The Influence of Surface Tension on the Effective Stiffness of Nanosize Plates

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The unique features of nanomaterials make them quite attractive for applications [1, 2]. One of the factors that is responsible for the anomalous behavior of nanomaterials is due to the surface effects. In particular, taking into account the surface tension allows us to describe the observed finite size effect, i.e., the dependence of the properties of the material, e.g., its Young modulus, on the sizes of the specimen. Experimental and theoretical studies of the influence of the surface tension on effective properties of nanomaterials (nanoporous media, nanofiners, etc.) were described in a significant number of papers (see, e.g., [3–6]). In the present work, we consider the problem of bending of a nanosize plate, taking into account the action of the surface effects. We obtain expressions for the effective stiffnesses.

1. RELATIONS OF ELASTICITY THEORY IF ONE TAKES INTO ACCOUNT THE SURFACE EFFECTS

Let us consider the deformation of a linearly elastic body, which occupies the region

$$V = \left\{(x, y, z): (x, y) \in \Omega \subset \mathbb{R}^2, z \in \left[-\frac{h}{2}, \frac{h}{2}\right]\right\}.$$

Here, x, y, z are Cartesian coordinates and h is the thickness of the plate. We will take into account the action of the surface effects on the upper and lower surfaces (at $z = \pm h/2$). The equilibrium equation and the corresponding boundary conditions will take the form

$$\nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{f} = 0, \quad \mathbf{n} \cdot \boldsymbol{\sigma}\big|_{z = \pm\frac{h}{2}} = \mathbf{t}_\pm. \tag{1.1}$$

Here, $\boldsymbol{\sigma}$ is the stress tensor, ∇ is the three-dimensional nabla (Hamilton) operator, ρ is the density, $\mathbf{f}$ are mass forces, $\mathbf{n}$ is the exterior normal vector to the surfaces $z = \pm h/2$, and $\mathbf{t}_\pm$ are the surface loadings. Taking into account the surface effects, one can represent $\mathbf{t}_\pm$ as the sum of two terms $\mathbf{t}_\pm = \mathbf{t}^0_\pm + \mathbf{t}^S_\pm$. The first term describes the exterior surface loadings, which are supposed to be given, while the second term expresses the action of the surface effects and can be represented in the following way [7–9]:

$$\mathbf{t}^S_\pm = -\nabla_S \cdot \boldsymbol{\tau}_\pm, \tag{1.2}$$

where ∇_S is the surface nabla operator, which is related to ∇ through the equation $\nabla_S = \nabla - \mathbf{i}_3 \frac{\partial}{\partial z}$; $\mathbf{i}_k$, $k = 1, 2, 3$, are the Cartesian coordinates of the basis vectors; and $\boldsymbol{\tau}_\pm$ is the surface tension tensor. The surface conditions (1.2) can be obtained via consideration of the corresponding surface energy [7]. Below we restrict ourselves to the case of a homogeneous material. The equation of state for $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ can be written in the form

$$\begin{gathered}\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda\mathbf{I}\operatorname{tr}\boldsymbol{\varepsilon}, \quad \boldsymbol{\tau}_\pm = 2\mu^S_\pm\boldsymbol{\epsilon}_\pm + \lambda^S_\pm\mathbf{A}\operatorname{tr}\boldsymbol{\epsilon}_\pm, \\ \boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T), \quad \boldsymbol{\epsilon}_\pm = \frac{1}{2}(\nabla_S\boldsymbol{v}_\pm + (\nabla_S\boldsymbol{v}_\pm)^T),\end{gathered} \tag{2}$$

where $\mathbf{I}$, $\mathbf{A} \equiv \mathbf{I} - \mathbf{i}_3 \otimes \mathbf{i}_3$ are the three- and two-dimensional unit tensors, λ and μ are Lamé constants; $\lambda^S_\pm$ and $\mu^S_\pm$ are the elasticity coefficients, which characterize the elasticity properties of the surface layer (the surface analogs of the Lamé constants); $\boldsymbol{\varepsilon}$ is the deformation tensor; $\mathbf{u} = \mathbf{u}(x, y, z)$ are the shift vectors; and $\boldsymbol{\epsilon}_\pm$ are the tensors of the tangential deformations, which are defined on the tangential displacements $\boldsymbol{v}_\pm = \boldsymbol{v}_\pm(x, y) \equiv \mathbf{A} \cdot \mathbf{u}(x, y, \pm h/2)$.

2. TRANSITION TO THE THEORY OF PLATES

To obtain the two-dimensional equilibrium equations in the theory of plates, let us integrate Eq. (1) over

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the thickness (see e.g., [10]). Integrating equation (1.1) over z and taking into account (1.2), we obtain

$$\nabla_S \cdot \langle \boldsymbol{\sigma} \rangle + \mathbf{t}_+ + \mathbf{t}_- + \langle \rho \mathbf{f} \rangle = 0, \quad \langle (\ldots) \rangle = \int_{-h/2}^{h/2} (\ldots) dz. \tag{3}$$

Taking the vector product of (1.1) with $z\mathbf{i}_3$ from the left and integrating over the thickness, we will obtain the second equilibrium equation:

$$-\nabla_S \cdot \langle z \boldsymbol{\sigma} \times \mathbf{i}_3 \rangle - \langle \mathbf{i}_3 \times \boldsymbol{\sigma} \cdot \mathbf{i}_3 \rangle + \frac{h}{2}\mathbf{i}_3 \times (\mathbf{t}_+ + \mathbf{t}_-) + \langle \rho z \mathbf{i}_3 \times \mathbf{f} \rangle = 0. \tag{4}$$

Introducing the force and momentum tensors using the equations $\mathbf{T} = \mathbf{A} \cdot \langle \boldsymbol{\sigma} \rangle$, $\mathbf{M} = -\mathbf{A} \cdot \langle z \boldsymbol{\sigma} \times \mathbf{i}_3 \rangle$ [10, 11], Eqs. (3), (4) can be written in the following form:

$$\begin{aligned} \nabla_S \cdot \mathbf{T} - \nabla_S \cdot \boldsymbol{\tau}_+ - \nabla_S \cdot \boldsymbol{\tau}_- + \mathbf{q} &= 0, \\ \nabla_S \cdot \mathbf{M} + \mathbf{T}_\times - \frac{h}{2}\mathbf{i}_3 \times (\nabla_S \cdot \boldsymbol{\tau}_+ - \nabla_S \cdot \boldsymbol{\tau}_-) + \mathbf{m} &= 0, \end{aligned} \tag{5}$$

where $\mathbf{q} = \mathbf{t}_+^0 + \mathbf{t}_-^0 + \langle \rho \mathbf{f} \rangle$, $\mathbf{m} = \mathbf{i}_3 \times (\mathbf{t}_+^0 + \mathbf{t}_-^0)\frac{h}{2} + \langle \rho z \mathbf{i}_3 \times \mathbf{f} \rangle$ are the external forces and moments, which are acting on the plate, and $\mathbf{T}_\times$ is the vector invariant of the tensor $\mathbf{T}$. Let us point out, that the existence of the terms, related to $\boldsymbol{\tau}_\pm$, distinguishes the equilibrium equations (5) from the ones that are known thus far in the theory of plates and shells [10–12]. The structure of equations (5) allows us to introduce the effective effect and momenta tensors

$$\mathbf{T}^* = \mathbf{T} - \boldsymbol{\tau}_+ - \boldsymbol{\tau}_-, \quad \mathbf{M}^* = \mathbf{M} + \frac{h}{2}(\boldsymbol{\tau}_+ - \boldsymbol{\tau}_-) \times \mathbf{i}_3. \tag{6}$$

Then equilibrium equations (5) take the form

$$\nabla_S \cdot \mathbf{T}^* + \mathbf{q} = 0, \quad \nabla_S \cdot \mathbf{M}^* + \mathbf{T}^*_\times + \mathbf{m} = 0. \tag{7}$$

Let us fetch the equations of state for $\mathbf{T}^*$ and $\mathbf{M}^*$ assuming the smallness of the deformations of plates. To describe the deformations of the plates, we assume an approximation for the field of displacements that is linear in thickness, which is used in the theory of shear deformable plates [10, 13]:

$$\mathbf{u}(x, y, z) = \mathbf{w}(x, y) - z\boldsymbol{\vartheta}(x, y), \quad \mathbf{i}_3 \cdot \boldsymbol{\vartheta} = 0. \tag{8}$$

Here we assume that the rotation vector $\boldsymbol{\vartheta}$ kinematically does not depend on the displacement vector of the middle surface of the plate $\mathbf{w}$. From (8) we obtain the following equations:

$$\boldsymbol{v}_\pm = \boldsymbol{v} \mp \frac{h}{2}\boldsymbol{\vartheta}, \quad \boldsymbol{\epsilon}_\pm = \boldsymbol{\epsilon} \mp \frac{h}{2}\boldsymbol{\kappa},$$

where $\boldsymbol{v} = \mathbf{w} \cdot \mathbf{A}$ and $\boldsymbol{\epsilon} = (1/2)(\nabla_S \boldsymbol{v} + (\nabla_S \boldsymbol{v})^T)$ and $\boldsymbol{\kappa} = (1/2)(\nabla_S \boldsymbol{\vartheta} + (\nabla_S \boldsymbol{\vartheta})^T)$ are two-dimensional tensors of strain and twist-torsion. Using (2), we will obtain the following expressions for the surface tensions $\boldsymbol{\tau}_\pm$:

$$\boldsymbol{\tau}_\pm = \lambda_\pm^S \mathbf{A} \operatorname{tr} \boldsymbol{\epsilon} + 2\mu_\pm^S \boldsymbol{\epsilon} \mp \frac{h}{2}(\lambda_\pm^S \mathbf{A} \operatorname{tr} \boldsymbol{\kappa} + 2\mu_\pm^2 \boldsymbol{\kappa}).$$

From now on we restrict ourselves to the case of the symmetrical plates and accept that the surface properties of the boundary faces of the plate are the same:

$$\lambda_+^S = \lambda_-^S = \lambda^S, \quad \mu_+^S = \mu_-^S = \mu^S.$$

Then

$$\begin{aligned} \boldsymbol{\tau}_+ + \boldsymbol{\tau}_- &= 2\lambda^S \mathbf{A} \operatorname{tr} \boldsymbol{\epsilon} + 4\mu^S \boldsymbol{\epsilon}, \\ \boldsymbol{\tau}_+ - \boldsymbol{\tau}_- &= -h[\lambda^S \mathbf{A} \operatorname{tr} \boldsymbol{\kappa} + 4\mu^S \boldsymbol{\kappa}]. \end{aligned} \tag{9}$$

Considering the plate to be homogeneous in the thickness direction, for $\mathbf{T}$ and $\mathbf{M}$ we will accept the following equations of state [10, 12]

$$\begin{aligned} \mathbf{T} \cdot \mathbf{A} &= C[(1-\nu)\boldsymbol{\epsilon} + \nu \mathbf{A} \operatorname{tr} \boldsymbol{\epsilon}], \\ \mathbf{M} &= D[(1-\nu)\boldsymbol{\kappa} + \nu \mathbf{A} \operatorname{tr} \boldsymbol{\kappa}] \times \mathbf{i}_3, \\ \mathbf{T} \cdot \mathbf{i}_3 &= \Gamma \boldsymbol{\gamma}, \quad C = \frac{Eh}{1-\nu^2}, \quad D = \frac{Eh^3}{12(1-\nu^2)}, \\ \Gamma &= k\mu h, \end{aligned} \tag{10}$$

where C, D are the tangential and twist stiffnesses of the plate, Γ is the stiffness with respect to the transverse shear, $E = 2\mu(1+\nu)$, $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ are the Young modulus and Poisson ratio of the material from which the plate is made, k is the coefficient of the transverse shear [10, 12, 13], $\boldsymbol{\gamma} = \nabla_S w - \boldsymbol{\vartheta}$ is the vector of the transverse shear, and $w = \mathbf{w} \cdot \mathbf{i}_3$ is the normal deflection of the plate.

Taking into account Eqs. (9), (10), one can deduce from (6) that $\mathbf{T}^*$ and $\mathbf{M}^*$ depend on $\boldsymbol{\epsilon}$ and $\boldsymbol{\kappa}$, similarly to $\mathbf{T}$ and $\mathbf{M}$, but with different moduli:

$$\begin{aligned} \mathbf{T}^* &= C_1 \boldsymbol{\epsilon} + C_2 \mathbf{A} \operatorname{tr} \boldsymbol{\epsilon} + \Gamma \boldsymbol{\gamma} \otimes \mathbf{i}_3, \\ \mathbf{M}^* &= [D_1 \boldsymbol{\kappa} + D_2 \mathbf{A} \operatorname{tr} \boldsymbol{\kappa}] \times \mathbf{i}_3, \end{aligned} \tag{11}$$

where

$$C_1 = C(1-\nu) - 4\mu^S, \quad C_2 = C\nu - 2\lambda^S,$$

$$D_1 = D(1-\nu) - h^2\mu^S, \quad D_2 = D\nu - \frac{h^2\lambda^S}{2}.$$

Note that the surface effects do not influence the transverse shear forces: $\mathbf{T}^* \cdot \mathbf{i}_3 = \mathbf{T} \cdot \mathbf{i}_3$.

The equations of state (11) allow us to write the equilibrium equations for the plate, with the surface tension being taken into account (7), in terms of the displacements $\boldsymbol{v}$ and rotations $\boldsymbol{\vartheta}$. We have to stress that the equations of state for the effective forces and moments (11) are qualitatively different from the equations of state for the isotropic homogeneous plates (10). The dependence of $\mathbf{T}$ and $\mathbf{M}$ on deformations is defined by the four elastic constants: C, D, ν, Γ. At the same time, the dependencies $\mathbf{T}^*$ and $\mathbf{M}^*$ are defined by the five independent elastic constants: C_1, C_2, D_1, D_2, Γ.

The application of the two-dimensional equations of the linear theory of plates to nanosize plates seems to be quite plausible in the case of small deformations, if one considers the equations of state of type (11) in the framework of the direct approach to the construction of the theory of plates and shells [10–12]. Here the equations of equilibrium and state are formulated directly for the two-dimensional medium, while the determination of the corresponding elastic constants should be based directly on the experimental results.

3. CONCLUSIONS

Thus, we have obtained the two-dimensional equations of equilibrium for plates taking into account the transverse shear and in the presence of surface effects. We have presented the relations between the force and moment tensors and found the expressions for the effective stiffnesses of the plate. In particular, we have shown that, if one takes into account surface effects, the stiffnesses of the plate do change significantly, which agrees with the results of [4, 6, 14, 15]. The obtained results can also be applied to the case of multilayer nanosize plates, where one can expect a greater influence of the surface effects, as well as to the case of nanosize shells, including isotropic shells.

REFERENCES

1. M. Yu. Gutkin and I. A. Ovid'ko, *Physical Mechanics of the Deformable Nanostructures* (Yanus, St. Petersburg, 2003) [in Russian].
2. *Springer Handbook of Nanotechnology*, Ed. by B. Bhushan et al. (Springer, 2007).
3. H. L. Duan, J. Wang, Z. P. Huang, and B. L. Karihaloo, J. Mech. Phys. Solids **53**, 1574 (2005).
4. H. L. Duan, J. Wang, B. L. Karihaloo, and Z. P. Huang, Acta Mater. **54**, 2983 (2006).
5. H. L. Duan and B. L. Karihaloo, J. Mech. Phys. Solids **55**, 1036 (2007).
6. G. Y. Jing, H. L. Duan, X. M. Sun, et al., Phys. Rev. **73**, 235409.
7. Ya. S. Podstrigach and Yu. Z. Povstenko, *Introduction to the Mechanics of Surface Phenomena in Deformable Solids* (Naukova Dumka, Kiev, 1985) [in Russian].
8. Y. Z. Povstenko, J. Mech. Phys. Solids **41**, 1499 (1993).
9. M. E. Gurtin and A. I. Murdoch, Arch. Ration. Mech. Anal. **57**, 291 (1975).
10. P. A. Zhilin, *Applied Mechanics: Principles of the Theory of Shells* (Izdat. Politekhn. Univ., St. Petersburg, 2006) [in Russian].
11. H. Altenbach and P. A. Zhilin, Usp. Mekh., No. 4, 107 (1988).
12. H. Altenbach and V. A. Eremeyev, Arch. Appl. Mech. **78** (10), 775 (2008).
13. E. I. Grigolyuk and I. T. Selezov, *Non-Classical Theories of the Vibrations of Rods, Plates and Shells* (Ser. Advances in Science and Technology. Mechanics of Deformed Solids) (VINITI, Moscow, 1973) [in Russian].
14. A. M. Krivtsov and N. F. Morozov, Dokl. Akad. Nauk **381** (3), 345 (2001).
15. E. A. Ivanova, A. M. Krivtsov, and N. F. Morozov, Dokl. Akad. Nauk **385** (4), 494 (2002).

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