# Perturbation methods for nonlinear autonomous discrete-time dynamic systems

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**Abstract.** Two perturbation methods for nonlinear autonomous discrete-time dynamical systems are presented. They generalize the classical Lindstedt–Poincaré and multiple scale perturbation methods that are valid for continuous-time systems. The Lindstedt–Poincaré method allows determination of the periodic or almost-periodic orbits of the nonlinear system (limit cycles), while the multiple scale method also permits analysis of the transient state and the stability of the limit cycles. An application to the discrete Van der Pol equation is also presented, for which the asymptotic solution is shown to be in excellent agreement with the exact (numerical) solution. It is demonstrated that, when the sampling step tends to zero the asymptotic transient and steady-state discrete-time solutions correctly tend to the asymptotic continuous-time solutions.

**Key words:** Perturbation methods, discrete maps, Neimark bifurcation, almost-periodic solutions.

## 1. Introduction

Many physical problems are intrinsically discrete and can be studied by discrete-time dynamical models. Examples are demographic or ecological systems, or mechanical and electromagnetic impulsive systems, encountered for example in repeated impact problems or in pulse-radar bearing problems. Other problems are instead continuous, but can be studied approximately as discrete problems. As an example of a physical device, an analog-to-digital convertor is employed to transform a continuous-time signal into a discrete one, so that it can be processed numerically; as an example of a mathematical model an integro-differential equation can be transformed into a difference equation by using finite-difference and quadrature formulas. Finally, a mathematical tool that – in principle – transforms an $m+1$-dimensional continuous system into an $m$-dimensional discrete system without approximation is the Poincaré map widely used, for instance to analyze bifurcation and stability of cycles [1, 2].

Discrete-time systems are governed by difference equations. The free evolution of a system following disturbance of an equilibrium state depends on the eigenvalues of the linearized map. If all the eigenvalues are contained in the unitary circle of the complex plane, the state tends to the equilibrium state; if one or more eigenvalues lies out of the circle, the state diverges. A critical condition occurs when, by varying a control parameter, a pair of complex conjugate eigenvalues crosses the circle; in this condition a secondary Hopf (or Neimark) bifurcation occurs and a steady-state solution lying on an invariant closed curve arises.

If the discrete-time system is obtained as a Poincaré map of a flow, an invariant torus is generated by the instability of a limit cycle. In fact, the unstable equilibrium point and the invariant closed curve for the map correspond to the cycle and to the torus cross-section for the flow, respectively. Bifurcated solutions are in general almost-periodic and fill the torus surface; correspondingly, the orbits of the map are made of an infinite number of point filling the invariant curve. However, as a very special case, the cycle can bifurcate in another periodic trajectory that lies on the torus but does not fill the whole surface. In this case the orbits of the map are made of a finite number of points. This phenomenon is known as "strong resonance" or "phase locking" [1] and has been studied for the flow by Iooss and co-workers [3]. Here, instead, attention is focused on the generic nonresonant case, and reference is made directly to maps. The task is to determine the nonlinear almost-periodic solutions of a discrete-time system for small values of a perturbation parameter, once a linear almost-periodic solution is known for the zero value of the parameter.

The problem of finding periodic solutions of nonlinear continuous-time systems has been widely dealt with in the literature through perturbation methods. For these systems the Hopf bifurcation analysis can be performed by using a modified version of the classical Lindstedt–Poincaré method [4, 5], assuming the amplitude as perturbation parameter and expanding in series both the frequency and the control parameter [3, 6]. The algorithm leads to linear equations in the derivatives of the frequency and the control parameter. Otherwise, the classical method can be applied, by fixing the control parameter and solving a nonlinear equation for the amplitude.

As an alternative approach, the multiple scale perturbation method [5] can be used to obtain partial differential equations in the amplitudes and phases. Thus, a limit cycle can be determined as a fixed point of an autonomous system and its stability ascertained simply by investigating the equilibrium quality at the fixed point. In addition the method allows determination of the evolution of the system from unstable equilibrium points to stable limit cycles.

However, application of these perturbation methods to discrete-time systems is far from simple. In fact there are difficulties due to the discrete nature of the independent variable and, consequently, to the fact that the problem is governed by a difference equation, rather than a differential equation. Examples of these difficulties can be found in the literature. Indeed, Minorsky [4], after having reduced a continuous system to a discrete one by a stroboscopic method (i.e. via a Poincaré map), replaces the resulting difference equations with a differential equation (the so-called stroboscopic differential equation) going back to a continuous system. Then, Guckenheimer and Holmes [7], by dealing with secondary Hopf bifurcations of maps, affirm that the question of describing the dynamics is a difficult problem that requires use of global bifurcation analysis.

The author is not aware of any analytical studies on the subject, except for the recent paper by Atadan [8], where the intrinsic method of harmonic balancing is adapted to a two-dimensional, two-parameter, discrete-time system. However, in that paper the focus is on bifurcating periodic solutions only, no attention being paid to almost-periodic and transient solutions.

In this paper, the Lindstedt–Poincaré and the multiple scale methods are extended to nonlinear autonomous discrete-time systems. An example relative to a second-order discrete system with cubic nonlinearities is worked out and the results are compared with known continuous-time perturbation solutions and discrete-time numerical results.

## 2. Periodic and Almost-Periodic Solutions of Discrete-Time Systems

A general continuous-time nonlinear autonomous system is governed by the $m$ differential first-order equations:

$$\dot{x}(t) = Tx(t) + \varepsilon f[x(t), \dot{x}(t)], \tag{1}$$

where $x(t) \in \mathbb{R}^m$ is the state-vector, depending on the time $t$, $T \in \mathbb{R}^m \times \mathbb{R}^m$ is the dynamic matrix, $f \in \mathbb{R}^m$ is a nonlinear function, $\varepsilon \ll 1$ a perturbation parameter and the dot denotes $t$-differentiation.

The equations governing the evolution of a general discrete-time nonlinear autonomous system are instead difference equations. They read:

$$x(n+1) = Tx(n) + \varepsilon f[x(n), x(n-1)], \tag{2}$$

where $x(n) \in \mathbb{R}^m$ is the state-vector at the discrete time $n$, $T \in \mathbb{R}^m \times \mathbb{R}^m$ is a transfer matrix, $f \in \mathbb{R}^m$ is a nonlinear function and $\varepsilon \ll 1$ a perturbation parameter. By introducing the shift operator $E_n : E_n x(n) = x(n+1)$, equation (2) can be rewritten as:

$$E_n x = Tx + \varepsilon f[x, E_n^{-1} x] \tag{3}$$

in which the independent variable $n$ has been omitted.

When $\varepsilon = 0$, equation (3) reduces to the linear problem

$$E_n x_0 = Tx_0. \tag{4}$$

Equation (4) admits $m$ solutions $x_{0j} = \lambda_j^n u_j$ where $(\lambda_j, u_j)$ is an eigensolution of $T$:

$$(T - \lambda_j I)u_j = 0 \qquad (j = 1, 2, \ldots, m). \tag{5}$$

By hypothesis, let matrix $T$ have eigenvalues of modulus less than one, except for a pair of complex conjugate eigenvalues of unitary modulus, $\lambda_j = \exp(i\theta_0)$ and $\lambda_{j+1} = \exp(-i\theta_0)$, with $\theta_0 \in [0, \pi]$; in addition, let $u_j$ and $u_{j+1} = \bar{u}_j$, be the associated, complex conjugate, eigenvectors. The linear problem (4) then admits the following steady-state solution:

$$\begin{aligned} x_0(n) &= Au_j \exp(i\theta_0 n) + \text{c.c.} \\ &= a[u_{jR} \cos(\theta_0 n + \phi) - u_{jI} \sin(\theta_0 n + \phi)], \end{aligned} \tag{6}$$

where $A = (1/2)a \exp(i\phi)$ is the complex amplitude, c.c. stands for complex conjugate and $u_j = u_{jR} + iu_{jI}$.

Equation (6) is formally similar to the linear solution of the continuous-time system (1), if this admits the unique undamped linear frequency $\omega_0$; in fact, the continuous solution is obtained from equation (6) by replacing $\theta_0 n$ by $\omega_0 t$. However, there are profound differences between the two solutions. Namely: (a) The continuous solution is periodic for any real $\omega_0$ while the discrete solution is periodic only if the $\theta_0/2\pi$ ratio is rational and it is almost-periodic if the ratio is irrational. In the first case phase locking occurs, and the orbit is made of a finite number of points lying on an ellipse on the $(u_{jR}, u_{jI})$-plane; in the second case the orbit is made of an infinite number of points that densely fill the curve. (b) If the amplitude $a$ is fixed and the phase $\phi$ is varied, the solution remains essentially unaltered in the continuous case, since the change corresponds to a shift of the time origin. In the discrete case this is true only if $\phi = \pm k\theta_0$, for integer $k$; otherwise the solution is substantially modified despite the fact that the system is autonomous. Obviously these differences depend on the discrete nature of the problem.

Equation (3) admit periodic solutions only in exceptional cases. In fact, even if the solution of the linear problem (4) is periodic, the solution of the nonlinear problem (3) is almost-periodic, since nonlinearities generally modify the period. Indeed, in [8] nonlinear bifurcating periodic solutions of a digital control system are found. However, in the problem dealt with there, both the control parameter and the sampling period are changed with the amplitude, so a periodic solution can occur. In a general case, instead, strictly periodic solutions do not exist; consequently, classical perturbation methods like the Lindstedt–Poincaré method seem to be inapplicable at first sight.

The basic idea to overcome the problem is searching for the continuous invariant closed curves of the map, instead for its discrete orbits. In fact, the closed curves have parametric equations $x = x(s)$ and satisfy the periodicity condition $x(s+S) = x(s)$, with $S$ the unknown period, as the trajectories $x(t)$ of the continuous-time system do. The search of such closed curves can be accomplished by replacing the original discrete variable $n \in \mathbb{N}$ by the new continuous variable $s \in \mathbb{R}$ (or, equivalently, by letting $n$ to assume any real value) and then by solving the difference equation under the periodicity condition. After a curve $x(s)$, $s \in \mathbb{R}$, has been determined, the discrete orbits $x(n)$, $n \in \mathbb{N}$, are calculated by sampling $x(s)$ by a constant step, i.e., by allowing $s$ to assume discrete values only. Thus, for example, the linear discrete solution (6) is obtained by sampling the $2\pi$-periodic continuous functions $\cos(s)$ and $\sin(s)$ at the discrete instants $s = \theta_0 n + \phi$. If the sampling step is in irrational ratio to the period $S$, $x(n)$ is almost-periodic; if several samplings are performed by varying the initial phase, different discrete solutions are obtained, each corresponding to shifted sequences of instants. Finally, it should be noted that the use of the continuous independent variable $s$ allows derivation and integration of the dependent variables, i.e., performance of the operations necessary for the perturbation analysis, as will be seen ahead.

The search for almost-periodic discrete solutions is thus transformed into the usual search for periodic solutions of a continuous variable. However, the main difficulty of the analysis lies in the fact that the problem is not governed by a differential equation, but rather by a difference equation, so that standard techniques cannot be applied straightforwardly. Specific methods are developed here.

## 3. The Lindstedt–Poincaré Method

The basic step of the Lindstedt–Poincaré method requires the independent variable to be strained in order to transform the unknown period into the known period $2\pi$. For the continuous-time system (1), by denoting the unknown frequency with $\omega(\varepsilon)$, the goal is achieved through the change of variable $\tau = \omega(\varepsilon)t$, that transforms the equation as follows:

$$\omega(\varepsilon)\dot{x}(\tau) = Tx(\tau) + \varepsilon f[x(\tau), \omega(\varepsilon)\dot{x}(\tau)], \tag{7}$$

the dot now denoting $\tau$-differentiation; in addition $x(\tau + 2\pi) = x(\tau)$ is required. For the discrete-time system (2), by performing an analogous change of variable

$$s = \theta(\varepsilon)n \tag{8}$$

with $\theta(\varepsilon)$ unknown frequency, the equation reads

$$x(s + \theta(\varepsilon)) = Tx(s) + \varepsilon f[x(s), x(s - \theta(\varepsilon))]. \tag{9}$$

The new independent variable $s$ is allowed to assume any real value. Correspondingly, the solutions to equation (9) satisfying the periodicity condition $x(s + 2\pi) = x(s)$ describe invariant closed curves $x(s)$, $s \in \mathbb{R}$, on which all the steady-state discrete orbits $x(n)$, $n \in \mathbb{N}$, lie, each differing from initial phase.

It should be noted that in equation (9), unlike $\omega(\varepsilon)$ in equation (7), frequency $\theta(\varepsilon)$ does not appear explicitly in the equation, but only appears as an argument of $x$. This circumstance stems from the fact that, since derivatives are not present in a difference equation, the chain rule $\mathrm{d}/\mathrm{d}t = \omega(\varepsilon)\,\mathrm{d}/\mathrm{d}\tau$, that makes $\omega(\varepsilon)$ explicit in equation (7), does not work in equation (9). To render the dependence on $\theta(\varepsilon)$ explicit in equation (9), one can note that $\theta(\varepsilon)$ is the unknown $\varepsilon$-dependent sampling step on the continuous-time scale $s$; it reduces to the known value $\theta_0 = \theta(0)$ when $\varepsilon \to 0$. Now, the key of the procedure is to express $x(s+\theta(\varepsilon))$, i.e. the dependent variable at the next step in the nonlinear problem, as a small correction of order $\varepsilon$ of $x(s+\theta_0)$, i.e. the dependent variable at the next step in the linear problem. This can be accomplished by expanding $x(s \pm \theta(\varepsilon))$ in a power series of $\varepsilon$ around $x(s \pm \theta(0))$. After having expanded the frequency $\theta(\varepsilon)$ as well:

$$\theta(\varepsilon) = \theta_0 + \varepsilon\theta_1 + \varepsilon^2\theta_2 + \ldots \tag{10}$$

one gets:

$$\begin{aligned} x(s \pm \theta(\varepsilon)) &= x(s \pm \theta_0) \pm \varepsilon\theta_1 x'(s \pm \theta_0) \\ &\quad + \frac{1}{2}\,\varepsilon^2[\theta_1^2 x''(s \pm \theta_0) \pm 2\theta_2 x'(s \pm \theta_0)] + \ldots, \end{aligned} \tag{11}$$

where the prime denotes $s$-differentiation. Equation (11) can be rewritten in more compact form by introducing the (forward) shift operator $E_s$, defined as follows:

$$E_s x(s) = x(s + \theta_0) \tag{12}$$

that allows one to write:

$$x(s+\theta(\varepsilon)) = E_s\left[1 + \varepsilon\theta_1\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}\,\varepsilon^2\left(\theta_1^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} + 2\theta_2\frac{\mathrm{d}}{\mathrm{d}s}\right) + \ldots\right] x(s). \tag{13}$$

Thus, between the two shift operators $E_n$ and $E_s$ the following relationship holds:

$$E_n = E_s\left[1 + \varepsilon\theta_1\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}\,\varepsilon^2\left(\theta_1^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} + 2\theta_2\frac{\mathrm{d}}{\mathrm{d}s}\right) + \ldots\right]. \tag{14}$$

Equation (14) is the counterpart for discrete-time systems of the chain rule valid for continuous-time systems.

Between the two backward shift operators $E_n^{-1}$ and $E_s^{-1}$ an analogous relationship follows from equation (11):

$$E_n^{-1} = E_s^{-1}\left[1 - \varepsilon\theta_1\frac{\mathrm{d}}{\mathrm{d}s} + \frac{1}{2}\,\varepsilon^2\left(\theta_1^2\frac{\mathrm{d}^2}{\mathrm{d}s^2} - 2\theta_2\frac{\mathrm{d}}{\mathrm{d}s}\right) + \ldots\right] \tag{15}$$

that could also be obtained by formally expanding $1/E_n(\varepsilon)$ around $\varepsilon = 0$.

By expanding $x(s)$ as well in power series:

$$x(s) = x_0(s) + \varepsilon x_1(s) + \varepsilon^2 x_2(s) + \ldots \tag{16}$$

by substituting equations (14) and (15) in (3) and equating to zero terms with the same powers of $\varepsilon$, the following perturbation equations are obtained:

$$\varepsilon^0: \quad E_s x_0 - T x_0 = 0 \tag{17a}$$

$$\varepsilon^1: \quad E_s x_1 - T x_1 = -\theta_1 E_s x_0' + f[x_0, E_s^{-1}x_0] \tag{17b}$$

$$\begin{aligned} \varepsilon^2: \quad E_s x_2 - T x_2 &= -E_s\left(\theta_1 x_1' + \frac{1}{2}\,\theta_1^2 x_0'' + \theta_2 x_0'\right) \\ &\quad + f_n[x_0, E_s^{-1}x_0]x_1 + f_{n-1}[x_0, E_s^{-1}x_0]E_s^{-1}(x_1 - \theta_1 x_0'), \end{aligned} \tag{17c}$$

where $f_k = \partial f/\partial x(k)$ $(k = n, n-1)$ and $x_h = x_h(s)$ $(h = 0, 1, 2, \ldots)$.

It is worth noting that because of equations (14) and (15), the difference equation (3) is transformed into a delay differential equation. However, the $s$-derivatives appear only at $\varepsilon$- or higher order, so that they are known terms in the perturbation equations (17). Since the operator acting on the unknowns $x_h(s)$ is a difference operator, the perturbation equations (17) are difference equations as well, like the original equation (3), although they involve now unknowns that are functions of the continuous variable $s$ instead of the discrete variable $n$. In addition, it should be noted that derivatives in equations (17) appear merely because $\theta(\varepsilon)$ has been expanded in equation (9); they do *not* indicate any substitution of the discrete system by a continuous system and, consequently, the replacement of a discrete equation by a differential equation.

Equation (17a) admits the periodic generating solution

$$x_0(s) = A u_j \exp(is) + \text{c.c.} \tag{18}$$

Substituting in equation (17b) leads to

$$\begin{aligned} E_s x_1 - T x_1 &= -\theta_1(iAu_j \exp[i(s+\theta_0)] + \text{c.c.}) \\ &\quad + f(Au_j \exp(is) + \text{c.c.}, Au_j \exp[i(s-\theta_0)] + \text{c.c.}). \end{aligned} \tag{19}$$

Since $f$ is $2\pi$-periodic on $s$ it can be expanded in a Fourier series:

$$f = \sum_{k=-\infty}^{\infty} F_k(A, \bar{A}) \exp(iks), \qquad F_k(A, \bar{A}) = \frac{1}{2\pi} \int_0^{2\pi} f \exp(-iks)\, \mathrm{d}s. \tag{20}$$

In order for $x_1$ to be $2\pi$-periodic the known terms affected by the factors $\exp(\pm is)$ must be made orthogonal to the left eigenvectors $v_j$ and $v_{j+1} = \bar{v}_j$, respectively, which are solutions of the adjoint eigenvalue problem:

$$(T - \lambda_j I)^{\mathrm{H}} v_j = 0 \qquad (j = 1, 2, \ldots, m), \tag{21}$$

where the H superscript denotes transpose conjugate. Since the known terms are themselves complex conjugate, it is sufficient to satisfy the unique (solvability) condition

$$-i\theta_1 A \exp(i\theta_0) + v_j^{\mathrm{H}} F_1 = 0 \tag{22}$$

having normalized $v_j$ in such a way that $v_j^{\mathrm{H}} u_j = 1$. By using the polar form $A = (1/2)a \exp(i\phi)$ and equating separately to zero the real and imaginary part, equation (22) yields:

$$0 = \frac{1}{\pi} \int_0^{2\pi} v_j^{\mathrm{H}} f[a u_j \cos\psi, a u_j \cos(\psi - \theta_0)] \cos(\psi + \theta_0)\, \mathrm{d}\psi \tag{23a}$$

$$\theta_1 = -\frac{1}{\pi a} \int_0^{2\pi} v_j^{\mathrm{H}} f[a u_j \cos\psi, a u_j \cos(\psi - \theta_0)] \sin(\psi + \theta_0)\, \mathrm{d}\psi \tag{23b}$$

in which $\psi = s + \phi$ has been posed. Equation (23a) is a nonlinear algebraic equation in the unknown amplitude $a$. Its real solutions are the amplitudes of the periodic solutions. Equation (23b) furnishes the associated frequency corrections.

To first approximation, by coming back to the discrete variable $n$, the almost-periodic solution of equation (2) reads:

$$\begin{aligned} x(n) &= a[u_{jR}\cos\psi(n) - u_{jI}\sin\psi(n)] \\ \psi(n) &= (\theta_0 + \varepsilon\theta_1)n + \phi, \end{aligned} \tag{24}$$

where the initial phase $\phi$ depends on the initial conditions.

To go to second approximation, equation (19) must be solved. One gets:

$$\begin{aligned} x_1 &= \sum_{k=-\infty}^{\infty}\sum_{h=1}^{m} \frac{v_h^{\mathrm{H}} F_k}{\exp(ik\theta_0) - \lambda_h} u_h \exp(iks) + (Bu_j \exp(is) + \text{c.c.}) \\ &\qquad (k,h) \neq (1,j) \text{ and } (-1, j+1), \end{aligned} \tag{25}$$

where $B = (1/2)b\exp(i\beta)$ is a complex arbitrary constant. It should be noted that equation (25) holds only if $\exp(ik\theta_0) \equiv \lambda_j^k \neq \lambda_h\ \forall, h$. When $h \neq j$ this entails that no other eigenvalues of $T$ lie on the unitary circle, so that no internal resonance conditions occur; when $h = j$ this entails that the $\theta_0/2\pi$ ratio is irrational, i.e. no strong resonance occurs. By using equations (18) and (25) in equation (17c) and zeroing secular terms, two equations in the unknowns $b$ and $\theta_2$ are obtained, while the phase $\beta$ remains undetermined.

## 4. The Multiple Scale Perturbation Method

When this method is applied to the continuous-time system (1), it is assumed that the variables $x(t)$ depend on different temporal scales $T_0 = t$, $T_1 = \varepsilon t$, $T_2 = \varepsilon^2 t, \ldots$. A uniformly valid expansion can be obtained by expanding the derivatives as well as the dependent variables in powers of the small parameter $\varepsilon$ [5]. This is achieved by applying the chain rule:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} = \frac{\partial}{\partial T_0} + \varepsilon\frac{\partial}{\partial T_1} + \varepsilon^2\frac{\partial}{\partial T_2} + \cdots \tag{26}$$

Therefore, the total time-derivative is expressed as the partial derivative with respect to the fast scale, plus higher order contributions associated with the partial derivatives with respect to the slow scales.

By following the same procedure for the discrete-time system (2), it is assumed that the dependent variables $x(n;\varepsilon)$ depend on $n$, $\varepsilon n$, $\varepsilon^2 n, \ldots$. The different continuous scales

$$r_0 = n, \quad r_1 = \varepsilon n, \quad r_2 = \varepsilon^2 n, \ldots \tag{27}$$

are introduced and

$$x(n;\varepsilon) = x(r_0, r_1, r_2, \ldots) \tag{28}$$

is posed. Equation (2) is then rewritten as:

$$\begin{aligned} &x(r_0 + 1, r_1 + \varepsilon, r_2 + \varepsilon^2, \ldots) \\ &\qquad = Tx(r_0, r_1, r_2, \ldots) + \varepsilon f[x(r_0, r_1, r_2, \ldots), x(r_0 - 1, r_1 - \varepsilon, r_2 - \varepsilon^2, \ldots). \end{aligned} \tag{29}$$

As for the continuous-time systems, the key point of the algorithm consists in expanding $x(r_0 \pm 1, r_1 \pm \varepsilon, r_2 \pm \varepsilon^2, \ldots)$ in power series of $\varepsilon$:

$$\begin{aligned} &x(r_0 \pm 1, r_1 \pm \varepsilon, r_2 \pm \varepsilon^2, \ldots) = x(r_0 \pm 1, r_1, r_2, \ldots) \\ &\qquad \pm \varepsilon D_1 x(r_0 \pm 1, r_1, r_2, \ldots) + \frac{1}{2}\,\varepsilon^2(D_1^2 \pm 2D_2)x(r_0 \pm 1, r_1, r_2, \ldots) + \ldots, \end{aligned} \tag{30}$$

where $D_k = \partial/\partial r_k$ $(k = 1, 2, \ldots)$. Equation (30) can be rewritten in more compact form by defining a (forward) partial shift operator $\mathcal{E}_0$ with respect to the fast variable $r_0$:

$$\mathcal{E}_0 x(r_0, r_1, r_2, \ldots) = x(r_0 + 1, r_1, r_2, \ldots). \tag{31}$$

By using (31), equation (30) is written as:

$$x(r_0+1, r_1+\varepsilon, r_2+\varepsilon^2, \ldots) = \mathcal{E}_0 \left[1 + \varepsilon D_1 + \frac{1}{2}\varepsilon^2(D_1^2 + 2D_2) + \ldots\right] x(r_0, r_1, r_2, \ldots). \tag{32}$$

Therefore, the total shift operator $E_n$ can be expressed as

$$E_n = \mathcal{E}_0 \left[1 + \varepsilon D_1 + \frac{1}{2}\,\varepsilon^2(D_1^2 + 2D_2) + \ldots\right] \tag{33}$$

and, similarly, its inverse as:

$$E_n^{-1} = \mathcal{E}_0^{-1} \left[1 - \varepsilon D_1 + \frac{1}{2}\,\varepsilon^2(D_1^2 - 2D_2) + \ldots\right]. \tag{34}$$

Equation (33) is the counterpart for discrete-time systems of the chain rule (26) valid for continuous-time systems. In a similar way it expresses the total shift operator $E_n$ as the partial shift operator on the fast scale $\mathcal{E}_0$, plus small differential quantities associated with the slow scales.

By expanding $x(r_0, r_1, r_2, \ldots)$ in series of $\varepsilon$,

$$x(r_0, r_1, r_2, \ldots) = \sum_{k=0}^{\infty} \varepsilon^k x_k(r_0, r_1, r_2, \ldots) \tag{35}$$

and using equations (33) and (34), the following perturbations equations are obtained from (3):

$$\varepsilon^0: \quad \mathcal{E}_0 x_0 - T x_0 = 0 \tag{36a}$$

$$\varepsilon^1: \quad \mathcal{E}_0 x_1 - T x_1 = -\mathcal{E}_0 D_1 x_0 + f[x_0, \mathcal{E}_0^{-1} x_0] \tag{36b}$$

$$\begin{aligned}\varepsilon^2: \quad \mathcal{E}_0 x_2 - T x_2 &= -\mathcal{E}_0 \left(D_1 x_1 + \frac{1}{2}\,(D_1^2 + 2D_2) x_0\right) \\ &\quad + f_n[x_0, \mathcal{E}_0^{-1} x_0] x_1 + f_{n-1}[x, \mathcal{E}_0^{-1} x_0] \mathcal{E}_0^{-1}(x_1 - D_1 x_0)\end{aligned} \tag{36c}$$

in which $x_h = x_h(r_0, r_1, r_2, \ldots)$ $(h = 0, 1, 2, \ldots)$.

Similarly to what was observed regarding the Lindstedt–Poincaré method, the difference equation (3) is transformed by equations (28) and (33) into a delay partial differential equation. However, the shift operator affects the fast variable $r_0$ only, while the partial derivatives operators affect the slow variables $r_1$, $r_2$, …, only; moreover the derivatives appear only at the $\varepsilon$- or higher order, whereas the order one part of the equation is an ordinary difference equation in the fast variable $r_0$. This implies that the perturbation equations (36) are still difference equations, like the original equation (3).

Equation (36a) admits the solution

$$x_0 = A(r_1, r_2, \ldots) u_j \exp(i\theta_0 r_0) + \text{c.c.} \tag{37}$$

that is periodic on the scale $r_0$ of period $2\pi/\theta_0$, with amplitude and phase slowly modulated on the other scales. Hence equation (36b) becomes:

$$\begin{aligned}\mathcal{E}_0 x_1 - T x_1 &= -(D_1 A u_j \exp[i\theta_0(r_0+1)] + \text{c.c.} \\ &\quad + f(A u_j \exp(i\theta_0 r_0) + \text{c.c.}, A u_j \exp[i\theta_0(r_0-1)] + \text{c.c.}).\end{aligned} \tag{38}$$

The periodic (on the scale $r_0$) function $f$ can be expanded in a Fourier series:

$$f = \sum_{k=-\infty}^{\infty} F_k(A,\bar{A}) \exp(ik\theta_0 r_0), \quad F_k(A,\bar{A}) = \frac{\theta_0}{2\pi} \int_0^{2\pi/\theta_0} f \exp(-ik\theta_0 r_0)\, \mathrm{d}r_0. \tag{39}$$

Elimination of secular terms necessitates satisfying the following condition:

$$-D_1 A \exp(i\theta_0) + v_j^{\mathrm{H}} F_1 = 0. \tag{40}$$

By using the polar form $A = (1/2)a\exp(i\phi)$ and separating real and imaginary parts, two differential equations are obtained:

$$\frac{\partial a}{\partial r_1} = \frac{1}{\pi} \int_0^{2\pi} v_j^{\mathrm{H}} f[a u_j \cos\psi, a u_j \cos(\psi - \theta_0)] \cos(\psi + \theta_0)\, \mathrm{d}\psi \tag{41a}$$

$$\frac{\partial \phi}{\partial r_1} = -\frac{1}{\pi a} \int_0^{2\pi} v_j^{\mathrm{H}} f[a u_j \cos\psi, a u_j \cos(\psi - \theta_0)] \sin(\psi + \theta_0)\, \mathrm{d}\psi \tag{41b}$$

in which $\psi = \theta_0 r_0 + \phi$. To first approximation, by coming back to the discrete variable $n$, the solution to equation (2) reads:

$$\begin{aligned}x(n) &= a(\varepsilon n)[u_{jR}\cos\psi(n) - u_{jI}\sin\psi(n)] \\ \psi(n) &= \theta_0 n + \phi(\varepsilon n),\end{aligned} \tag{42}$$

where $a$ and $\phi$ are solutions of the differential equations (41). It can be noted that the steady-state solutions $a = \text{const}$, $\phi = \theta_1 r_1 + \text{const}$ of equations (41) satisfy the same equations (23) furnished by the Lindstedt–Poincaré method; in this case solutions (42) and (24) coincide. Thus the multiple scale perturbation method include the Lindstedt–Poincaré method, as in the continuous case.

Equation (41a) can be used to analyze the orbital stability of the steady-state solutions, by linearizing it around the fixed point.

To determine the second approximation, equation (38) must be solved. If internal resonance conditions do not occur and $\theta_0/2\pi$ is irrational, the following solution holds:

$$\begin{aligned}x_1 &= \sum_{k=-\infty}^{\infty} \sum_{h=1}^{m} \frac{v_h^{\mathrm{H}} F_k}{\exp(ik\theta_0) - \lambda_h} u_h \exp(ik\theta_0 r) + (B u_j \exp(i\theta_0 r_0) + \text{c.c.}) \\ &\qquad (k,h) \neq (1,j) \text{ and } (-1, j+1)\end{aligned} \tag{43}$$

in which $B = B(r_1, r_2, \ldots)$ is an arbitrary function. By eliminating secular terms in equation (36c), differential equations in $A(r_2)$ and $B(r_1)$ are obtained, as for the continuous-time systems.

## 5. An Illustrative Example: The Discrete Van der Pol Equation

The perturbation methods previously developed are now applied to a sample example. Numerical results are also given.

Let us consider the following discrete-time system:

$$(E_n - 2\cos\theta_0 + E_n^{-1})x = \varepsilon(1 - x^2)(E_n - E_n^{-1})x. \tag{44}$$

When $\varepsilon = 0$, equation (44) admits the periodic or almost-periodic solution $x(n) = a\cos(\theta_0 n + \phi)$ of frequency $\theta_0$ and arbitrary amplitude and phase. Rational values of $\theta_0/2\pi$ that would produce secular terms in the perturbation procedure will be excluded.

Equation (44) can be considered as a central finite difference approximation of the Van der Pol equation

$$\ddot{x} + \omega_0^2 x = \mu(1 - x^2)\dot{x}. \tag{45}$$

In this case $x(n)$ is the value of $x(t)$ at the instant $t_n = nh$, where $h$ is the discretization time step; in addition

$$\varepsilon = \mu h/2 \tag{46a}$$

$$\cos\theta_0 = 1 - (h\omega_0)^2/2 \tag{46b}$$

having assumed $h \leq (2/\omega_0)$.

The second-order difference equation (44) could be rewritten in the form (3) as two first-order equations and formulas previously obtained could be applied. However, as an example, the analysis will be developed here just by reference to the second-order equation.

### 5.1. The Lindstedt–Poincaré Method

By changing the independent variable according to equations (8) and (10), and using (14) to (16) in equations (44), the following perturbation equations are obtained:

$$\varepsilon^0: \quad Lx_0 = 0 \tag{47a}$$

$$\varepsilon^1: \quad Lx_1 = -\theta_1(E_s - E_s^{-1})x_0' + (1 - x_0)^2(E_s - E_s^{-1})x_0 \tag{47b}$$

$$\begin{aligned}\varepsilon^2: \quad Lx_2 = {} & -\theta_1(E_s - E_s^{-1})x_1' - \frac{1}{2}\,\theta_1^2(E_s + E_s^{-1})x_0'' - \theta_2(E_s - E_s^{-1})x_0' \\ & - 2x_0x_1(E_s - E_s^{-1})x_0 + (1 - x_0^2)(E_s - E_s^{-1})(x_1 + \theta_1 x_0'),\end{aligned} \tag{47c}$$

where $L = E_s - 2\cos\theta_0 + E_s^{-1}$.

The generating solution at the $\varepsilon^0$ order is

$$x_0 = A\exp(is) + \text{c.c.} \tag{48}$$

from which the $\varepsilon$-order equation reads:

$$Lx_1 = 2\sin\theta_0\{[\theta_1 A + i(A - A^2\bar{A})]\exp(is) - iA^3\exp(3is)\}. \tag{49}$$

To eliminate secular terms the coefficient of $\exp(is)$ must vanish. By using the polar form for $A$ and solving

$$\theta_1 = 0, \tag{50a}$$

$$a = 2 \tag{50b}$$

are found. Therefore, only one closed orbit (limit cycle) exists, with no frequency correction, to first approximation, with respect to the linear solution $\varepsilon = 0$. By solving equation (49) one obtains

$$x_1 = C\exp(3is) + B\exp(is) + \text{c.c.}, \tag{51}$$

where $B = (1/2)b\exp(i\beta)$ is an arbitrary constant and

$$C = i\frac{\sin\theta_0}{\cos\theta_0 - \cos 3\theta_0}A^3. \tag{52}$$

By reminding that $\theta_0 \in [0, \pi]$, the amplitude $C$ in equation (51) is finite only if $\theta_0 \neq \pi$, i.e., if no strong resonance occurs. This is a consequence of the fact that the last term in equation (49) also produces secular terms if $\theta_0 = \pi$.

Substituting the previous results in equation (47c) and eliminating terms that produce secular terms leads to:

$$2\theta_2 A\sin\theta_0 + 2i\sin\theta_0(B - 2A\bar{A}B - A^2\bar{B} + 2C\bar{A}^2) - 2iC\bar{A}^2\sin 3\theta_0 = 0. \tag{53}$$

Hence, by separating real and imaginary parts and accounting for equation (50b), equation (53) furnishes:

$$b = 0, \tag{54a}$$

$$\theta_2 = \frac{2\sin\theta_0 - \sin 3\theta_0}{\cos\theta_0 - \cos 3\theta_0}. \tag{54b}$$

To second approximation, equation (44) admits the periodic solution

$$x(n) = 2\left[\cos\psi(n) - \varepsilon\frac{\sin\theta_0}{\cos\theta_0 - \cos 3\theta_0}\sin 3\psi(n)\right] \tag{55a}$$

$$\psi(n) = (\theta_0 + \varepsilon^2\theta_2)n + \phi \tag{55b}$$

with $\theta_2$ given by equation (54b).

It is worth comparing the perturbative steady-state solution (55) of the discrete Van der Pol equation (44) with the perturbative steady-state solution of the continuous equation (45), by performing the limit $h \to 0$. From equation (46b) it is seen that when $h \to 0$, $\theta_0 \to 0$ too; in addition, by comparing (46b) with the two-terms MacLaurin expansion of $\cos\theta_0$, $\theta_0/h \to \omega_0$ follows. From equation (54b), by accounting for equation (46a), it ensues that $\varepsilon^2\theta_2/\theta_0 \to -\mu^2/(16\omega_0^2)$. Therefore, at the instant $t_n = nh$, equations (55) read:

$$\begin{aligned}\lim_{h\to 0} x(t_n) &= 2\cos\psi(t_n) - \frac{\mu}{4\omega_0}\sin 3\psi(t_n)\\ \lim_{h\to 0}\psi(t_n) &= \omega_0\left(1 - \frac{\mu^2}{16\omega_0^2}\right)t_n + \phi.\end{aligned} \tag{56}$$

They coincide with the sampled steady-state asymptotic solution of the continuous-time Van der Pol equation, that can be found, e.g., in [5].

By introducing the scales $r_0$, $r_1$ and $r_2$, and using equations (33) to (35), the following perturbation equations are drawn from equation (44):

$$\varepsilon^0: \quad \mathcal{L}x_0 = 0 \tag{57a}$$

$$\varepsilon^1: \quad \mathcal{L}x_1 = -(\mathcal{E}_0 - \mathcal{E}_0^{-1})D_1x_0 + (1 - x_0)^2(\mathcal{E}_0 - \mathcal{E}_0^{-1})x_0 \tag{57b}$$

$$\begin{aligned}\varepsilon^2: \quad \mathcal{L}x_2 = {} & -(\mathcal{E}_0 - \mathcal{E}_0^{-1})(D_1x_1 + D_2x_0) - \frac{1}{2}(\mathcal{E}_0 + \mathcal{E}_0^{-1})D_1^2x_0 - 2x_0x_1(\mathcal{E}_0 - \mathcal{E}_0^{-1})x_0 \\ & + (1 - x_0^2)[(\mathcal{E}_0 - \mathcal{E}_0^{-1})x_1 + (\mathcal{E}_0 + \mathcal{E}_0^{-1})D_1x_0],\end{aligned} \tag{57c}$$

where $\mathcal{L} = \mathcal{E}_0 - 2\cos\theta_0 + \mathcal{E}_0^{-1}$.

The generating solution is:

$$x_0 = A(r_1, r_2)\exp(i\theta_0 r_0) + \text{c.c.} \tag{58}$$

and equation (57b) becomes:

$$\mathcal{L}x_1 = -2i\sin\theta_0[(D_1A - A + A^2\bar{A})\exp(i\theta_0 r_0) + A^3\exp(3i\theta_0 r_0)] + \text{c.c.} \tag{59}$$

By zeroing the coefficient of $\exp(i\theta_0 r_0)$ and separating real and imaginary parts, it follows:

$$\frac{\partial a}{\partial r_1} = \left(1 - \frac{1}{4}a^2\right)a, \tag{60a}$$

$$\frac{\partial \phi}{\partial r_1} = 0. \tag{60b}$$

By integrating and remembering that $r_1 = \varepsilon n$ one obtains:

$$a = \frac{2}{[1 + c(r_2)\exp(-2\varepsilon n)]^{1/2}}, \tag{61a}$$

$$\phi = \phi(r_2). \tag{61b}$$

To first approximation, the amplitude of the periodic solution (58) tends exponentially to the steady-state solution, while no modulation of phase occurs. The fixed point of equation (60a) is stable, so that the periodic orbit is also stable.

By solving equation (59) and omitting for simplicity the homogeneous solution, one has:

$$x_1 = C\exp(3i\theta_0 r_0) + \text{c.c.}, \tag{62}$$

where $C$ is still given by equation (52). Substituting in equation (57c) leads to the following solvability condition:

$$\begin{aligned}& -\cos\theta_0 D_1^2A - 2i\sin\theta_0 D_2A + 2\cos\theta_0(D_1A - A^2D_1\bar{A}) - 2A\bar{A}D_1A \\ & \quad + 2i(2\sin\theta_0 - \sin 3\theta_0)C\bar{A}^2 = 0.\end{aligned} \tag{63}$$

In equation (63) $D_1A$ and $D_1^2A$ can be expressed as a function of $A$ by the solvability condition at the $\varepsilon$-order. By substituting for $C$ equation (52) and passing to the polar form, the following two differential equations are drawn:

$$\frac{\partial a}{\partial r_2} = 0, \tag{64a}$$

$$a\frac{\partial \phi}{\partial r_2} = -\frac{1}{2\sin\theta_0}\left(a\cos\theta_0 - a^3 + \frac{1-k}{4}a^5\right), \tag{64b}$$

where $k$ is a constant, depending on $\theta_0$:

$$k = \frac{1}{4}\left[\cos\theta_0 + 2\sin\theta_0\frac{2\sin\theta_0 - \sin 3\theta_0}{\cos\theta_0 - \cos 3\theta_0}\right]. \tag{65}$$

Equation (64b) can be more simply integrated if use is made of equation (60a) to rewrite it as

$$\frac{\mathrm{d}\phi}{\mathrm{d}r_2} = \frac{1}{2\sin\theta_0}\left[a(1-k) - 4\frac{k}{a}\right]\frac{\mathrm{d}a}{\mathrm{d}r_1} + \theta_2 \tag{66}$$

having used position (54b).

It should be observed that equation (66) contains one known function of the $r_1$ variable and one unknown function of the $r_2$ variable. Therefore, if the two variables are considered to be independent the equation cannot be satisfied. This is a consequence of having omitted the homogeneous solution in equation (62). In fact, the availability of an unknown function $B(r_1)$ allows separation of the functions of the two variables, as done by Nayfeh ([5], p. 245) in the analysis of the continuous Van der Pol equation. However, equation (66) can be equally integrated if the functions $\phi$ and $a$ are referred to the same scale $r_0$. In this way an equation similar to that furnished by the Krylov–Bogolioubov–Mitropolsky technique is obtained (see, e.g. [5], p. 176). By operating in this way one obtains:

$$\phi(n) = \varepsilon^2\theta_2 n + \varepsilon\frac{1}{2\sin\theta_0}\left[\frac{a^2}{2}(1-k) - 4k\ln a\right] + \phi_0, \tag{67}$$

where $\phi_0$ denotes a constant and the discrete variable $n = r_0$ has again been introduced. It should be noted incidentally that $\phi$ does not depend on $r_2 \equiv \varepsilon^2 n$ alone, but, in contrast with equation (60b), it depends on $r_1$ also, since $a = a(r_1)$; this is due to the fact that $\phi$ accounts also for the omitted complex constant $B(r_1)$. To second approximation the solution is:

$$\begin{aligned} x(n) &= a(n)\cos\psi(n) - \frac{\varepsilon}{4}\frac{\sin\theta_0}{\cos\theta_0 - \cos 3\theta_0}a^3(n)\sin 3\psi(n) \\ \psi(n) &= \theta_0 n + \phi(n), \end{aligned} \tag{68}$$

where $a(n)$ is given by equation (61a), with $c(r_2) = 4/a^2(0) - 1 = \text{const}$, and $\phi(n)$ by equation (68). The latter shows that, after the transient state has been exhausted and the amplitude gets the limit value $a \to 2$, $\phi(n) \to \varepsilon^2\theta_2 n + \text{const}$; therefore equation (68b) tends to (55b) and (68a) to (55a). Thus, the steady-state solution previously obtained by the Lindstedt–Poincaré method is recovered. However, since the constant part of the phase $\phi$ depends on the initial conditions, the limit cycle is sampled in different ways for different initial conditions.

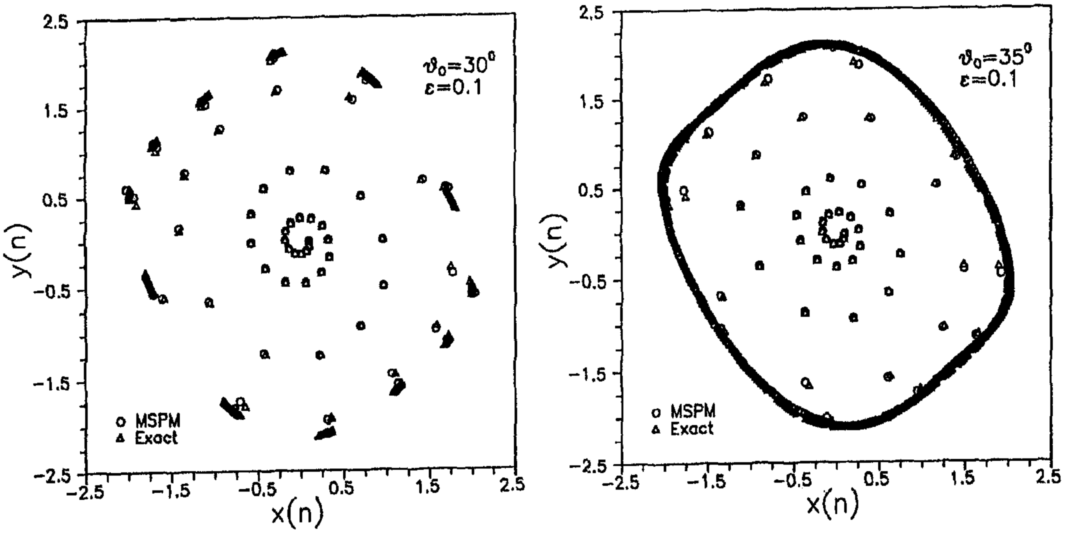


*Figure 1.* Orbits for the discrete-time Van der Pol oscillator.

Finally the limit of equations (68) for $h \to 0$ is studied; since $k \to 1/8$ the solution tends to:

$$\lim_{h\to 0} x(t_n) = a(t_n)\cos\psi(t_n) - \frac{1}{32}\frac{\mu}{\omega_0}\, a^3(t_n)\sin 3\psi(t_n)$$

$$\lim_{h\to 0} \psi(t_n) = \omega_0\left(1 - \frac{\mu^2}{16\omega_0^2}\right)t_n + \frac{\mu}{\omega_0}\left[\frac{7}{64}\, a^2(t_n) - \frac{1}{8}\,\ln a(t_n)\right] + \phi_0, \tag{69}$$

where

$$a(t_n) = \frac{2}{[1 + c\exp(-\mu t_n)]^{1/2}}\,. \tag{70}$$

Equations (69) and (70) coincide with the sampled asymptotic solution of the continuous-time Van der Pol equation (see, e.g. [5], p. 248).

### 5.3. Numerical Results

The validity of the methods proposed has been checked analytically for $\theta_0 \to 0$. However, when $\theta_0 = 0(1)$ equation (44) cannot be considered as a good finite difference approximation of the continuous Van der Pol equation; therefore the asymptotic solution (67) has been directly compared with the exact (numerical) solution of equation (44). The exact solution can be easily obtained by writing the equation in the form of a recurrence equation, $x(n+1) = \varphi[x(n), x(n-1); \varepsilon]$ and by assigning the initial conditions $x(0) = x^{(0)}$, $x(1) = x^{(1)}$.

By fixing $\varepsilon = 0.1$, two different values $\theta_0 = 30°$ and $\theta_0 = 35°$ have been considered, for which the generating solution (58) is strictly periodic, with a short or long period, respectively. For these values of $\theta_0$ resonance conditions appears only in higher-order perturbation equations. Results have been plotted in Figure 1 on the phase-plane, by assuming $x(n)$ and $y(n) = (x(n+1) - x(n))/\theta_0$ as state variables, and varying $n$ in the interval $[0, 500]$ in both cases.

It is apparent that the asymptotic solution is an excellent approximation of the exact solution, both during the transient state and the steady condition. Since nonlinearities modify the frequency $\theta_0$, in both cases considered the nonlinear steady response is almost-periodic, and the limit cycle is densely filled with points. However, since the frequency correction is small of the order $\varepsilon^2$, the limit cycle is covered much more slowly in the $\theta_0 = 30°$ case than in the $\theta_0 = 35°$ case.

## 6. Conclusions

The classical Lindstedt–Poincaré and multiple scale perturbation methods have been adapted to deal with nonlinear discrete-time dynamical systems. The methods allow determination of the almost-periodic nonlinear solutions bifurcating from the equilibrium state when a generic secondary Hopf critical condition occurs, namely when a pair of complex conjugate eigenvalues of the linearized map crosses the unitary circle at an irrational point. The transient evolution from the unstable fixed point to the stable limit cycle can also be analyzed in the supercritical case by the multiple scale method.

Very simple shift operators transformation laws have been determined in the two methods; they can be considered to be the counterpart for discrete systems of the chain rules valid for continuous systems and allow the perturbation equation to be derived in a systematic way.

The perturbation procedures have been tested by reference to a discretized Van der Pol equation. It has been shown that the discrete asymptotic solutions are in excellent agreement with the numerical solutions; moreover, they correctly tend to the asymptotic continuous solutions when the time step tends to zero.

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## References

1. Thompson, J. M. T. and Stewart, H. B., *Nonlinear Dynamics and Chaos*, Wiley, Chichester, 1986.
2. Parker, T. S. and Chua, L. O., *Practical Numerical Algorithms for Chaotic Systems*, Springer, New York, 1989.
3. Iooss, G. and Joseph, D. D., *Elementary Stability and Bifurcation Theory*, Springer, New York, 1980.
4. Minorsky, N., *Nonlinear Oscillations*, Van Nostrand, New York, 1962.
5. Nayfeh, A., *Perturbation Methods*, Wiley, New York, 1973.
6. Sethna, P. R. and Shapiro, S. M., 'Nonlinear behavior of flutter unstable dynamical systems with gyroscopic and circulatory forces', *Journal of Applied Mechanics* **12**, 1977, 755–782.
7. Guckenheimer, J. and Holmes, P., *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer, New York, 1983.
8. Atadan, S., 'Periodic behaviour in nonlinear, discrete-time systems and its application to a digital control system', *Dynamics and Stability of Systems* 7(3), 1992, 151–161.