bar{B}_2 &= \frac{4}{c_2c_3}(\mu_1 - i\nu_1)(\mu_3 + i\nu_3) + i\frac{2}{c_1}(\mu_1 - i\nu_1)\frac{A_0}{B_3}, \\
B_3\bar{B}_3 &= \frac{4}{c_1c_2}(\mu_1 - i\nu_1)(\mu_2 + i\nu_2).
\end{aligned} \tag{32}
$$

By separating in Equations (32_3) the real and imaginary parts, two real equations are obtained

$$
\begin{aligned}
\frac{1}{4}|B_3|^2 &= \frac{4}{c_1c_2}(\mu_1\mu_2 + \nu_1\nu_2), \\
0 &= \frac{4}{c_1c_2}(\mu_1\nu_2 - \nu_1\mu_2).
\end{aligned} \tag{33}
$$

Equations (33_2), together with Equations (29), univocally determine the parameters ν_n; Equation (33_1) then furnishes the modulus of B_3, while its phase φ_3 is still unknown. From Equation (32_2), however, φ_3 and $|B_2|$ are successively determined and, finally, from Equation (32_1) a_1 and a relation between φ_1 and φ_2 are drawn. Therefore one phase remains undetermined.

In particular, if $A_0 = 0$ (free self-excited oscillations), also Equations (32_2) leads to equations similar to (33). The two homogenous equations, together with the only resonance condition (29_1), again univocally determine the parameters ν_n. A unique relation among φ_1, φ_2 and φ_3 is found from Equation (32_1), so that two phases remain undetermined.

If, in a second particular case, $\mu_n = 0$ while $A_0 \neq 0$ (undamped forced system), Equation (32_2) identically vanishes and one of the ν_m's ($m = 1, 2$) remains undetermined. From Equation (33_1) and Equation (32_2) $|B_1|$, $|B_2|$ and φ_3 are obtained as functions of the undetermined ν_m ($m = 1, 2$); finally, from Equation (31_1) $|B_1|$ and a relation between φ_1 and φ_2 are drawn, also as functions of the undetermined ν_m. Therefore ∞^2 solutions exist.

In a third particular case, if $\mu_n = 0$ and $A_0 = 0$ (free undamped oscillations), both the ν_m's ($m = 1, 2$) remain undetermined. Equations ($32_{2,3}$) furnish only the modulus of B_2 and B_3, while from Equation (32_1) a_1 and a relation between φ_1, φ_2 and φ_3 are obtained, all of them as functions of the undetermined ν_m's. Therefore ∞^4 solutions exist.

These cases are examples of the validity of the statements of Section 2 regarding the degree of arbitrariness of the periodic solutions of forced and unforced conservative systems.

Appendix B: Stability Analysis of Incomplete Motions by the Rotating Form of AME: An Illustrative Example

Let us consider a system with quadratic nonlinearities in which $N = 4$ modes are involved in $R = 2$ independent resonances condition $\omega_2 = 2\omega_1 + \sigma_1$, $\omega_2 = \omega_3 + \omega_4 + \sigma_2$, that entail the third order (dependent) resonance condition $2\omega_1 = \omega_3 + \omega_4 + \sigma_2 - \sigma_1$. Up to order-three, by neglecting terms as $A_m(A_n\bar{A}_n)$, the AME read

$$
A_1' + ic_1A_2\bar{A}_1\,\mathrm{e}^{i\sigma_1 t} + A_3A_4\bar{A}_1\,\mathrm{e}^{i(\sigma_1-\sigma_2)t} = 0,
$$

$$A_2' + ic_2 A_1^2\, \mathrm{e}^{-i\sigma_1 t} + ic_3 A_3 A_4\, \mathrm{e}^{-i\sigma_2 t} = 0,$$
$$A_3' + ic_4 A_2 \bar{A}_4\, \mathrm{e}^{i\sigma_2 t} + A_1^2 \bar{A}_4\, \mathrm{e}^{i(\sigma_2-\sigma_1)t} = 0,$$
$$A_4' + ic_5 A_2 \bar{A}_3\, \mathrm{e}^{i\sigma_2 t} + A_1^2 \bar{A}_3\, \mathrm{e}^{i(\sigma_2-\sigma_1)t} = 0, \tag{34}$$

where the c_i coefficients are assumed to be real, as occurs in conservative systems. Equations (34) admit the monomodal solution $(A_1^0, A_2^0, A_3^0, A_4^0) = (0, (1/2)a_2, 0, 0)$ with frequency correction $\nu_2 = 0$. Its stability is governed by the variational equation

$$\delta A_1' + \frac{1}{2} ic_1 a_2 \delta \bar{A}_1\, \mathrm{e}^{i\sigma_1 t} = 0,$$
$$\delta A_2' = 0,$$
$$\delta A_3' + \frac{1}{2} ic_4 a_2 \delta \bar{A}_4\, \mathrm{e}^{i\sigma_2 t} = 0,$$
$$\delta A_4' + \frac{1}{2} ic_5 a_2 \delta \bar{A}_3\, \mathrm{e}^{i\sigma_2 t} = 0. \tag{35}$$

The rotating form of the variational equation is obtained by performing the change of variable $\delta A_n = \delta B_n\, \mathrm{e}^{i\nu_n t}$ $(n = 1, \ldots, 4)$ where the coefficient ν_n satisfies the following conditions

$$\nu_2 = 2\nu_1 - \sigma_1, \qquad \nu_2 = \nu_3 + \nu_4 - \sigma_2. \tag{36}$$

Since $\nu_2 = 0$, Equations (36) furnish $\nu_1 = \sigma_1/2$ and $\nu_3 = \sigma_2 - \nu_4$, with ν_4 being arbitrary. The rotating form of (35) then reads

$$\delta B_1' + \frac{1}{2} i\sigma_1 \delta B_1 + \frac{1}{2} ic_1 a_2 \delta \bar{B}_1 = 0,$$
$$\delta B_3' + i(\sigma_2 - \nu_4)\delta B_3 + \frac{1}{2} ic_4 a_2 \delta \bar{B}_4 = 0,$$
$$\delta B_4' + i\nu_4 \delta B_4 + \frac{1}{2} ic_5 a_2 \delta \bar{B}_3 = 0, \tag{37}$$

having ignored the trivial Equation (35_2). By letting $\delta B_n = (x_n + iy_n)\, \mathrm{e}^{\lambda t}$, two uncoupled eigenvalue problems follow, governing stability to A_1- or (A_3, A_4)-perturbations of A_2^0, respectively

$$\begin{bmatrix} \lambda & \frac{1}{2}(c_1 a_2 - \sigma_1) \\ \frac{1}{2}(c_1 a_2 + \sigma_1) & \lambda \end{bmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{38}$$

$$\begin{bmatrix} \lambda & \nu_4 - \sigma_2 & 0 & \frac{1}{2}c_4 a_2 \\ \sigma_2 - \nu_4 & \lambda & \frac{1}{2}c_4 a_2 & 0 \\ 0 & \frac{1}{2}c_5 a_2 & \lambda & -\nu_4 \\ \frac{1}{2}c_5 a_2 & 0 & \nu_4 & \lambda \end{bmatrix} \begin{pmatrix} x_3 \\ y_3 \\ x_4 \\ y_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{39}$$

From Equation (38), $\lambda_{1,2} = \pm\sqrt{(c_1^2 a_2^2 - \sigma_1^2)/4}$ is obtained, from which the monomodal solution is stable if $a_2 < a_{2c} := \sigma_1/c_1$. From Equation (39)

$$\lambda_{1,2} = i\left(\nu_4 - \frac{\sigma_2}{2} \pm \frac{1}{2}\sqrt{\sigma_2^2 - c_4 c_5 a_2^2}\right), \qquad \lambda_{3,4} = \bar{\lambda}_{1,2} \tag{40}$$

is drawn. When $a_2 \le a_{2c} := \sigma_2/(\sqrt{c_4 c_5})$ the four eigenvalues lie on the imaginary axis, and the solution is stable to (A_3, A_4)-perturbations. At $a_2 = a_{2c}$ two couples of eigenvalues coalesce, after which they become complex conjugate, two of them with positive real parts. Therefore, the monomodal solution loses stability at $a_2 = a_{2c}$, *irrespective of the value fixed for the arbitrary parameter* ν_4. This, therefore, can be put equal to zero, since the change of variable in the amplitudes is performed.

It can be shown, that according to the comments of Section 3, the evolution of the perturbations $\delta A_n = \delta B_n \, \mathrm{e}^{i\nu_n t}$ $(n = 3, 4)$ does not depend on ν_4. The question is not trivial, since δA_n vary in time according to exponents $\exp[(\lambda_m + i\nu_n)t]$, some of which *depend on* ν_4. Therefore it is to be expected that there exists a mechanism involving the eigenvectors $\boldsymbol{\phi}_m$ of the variational matrix, which cancels the effects of such exponents. The general solution of the variation equation reads

$$\left\{ \begin{array}{c} \delta A_3 \\ \delta A_4 \end{array} \right\} = \sum_{k=1}^{2} \left\{ c_k \begin{pmatrix} (\phi_{k1} + i\phi_{k2}) \, \mathrm{e}^{i\nu_3 t} \\ (\phi_{k3} + i\phi_{k4}) \, \mathrm{e}^{i\nu_4 t} \end{pmatrix} \mathrm{e}^{\lambda_k t} + d_k \begin{pmatrix} (\bar{\phi}_{k1} + i\bar{\phi}_{k2}) \, \mathrm{e}^{i\nu_3 t} \\ (\bar{\phi}_{k3} + i\bar{\phi}_{k4}) \, \mathrm{e}^{i\nu_4 t} \end{pmatrix} \mathrm{e}^{\bar{\lambda}_k t} \right\}, \tag{41}$$

where ϕ_{kj} is the j-th component of $\boldsymbol{\phi}_k$ and c_k, d_k are arbitrary constants. Since $\nu_3 + \nu_4 = \sigma_2$, $\lambda_k + i\nu_3$ and $\bar{\lambda}_k + i\nu_4$ are independent of ν_4; in contrast, $\nu_3 + \bar{\lambda}_k$ and $\nu_4 + \lambda_k$ depend on ν_4. However, since the eigenvectors are found to be of the following type

$$\boldsymbol{\phi}_k = (\alpha_k + i\beta_k \quad \beta_k - i\alpha_k \quad \gamma_k - i\delta_k \quad \delta_k + i\gamma_k)^T, \quad \boldsymbol{\phi}_{k+2} = \bar{\boldsymbol{\phi}}_k, \quad k = 1, 2, \tag{42}$$

it follows that $\phi_{k1} + i\phi_{k2} \neq 0$, $\bar{\phi}_{k3} + i\bar{\phi}_{k4} \neq 0$ while $\bar{\phi}_{k1} + i\bar{\phi}_{k2} = 0$, $\phi_{k3} + i\phi_{k4} = 0$, i.e. the coefficients of the exponents depending on ν_4 vanish.

Appendix C: Standard Form of AME for a 1:2 Internally Resonant System

An example of the procedure illustrated in Section 4 is given here. The AME governing the free vibrations of a system in the 1:2 resonance condition $\omega_2 = 2\omega_1 + \varepsilon\sigma$ are [1]

$$\begin{aligned} &-2i(A_1' + \mu_1 A_1) + c_1 A_2 \bar{A}_1 \, \mathrm{e}^{i\sigma t} = 0, \\ &-2i(A_2' + \mu_2 A_2) + c_2 A_1^2 \, \mathrm{e}^{-i\sigma t} = 0. \end{aligned} \tag{43}$$

Since $N = 2$, $R = 1$, it is necessary to choose $L = N - R = 1$ principal amplitudes. From Equation (43), it follows that $A_2 \neq 0$ in any motion, while the same property does not hold for A_1. Therefore A_2 must be taken as the principal amplitude.

Equation (22) reads $\alpha_2 - 2\alpha_1 + \sigma = 0$. By taking $\alpha_2 = \vartheta_2$, $\alpha_1 = (\vartheta_2 + \sigma)/2$ follows. Therefore, the mixed representation Equation (22) to be used is

$$A_1 = \frac{1}{2}(u_1 + iv_1) \, \mathrm{e}^{(i/2)(\vartheta_2 + \sigma t)}, \quad A_2 = \frac{1}{2} a_2 \, \mathrm{e}^{i\vartheta_2}. \tag{44}$$

By substituting it in Equations (43), four real equations in the unknowns $(a_2, \vartheta_2, u_1, v_1)$ are drawn. The equation governing the ϑ_2-evolution reads

$$\dot{\vartheta}_2 = -\frac{c_2}{4a_2}(u_1^2 - v_1^2), \tag{45}$$

which, after substitution into the other three equations, leads to the following standard form AME

$$
\begin{aligned}
\dot{a}_2 &= -\frac{1}{2}(2\mu_2 a_2 - c_2 u_1 v_1), \\
\dot{u}_1 &= -\frac{1}{8a_2}(8\mu_1 a_2 u_1 - 4\sigma a_2 v_1 + 2c_1 a_2^2 v_1 + c_2 u_1^2 v_1 - c_2 v_1^3), \\
\dot{v}_1 &= -\frac{1}{8a_2}(4\sigma a_2 u_1 + 2c_1 a_2^2 u_1 - 2c_2 u_1^3 + 8\mu_1 a_2 v_1 + c_2 u_1 v_1^2).
\end{aligned}
\tag{46}
$$

Although $a_2 \neq 0$, if it is small in some motions, small denominators appear in Equations (46), as a consequence of Equation (45). To eliminate these terms, the following alternative representation is adopted

$$
A_1 = \frac{1}{2}a_2(u_1 + iv_1)\,\mathrm{e}^{(i/2)(\vartheta_2+\sigma t)}, \quad A_2 = \frac{1}{2}a_2\,\mathrm{e}^{i\vartheta_2},
\tag{47}
$$

in which A_1 plays the role of *slave* amplitude and A_2 that of *master* amplitude, since A_1 cannot exist without A_2. From Equations (43) and (47)

$$
\dot{\vartheta}_2 = -\frac{c_2 a_2}{4}(u_1^2 - v_1^2)
\tag{48}
$$

and

$$
\begin{aligned}
\dot{a}_2 &= -\frac{1}{2}(2\mu_2 a_2 - c_2 a_2^2 u_1 v_1), \\
\dot{u}_1 &= -\frac{1}{8}(8\mu_1 u_1 - 8\mu_2 u_1 - 4\sigma v_1 + 2c_1 a_2 v_1 + 5c_2 a_2 u_1^2 v_1 - c_2 a_2 v_1^3), \\
\dot{v}_1 &= -\frac{1}{8}(4\sigma u_1 + 2c_1 a_2 u_1 - c_2 a_2 u_1^3 + 8\mu_1 v_1 - 8\mu_2 v_1 + 5c_2 a_2 u_1 v_1^2)
\end{aligned}
\tag{49}
$$

follow, as counterparts of Equations (45) and (46). Equations (49) do not suffer the numerical problems of Equations (46); however, it is necessary to use variables (u_1, v_1) and a_2, which in general are not of the same order of magnitude.

The procedure illustrated here should be generalized for more complex problems. It has already been adopted in [10], where, however, its use was suggested an account of the nature of the problem, rather than for reasons of mathematical convenience.

Appendix D: Standard Form of AME for a 1:2 Internally and a 1:1 Externally Resonant System

The previous system is considered again, with the mode-2 now excited by a sinusoidal force of frequency $\omega_0 \simeq \omega_2$. The resonance conditions are $\omega_2 = 2\omega_1 + \varepsilon\sigma_1$ and $\omega_0 = \omega_2 + \varepsilon\sigma_2$. The AME (43) modify as follows [1]

$$
\begin{aligned}
&-2i(A_1' + \mu_1 A_1) + c_1 A_2 \bar{A}_1\,\mathrm{e}^{i\sigma_1 t} = 0, \\
&-2i(A_2' + \mu_2 A_2) + c_2 A_1^2\,\mathrm{e}^{-i\sigma t} + \frac{1}{2}A_0\,\mathrm{e}^{i\sigma_2 t} = 0,
\end{aligned}
\tag{50}
$$

since, as in the example of the Appendix A, the dependent resonance conditions are of higher order. Since $R = N = 2$, it is $L = N - R = 0$, so that no principal amplitudes need be sought, i.e. all the complex amplitudes must be expressed in Cartesian form. Equation (22) read: $\alpha_2 - 2\alpha_1 + \sigma_1 = 0$, -$\alpha_2 + \sigma_2 = 0$. By solving them $\alpha_1 = (\sigma_1 + \sigma_2)/2$, $\alpha_2 = \sigma_2$ are found and, therefore, the change of variable to be used is

$$A_1 = \frac{1}{2}(u_1 + i v_1)\, \mathrm{e}^{(i/2)(\sigma_1 - \sigma_2)t}, \quad A_2 = \frac{1}{2}(u_2 + i v_2)\, \mathrm{e}^{i\sigma_2 t}. \tag{51}$$

From Equations (50) and (51), the standard form equations follow

$$\begin{aligned}
u_1' &= -\frac{1}{4}(4\mu_1 u_1 - 2\sigma_1 v_1 - 2\sigma_2 v_1 + c_1 u_2 v_1 - c_1 u_1 v_2), \\
v_1' &= -\frac{1}{4}(2\sigma_1 u_1 + 2\sigma_2 u_1 + c_1 u_1 u_2 + 4\mu_1 v_1 + c_1 v_1 v_2), \\
u_2' &= -\frac{1}{2}(2\mu_2 u_2 - c_2 u_1 v_1 - 2\sigma_2 v_2), \\
v_2' &= -\frac{1}{4}(a_0 + c_2 u_1^2 + 4\sigma_2 u_2 - c_2 v_1^2 + 4\mu_2 v_2).
\end{aligned} \tag{52}$$

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