Just in Time Control of Timed Event Graphs: Update of Reference Input, Presence of Uncontrollable Input

Menguy E., Boimond J.-L., Hardouin L. and Ferrier J.-L.

Abstract

A linear system theory has been developed for the class of discrete event systems subject to synchronization. This paper presents the just in time control of such systems when reference input is updated and/or in the presence of uncontrollable input(s). The proposed controls are the solutions to an optimization problem under equality constraint.

Keywords

Discrete Event Systems, Dioid, Residuation, Timed Event Graphs, Just in Time Control

I. Introduction

Timed Event Graphs (TEGs) are a subclass of Petri nets well adapted to modeling synchronization phenomena, moreover they can be seen as linear dynamic systems in dioid algebra [1, §2]. We deal with the just in time control of such systems: given desired output transition firing dates $z = \{z(k)\}$ (also called reference input), find the *latest* input transition firing dates $u = \{u(k)\}$ such that output transition firing dates occur *before* the given ones. In a production context, it amounts to satisfying the customer demand while minimizing the stocks. This tracking problem [3], [1, §5.6] is solved *via* Residuation theory [2]. It formally consists in finding

Laboratoire d'Ingénierie des Systèmes Automatisés, 62, avenue Notre-Dame du Lac, 49000 ANGERS, FRANCE, Tel: (33) 2 41 36 57 33, Fax: (33) 2 41 36 57 35. E-mail: [menguy, boimond, hardouin, jean-louis.ferrier]@istia.univ-angers.fr

the greatest solution to $G(u) \preceq z$, where G is a (max, +)-linear map in dioid algebra. Its computation *a priori* requires the availability of z. Here, we successively consider two assumptions:

- Reference input can be updated. For example in a manufacturing context, a customer demand occurrence may lead to the modification of the planned production;
- Firing dates of some input transitions cannot be modified. These inputs may represent the supply in raw materials.

The control synthesis leads to solving a tracking problem under an equality constraint. It formally consists in finding the greatest solution to $G(u) \preceq z$ which satisfies equality $F(u) = c$, where F is a (max, +)-linear map in dioid algebra and $c = \{c(k)\}$ is a fixed trajectory. The solution is also based on Residuation theory. To guarantee the existence of a solution, it may be necessary to modify the reference input which weakens the just in time objective: the firing dates of output transition occur *as close as possible to* (eventually after) the given ones.

The paper is organized as follows. Section 2 briefly deals with a few algebraic results of dioids, Residuation theory and TEGs. After a brief presentation of the tracking problem and its optimal solution, we study the tracking problem under an equality constraint in section 3. It is successively applied to the two considered assumptions in section 4. An illustrative example is provided in section 5.

II. Dioids, Residuation Theory and Timed Event Graphs

For an exhaustive presentation of dioids see [1], [4] and for Residuation theory see [2], [1, §4].

A dioid $(\mathcal{D}, \oplus, \otimes)$ is a semiring in which $\oplus$ is idempotent ($\forall a,\ a \oplus a = a$), neutral elements of $\oplus$, $\otimes$ are noted ε, e respectively. Due to idempotency of $\oplus$, a natural order relation is defined by $a \preceq b \Longleftrightarrow b = a \oplus b$ (the least upper bound of $\{a, b\}$ is then equal to $a \oplus b$).

A dioid $\mathcal{D}$ is *complete* if every subset A of $\mathcal{D}$ has a least upper bound equal to $\bigoplus_{x \in A} x$, and if $\otimes$ distributes over infinite sums. The greatest element, noted $\top$, of a complete dioid $\mathcal{D}$ is equal to

$\bigoplus_{x\in\mathcal{D}} x$. The largest lower bound of every subset X of a complete dioid $\mathcal{D}$ always exists and is noted $\bigwedge_{x\in X} x$. A dioid $\mathcal{D}$ is *distributive* if it is complete and if $\forall C \subset \mathcal{D}$, $\forall a \in \mathcal{D}$, $(\bigwedge_{c\in C} c) \oplus a = \bigwedge_{c\in C}(c \oplus a)$ and $(\bigoplus_{c\in C} c) \wedge a = \bigoplus_{c\in C}(c \wedge a)$.

Starting from a 'scalar' dioid $\mathcal{D}$, consider $p \times p$ matrices with entries in $\mathcal{D}$. Sum and product of matrices are defined conventionally from the sum and product of scalars in $\mathcal{D}$. This set of matrices endowed with these two operations is a dioid noted $\mathcal{D}^{p\times p}$.

Residuation is a general notion in lattice theory which allows providing an answer to the problem of 'solving' equations of the form $f(x) = b$ where f is isotone ($a \preceq b \Rightarrow f(a) \preceq f(b)$). We consider here the notion of 'subsolution' as an alternative to that of 'solution'.

Definition 1: An isotone map $f : \mathcal{D} \mapsto \mathcal{C}$ where $\mathcal{D}$ and $\mathcal{C}$ are ordered sets is *residuated* if there exists an isotone map $h : \mathcal{C} \mapsto \mathcal{D}$ such that $f \circ h \preceq Id_{\mathcal{C}}$ and $h \circ f \succeq Id_{\mathcal{D}}$ ($Id_{\mathcal{C}}$ and $Id_{\mathcal{D}}$ are identity maps of $\mathcal{C}$ and $\mathcal{D}$ respectively). Map h, also noted $f^{\sharp}$, is unique and is called *the residual* of f.

If f is residuated then $\forall y \in \mathcal{C}$, the least upper bound of subset $\{x \in \mathcal{D} \mid f(x) \preceq y\}$ exists and belongs to this subset. This *greatest* 'subsolution' is equal to $f^{\sharp}(y)$.

Remark 1: If there exists a solution (possibly nonunique) to $f(x) = y$, then $f^{\sharp}(y)$ is the greatest solution to this *equation.*

Remark 2: If map f has an inverse map, it is equal to $f^{\sharp}$.

Let $\mathcal{D}$ a complete dioid, consider the isotone map $L_a : x \mapsto a \otimes x$ from $\mathcal{D}$ into $\mathcal{D}$. The greatest solution to $a \otimes x \preceq b$ exists and is equal to $L_a^{\sharp}(b)$, also noted $\frac{b}{a}$. Some formulæ and properties which will be used later on are recalled in the following.

Property 1: 1. $\forall a, b \in \mathcal{D}, \quad a \otimes \frac{b}{a} \preceq b \quad$ and $\quad \frac{a\otimes b}{a} \succeq b \quad (L_a \circ L_a^{\sharp} \preceq Id_{\mathcal{D}},\ L_a^{\sharp} \circ L_a \succeq Id_{\mathcal{D}})$.

2. $\forall a \in \mathcal{D}$, the map $L_a^{\sharp} : x \mapsto \frac{x}{a}$ from $\mathcal{D}$ into $\mathcal{D}$ is isotone (the residual is an isotone map).

3. Let $A \in \mathcal{D}^{n\times p}$ and $B \in \mathcal{D}^{n\times q}$, $\frac{B}{A} \in \mathcal{D}^{p\times q}$ and $(\frac{B}{A})_{ij} = \bigwedge_{l=1}^{n} \frac{b_{lj}}{a_{li}}, \quad 1 \le i \le p,\ 1 \le j \le q$. This point relates the residuation of matrices to the residuation of scalars.

Event graphs are a particular class of Petri nets in which there are a single transition upstream, and a single transition downstream in every place [1, §2],[3]. With each transition X_i of a TEG, we associate a *dater* x_i which is a non decreasing map from $\mathbb{Z}$ into $\mathbb{Z} \cup \{\pm\infty\}$ where $x_i(k) = t$ means that the firing numbered k of transition X_i occurs at time t. A linear system theory has been developed for TEGs [1, §6]: a SISO TEG with input transition U and output transition Y can be represented by the relation $y(k) = (g \otimes u)(k) = \underset{s\in\mathbb{Z}}{Sup}\{\, g(k-s) + u(s)\,\}$ where dater g is called *impulse response*. Operation $\otimes$ is nothing but the sup-convolution which has the role of convolution in this theory. The set of daters, noted Σ, endowed with operations pointwise maximum ($\oplus$) and sup-convolution ($\otimes$) is a distributive dioid with neutral elements ε and e: $\varepsilon(k) = -\infty\ \forall k \in \mathbb{Z}$; $e(k) = -\infty\ \forall k < 0$ and 0 otherwise. The greatest element $\top$ is such that $\top(k) = +\infty,\ \forall k \in \mathbb{Z}$, moreover we have $u \preceq v \Longleftrightarrow u(k) \leq v(k) \quad \forall k \in \mathbb{Z}$.

The residuation of product [1, §4] is defined by $\forall u, v \in \Sigma, \forall k \in \mathbb{Z}$, $\frac{u}{v}(k) = \underset{s\in\mathbb{Z}}{Inf}\{\, u(s) - v(s-k)\,\}$.

III. Optimal Control under Equality Constraint

The just in time control consists in firing input transition *at the latest* so that the firing dates of output transition occur *at the latest* before the desired ones. Let us define $z = \{z(k)\}_{(k\in\mathbb{Z})}$ as the dater of desired outputs: $z(k) = t$ means that the firing numbered k of output transition is desired at time t at the latest. We search for the *greatest* solution to $y = g \otimes u \preceq z$, this optimal control, noted u_{opt}, is given by Residuation theory (cf. section 2):

$u_{opt} \overset{def}{=} \bigoplus_{\{\, u\, \in\Sigma\ \mid\ g\otimes u \preceq z\,\}} u = \frac{z}{g}$ (consider map $L_g : x \mapsto g \otimes x$). The two applications considered in next section lead to solving the previous tracking problem while satisfying the equality constraint $F(u) = c$, where F is a residuated map and c is a fixed dater. In other words, we search for the *greatest* element of set

$$S_1 = \{\, u \in \Sigma \mid g \otimes u \preceq z \quad \text{and} \quad F(u) = c\,\}.$$

Its expression is deduced from the following theorem.

Theorem 1: Let $\mathcal{D}$ and $\mathcal{C}$ two complete dioids, $F, G : \mathcal{D} \to \mathcal{C}$ two residuated maps and $c_1, c_2 \in \mathcal{C}$. Let us consider sets

$$S_a = \{ x \in \mathcal{D} \mid G(x) \preceq c_1 \quad \text{and} \quad F(x) = c_2 \} \quad \text{and} \quad S_b = \{ x \in \mathcal{D} \mid G(x) \preceq c_1 \quad \text{and} \quad F(x) \preceq c_2 \}.$$

Set S_a is supposed not empty and we note $x_a = \bigoplus_{x \in S_a} x$, $x_b = \bigoplus_{x \in S_b} x$, then $x_a = x_b$ with $x_b = G^\sharp(c_1) \wedge F^\sharp(c_2)$.

Proof: The expression of the greatest element (x_b) of set S_b is classical. Moreover, it is obvious that if the equality constraint is supposed achievable then the solution provided by residuation will also achieve equality constraint (cf. remark 1). ■

So the greatest element of set S_1 *if it is not empty* is equal to $\frac{z}{g} \wedge F^\sharp(c)$. We will see in the two considered applications that equality constraint (defined by F and c) cannot be modified and can lead to an empty set S_1. We modify z additively (in sense of $\oplus$) to always have a solution. We consider the least possible modification to have output transition firing dates *as close as possible* to the desired ones defined by z. This approach leads us to consider a new set S_2 defined from the next property.

Property 2: Let $\mathcal{D}$ and $\mathcal{C}$ two complete dioids, $F, G : \mathcal{D} \to \mathcal{C}$ two residuated maps and $c_1, c_2 \in \mathcal{C}$. We assume that the largest lower bound of the subset $\{ x \in \mathcal{D} \mid F(x) = c_2 \}$, denoted $\underline{x}$, belongs to this subset. Then $c_1 \oplus G(\underline{x})$ is the least dater of the form $c_1 \oplus \Delta c_1$ such that set $S(\Delta c_1) = \{ x \in \mathcal{D} \mid G(x) \preceq c_1 \oplus \Delta c_1 \quad \text{and} \quad F(x) = c_2 \}$ is not empty.

Proof: $S(G(\underline{x}))$ is not empty since it contains $\underline{x}$. If $S(\Delta c_1)$ is not empty then $c_1 \oplus G(\underline{x}) \preceq c_1 \oplus \Delta c_1$. Indeed, let $y \in S(\Delta c_1)$, as $\underline{x} \preceq y$ and G is isotone, we have $G(\underline{x}) \preceq G(y) \preceq c_1 \oplus \Delta c_1$. We deduce that $c_1 \oplus G(\underline{x}) \preceq c_1 \oplus \Delta c_1$. ■

Remark 3: We easily prove by isotony of G that $G(\underline{x}) = Inf\{ G(x) \in \mathcal{C} \mid F(x) = c_2 \}$.

In the two considered applications, we will show that $\underline{u} = Inf\{ u \in \Sigma \mid F(u) = c \}$ is a solution to the equality constraint $F(u) = c$. By using previous property, we will then search for the greatest element, noted u'_{opt}, of set

$$S_2 = \{\ u \in \Sigma \mid g \otimes u \preceq z \oplus g \otimes \underline{u} \quad \text{and} \quad F(u) = c\ \}.$$

$z \oplus g \otimes \underline{u}$ is the least dater of the form $z \oplus \Delta z$ such that set S_2 is not empty. We deduce from theorem 1 that

$$u'_{opt} = \frac{z \oplus g \otimes \underline{u}}{g} \wedge F^{\sharp}(c). \tag{1}$$

The reference input *effectively* tracked is $z \oplus g \otimes \underline{u}$. Let us note that $z \oplus g \otimes \underline{u} = z$ when S_1 is not empty, indeed S_1 not empty implies that $g \otimes \underline{u} \preceq z$.

IV. Applications

For the two considered applications, we define set S_1 and successively give the expressions of map F, its residual $F^{\sharp}$, daters c, $\underline{u}$ and set S_2.

A. Update of Reference Input

We are interested in the tracking problem when reference input z is updated. For example, z can be updated at a time t' to take into account a new production objective when a customer demand occurs at this time. See [5, §5] for the multi-input, multi-output (MIMO) case. Let $u_{opt} = \frac{z}{g}$ the optimal control associated with reference input z (cf. section III). A new reference input z' is defined at time t' which leads to considering the greatest control of set

$$\mathcal{F}_1 = \{\ u \in \Sigma \mid g \otimes u \preceq z' \quad \text{and} \quad u(k) = u_{opt}(k),\ \forall k \leq K\ \} \text{ with } K = Sup\,\{\, k \in \mathbb{Z} \mid u_{opt}(k) \leq t'\ \}.$$

The computation of this control takes into account both the track of z' ($g \otimes u \preceq z'$) and the constraint due to past controls applied to the system input ($u(k) = u_{opt}(k),\ \forall k \leq K$).

1. *Expressions of maps F, $F^{\sharp}$ and dater c.*

Let us define map r_K^- from Σ into Σ: $[r_K^-(u)](k) = \begin{cases} u(k) \ if\ k \leq K \\ u(K) \ if\ k > K \end{cases}$, which allows expressing constraint $(u(k) = u_{opt}(k),\ \forall k \le K)$ as $r_K^-(u) = r_K^-(u_{opt})$. $\mathcal{F}_1$ has the same form as S_1 with $F = r_K^-$ and $c = r_K^-(u_{opt})$.

Remark 4: We can easily verify that map r_K^- is isotone, is a projector ($r_K^- \circ r_K^- = r_K^-$) and is a $\oplus$-morphism ($\forall u, v \in \Sigma$, $r_K^-(u \oplus v) = r_K^-(u) \oplus r_K^-(v)$), moreover $r_K^- \preceq Id_\Sigma$.

Let us prove that r_K^- is a residuated map and $(r_K^-)^\sharp$ is the map $x \mapsto r_K^-(x) \oplus d_K$ where d_K is the dater defined by $d_K(k) = \begin{cases} -\infty & if \quad k \le K \\ +\infty & if \quad k > K \end{cases}$.

Let us note $h : x \mapsto r_K^-(x) \oplus d_K$. From definition of r_K^- and its properties, we have

(a) h isotone: if $x \preceq y$ then $r_K^-(x) \preceq r_K^-(y)$ and $r_K^-(x) \oplus d_K \preceq r_K^-(y) \oplus d_K$.

(b) $(r_K^- \circ h)(x) = r_K^-(r_K^-(x) \oplus d_K) = r_K^-(r_K^-(x)) \oplus r_K^-(d_K) = r_K^-(r_K^-(x)) \oplus \varepsilon = r_K^-(x) \preceq x$.

(c) $(h \circ r_K^-)(x) = h(r_K^-(x)) = r_K^-(r_K^-(x)) \oplus d_K = r_K^-(x) \oplus d_K \succeq x$.

From definition 1, we deduce that $h = (r_K^-)^\sharp$ (the residual of r_K^-).

2. *Expressions of dater $\underline{u}$ and set $\mathcal{F}_\in$.*

We easily verify that $\underline{u} \overset{def}{=} Inf\,\{\, u \in \Sigma \,|\, r_K^-(u) = r_K^-(u_{opt}) \,\} = r_K^-(u_{opt})$ and $F(\underline{u}) = c$ which leads to considering set $\mathcal{F}_2 = \{\ u \in \Sigma \mid g \otimes u \preceq z' \oplus g \otimes r_K^-(u_{opt}) \quad \text{and} \quad r_K^-(u) = r_K^-(u_{opt})\ \}$. This set being not empty (cf. property 2), the next property gives the formal expression of the greatest element, noted u'_{opt}, of $\mathcal{F}_2$ by using expression (1).

Property 3: Let r_K^+ the map from Σ into Σ defined by $r_K^+(x) = x \wedge d_K$. We have

$$u'_{opt} = r_K^-(u_{opt}) \oplus r_K^+(\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g}).$$

Proof: From expression (1), we have $u'_{opt} = \frac{z' \oplus g \otimes r_K^-(u_{opt})}{g} \wedge (r_K^-)^\sharp(r_K^-(u_{opt}))$. From the expression of $(r_K^-)^\sharp$ and since r_K^- is a projection, $(r_K^-)^\sharp(r_K^-(u_{opt})) = r_K^-(r_K^-(u_{opt})) \oplus d_K = r_K^-(u_{opt}) \oplus d_K$. Then

$$\begin{aligned} u'_{opt} &= \frac{z' \oplus g \otimes r_K^-(u_{opt})}{g} \wedge (r_K^-(u_{opt}) \oplus d_K) \\ &= (\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g} \wedge r_K^-(u_{opt})) \oplus (\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g} \wedge d_K) \quad (\Sigma \text{ is a distributive dioid}) \\ &= (\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g} \wedge r_K^-(u_{opt})) \oplus r_K^+(\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g}). \end{aligned}$$

As $z' \oplus g \otimes r_K^-(u_{opt}) \succeq g \otimes r_K^-(u_{opt})$ and by using properties 1.2 and 1.1, we have $\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g} \succeq \frac{g \otimes r_K^-(u_{opt})}{g} \succeq r_K^-(u_{opt})$. We deduce that $\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g} \wedge r_K^-(u_{opt}) = r_K^-(u_{opt})$. ■

Control u'_{opt} is the sum (in the sense of $\oplus$) of signal $r_K^-(u_{opt})$ expressing the control applied until event K and signal $r_K^+(\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g})$ expressing the control applied beyond event K. Such a control allows firing the system input *at the latest* dates so that the output firing dates occur *as close as possible* to the ones defined by updated reference input z'.

Remark 5: It is proved in [5, §5] that $r_K^+(\frac{z' \oplus g \otimes r_K^-(u_{opt})}{g}) = \frac{r^+_{K+val\,g}(z' \oplus g \otimes r_K^-(u_{opt}))}{g}$ with $val\,g = Inf\{\, k \in \mathbb{Z} \,|\, g(k) > -\infty \,\}$. Thus u'_{opt} is such that firing date $y(k)$ occurs *at the latest* before $z(k)$ for $k \leq K + val\,g$ and *at the latest* before $(z' \oplus g \otimes r_K^-(u_{opt}))(k)$ for $k > K + val\,g$.

B. Uncontrollable Input

We are interested in the tracking problem when the firing dates of some input transitions are not controlled. For example, the supply in raw materials can be uncontrollable in a production context.

Definition 2: An input transition of a TEG is said to be controllable if we have the control of all its firing dates, otherwise it is said to be uncontrollable.

The aim is to control the system *via* its controllable inputs under the constraint due to uncontrollable inputs. We consider a MIMO TEG with p input transitions and q output transitions; let $u = (u_1, \ldots, u_p)^t$ the input dater vector, only the n-first inputs $u_1, \ldots, u_n$, $n < p$, are supposed controllable. We define the following partitions: $u = \begin{bmatrix} u_c \\ u_{\overline{c}} \end{bmatrix}$ and $g = [g_c \,,\; g_{\overline{c}}]$, where $u_c = (u_1, \ldots, u_n)^t$ is the controllable component of u and $u_{\overline{c}} = (u_{n+1}, \ldots, u_p)^t$ its uncontrollable part; g_c and $g_{\overline{c}}$ are the parts of g linked up to u_c and $u_{\overline{c}}$ respectively. Every system input u will verify condition $u_{\overline{c}} = v_{\overline{c}}$ where $v_{\overline{c}}$ is *a priori* fixed dater vector of $\Sigma^{(p-n)\times 1}$. This constraint leads to considering the greatest control of set $\mathcal{G}_1 = \{\, u \in \Sigma^{p\times 1} \mid g \otimes u \preceq z \quad \text{and} \quad u_{\overline{c}} = v_{\overline{c}} \,\}$. The computation of this control takes into account both the track of z ($g \otimes u \preceq z$) and the constraint due to uncontrollable inputs ($u_{\overline{c}} = v_{\overline{c}}$).

1. *Expressions of maps F, $F^\sharp$ and dater c.*

Condition $u_{\overline{c}} = v_{\overline{c}}$ is equivalent to $[\varepsilon_{(p-n)\times n}\ Id_{(p-n)\times(p-n)}]\, u = v_{\overline{c}}$ where $\varepsilon_{(p-n)\times n}$ is the null $(p-n)\times n$ matrix and $Id_{(p-n)\times(p-n)}$ is the identity $(p-n)\times(p-n)$ matrix. $\mathcal{G}_1$ has the same form as S_1 with $F : \Sigma^{p\times 1} \to \Sigma^{(p-n)\times 1}$, $x \mapsto [\varepsilon_{(p-n)\times n}\ Id_{(p-n)\times(p-n)}]\, x$ and $c = v_{\overline{c}} \in \Sigma^{(p-n)\times 1}$. F is clearly residuated and its residual is defined by

$$\forall x \in \Sigma^{(p-n)\times 1},\ F^\sharp(x) = \frac{x}{[\varepsilon_{(p-n)\times n}\ Id_{(p-n)\times(p-n)}]} = \begin{bmatrix} \frac{x}{\varepsilon_{(p-n)\times n}} \\ \frac{x}{Id_{(p-n)\times(p-n)}} \end{bmatrix} = \begin{bmatrix} \top_{n\times 1} \\ x \end{bmatrix}.$$

The result of Residuation given by property 1.3 can be extended if we consider a_{li} and b_{lj} as matrices with compatible dimensions. Moreover, we easily verify that if $\mathcal{D}$ is a complete dioid, then $\forall a \in \mathcal{D}$, $\frac{a}{\varepsilon} = \top$ and $\frac{a}{e} = a$. We deduce that $\frac{x}{\varepsilon_{(p-n)\times n}} = \top_{n\times 1}$ where $\top_{n\times 1}$ is the $n\times 1$ vector whose components are equal to dater $\top$ and, $\frac{x}{Id_{(p-n)\times(p-n)}} = x$.

2. *Expressions of dater $\underline{u}$ and set $\mathcal{G}_\in$.*

The least dater satisfying constraint $u_{\overline{c}} = v_{\overline{c}}$ is clearly $\underline{u} = [\varepsilon_{n\times 1},\ v_{\overline{c}}]^t$. We easily verify that $F(\underline{u}) = c$ and $g \otimes \underline{u} = g_{\overline{c}}$. We then consider set

$\mathcal{G}_2 = \{u \in \Sigma^{p\times 1} \mid g \otimes u \preceq z \oplus g_{\overline{c}} \otimes v_{\overline{c}} \text{ and } [\varepsilon_{(p-n)\times n}\ Id_{(p-n)\times(p-n)}]\, u = v_{\overline{c}}\}$.

This set being not empty (cf. property 2), the next property gives the expression of the greatest control, noted u'_{opt}, of $\mathcal{G}_2$ by using expression (1).

Property 4: We have $u'_{opt} = \begin{bmatrix} \frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_c} \\ v_{\overline{c}} \end{bmatrix}$.

Proof: From expression (1), we have

$$u'_{opt} = \frac{z \oplus g_{\overline{c}} \otimes v_{\overline{c}}}{g} \wedge F^\sharp(v_{\overline{c}}) = \begin{bmatrix} \frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_c} \\ \frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_{\overline{c}}} \end{bmatrix} \wedge \begin{bmatrix} \top_{n\times 1} \\ v_{\overline{c}} \end{bmatrix} = \begin{bmatrix} \frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_c} \\ \frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_{\overline{c}}} \wedge v_{\overline{c}} \end{bmatrix},$$

and $\frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_{\overline{c}}} \wedge v_{\overline{c}} = v_{\overline{c}}$ since we have $\frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_{\overline{c}}} \succeq \frac{g_{\overline{c}}\otimes v_{\overline{c}}}{g_{\overline{c}}} \succeq v_{\overline{c}}$ by using properties 1.2 and 1.1. ■

Component $\frac{z\oplus g_{\overline{c}}\otimes v_{\overline{c}}}{g_c}$ results from the tracking objective and component $v_{\overline{c}}$ results from the equality constraint. Control u'_{opt} allows firing controllable input u_c *at the latest* dates so that the firing dates of output occur *as close as possible* to the ones defined by reference input z (more precisely *at the latest* before $z \oplus g_{\overline{c}} \otimes v_{\overline{c}}$).

V. Example

Let us illustrate the control design presented in section IV-A. We consider the TEG of Fig. 1, its impulse response is equal to $g(k) = \begin{cases} -\infty & \text{if } k < 0 \\ 2 + 2\lfloor \frac{k}{2} \rfloor & \text{otherwise} \end{cases}$, where $\lfloor x \rfloor$ is the truncated part of x; we have $val\, g = 0$.

We initially consider reference input z described in figure 2, it is updated at time $t' = 6$. Until this time, optimal controls $u_{opt}(k) = \frac{z}{g}(k)$, $k \leq K$ with $K = Sup\{\ k \in \mathbb{Z} \mid u_{opt}(k) \leq t'\ \} = 4$, are applied to the system. At time t', we consider a new reference input z' (we note that $z'(k) = z(k)$ for $k \leq 4$ (see figure 2)). The effect of applied past controls $u_{opt}(k), k \leq 4$ necessarily implies that any output will be greater than dater $g \otimes r_4^-(u_{opt})$ (cf. remark 3) (see figure 2). The greatest control u'_{opt} of set $\mathcal{F}_2 = \{\ u \in \Sigma \mid g \otimes u \preceq z' \oplus g \otimes r_4^-(u_{opt}) \quad \text{and} \quad r_4^-(u) = r_4^-(u_{opt})\ \}$ is equal to $r_4^-(u_{opt}) \oplus r_4^+(\frac{z' \oplus g \otimes r_4^-(u_{opt})}{g})$ (cf. property 3) and is represented in Fig. 3. Before time t', we apply $u'_{opt}(k) = u_{opt}(k)$, $\forall k \leq 4$, to track reference input z. Beyond time t', we apply $u'_{opt}(k) = \frac{z' \oplus g \otimes r_4^-(u_{opt})}{g}(k)$, $\forall k > 4$; the reference input effectively tracked is $(z' \oplus g \otimes r_4^-(u_{opt}))(k)$, $k > 4$. Outputs are greater than desired outputs (see areas into ellipses in Fig. 4) when $(g \otimes r_4^-(u_{opt}))(k) > z'(k)$: these outputs are equal to $(g \otimes r_4^-(u_{opt}))(k)$ and occur *at the earliest* after $z'(k)$, and *at the latest* before $(z' \oplus g \otimes r_4^-(u_{opt}))(k)$, $\forall k > 4$.

(Possible location for fig. 1, 2, 3 and 4)

VI. Conclusion

We have considered the just in time control problem when reference input is updated and/or in the presence of uncontrollable input transitions. Synthesis of optimal controls is based on Residuation theory. An additive modification of reference input can be necessary to ensure the existence of a solution. This principle of optimal control under an equality constraint has been also used for construction of an adaptive controller in dioid algebra [6], [5].

References

[1] F. Baccelli, G. Cohen, G.J. Olsder, and J.P. Quadrat. *Synchronization and Linearity.* Wiley, 1992.

[2] T.S. Blyth and M.F. Janowitz. *Residuation Theory.* Pergamon Press, 1972.

[3] G. Cohen, P. Moller, J.P. Quadrat, and M. Viot. Algebraic Tools for the Performance Evaluation of Discrete Event Systems. In *IEEE*, number 1, pages 39–58, 1989.

[4] R.A. Cuninghame-Green. *Minimax Algebra.* Number 166. Springer Verlag, Berlin, 1979.

[5] E. Menguy. *Contribution à la commande des systèmes linéaires dans les dioïdes.* PhD thesis, Univ. Angers, 1997.

[6] E. Menguy, J.L. Boimond, and L. Hardouin. Adaptive Control for Linear Systems in Max-Algebra. In *International Workshop on Discrete Event Systems (Wodes'98)*, Cagliari. Italy, August 1998.