About Brézis-Merle Problem in dimension 2.

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A new proof for Brezis-Merle Problem with Lipschitz condition

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Abstract

We give a quantization analysis to Brezis-Merle problem with Dirichlet condition. An application, we have a new proof to Brezis-Merle Problem with Lipschitz condition.

Keywords: quantization, blow-up, boundary, a priori estimate, Lipschitz condition.
MSC: 35J60, 35B44, 35B45

1 Introduction and Main Results

We set $\Delta = -\partial_{11} - \partial_{22}$ on open set Ω of $\mathbb{R}^2$ with a smooth boundary.

We consider the following equation:

$$(P)\begin{cases}\Delta u = Ve^u \text{ in } \Omega\\ u = 0 \text{ on } \partial\Omega.\end{cases}$$

The previous equation was studied by many authors, with or without the boundary condition, also for Riemann surfaces, see [1,18], we can find some existence and compactness results.

Among other results, we can see in [4] the following important Theorem,

Theorem. *(Brezis-Merle [4]).If $(u_i)_i$ and $(V_i)_i$ are two sequences of functions relatively to the problem (P) with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set K of Ω,*

$$\sup_K u_i \leq c = c(a, b, m, K, \Omega).$$

We can find in [4] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of e^{u_i}.

If we assume V with more regularity, we can have another type of estimates, a $\sup + \inf$ type inequalities. It was proved by Shafrir see [15], that, if $(u_i)_i, (V_i)_i$ are two sequences of functions solutions of the previous equation without assumption on the boundary and, $0 < a \leq V_i \leq b < +\infty$, then we have the following interior estimate:

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$$C\left(\frac{a}{b}\right)\sup_K u_i + \inf_\Omega u_i \leq c = c(a,b,K,\Omega).$$

We can see in [7] an explicit value of $C\left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}$. In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [2]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose $(V_i)_i$ uniformly Lipschitzian with A the Lipschitz constant, then, $C(a/b) = 1$ and $c = c(a,b,A,K,\Omega)$, see Brézis-Li-Shafrir [5]. This result was extended for Hölderian sequences $(V_i)_i$ by Chen-Lin, see [7]. Also, we can see in [12], an extension of the Brezis-Li-Shafrir result to compact Riemann surface without boundary. We can see in [13] explicit form, ($8\pi m, m \in \mathbb{N}^*$ exactly), for the numbers in front of the Dirac masses when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [8] and [18] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

Here we give the behavior of the blow-up points on the boundary and a new proof of Brezis-Merle conjecture with Lipschitz condition.

The Brezis-Merle Conjecture (see [4]) is:

Conjecture. Suppose that $V_i \to V$ in $C^0(\bar{\Omega})$with $0 \leq V_i \leq b$ for some positive constant b. Also, we consider a sequence of solutions (u_i) of (P) relatively to (V_i) such that,

$$\int_\Omega e^{u_i} dx \leq C,$$

is it possible to have:

$$||u_i||_{L^\infty} \leq C = C(b,\Omega,C)?$$

Here, we give a caracterization of the behavior of the blow-up points on the boundary and also a new proof of Ma-Wei theorem. For the behavior of the blow-up points on the boundary, the following condition is enough,

$$0 \leq V_i \leq b,$$

The condition $V_i \to V$ in $C^0(\bar{\Omega})$ is not necessary.

But for the new proof for the Brezis- Merle problem we assume that:

$$||\nabla V_i||_{L^\infty} \leq A.$$

We have the following caracterization of the behavior of the blow-up points on the boundary.

Theorem 1.1 *Assume that* $\max_\Omega u_i \to +\infty$*, Where* (u_i) *are solutions of the probleme* (P) *with:*

$$0 \leq V_i \leq b, \text{ and } \int_\Omega e^{u_i} dx \leq C, \;\; \forall \; i,$$

then, after passing to a subsequence, there is a finction u, there is a number $N \in \mathbb{N}$ and N points $x_1, \ldots, x_N \in \partial\Omega$, such that,

$$\partial_\nu u_i \to \partial_\nu u + \sum_{j=1}^{N} \alpha_j \delta_{x_j}, \ \ \alpha_j \geq 4\pi, \ \text{weakly in the sens of measure } L^1(\partial\Omega).$$

$$u_i \to u \ \text{ in } \ C^1_{loc}(\bar{\Omega} - \{x_1, \ldots, x_N\}).$$

In the following theorem, we have a new proof for the global a priori estimate which concern the problem (P). The proof of Chen-Li and Ma-Wei [6, 14], use the moving-plane method.

Theorem 1.2 *Assume that (u_i) are solutions of (P) relatively to (V_i) with the following conditions:*

$$||\nabla V_i||_{L^\infty} \leq A \ \text{ and } \ \int_\Omega e^{u_i} \leq C,$$

We have,

$$||u_i||_{L^\infty} \leq c(b, A, C, \Omega),$$

2 Proof of the theorems

Proof of theorem 1.1:

We have,

$$\int_{\partial\Omega} \partial_\nu u_i d\sigma \leq C,$$

Without loss of generality, we can assume that $\partial_\nu u_i > 0$. Thus, (using the weak convergence in the space of Radon measures), we have the existence of a positive Radon measure μ such that,

$$\int_{\partial\Omega} \partial_\nu u_i \phi d\sigma \to \mu(\phi), \ \ \forall \ \phi \in C^0(\partial\Omega).$$

We take an $x_0 \in \partial\Omega$ such that, $\mu(x_0) < 4\pi$. Without loss of generality, we can assume that the following curve, $B(x_0, \epsilon) \cap \partial\Omega := I_\epsilon$ is an interval.(In this case, it is more simple to construct the following test function η_ϵ). We choose a function η_ϵ such that,

$$\begin{cases} \eta_\epsilon \equiv 1, \ \text{ on } \ I_\epsilon, \ \ 0 < \epsilon < \delta/2, \\ \eta_\epsilon \equiv 0, \ \text{ outside } \ I_{2\epsilon}, \\ 0 \leq \eta_\epsilon \leq 1, \\ ||\nabla \eta_\epsilon||_{L^\infty(I_{2\epsilon})} \leq \dfrac{C_0(\Omega, x_0)}{\epsilon}. \end{cases}$$

We take a $\tilde{\eta}_\epsilon$ such that,

$$\begin{cases} \Delta \tilde{\eta}_\epsilon = 0 \text{ in } \Omega \\ \tilde{\eta}_\epsilon = \eta_\epsilon \text{ on } \partial\Omega. \end{cases}$$

We use the following estimate, see [3, 11, 17],

$$||\nabla u_i||_{L^q} \leq C_q, \ \forall \ i \text{ and } 1 < q < 2.$$

We deduce from the last estimate that, (u_i) converge weakly in $W_0^{1,q}(\Omega)$, almost everywhere to a function $u \geq 0$ and $\int_\Omega e^u < +\infty$ (by Fatou lemma). Also, V_i weakly converge to a nonnegative function V in L^∞. The function u is in $W_0^{1,q}(\Omega)$ solution of :

$$\begin{cases} \Delta u = V e^u \in L^1(\Omega) \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

According to the corollary 1 of Brezis-Merle result, see [4], we have $e^{ku} \in L^1(\Omega), k > 1$. By the elliptic estimates, we have $u \in C^1(\bar{\Omega})$.

We can write,

$$\Delta((u_i - u)\tilde{\eta}_\epsilon) = (V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon - 2 < \nabla(u_i - u)|\nabla\tilde{\eta}_\epsilon > . \qquad (1)$$

We use the interior esimate of Brezis-Merle, see [4],

<u>*Step 1:*</u> Estimate of the integral of the first term of the right hand side of (1).

We use the Green formula between $\tilde{\eta}_\epsilon$ and u, we obtain,

$$\int_\Omega V e^u \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u \eta_\epsilon \leq 4\epsilon ||\partial_\nu u||_{L^\infty} = C\epsilon \qquad (2)$$

We have,

$$\begin{cases} \Delta u_i = V_i e^{u_i} \text{ in } \Omega \\ u_i = 0 \text{ on } \partial\Omega. \end{cases}$$

We use the Green formula between u_i and $\tilde{\eta}_\epsilon$ to have:

$$\int_\Omega V_i e^{u_i} \tilde{\eta}_\epsilon dx = \int_{\partial\Omega} \partial_\nu u_i \eta_\epsilon d\sigma \to \mu(\eta_\epsilon) \leq \mu(I_{2\epsilon}) \leq 4\pi - \epsilon_0, \ \ \epsilon_0 > 0 \qquad (3)$$

From (2) and (3) we have for all $\epsilon > 0$ there is $i_0 = i_0(\epsilon)$ such that, for $i \geq i_0$,

$$\int_\Omega |(V_i e^{u_i} - V e^u)\tilde{\eta}_\epsilon| dx \leq 4\pi - \epsilon_0 + C\epsilon \qquad (4)$$

<u>Step 2:</u> Estimate of integral of the second term of the right hand side of (1).

Let $\Sigma_\epsilon = \{x \in \Omega, d(x, \partial\Omega) = \epsilon^2\}$ and $\Omega_{\epsilon^2} = \{x \in \Omega, d(x, \partial\Omega) \geq \epsilon^2\}$, $\epsilon > 0$. Then, for ϵ small enough, Σ_ϵ is hypersurface.

The measure of $\Omega - \Omega_{\epsilon^2}$ is $k_2\epsilon^2 \leq \mu_L(\Omega - \Omega_{\epsilon^2}) \leq k_1\epsilon^2$.

Remark: for the unit ball $\bar{B}(0,1)$, our new manifold is $\bar{B}(0, 1-\epsilon^2)$.

We write,

$$\int_\Omega | < \nabla(u_i - u) | \nabla\tilde{\eta}_\epsilon > | dx = \int_{\Omega_{\epsilon^2}} | < \nabla(u_i - u) | \nabla\tilde{\eta}_\epsilon > | dx + \int_{\Omega - \Omega_{\epsilon^2}} < \nabla(u_i - u) | \nabla\tilde{\eta}_\epsilon > | dx. \quad (5)$$

<u>*Step 2.1:*</u> Estimate of $\int_{\Omega - \Omega_{\epsilon^2}} | < \nabla(u_i - u) | \nabla\tilde{\eta}_\epsilon > | dx$.

First, we know from the elliptic estimates that $||\nabla\tilde{\eta}_\epsilon||_{L^\infty} \leq C_1/\epsilon$, C_1 depends on Ω

We know that $(|\nabla u_i|)_i$ is bounded in $L^q, 1 < q < 2$, we can extract from this sequence a subsequence which converge weakly to $h \in L^q$. But, we know that we have locally the uniform convergence to $|\nabla u|$ (by Brezis-Merle theorem), then, $h = |\nabla u|$ a.e. Let q' be the conjugate of q.

We have, $\forall f \in L^{q'}(\Omega)$

$$\int_\Omega |\nabla u_i| f dx \to \int_\Omega |\nabla u| f dx$$

If we take $f = 1_{\Omega - \Omega_{\epsilon^2}}$, we have:

$$\text{for } \epsilon > 0 \; \exists \; i_1 = i_1(\epsilon) \in \mathbb{N}, \;\; i \geq i_1, \; \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \leq \int_{\Omega - \Omega_{\epsilon^2}} |\nabla u| + \epsilon^2.$$

Then, for $i \geq i_1(\epsilon)$,

$$\int_{\Omega - \Omega_{\epsilon^2}} |\nabla u_i| \leq mes(\Omega - \Omega_{\epsilon^2}) ||\nabla u||_{L^\infty} + \epsilon^2 = \epsilon^2(k_1 ||\nabla u||_{L^\infty} + 1).$$

Thus, we obtain,

$$\int_{\Omega - \Omega_{\epsilon^2}} | < \nabla(u_i - u) | \nabla\tilde{\eta}_\epsilon > | dx \leq \epsilon C_1 (2k_1 ||\nabla u||_{L^\infty} + 1) \quad (6)$$

The constant C_1 does not depend on ϵ but on Ω.

<u>*Step 2.2:*</u> Estimate of $\int_{\Omega_{\epsilon^2}} | < \nabla(u_i - u) | \nabla\tilde{\eta}_\epsilon > | dx$.

We know that, $\Omega_\epsilon \subset\subset \Omega$, and (because of Brezis-Merle's interior estimates) $u_i \to u$ in $C^1(\Omega_{\epsilon^2})$. We have,

$$||\nabla(u_i-u)||_{L^\infty(\Omega_{\epsilon^2})}\leq \epsilon^2, \text{ for } i\geq i_3=i_3(\epsilon).$$

We write,

$$\int_{\Omega_{\epsilon_2}} |<\nabla(u_i-u)|\nabla\tilde{\eta}_\epsilon>|dx \leq ||\nabla(u_i-u)||_{L^\infty(\Omega_{\epsilon^2})}||\nabla\tilde{\eta}_\epsilon||_{L^\infty}\leq C_1\epsilon \text{ for } i\geq i_3,$$

For $\epsilon>0$, we have for $i\in\mathbb{N}$, $i\geq\max\{i_1,i_2,i_3\}$,

$$\int_\Omega |<\nabla(u_i-u)|\nabla\tilde{\eta}_\epsilon>|dx\leq \epsilon C_1(2k_1||\nabla u||_{L^\infty}+2) \qquad (7)$$

From (4) and (7), we have, for $\epsilon>0$, there is $i_3=i_3(\epsilon)\in\mathbb{N}, i_3=\max\{i_0,i_1,i_2\}$ such that,

$$\int_\Omega |\Delta[(u_i-u)\tilde{\eta}_\epsilon]|dx\leq 4\pi-\epsilon_0+\epsilon 2C_1(2k_1||\nabla u||_{L^\infty}+2+C) \qquad (8)$$

We choose $\epsilon>0$ small enough to have a good estimate of (1).

Indeed, we have:

$$\begin{cases}\Delta[(u_i-u)\tilde{\eta}_\epsilon]=g_{i,\epsilon} \text{ in } \Omega,\\ (u_i-u)\tilde{\eta}_\epsilon=0 \text{ on } \partial\Omega.\end{cases}$$

with $||g_{i,\epsilon}||_{L^1(\Omega)}\leq 4\pi-\epsilon_0$.

We can use Theorem 1 of [4] to conclude that there is $q>1$ such that:

$$\int_{V_\epsilon(x_0)} e^{q(u_i-u)}dx\leq\int_\Omega e^{q(u_i-u)\tilde{\eta}_\epsilon}dx\leq C(\epsilon,\Omega).$$

where, $V_\epsilon(x_0)$ is a neighberhooh of x_0 in $\bar{\Omega}$.

Thus, for each $x_0\in\partial\Omega-\{\bar{x}_1,\ldots,\bar{x}_m\}$ there is $\epsilon_{x_0}>0, q_{x_0}>1$ such that:

$$\int_{B(x_0,\epsilon_{x_0})} e^{q_{x_0}u_i}dx\leq C, \ \forall\ i.$$

Now, we consider a cutoff function $\eta\in C^\infty(\mathbb{R}^2)$ such that:

$$\eta\equiv 1 \text{ on } B(x_0,\epsilon_{x_0}/2) \text{ and } \eta\equiv 0 \text{ on } \mathbb{R}^2-B(x_0,2\epsilon_{x_0}/3).$$

We write,

$$\Delta(u_i\eta)=V_ie^{u_i}\eta-2<\nabla u_i|\nabla\eta>+u_i\Delta\eta.$$

By the elliptic estimates, $(u_i\eta)_i$ is uniformly bounded in $W^{2,q_1}(\Omega)$ and also, in $C^1(\bar{\Omega})$.

Finaly, we have, for some $\epsilon>0$ small enough,

$$||u_i||_{C^{1,\theta}[B(x_0,\epsilon)]} \leq c_3 \;\; \forall \;\; i.$$

We have proved that, there is a finite number of points $\bar{x}_1, \ldots, \bar{x}_m$ such that the squence $(u_i)_i$ is locally uniformly bounded in $\bar{\Omega} - \{\bar{x}_1, \ldots, \bar{x}_m\}$.

Proof of theorem 1.2:

The Pohozaev identity gives :

$$\int_{\partial\Omega} [(\partial_\nu u_i)\nabla u_i - \frac{1}{2}||\nabla u_i|^2 \nu] dx = \int_\Omega \nabla V_i e^{u_i} - \int_{\partial\Omega} V_i e^{u_i} \nu,$$

We use the boundary condition to have:

$$\int_{\partial\Omega} (\partial_\nu u_i)^2 dx \leq c_0(b, A, C, \Omega).$$

Thus we can use the weak convergence in $L^2(\partial\Omega)$ to have a subsequence $\partial_\nu u_i$, such that:

$$\int_{\partial\Omega} \partial_\nu u_i \phi dx \to \int_{\partial\Omega} \partial_\nu u \phi dx, \;\; \forall \;\; \phi \in L^2(\partial\Omega),$$

Thus, $\alpha_j = 0$, $j = 1, \ldots, N$ and (u_i) is uniformly bounded.

Remarks:

The most important application of the hypothesis that the functions V_i are uniformly lipschitzian is the fact that we can bound the L^2 (on $\partial\Omega$) norm of the sequence $\partial_\nu u_i$, and the consequence of this fact is, because $L^2(\partial\Omega)$ is reflexive, the weak convergnece to a new function.

The weak convergence of $V_i e^{u_i}$ is true, because we have proved that (this is a consequence of the proof of the first theorem and by integration by part):

$$\int_\Omega V_i e^{u_i} \phi dx \to \int_\Omega V e^u \phi dx + \sum_{i=1}^m \alpha_i \phi(x_i)$$

in particular, because $x_i \in \partial\Omega$, if we take $\phi \in C_0^\infty(\Omega)$ it vanish on the boundary and thus $\phi(x_i) = 0$. But it is not enough to remove the concentration points.

The goal is to know what happens on the boundary because the concentration points are on $\partial\Omega$ and not in Ω. And we want to remove them.

This is a reason why we are interested by the convergence of $\partial_\nu u_i$ (and thus, in $L^2(\partial\Omega)$), and not $V_i e^{u_i}$ (on $L^1(\Omega)$). Because, since the begining we took $\partial_\nu u_i$ and we work with $\partial_\nu u_i$, on the boundary.(the proof of the first theorem use also the interior estimate).

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