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Elimination of extremal index zeroes from generic paths of closed 1-forms

C. Moraga Ferrándiz

March 23, 2013

Abstract

Let α be a Morse closed 1-form of a smooth n -dimensional manifold M . The zeroes of α of index 0 or n are called *centers*. It is known that every non-vanishing de Rham cohomology class u contains a Morse representative without centers. The result of this paper is the one-parameter analogue of the last statement: every generic path $(\alpha_t)_{t \in [0,1]}$ of closed 1-forms in a fixed class $u \neq 0$ such that α_0, α_1 have no centers, can be modified relatively to its extremities to another such path $(\alpha'_t)_{t \in [0,1]}$ having no center at all.

1 Introduction and main result

Let M be a closed smooth manifold of dimension n and u be a non-zero de Rham cohomology class of degree 1 of M . We are considering $(\alpha_t)_{t \in [0,1]}$ a path of closed 1-forms where $[\alpha_t] = u$ for every t . Generically, such a path only consists of Morse 1-forms but in a finite set of times $\{t_i\}_{i=1}^s$ where the path crosses transversely the co-dimension one strata of birth/elimination closed 1-forms. Namely, such an α_{t_i} presents a unique degenerate zero $p \in M$ such that for t near to t_i we have:

$$\alpha_t|_U = d(x_n^3 \pm (t - t_i)x_n + Q(x_1, \dots, x_{n-1})).$$

Here U is some coordinate neighbourhood of p and Q is a non-degenerate quadratic form in the first $n - 1$ variables. The birth/elimination strata are naturally co-oriented: we will say that α_{t_i} is of birth-type if the orientation induced by the path represents the fixed co-orientation of the strata; it corresponds to the case when two critical points of Morse type have appeared when t increases over t_i .

The genericness of these properties for a path is a direct consequence of Thom's transversality theorem in jet spaces ([Tho56] or [GG73] for a more educational presentation), as it was shown by Cerf in [Cer68] for the case $u = 0$ (see also [Lau10]).

Among the zeroes of a Morse closed 1-form α , which we denote by $Z(\alpha)$, those of extremal index are called *centers*. The idea of cancelling pairs of critical points

of consecutive index which are connected by a unique (pseudo)gradient trajectory, so much exploited in proving Smale's theorem of h -cobordism (see [Mil65]), is used in the paper [AL86] to construct α' , a Morse closed 1-form cohomologous to α which does not have any center. Their only restriction is to ask the cohomology class u not to be rational ($\text{rk}(u) > 1$). Latour gives a more topological proof without any requirement to $u \neq 0$ in [Lat94]. We deal with the one-parameter version of this result and obtain the following theorem:

Theorem 1. *Let $\dim(M) \geq 3$. Every generic path of cohomologous closed 1-forms $(\alpha_t)_{t \in [0,1]}$ such that α_0, α_1 have no centers can be deformed into a path $(\beta_t)_{t \in [0,1]}$ of the same kind which has the same extremities and no center at all.*

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2 Morse-Novikov homology

Take α a Morse closed 1-form and choose $B_* \subset \widetilde{M}$ a lifting of $Z(\alpha)$ to the universal cover $\widetilde{M} \xrightarrow{\pi} M$, where $P \in B_*$ corresponds bijectively to $p \in Z(\alpha)$. We remark that B_* is graduated by the index. From that, we derive a graded module $C_*(\alpha)$ freely generated by B_* over the Novikov ring Λ_{-u} associated to the cohomology class of α . We recall that Λ_{-u} is the completion of the group ring $\Lambda := \mathbb{Z}[\pi_1 M]$ given by

$$\Lambda_{-u} = \left\{ \sum_{g \in \pi_1 M} n_g g, n_g \in \mathbb{Z} \mid \begin{array}{l} n_g = 0 \text{ except for finitely many } g \\ \text{or } \lim_{n_g \neq 0} u(g) = -\infty \end{array} \right\}.$$

As for Morse functions, a differential for $C_*(\alpha)$ can be obtained if we give us ξ , a special kind of vector field adapted to α .

Definition 2. *A vector field ξ is a pseudo-gradient of a Morse closed 1-form α if the function $\alpha(\xi) \in \mathcal{C}^\infty(M)$*

- (1) *is strictly negative outside $Z(\alpha)$ and*
- (2) *$Z(\alpha)$ are non-degenerate maxima of $\alpha(\xi)$.*

Such a vector field vanishes only at $Z(\alpha)$. Each zero $p \in Z(\alpha)$ has a local stable and unstable manifold determined by ξ ; they are denoted respectively by $W_{\text{loc}}^s(p), W_{\text{loc}}^u(p)$ and are diffeomorphic to Euclidean spaces of complementary dimension. Their intersection is transverse and reduced to p .

The global stable and unstable manifolds of $p \in Z(\alpha)$ are defined by

$$W^s(p) := \left\{ x \in M \mid \lim_{t \rightarrow +\infty} \xi^t(x) = p \right\} ; W^u(p) := \left\{ x \in M \mid \lim_{t \rightarrow -\infty} \xi^t(x) = p \right\}$$

where $(\xi^t)_{t \in [0,1]}$ denotes the flow of ξ . Given P a lifting of p , we denote $W^{s/u}(P)$ the connected component of $\pi^{-1}(W^{s/u}(p))$ containing P .

We denote the set of orbits of ξ going from p to q by $\mathcal{L}(p, q)$. The image of these orbits is clearly contained on $W^u(p) \cap W^s(q)$. The choice B_* allows one to define the *enwrapment*, which is a map $\mathcal{L}(p, q) \xrightarrow{e} \pi_1 M$. Remark that every $\ell \in \mathcal{L}(p, q)$ has a unique lifting $\tilde{\ell}$ starting from P ; $\tilde{\ell}$ goes so to $g_\ell Q$ for a unique $g_\ell \in \pi_1 M$. We set $e(\ell) := g_\ell$.

Definition 3. *A pseudo-gradient ξ of a Morse closed 1-form α is Morse-Smale if its global stable and unstable manifolds intersect transversely:*

$$W^u(p) \pitchfork W^s(q), \quad \text{for all } p, q \in Z(\alpha).$$

This class of vector fields allows one to count orbits from p to q , two zeroes of α such that $\text{ind}(p) = \text{ind}(q) + 1$. The orbits $\ell \in \mathcal{L}(p, q)$ depend on the choice of such a ξ and their enwrapment $e(\ell)$ on the choice of the lifting B_* . A sign $s(\ell) = \pm 1$ can be computed if an additional choice of orientation of the unstable manifolds is made. We obtain the incidences $\langle P, Q \rangle^{\xi, B} := \sum_{\ell \in \mathcal{L}(p, q)} s(\ell) e(\ell) \in \Lambda_{-u}$. The property of being Morse-Smale for a pseudo-gradient is generic as it was proven in [Kup63]; the interested reader is sent to [Paj06]. The next theorem is due to Novikov:

Theorem 4 ([Nov82]). *The Λ_{-u} -linear extension of the map*

$$\begin{aligned} \partial_{*+1}^{\xi, B} : C_{*+1}(\alpha) &\longrightarrow C_*(\alpha) \\ P &\longmapsto \sum_{Q \in B_*(\alpha)} \langle P, Q \rangle^{\xi, B} Q \end{aligned}$$

is a differential for the graded Λ_{-u} -module $C_(\alpha)$.*

The complex $C_*(\alpha, \partial_*^{\xi, B})$, which is called the *Morse-Novikov complex*, does not depend on the choices we have made, up to simple-equivalence (see [Lat94]). We denote its homology by $H_*(M, u)$.

3 Connecting saddles and elimination of centers

We are going to consider *sinks* (centers of index 0). The case of *sources* (centers of index n) can be treated in an analogous way.

A Morse closed 1-form α induces a singular foliation \mathcal{F}_α in M : two points x, y belong to the same leaf F if there exists a smooth path $[0, 1] \xrightarrow{\gamma} M$ joining x to y and such that $\alpha(\gamma'(t)) = 0$ for all $t \in [0, 1]$. A leaf is called singular if it contains some zero of α .

Consider now ξ a pseudo-gradient for α . The orbits $\ell \in \mathcal{L}^\xi(p, q)$ of ξ going from p to q , two zeroes of α , have an associated *transverse length* $L(\ell)$. This length is given by the integral

$$L(\ell) := - \int \ell^*(\alpha),$$

which is positive thanks to the condition (1) of the definition 2 of a pseudo-gradient.

Remark that u induces a group morphism $\pi_1 M \xrightarrow{u} \mathbb{R}$ given by integrating α along a representative γ of a loop $g = [\gamma]$. As $\pi^*\alpha$ is exact, let us take a *primitive* $\widetilde{M} \xrightarrow{h} \mathbb{R}$, which verifies $dh = \pi^*\alpha$. An easy calculation shows that for $\ell \in \mathcal{L}(p, q)$ we have the equality:

$$L(\ell) = h(P) - h(gQ) = h(P) - h(Q) - u(g), \quad (1)$$

which relates numerically the choice of liftings with the enwrapment of orbits.

We call *saddles* $\mathcal{S} := Z_1(\alpha)$ the index 1 zeroes of α . The unstable manifold of a saddle s always decomposes in two different orbits ℓ^+, ℓ^- , that we called separatrices. A well known fact about Morse-Novikov homology is that $H_0(M, u) = 0$ for every $u \neq 0$ (see [Far04, Cor. 1.33] for example). So, for ξ Morse-Smale, it implies in particular that the set \mathcal{S}_c of saddles with at least one separatrix going to c is non-empty: otherwise the lifting C has no chance to be in $\text{Im}(\partial_1^\xi)$ and we would have $0 \neq [C] \in H_0(M, u)$. We distinguish two kinds of saddles $s \in \mathcal{S}_c$.

The saddle $s \in \mathcal{S}_c$ is said to be of type *ker* if $\mathcal{L}(s, c) = \{\ell^+, \ell^-\}$ and $L(\ell^+) = L(\ell^-)$. Any lifting of $W^u(s)$ determines two liftings of c related by an element $g \in \ker(u)$ as in figure 1: this is a direct consequence of relation (1). These collection of saddles is denoted by $\mathcal{S}_c^{\text{ker}}$. Saddles in the complementary set $\mathcal{S}_c^n := \mathcal{S}_c \setminus \mathcal{S}_c^{\text{ker}}$ are called *normal saddles*. Remark that if $C \in B_*$ is the chosen lifting of a sink c , we can find a lifting S of $s \in \mathcal{S}_c$ such that $W^u(S)$ is as in figure 1 or 2. However, there is a priori no reason to have $S \in B_*$.

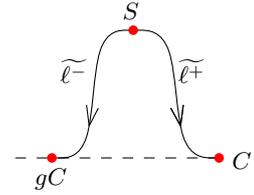


Figure 1: $s \in \mathcal{S}_c^{\text{ker}}$

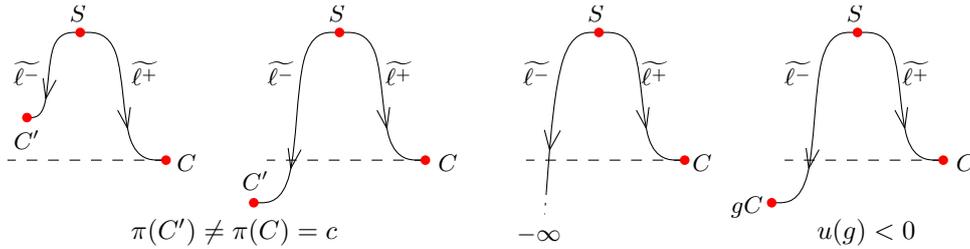


Figure 2: Possible situations of a lifting of $W^u(s)$ to \widetilde{M} for $s \in \mathcal{S}_c^n$.

If a normal saddle $s \in \mathcal{S}_c^n$ has its two orbits going to c , their lengths are different; for any normal saddle s , we note by ℓ^+ the shortest orbit joining s to c and call it *the connecting orbit* of s . We note by \mathcal{L}_c^+ the finite set of connecting orbits from normal saddles $s \in \mathcal{S}_c^n$ to c .

Definition 5. A *connecting saddle for the sink c* is a normal saddle $s \in \mathcal{S}_c^n$ minimizing the length of connecting orbits $\mathcal{L}_c^+ \xrightarrow{L} \mathbb{R}^+$. In other words: $L(\ell_s^+) \leq L(\ell_{s'}^+)$ for every $s' \in \mathcal{S}_c^n$.

Connecting saddles are going to be the main tool to eliminate sinks. We justify their existence by a purely algebraic argument in lemma 6 below.

Lemma 6. *Let α be a Morse closed 1-form together with a Morse-Smale pseudo-gradient ξ . Then, every sink $c \in Z_0(\alpha)$ admits a connecting saddle.*

Proof. Choose liftings of c and of saddles \mathcal{S} ; for saddles in \mathcal{S}_c we do that as in figures 1 and 2. Let us see that $\mathcal{S}_c = \mathcal{S}_c^{\ker}$ would imply that $C \notin \text{Im}(\partial_1^\xi)$, leading to the same contradiction as before. By identifying saddles with their chosen liftings, we clearly have $C \notin \partial_1^\xi(\Lambda_{-u} \otimes (\mathcal{S} \setminus \mathcal{S}_c))$. An element of $\text{Im}(\partial_1^\xi|_{\Lambda_{-u} \otimes (\mathcal{S}_c^{\ker})})$ is written as a finite sum times C :

$$\mu \cdot C := \sum_{s_i \in \mathcal{S}_c^{\ker}} \lambda_i(\pm 1 \pm g_i) \cdot C,$$

where $g_i \in \ker(u)$ and $\lambda_i \in \Lambda_{-u}$. Let us consider the map

$$\begin{aligned} (\cdot)^* : \Lambda_{-u} &\longrightarrow \mathbb{Z}[\ker(u)] \\ \lambda &\longmapsto \sum_{g \in \ker(u)} n_g g \end{aligned} ;$$

since $(\pm 1 \pm g_i) = (\pm 1 \pm g_i)^*$ for all i , we have $\mu^* = \sum \lambda_i^*(\pm 1 \pm g_i)$. Consider I the kernel of the augmentation morphism of rings $\mathbb{Z}[\ker(u)] \xrightarrow{\varepsilon_2} \mathbb{Z}_2$ given by $ng \mapsto n \pmod{2}$. As the terms $(\pm 1 \pm g_i)$ belong to the augmentation ideal, also does μ^* . Then μ cannot be 1. \square

Definition 7. *A Morse closed 1-form α is said to be 0-excellent if there exists a pseudo-gradient ξ for α such that for every sink $c \in Z_0(\alpha)$, the map $\mathcal{L}_c^+ \xrightarrow{L} \mathbb{R}^+$ is injective.*

Remark 8. The property of 0-excellence guarantees the *uniqueness* of connecting saddles: the connecting saddle of a sink c is the only $s \in \mathcal{S}_c^n$ such that ℓ_s^+ realizes the minimum of $\mathcal{L}_c^+ \xrightarrow{L} \mathbb{R}^+$. Moreover, 0-excellence is generic for α because any Morse closed 1-form can be slightly perturbed in order to obtain a cohomologous 0-excellent 1-form by application of lemma 9 to saddles.

Lemma 9 (REARRANGEMENT LEMMA). *Let p be a zero of a Morse closed 1-form α_0 of index k , equipped with a primitive $\widetilde{M} \xrightarrow{h_0} \mathbb{R}$ and a pseudo-gradient ξ ; consider P a lifting of p to \widetilde{M} . If $K > K' > 0$ are such that*

$$W^u(P) \cap h_0^{-1}([h_0(P) - K, +\infty)) \text{ is an embedded disk } \mathbb{D}^k,$$

there exists U a neighbourhood of p and a path $(\alpha_t)_{t \in [0,1]}$ of cohomologous Morse closed 1-forms with primitives $(h_t)_{t \in [0,1]}$ such that for all $t \in [0, 1]$ we have:

1. ξ is pseudo-gradient of α_t ,
2. $U \cap Z(\alpha_t) = p$,
3. $\alpha_t = \alpha_0$ on $M \setminus U$ and

$$4. h_1(P) = h_0(P) - K'.$$

Proof. Remark that sinks ($k = 0$) trivially verify the hypothesis about $K > 0$ for every such a K . There is no obstruction to decrease the values of a primitive as much as we want near local minima. This lemma is well known and can be found, for example, in [Far04, 9.5.1] under a slightly different presentation, or in [Mil65, 4.1] in the classical context for functions.

The proof of this lemma under this version can be found in [MF12, Lemme 2.2.34] for $k > 0$. The easy case $k = 0$ is presented now for the sake of completeness, since we deal principally with sinks.

Choose ξ a Morse-Smale pseudo-gradient for α_0 and call $m := h_0(C)$. Define $U := W_{\text{loc}}^s(c)$ and take V the connected component of $\pi^{-1}(U)$ containing C . We provide \mathbb{D}_ε^n , the closed n -disk of radius $\varepsilon > 0$, with polar coordinates $(\theta, r) \in \frac{\mathbb{S}^{n-1} \times [m, m+\varepsilon]}{\mathbb{S}^{n-1} \times \{m\}}$. By taking an $\varepsilon > 0$ small enough, we have a diffeomorphism $\Psi : \mathbb{D}_\varepsilon^n \rightarrow V \cap h_0^{-1}((-\infty, m + \varepsilon])$ such that:

1. $h_0(\Psi(\mathbb{S}^{n-1} \times \{r\})) = r$, for every $r \in [m, m + \varepsilon]$ and
2. $\Psi(\{\theta\} \times (m, m + \varepsilon])$ is a lifting of a trajectory of ξ , for all $\theta \in \mathbb{S}^{n-1}$.

Take $(\varphi)_{t \in [0,1]}$ an isotopy of \mathbb{R} such that for all $t \in [0, 1]$:

1. $\varphi_t|_{[m+\frac{\varepsilon}{2}, \infty)} = \text{Id}$,
2. $\varphi_t|_{(-\infty, m+\frac{\varepsilon}{4}]} = \text{Id} - tK$.

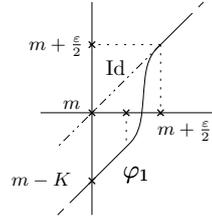


Figure 3: Graph of φ_0 and φ_1 .

The path $h_t := \varphi_t \circ h_0|_V$ extends $\pi_1 M$ -equivariantly to $V' := \bigcup_{g \in \pi_1 M} gV$ and then to the whole \widetilde{M} by taking $h_t = h_0$ on $\widetilde{M} \setminus V'$. The induced path of 1-forms $\alpha_t := \pi_*(dh_t)$ fulfills the required conditions.

Clearly ξ keeps the property of being a pseudo-gradient all along the path $(\alpha_t)_{t \in [0,1]}$ since the values of $\alpha_t(\xi)$ are those of $\alpha_0(\xi)$ eventually multiplied by $\varphi'_t > 0$. \square

Indeed, we only need the cases $k = 0, 1$ of lemma 9. We also want to eliminate pairs of Morse zeroes of these indexes; the next version of the Morse elimination lemma, which corresponds to [Lau, Theorem], is well adapted to our purpose.

Lemma 10 (MORSE ELIMINATION LEMMA). *Let $N \xrightarrow{h_0} \mathbb{R}$ be a Morse function equipped with a Morse-Smale pseudo-gradient ξ . Let p, q two critical points of h_0 such that $\text{ind}(p) = \text{ind}(q) + 1$. Suppose that there exists $\varepsilon > 0$ such that every orbit of $W^u(p)$ reaches the level $h^{-1}(h(q) - \varepsilon)$ but one which goes to q , then there exists V a neighbourhood of $\overline{W^u(p)} \cap h_0^{-1}([h_0(q) - \varepsilon, +\infty))$ and a generic path of functions $(\alpha_t)_{t \in [0,1]}$ such that:*

1. h_t is Morse for all $t \neq \frac{1}{2}$,
2. $h_t = h_0$ on $N \setminus V$ for all t and
3. $\text{Crit}(h_t|_V) = \emptyset$ for all $t > \frac{1}{2}$.

Lemma 11. *Let s be a connecting saddle for a sink c of a Morse closed 1-form α_0 . There exists a generic path $(\alpha_t)_{t \in [0,1]}$ of cohomologous 1-forms, beginning at α_0 , crossing the birth/elimination strata only once and such that $Z(\alpha_1) = Z(\alpha_0) \setminus \{s, c\}$.*

Proof. Equip α_0 with a Morse-Smale pseudo-gradient and consider $\widetilde{M} \xrightarrow{h_0} \mathbb{R}$ a primitive of α_0 . Take S, C liftings of s, c as in figure 2. If $|\mathcal{L}(s, c)| = 1$, the separatrix ℓ^- of $W^u(s)$ not going to c can be supposed such that $L(\ell^-) > L(\ell^+)$: equivalently, we are not under the first situation depicted on figure 2. This follows directly from lemma 9 applied to $P = C'$, the ending point of ℓ^- .

So the orbit ℓ^- goes under the level containing C , and the unstable manifold $W^u(S)$ verifies the hypothesis of the lemma 10. We find N a neighbourhood of $\overline{W^u(S)} \cap h_0^{-1}([h_0(C) - \varepsilon, \infty))$ for $\varepsilon > 0$ small enough so that $\pi|_N$ is injective. Apply now the elimination lemma 10 to $h_0|_N$ and the pair S, C ; we perform this deformation of h_0 in a $\pi_1 M$ -equivariant manner by imposing $h_t|_{gN} = h_t|_N + u(g)$ for every $g \in \pi_1 M$ and $t \in [0, 1]$. So the push-forward $\alpha_t := \pi_*(dh_t)$ for $t \in [0, 1]$, is a well defined path of 1-forms which has the required properties. \square

Note that lemmas 9, 10 and 11 also hold for smooth families of data when the assumptions are satisfied for every parameter. The next lemma is nothing but the adapted version for closed 1-forms of [Lau12, Elementary lips lemma (2.8)] where the author states that the proof is similar to that of [Lau12, Elementary swallow-tail lemma (2.6)]. We supply below the details of the elementary lips lemma (for any index i) because it is fundamental for our aim.

Lemma 12 (ELEMENTARY LIPS LEMMA). *Let $(\alpha_t, \xi_t)_{t \in [0,1]}$ be generic. Suppose that $(s_t, c_t)_{t \in [t_0, t_1]}$ are continuous paths of zeroes such that $s_{t_0} = c_{t_0}$, $s_{t_1} = c_{t_1}$ are respectively birth and elimination and that there exists an $\varepsilon > 0$ such that for every $t \in (t_0, t_1)$:*

1. the zeroes are Morse with $\text{ind}(s_t) - 1 = \text{ind}(c_t) = i$,
2. $W^u(s_t) \pitchfork W^s(c_t)$ along a unique orbit ℓ_t^+ ; moreover $L(\ell_t^+) + \varepsilon < L(\ell_t)$ for every orbit $\ell_t \subset W^u(s_t) \setminus \{\ell_t^+\}$.

Then, for all $\delta > 0$ there exists a generic $(\alpha'_t)_{t \in [0,1]}$ such that:

1. $\alpha_t = \alpha'_t$ for all $t \notin (t_0 - \delta, t_1 + \delta)$,
2. $Z(\alpha'_t) = Z(\alpha_t) \setminus \{s_t, c_t\}$ for all $t \in (t_0 - \delta, t_1 + \delta)$.

Proof. The situation that we are going to describe fits completely with that of [Lau, Theorem] for $t \in [t_0, t_1]$. Choose primitives $(h_t)_{t \in [0,1]}$ and consider a continuous lifting $(\widetilde{\ell}_t^+)_{t \in [t_0, t_1]}$ of the distinguished orbits, which join S_t with C_t , liftings of the considered zeroes. Consider $[0, 1] \xrightarrow{I_t} \widetilde{M}$ a smooth arc such that for all $t \in [t_0, t_1]$:

1. its image contains $\widetilde{\ell}_t^+$ and is contained in $W^u(S_t) \cup W^s(C_t)$,
2. the only critical points of $f_t := h_t|_{\text{Im}(I_t)}$ are S_t, C_t , which have same nature and index decreased by i ,
3. the extreme values are $f_t(0) = h_t(C_t) - \varepsilon$ and $f_t(1) = h_t(C_t) + \varepsilon$.

For $t \in [t_0 - \delta, t_0)$ we define I_t by the backward rescaled flow line of ξ_t starting at $I_{t_0}(0)$, of transverse length equal to 2ε ; namely $I_t(u) := \xi_t^{-2\varepsilon u}(I_{t_0}(0))$. For $t \in (t_1, t_1 + \delta]$, replace only t_0 by t_1 .

CLAIM: *There exists a smooth family $(N_t)_{t \in [t_0 - \delta, t_1 + \delta]}$ of tubular neighbourhoods of the arcs I_t in \widetilde{M} where π is injective together with coordinates $(u, y, z) \in [0, 1] \times \mathbb{R}^i \times \mathbb{R}^{n-i-1}$ such that $\text{Im}(I_t) = \{y = 0, z = 0\}$ and*

$$h_t(u, y, z) = f_t(u) - q_1(y) + q_2(z),$$

where q_1, q_2 are two positive definite quadratic forms.

For $t \in [t_0, t_1]$, this is just [Lau, Lemma 1], where $W_t := \{z = 0\} \cap \{h_t \geq h_t(C_t) - \varepsilon\}$ contains the portion of $W^u(S_t) \cup W^u(C_t)$ above the specified level and defines an i -disks fibration $W_t \rightarrow \text{Im}(I_t)$ pinched at $I_t(0)$.

We define now W_t for $t \in [t_0 - \delta, t_0)$ in such a manner that coordinates smoothly extend to N_t for these times and have analogous properties for $\text{Im}(I_t) \subset W_t$.

Consider the i -disk at the birth time given by $\partial^- W_{t_0} := W_{t_0} \cap \{h_t = h_t(C_t) - \varepsilon\}$. Define now W_t by taking the union of the backward rescaled flow lines of ξ_t starting at $\partial^- W_{t_0}$ of transverse length equal to 2ε ; in other terms

$$W_t := \xi_t^{[-2\varepsilon, 0]}(\partial^- W_{t_0}).$$

For all $t \in [t_0 - \delta, t_1 + \delta]$, choose a smooth function $g_t : I_t \rightarrow \mathbb{R}$ such that $g_t \leq f_t$, coincides with f_t near ∂I_t and has no critical points as figure 4 suggests.

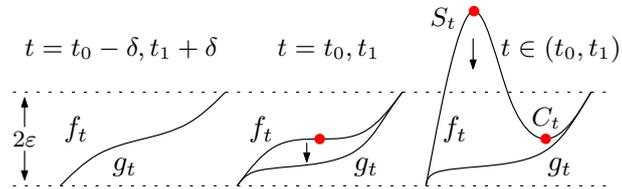


Figure 4

Clearly, the interpolation given by $k_t^s := f_t + s(g_t - f_t)$ transports f_t to g_t when s runs into $[0, 1]$. We have to multiply by a bump function $\omega : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ centered at 0 to avoid modifying h_t near ∂N_t . The two parameter family of functions

$$h_t^s(u, y, z) := f_t(u) + \omega(y, z) \cdot s(g_t(u) - f_t(u)) - q_1(y) + q_2(z),$$

has the same critical points as k_t^s since the (y, z) -derivatives vanish simultaneously only at $(y, z) = (0, 0)$ and $\omega \equiv 1$ near $(0, 0)$.

This deformation can be done $\pi_1 M$ -equivariantly by injectivity of π in N_t and we obtain the $s = 1$ extremity $(h_t^1)_{t \in [t_0 - \delta, t_1 + \delta]}$ which induces the claimed generic path of closed 1-forms $(\alpha'_t)_{t \in [0, 1]}$. \square

The next concept is useful to describe paths of closed 1-forms $(\alpha_t)_{t \in [0, 1]}$. It depends essentially on the choice of paths $B_*(\alpha_t) \subset \widetilde{M}$ lifting the continuous paths determined by the zeroes $Z(\alpha_t)$. Choose $B_*(\alpha_0)$, liftings of the initial zeroes together with liftings of the birth zeroes; this determines $B_*(\alpha_t)$, *continuous liftings* of the zeroes.

Definition 13. *Let $(\alpha_t)_{t \in [0, 1]}$ be a generic path of closed 1-forms. Take a family of primitives $(h_t)_{t \in [0, 1]}$ together with continuous liftings $B_*(\alpha_t)$. Associated with this data, we have the Cerf-Novikov graphic, which is given by the set:*

$$\text{Gr}(B_*) := \bigcup_{t \in [0, 1]} \{t\} \times h_t(B_*(\alpha_t)) \subset [0, 1] \times \mathbb{R}.$$

Example 14. The local change in the Cerf-Novikov graphic when we apply the elementary lips lemma 12 can be seen in figure 5 here opposite.

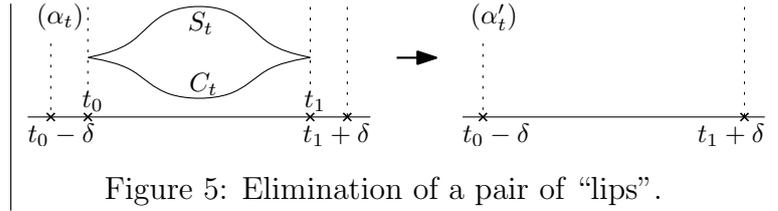


Figure 5: Elimination of a pair of “lips”.

To finish this section, we give a last lemma which is the analogue for closed 1-forms of [Lau12, Elementary swallow-tail lemma (2.6)]. The proof given in the cited paper can be adapted in the same way that we have done to prove lemma 12.

Lemma 15 (ELEMENTARY SWALLOW-TAIL LEMMA). *Let $(\alpha_t, \xi_t)_{t \in [0, 1]}$ be generic. Suppose that $(s_t, s'_t, c_t)_{t \in [t_0, t_1]}$ are continuous paths of zeroes such that $s_{t_0} = c_{t_0}$, $s'_{t_1} = c_{t_1}$ are respectively birth and elimination and that there exists an $\varepsilon > 0$ such that for every $t \in (t_0, t_1)$:*

1. *the zeroes are Morse with $\text{ind}(s_t) - 1 = \text{ind}(s'_t) - 1 = \text{ind}(c_t) = i$,*
2. *$W^u(s_t) \pitchfork W^s(c_t)$ along a unique orbit ℓ_t^+ ; moreover $L(\ell_t^+) + \varepsilon < L(\ell_t)$ for every orbit $\ell_t \subset W^u(s_t) \setminus \{\ell_t^+\}$,*
3. *$W^u(s'_t) \pitchfork W^s(c_t)$ along a unique orbit $(\ell'_t)^+$; moreover $L((\ell'_t)^+) + \varepsilon < L(\ell'_t)$ for every orbit $\ell'_t \subset W^u(s'_t) \setminus \{(\ell'_t)^+\}$.*

Then, for all $\delta > 0$ small enough, there exists a generic $(\alpha'_t)_{t \in [0, 1]}$ such that:

1. *$\alpha_t = \alpha'_t$ for all $t \notin (t_0 - \delta, t_1 + \delta)$,*
2. *$Z(\alpha'_t) = (Z(\alpha_t) \setminus \{s_t, s'_t, c_t\}) \cup \{s''_t\}$ for all $t \in (t_0 - \delta, t_1 + \delta)$, where $(s''_t)_{t \in [t_0 - \delta, t_1 + \delta]}$ is a continuous path of zeroes starting at $s'_{t_0 - \delta}$ and ending at $s_{t_1 + \delta}$.*

Example 16. Applying lemma 15 affects the graphic as it is depicted in figure 6. The enrwrapment of $(\ell_t)^+$ does not vary in (t_0, t_1) ; denote it by $g := e((\ell_t)^+)$.

Nothing guarantees a priori that $g = 1_{\pi_1 M}$, or in other words, that the chosen liftings S'_t of s'_t are those realizing the elimination of the C_t . The dashed line is just a translation by the vector $(0, u(g^{-1}))$ of the curve $h_t(S'_t)$ of the graphic.

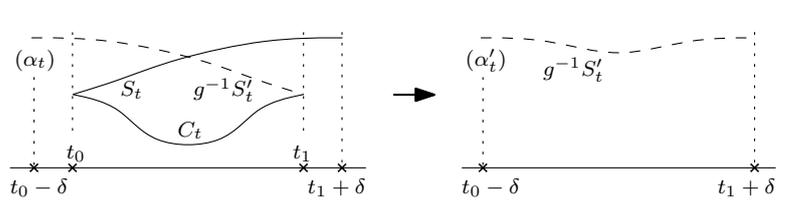


Figure 6: Elimination of a “swallow-tail”.

4 One-parameter elimination of centers

We want to eliminate a path of sinks $(c_t)_{t \in [t_0, t_1]}$ from a generic one-parameter family of closed 1-forms $(\alpha_t)_{t \in [0, 1]}$; here t_0, t_1 denote respectively the index 0 birth and elimination times associated with the considered path of sinks. We would be done if there was a path of saddles $(s_t)_{t \in [t_0, t_1]}$ satisfying the hypothesis of lemma 12 with respect to the specified sinks. We have defined connecting saddles in order to find such a candidate of saddles path $(s_t)_{t \in [t_0, t_1]}$, but lemma 6 does not apply to a one-parameter family since we cannot equip $(\alpha_t)_{t \in [0, 1]}$ with pseudo-gradients $(\xi_t)_{t \in [0, 1]}$ being Morse-Smale at every time. For a generic path $(\xi_t)_{t \in [0, 1]}$ of pseudo-gradients for $(\alpha_t)_{t \in [0, 1]}$, the set of times t where ξ_t is Morse-Smale, is only a dense subset of $[0, 1]$. We want a more general condition than Morse-Smale’s one, which assures existence of connecting saddles for generic paths. The next proposition goes in this direction.

Proposition 17. *Let $(\alpha_t)_{t \in [0, 1]}$ be a generic path of Morse closed 1-forms together with a path $(c_t)_{t \in [0, 1]}$ of sinks. For every generic path $(\xi_t)_{t \in [0, 1]}$ of pseudo-gradients for $(\alpha_t)_{t \in [0, 1]}$, consider*

$$\mathcal{T} := \{t \in [0, 1] \mid c_t \text{ admits a connecting saddle } s_t\};$$

then the map $\mathcal{T} \xrightarrow{L} \mathbb{R}^+$ given by $t \mapsto L(\ell_t^+)$ is bounded by some constant $K > 0$.

Proof. Since $(\xi_t)_{t \in [0, 1]}$ is generic, the set \mathcal{T} is at least dense in $[0, 1]$ by lemma 6. If the map L of the statement is not bounded, there exists $(t_i) \subset \mathcal{T}$ converging to some $\tau \notin \mathcal{T}$ such that the sequence $(L(t_i))_{t_i \in \mathcal{T}}$ diverges. Let us see that this can not happen.

Take $\tau \notin \mathcal{T}$ and η_τ , a Morse-Smale pseudo-gradient for α_τ . Consider Ω a small Morse neighbourhood of $Z(\alpha_\tau)$. We can find an isotopy $(\varphi_t)_t$ of $\varphi_\tau = \text{Id}_M$ such that $\varphi_t^*(\alpha_t)|_\Omega = \alpha_\tau|_\Omega$ for t near τ . Take a $\delta > 0$ small enough such that the condition $\langle \varphi_t^*(\alpha_t), \eta_\tau \rangle|_{M \setminus \Omega} < 0$ holds for every t such that $|t - \tau| < \delta$. The vector field $\eta_t := (\phi_t)_*(\eta_\tau)$ is still a pseudo-gradient for every $t \in [\tau - \delta, \tau + \delta]$ and clearly Morse-Smale since ϕ_t is a diffeomorphism. In particular, the sinks c_t have connecting saddles relative to $(\eta_t)_{t \in [\tau - \delta, \tau + \delta]}$, whose connecting orbits verify $L(\ell_{\eta_t}^+) < K$ for every $t \in [\tau - \delta, \tau + \delta]$ and some $K > 0$ since η_t is Morse-Smale in these times.

Consider now any Morse α together with a primitive $h : \widetilde{M} \rightarrow \mathbb{R}$ and ξ a Morse-Smale pseudo-gradient for α . If C is a lifting of a sink c of α , define for all $a > h(C)$, the basin $\text{Bas}(a) \subset \widetilde{M}$ given by the closure of $W^s(C) \cap h^{-1}((-\infty, a))$. If s is a connecting saddle for c of connecting orbit ℓ^+ , call S the initial extremity of the lifting $\widetilde{\ell^+}$ going to C . Remark that for every $\varepsilon > 0$, any critical point of h of index 1 contained in $\text{Bas}(h(S) - \varepsilon)$ corresponds to a saddle in $\mathcal{S}_c^{\text{ker}}$. In particular we have that $h(S)$ coincides with the value

$$D := \sup \{ a \in \mathbb{R} \mid \alpha|_{\pi(\text{Bas}(a))} \text{ is exact} \}.$$

The latter value only depends in α , and the length of the connecting orbit for any Morse-Smale pseudo-gradient ξ coincides with $D - h(C)$. We conclude that $L(\ell_{\xi_t}^+) = L(\ell_{\eta_t}^+)$ for all $t \in [\tau - \delta, \tau + \delta]$ where ξ_t is Morse-Smale. \square

We need the notion of truncated unstable manifold associated with some pseudo-gradient ξ : denote by γ_x^p the portion of orbit going from $p \in Z(\alpha)$ to some $x \in W^u(p)$. For every $K > 0$, we set

$$W_K^u(p) := \{ x \in W^u(p) \mid L(\gamma_x^p) \leq K \}.$$

Definition 18. Let $K > 0$. We say that a pseudo-gradient ξ for a closed 1-form α is K -transversal if

$$W_K^u(p_t) \pitchfork W^s(q_t), \text{ for every } p_t, q_t \in Z_*(\alpha_t).$$

Proposition 17 says that the length L of connecting orbits do not explode in one-parameter families. Take $K := K' + 1$ where $K' > 0$ is a bounding value for L . The condition of K -transversality of definition 18 concerns now a finite collection of compact subsets of M given by the truncated unstable manifolds of ξ ; so generically, one-parameter families of pseudo-gradients verify K -transversality in a cofinite set of times of $[0, 1]$, where the length of the connecting orbits ℓ_t^+ is bounded by K . The property of K -transversality is so enough to assure the existence of connecting saddles, while 0-excellence assures unicity and holds also generically in a cofinite set of times since it only concerns the finite collection of compact sets given by the unstable manifolds associated to $\mathcal{S}_{c_t}^n$ and ξ_t , truncated at length K . In this setting, call \mathcal{R} the *finite set* of times where K -transversality of ξ_t or 0-excellence of α_t fails.

The map $[t_0, t_1] \setminus \mathcal{R} \xrightarrow{s} M$ defined by $t \mapsto s_t$, where s_t is the unique connecting saddle s_t for c_t , is continuous. Let us study what happens near each $\tau \in \mathcal{R}$.

1st problem: α_τ is not 0-excellent for ξ_τ . We call τ a *competition time*, where two saddles s_τ^1, s_τ^2 compete to be the connecting saddle of c_τ . The map s presents a discontinuity as figure 7 suggests.

We can compare competition times with the fact that a generic path of functions $f_t : V \rightarrow \mathbb{R}$ from a compact manifold V has a finite set of times t_i where two critical points p_{t_i}, q_{t_i} cross their critical value (see [Cer70]). This holds so for the path $h_t|_V$

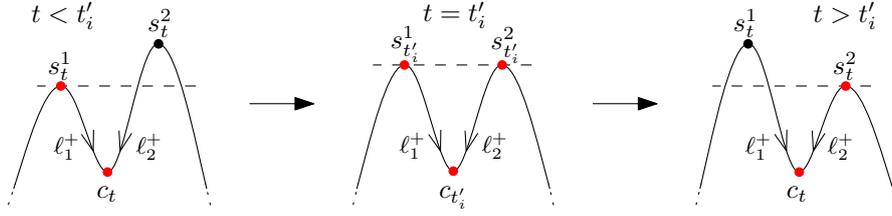


Figure 7: Two normal saddles competing on $t = \tau$.

where $(h_t)_{t \in [0,1]}$ are primitives of $(\pi^* \alpha_t)_{t \in [0,1]}$ and V is a compact neighbourhood containing $\bigcup_{t \in [t_0, t_1]} (W^s(C_t) \cap h^{-1}([h_t(C_t), h_t(C_t) + K]))$ for a continuous lifting $(C_t)_{t \in [t_0, t_1]}$ of the sinks.

2nd problem: ξ_τ is not K -transverse.

For generic paths, K -transversality fails in a finite set of times where we find a unique orbit ℓ connecting two not necessarily different zeroes of same index j . The only case which concerns connecting saddles is when $j = 1$ and the connecting orbits (ℓ_t^+) converge for $t \xrightarrow{t < \tau} \tau$ to a broken orbit $\ell * \ell'_\tau \in \mathcal{L}(s_\tau, s'_\tau) * \mathcal{L}(s'_\tau, c_\tau)$ as the figure 8 suggests. These times will be called of K -sliding.

Remark that the saddle s'_τ is necessarily in $\mathcal{S}_{c_\tau}^{\ker}$: the accident does not concern $W^u(s'_t)$ and s'_t conserves its type for t near τ ; if s'_τ is a normal saddle, s_t cannot be the connecting saddle for c_t when $t < \tau$ because the connecting orbit $(\ell_t^+)'$ of s'_t would be shorter than the connecting orbit ℓ_t^+ of s_t .

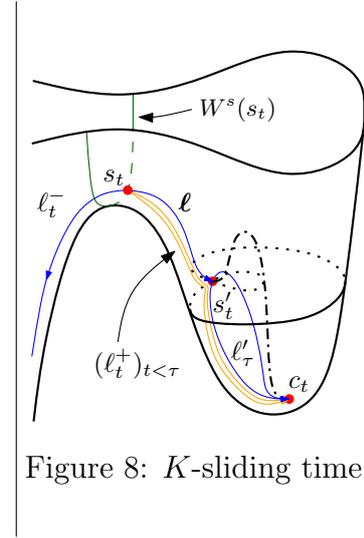


Figure 8: K -sliding time.

In particular, the enwrapments of the connecting orbits ℓ_t^+ for times just before and after $t = \tau$ are related by some $g \in \ker(u)$: the length of the connecting orbit ℓ_t^+ presents an avoidable discontinuity at $t = \tau$. We describe now a local modification of a generic path which allows one to transmute a K -sliding time into a pair of competition times.

Proposition 19. *In this setting $(s_t, c_t, \ell_t^+)_{t \in [0,1]}$ associated to a path of Morse closed 1-forms $(\alpha_t)_{t \in [0,1]}$ provided with a generic equipment $(\xi_t)_{t \in [0,1]}$ with no competition times and only one K -sliding time at $t = \tau$. There exists a deformation to a generic couple $(\alpha'_t, \xi'_t)_{t \in [0,1]}$ such that:*

1. nothing has changed in the complementary of some interval (t_0, t_1) containing τ ,
2. the only accidents of the equipment $(\xi'_t)_{t \in [t_0, t_1]}$ are two competition times $\tau^- < \tau^+$,
3. $Z_0(\alpha_t) = Z_0(\alpha'_t)$ for every time t .

Proof. Choose initial liftings $B_*(\alpha_0)$ of the zeroes and take $B_*(\alpha_t)$ the continuous path of liftings associated to it. We denote by $(C_t)_{t \in [0,1]}$ the lifting of the sinks $(c_t)_{t \in [0,1]}$ containing

c_τ , the sink involved with the K -sliding phenomena. The stated deformation is going to produce the local changes in the Cerf-Novikov graphic as it is depicted in figure 9.

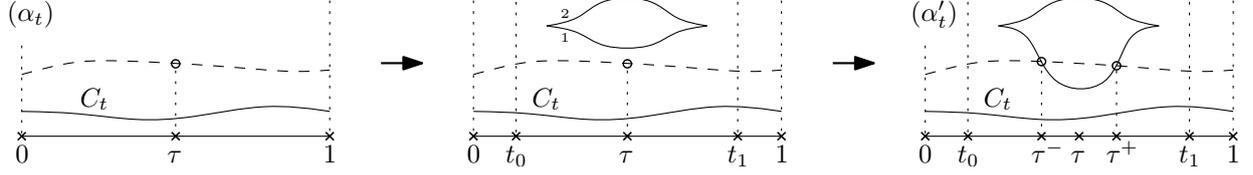


Figure 9

The connecting orbits ℓ_t^+ lift uniquely to orbits $\widetilde{\ell}_t^+$ going to C_t , and starting from $g^{-1}S_t$ where g is the enwrapment of ℓ_t and S_t is the chosen lifting of the connecting saddle s_t of c_t ; the dashed line represents the values $h_t(g^{-1}S_t)$, a translated curve of the graphic.

There exists an $\varepsilon > 0$ small enough such that the intersection of $W^s(S_t)$ with L_t , the level of $h_t(S_t) + \varepsilon$, is a $(n - 2)$ -sphere for all t near τ , which we denote by \mathbb{S}_t . Take K_t a relatively compact open neighbourhood of \mathbb{S}_t in \widetilde{M} such that $\pi|_{K_t}$ is injective. For every t near τ , choose an arc $I_t : [0, 1] \rightarrow L_t \cap K_t$ intersecting \mathbb{S}_t transversely only once at $\theta = \frac{1}{2}$. By the hypothesis on the equipment $(\xi_t)_{t \in [0,1]}$, one of the connected components of $I_t \setminus ([0, 1] \setminus \{\frac{1}{2}\})$, say $I_t^+ := I_t((\frac{1}{2}, 1])$, is entirely contained in $W^s(gC_t) \cup W^s(g'S'_t)$, where g' is the enwrapment of the accidental orbit $\ell \in \mathcal{L}(s_\tau, s'_\tau)$. These arcs can be chosen in order to have $I_t^+ \pitchfork W^s(g'S'_t)$; in particular, the extremities $I_t(1)$ can be taken into $W^s(gC_t)$. The other component I_t^- verifies

$$I_t^- \cap \bigcup_{u(k) \geq 0} W^s(kgC_t) = \emptyset : \quad (2)$$

if there was not the case, S_t would be in $\overline{W^s(kgC_t)}$ leading to a contradiction with the fact that s_t is a connecting saddle for c_t whose connecting orbit have enwrapment g .

For every t in an interval $[t_0, t_1]$ containing τ , we obtain new primitives $(h'_t)_{t \in [t_0, t_1]}$ by modifying $\pi_1 M$ -equivariantly the initial h_t in $K_t \cap h_t^{-1}((h_t(S_t), +\infty))$ by introducing a cancelling pair of critical points S''_t, R_t of respective index 1, 2. The new generic family of pseudo-gradients, which only differs from the original on $\pi(K_t)$, can be chosen such that $W^u(S''_t) \cap L_t = \{I_t(0), I_t(1)\}$ and $W^u(R_t) \cap L_t = I_t((0, 1))$: a new pair of zeroes of index 1, 2 appear now in times $t \in (t_0 + \delta, t_1 - \delta)$ for a small δ . The associated Cerf-Novikov looks as in the middle picture of figure 9. From genericness of $(\xi'_t)_{t \in [t_0, t_1]}$ together with (2), we deduce that the new saddles $s''_t := \pi(S''_t)$ belong indeed to $\mathcal{S}_{c_t}^n$, and this for every $t \in (t_0, t_1)$.

The separatrices of S''_t cross the level of S_t for every $t \in (t_0, t_1)$ and $s''_t := \pi(S''_t) \in \mathcal{S}_{c_t}^n$ by construction; we can so apply the rearrangement lemma 9 to the one-parameter family of S''_t to continuously decrease the value of S''_t under $h_t(S_t)$ for t near τ . The saddle

s_t'' becomes so the connecting saddle for c_t into the interval (τ^-, τ^+) of times t where $h_t'(S_t'') < h_t'(S_t)$. The times $t = \tau^-, \tau^+$ corresponds to competition times between s_t and s_t'' . □

We can now prove our main result, which is theorem 1 announced in the introduction.

Proof of theorem 1. We only eliminate the sinks; the sources can be treated similarly. Denote by $\nu \geq 0$ the amount of times t where α_t is of type birth of index 0. As the extremities have no center, we have necessarily ν elimination times of index 0. If $\nu = 0$ we are done because the extremities of $(\alpha_t)_{t \in [0,1]}$ are supposed to have no center. We can suppose that any birth time precedes any elimination time and moreover, that the i -th birth of index 0 corresponds to the $(\nu + 1 - i)$ -th elimination of index 0. This can be done by shifting birth times to the left in a judicious way (see [MF12, Lemme 3.1.3]), modification which only requires $\dim(M) > 1$. Denote by a the last birth time and b the first elimination time of $(\alpha_t)_{t \in [0,1]}$. We are going to produce a generic path $(\alpha_t^{(1)})_{t \in [0,1]}$ such that:

1. it coincides with the original $(\alpha_t)_{t \in [0,1]}$ outside of $(a - \varepsilon, b + \varepsilon)$,
2. $(\alpha_t^{(1)})_{t \in [0,1]}$ has $\nu - 1$ birth times of index 0.

The path $(\alpha_t^{(1)})_{t \in [0,1]}$ clearly share extremities with $(\alpha_t)_{t \in [0,1]}$. Iterating the construction ν times, we obtain a generic $(\alpha_t^{(\nu)})_{t \in [0,1]}$ which is the announced $(\beta_t)_{t \in [0,1]}$.

Choose continuous liftings $B_*(\alpha_t)$ and primitives $(h_t)_{t \in [0,1]}$ of $(\alpha_t)_{t \in [0,1]}$. Consider the path of local minima of $(h_t)_{t \in [0,1]}$ given by $(C_t)_{t \in [a,b]}$, the lifting of the inner path of sinks $(c_t)_{t \in [a,b]}$. Take $(\xi_t)_{t \in [0,1]}$ a generic equipment for $(\alpha_t)_{t \in [0,1]}$. We replace the finite amount of K -sliding times in the interval $[a, b]$ concerning our sinks by twice competition times by introducing trivial pairs of index $(1, 2)$ as in proposition 19. The dimension hypothesis is used here: the new zeroes of index 2 are not sources because $n \geq 3$. Let us denote by $\{t_i\}_{i=1}^r \subset (a, b)$ the finite set of competition times related to $(c_t)_{t \in [a,b]}$.

Suppose first that the simplest case holds: there are $r = 0$ competitions. Since there are neither competition nor K -sliding times between a and b , we have a continuous path of connecting saddles $(s_t)_{t \in [a,b]}$ for our sinks. Consider the set of times $\Delta \subset (a, b)$ where the family of non-connecting separatrices $(\ell_t^-)_{t \in [a,b]}$ do not verify $L(\ell_t^-) > L(\ell_t^+)$. For any $t \in \Delta$, s_t is the connecting saddle for c_t and ℓ_t^- has to go to a sink $c'_t \neq c_t$; we can so apply the one-parameter version of the rearrangement lemma 9 to c'_t for t in any interval containing Δ in order to have $L(\ell_t^-) > L(\ell_t^+)$ for all $t \in (a, b)$. The hypothesis of the elementary lips lemma 12 respectively to the couples $(s_t, c_t)_{t \in [a,b]}$ is now verified and we obtain the claimed family $(\alpha_t^{(1)})_{t \in [0,1]}$ by applying the mentioned lemma; the graphic changes as in figure 5.

The rest of the proof consists in decreasing r by one; we reduce so to the case $r = 0$ and we are done. Consider t_1 the first competition time and denote by t_2 the next singular time, which can correspond to another competition or eventually to the elimination time of the path of sinks that we are considering. The continuous path of saddles $(s_t)_t$ starting at $s_a = c_a$ is defined until $t < t_2$ at least; since there is no accident in (a, t_1) , the saddle s_t is *the* connecting saddle of c_t for every $t \in (a, t_1)$. In $t = t_1$, a connecting saddle $s'_{t_1} \neq s_{t_1}$ competes with s_{t_1} . In the same way, the continuous family (s'_t) is defined at least in (a, t_2) and s'_t is *the* connecting saddle of c_t for every $t \in (t_1, t_2)$. We perform a preliminary modification to $(\alpha_t)_{t \in [0,1]}$ as figure 10 suggests.

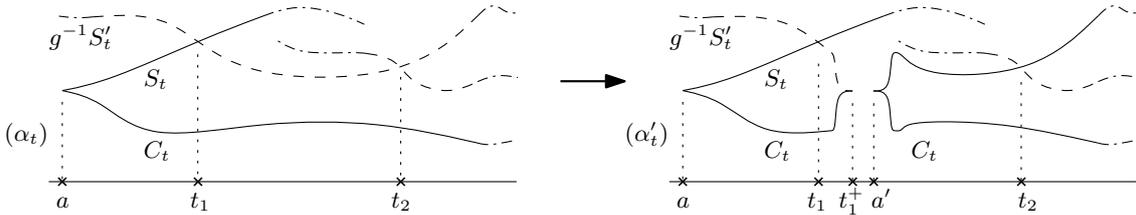


Figure 10: Effect in the Cerf-Novikov graphic of the preliminary modification on $(\alpha_t)_{t \in [0,1]}$.

The modified $(\alpha'_t)_{t \in [0,1]}$ is the generic path of 1-forms obtained by inserting a loop (γ_t^{1+}) in $(\alpha_t)_{t \in [0,1]}$, based at time $t_1 + \varepsilon$ for an $\varepsilon > 0$ small enough. This loop is constructed by following forwards then backwards the path realizing the elimination of $s_{t_1 + \varepsilon}$ with $c_{t_1 + \varepsilon}$ given by lemma 11. The new birth and elimination times of the considered path of sinks $(c_t)_t$ are denoted by: $t_1 < t_1^+ < a' < t_2$.

Remark that the output path of the elementary swallow-tail lemma 15 has one competition time less than its input path.

Since there is no accident in the interval $[a, t_1^+]$ other than the competition at $t = t_1$, the enwrapment of the orbits stays fixed. In particular, we can suppose that the length of the non-connecting orbit of $W^u(s_t)$ is bigger than $L(\ell_t^+)$ for every time since it is the case for times near a . The same reasoning applies to the non-connecting orbit of $W^u(s'_t)$ and we can apply the elementary swallow-tail lemma 15 to $(\alpha'_t)_{t \in [0,1]}$, relative to the zeroes $(s_t, s'_t, c_t)_{t \in [a, t_1^+]}$.

The resulting path $(\alpha_t^{(1)})_{t \in [0,1]}$, whose graphic is depicted in figure 11, is a generic one still containing ν births of index 0 but $r - 1$ competitions as we claimed. \square

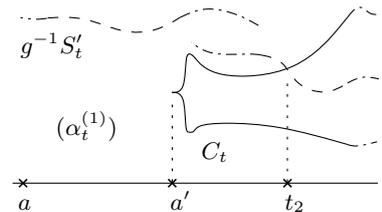


Figure 11: One competition less.

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