On some historical aspects of Riemann zeta function, 2

Giuseppe Iurato

On some historical aspects of Riemann zeta function, 2

Giuseppe Iurato
University of Palermo, IT

E-mail: giuseppe.iurato@unipa.it

Abstract. Starting to develop what sketchily and synoptically planned in the preprint (Iurato 2013), in this place we would like to carry out a first deeper historical study of some questions there outlined. To be precise, we here simply focus on some first historiographical details regarding the early key moments in the history of the entire function factorization theorem.

1. Introduction

In his celebrated unique 1959 number theory paper on distribution of prime numbers[1], Riemann stated the following function

$$\xi(t) \stackrel{\text{def}}{=} \left(\frac{1}{2}\Gamma\left(\frac{s}{2}\right)s(s-1)\pi^{-\frac{s}{2}}\zeta(s)\right)_{s=\frac{1}{2}+it}, \qquad (1)$$

later called *Riemann* ξ*-function*. It is an entire function. Riemann conjectured that $(\xi(t)=0) \Rightarrow (\Im(t)=0)$, that is to say, the famous *Riemann hypothesis* (RH), as it will be called later. Whereupon, he stated that

> *«This function $\xi(t)$ is finite for all finite values of t, and allows itself to be developed in powers of tt as a very rapidly converging series. Since, for a value of s whose real part is greater than 1, $log\,\zeta(s) = -\sum log(1-p^{-s})$ remains finite, and since the same holds for the logarithms of the other factors of $\xi(t)$, it follows that the function $\xi(t)$ can only vanish if the imaginary part of t lies between $i/2$ and $-i/2$. The number of roots of $\xi(t)=0$, whose real parts lie between 0 and T is approximately $=\frac{T}{2\pi}log\frac{T}{2\pi}-\frac{T}{2\pi}$; because the integral $\int d\,log\,\xi(t)$, taken in a positive sense around the region consisting of the values of t whose imaginary parts lie between $i/2$ and $-i/2$ and whose real parts lie between 0 and T, is (up to a fraction of the order of magnitude of the quantity $1/T$) equal to $\left(T\,log\frac{T}{2\pi}-T\right)i$; this integral however is equal to the number of roots of $\xi(t)=0$ lying within in this region, multiplied by $2\pi i$. One now finds indeed approximately this number of real roots within these limits, and it is very probable that all roots are real. Certainly one would wish for a stricter proof here; I have meanwhile temporarily put aside the search for this after some fleeting futile attempts, as it appears unnecessary for the next objective of my investigation.*
>
> *If one denotes by α all the roots of the equation $\xi(\alpha)=0$, one can express $log\,\xi(\alpha)$ as*
>
> $$\sum log\left(1-\frac{tt}{\alpha\alpha}\right)+log\,\xi(0) \qquad (2)$$

[1] This paper was presented by Riemann, after his nomination as full professor in July 1859, to the Berlin Academy for his election as a corresponding member. To be precise, following (Bottazzini 2003), due to this election, Riemann and Dedekind visited Berlin where they met E.E. Kummer, L. Kronecker and Weierstrass: according to Dedekind (1876), very likely, it was just from this meeting that sprung out of the celebrated 1859 Riemann number theory paper which, amongst other, was then sent to Weierstrass himself for submission to the Berlin Academy. Therefore, Weierstrass was aware of this Riemann work since the late 1850s.

for, since the density of the roots of the quantity t grows with t only as $log\,\frac{t}{2\pi}$, it follows that this expression converges and becomes for an infinite t only infinite as $t\,log\,t$; thus it differs from $log\,\xi(t)$ by a function of tt, that for a finite t remains continuous and finite and, when divided by tt, becomes infinitely small for infinite t. This difference is consequently a constant, whose value can be determined through setting $t = 0$. With the assistance of these methods, the number of prime numbers that are smaller than x can now be determined».

So, in his celebrated 1859 paper, Riemann himself already used an infinite product factorization of $\xi(t)$, namely the (2), which can be equivalently written as follows

(3) $$\log\xi(t) = \sum log\left(1-\frac{tt}{\alpha\alpha}\right) + log\,\xi(0) = \log\xi(0)\prod\left(1-\frac{t^2}{\alpha^2}\right)$$

from which it follows $\xi(t) = \xi(0)\prod\left(1-\frac{\rho^2}{\alpha^2}\right)$. Therefore, questions related to entire function factorizations had already played a certain notable role in Riemann works. Well, in this second paper, we wish to outline the main points concerning the very early history of entire function factorization theorems, having taken the 1859 Riemann paper as a occasional starting point of this historical question, in which, inter alia, a particular entire function factorization – i.e. the (3) – has been used. In short, this 1859 Riemann paper has been a precious $\kappa\alpha\iota\rho$ός to begin to undertake one of the many study's branch which may depart from this milestone of the history of mathematics, to be precise that branch concerning entire function theory, which, however, as delineated in the previous preprint (Iurato 2013), runs parallel to certain aspects of the theory of Riemann zeta function.

Edwards (1974, Chapter 1, Sections 1.8-1.19) affirms that the parts concerned with (2) are the most difficult portion of 1859 Riemann's paper. Their goal is essentially to prove that $\xi(s)$ can be expressed as an infinite product, stating that

«[…] any polynomial $p(t)$ can be expanded as a finite product $p(t) = p(0)\prod_\rho\left(1-\frac{t}{\rho}\right)$, where p ranges over the roots of the equation $p(\rho) = 0$ [except that the product formula for $p(t)$ is slightly different if $p(0) = 0$]; hence the product formula (3) states that $\xi(t)$ is like a polynomial of infinite degree. Similarly, Euler thought of $sin\,x$ as a ''polynomial of infinite degree'' when he conjectured, and finally proved, the formula $sin\,\pi x = \pi x\prod_{n\in\mathbb{N}}\left(1-\left(\frac{x}{n}\right)^2\right)$. On other hand, […] $\xi(t)$ is like a polynomial of infinite degree[2], of which a finite number of its terms gives a very good approximation in any finite part of the plane. […] Hadamard (in 1893) proved necessary and sufficient conditions for the validity of the product formula $\xi(t) = \xi(0)\prod_\rho\left(1-\frac{t}{\rho}\right)$ but the steps of the argument by which Riemann went from the one to the other are obscure, to say the very least».

The last sentence of this Edwards' quotation is historically quite interesting and deserves further attention and investigation. Furthermore, Edwards states that *«a recurrent theme in Riemann's work is the global characterization of analytic functions by their singularities. See, for example, the Inauguraldissertation, especially Article 20 of Riemann's Werke (pp. 37-39) or Part 3 of the introduction to the Riemann article ''Theorie der Abel'schen Functionen'', which is entitled ''Determination of a function of a complex variable by boundary values and singularities''. See*

[2] Following (Bottazzini & Gray 2013, Section 8.5.1), amongst the functions that behave very like a polynomial, is the Riemann ξ function. In this regards, see also what said in (Iurato 2013) about Lee-Yang theorems and, in general, the theory concerning the location of the zeros of polynomials.

also Riemann's introduction to Paper XI of the his collected works, where he writes about '' […] *our method, which is based on the determination of functions by means of their singularities (Umtetigkeiten und Unendlichwerden)* […] *''. Finally, see the textbook of Ahlfors (1979), namely the section 4.5 of Chapter 8, entitled ''Riemann's Point of View''»*, according to which Riemann was a strong proponent of the idea that an analytic function can be defined by its singularities and general properties, just as well as or perhaps better than through an explicit expression, in this regards showing, with Riemann, as the solutions of a hypergeometric differential equation can be characterized by properties of this nature. In short, all this strongly suggests the need for a deeper re-analysis of Riemann *œuvre* concerning these last arguments, as well as a historical search of the mathematical background which lay at the origins of his celebrated 1859 number theory paper. From what has just been said, it turns out clear that a look at the history of entire function theory, within the general and wider complex function theory framework, is needed to clarify some of the historical aspects of this influential seminal paper which, as Riemann himself recognized, presented some obscure points. In this regards, also Gabriele Torelli (1901, Chapter VIII, Sections 60-64) claimed upon this last aspect, pointing out, in particular, on the Riemann's *ansatz* according to which the entire function $\xi(t)$ is equal, via (3), to the Weierstrass' infinite product of primary factors without any exponential factor. As is known, this basic question will be brilliantly solved by J. Hadamard in his famous 1892 paper that, inter alia, will mark a crucial moment in the history of entire function theory – see (Maz'ya & Shaposhnikova 1998, Chapter 9, Section 9.2).

2. On the history of entire function factorization theorem, I

In this section, we wish to preliminarily follow the basic textbook on complex analysis of Giulio Vivanti (1859-1949), an Italian mathematician whose main research field was into complex analysis, becoming an expert on entire functions. He wrote some notable treatises on entire, modular and polyhedral analytic functions: a first edition of a prominent treatise on analytic functions appeared in 1901, under the title *Teoria delle funzioni analitiche*, published by Ulrico Hoepli in Milan, where the first elements of the theory of analytic functions, worked out in the late 19th-century quarter, are masterfully exposed into three main parts, giving a certain load to the Weierstrass' approach respect to the Cauchy's and Riemann's ones. The importance of this work immediately arose, so that a German edition was carried out, in collaboration with A. Gutzmer, and published in 1906 by B.G. Teubner in Leipzig, under the title *Theorie der eindeutigen analytischen funktionen. Umarbeitung unter mitwirkung des verfassers deutsch herausgegeben von A. Gutzmer*, which had to be considered as a kind of second enlarged and revised edition of the 1901 first Italian edition according to what Vivanti himself said in the preface to the 1928 second Italian edition, entitled *Elementi della teoria delle funzioni analitiche e delle trascendenti intere*, again published by Ulrico Hoepli in Milan, and that was written following the above German edition in which many new and further arguments and results were added, amongst other things, as regards entire functions. Almost all the Vivanti's treatises are characterized by the presence of a detailed and complete bibliographical account of the related literature, this showing the great historical attention that he always put in his works. Therefore, he also was a valid historian of mathematics besides to be an able researcher (see (Janovitz & Mercanti 2008, Chapter 1) and references therein), so that his works are precious sources for historical studies, in our case, as concerns entire functions. The above mentioned Vivanti's textbook on complex analysis has been one of the most influential Italian treatise on the argument. It has also had wide fame thanks to its German edition.

Roughly speaking, the transcendental entire functions may be formally considered as a generalization, in the complex field, of polynomial functions. Following (Vivanti 1928, Sections 134-135) (and also (Markuševič 1988, Capitolo VII) as well as (Pierpont 1914, Chapter VIII, Sections 127 and 140)), the great analogy subsisting between these two last function classes suggested the search for an equal formal analogy between the corresponding main properties. To be precise, the main properties of polynomials concerned either with the existence of zeros (*Gauss'* *theorem*) and the linear factor decomposition of a polynomial, so that it was quite obvious trying to see whether these could be, in a certain way, extended to entire functions. As regards the Gauss' theorem, it was immediately realized that it couldn't subsist because of the simple counterexample given by the fundamental elementary transcendental function e^x which does not have any zero in the whole of complex plane. On the other hand, just this last function will provide the basis for building up the most general entire function which is never zero, which has the general form $e^{G(x)}$, where $G(x)$ is an arbitrary entire function, and is said to be an *exponential factor*. Whereupon, the next problem consisted in finding those entire functions having zeros and then how it is possible to build up them from their zero set. In this regards, it is well-known that, if $P(z)$ is an arbitrary non-zero polynomial with zeros $z_1, \dots, z_n \in \mathbb{C}\backslash\{0\}$, having $z = 0$ as a zero with multiplicity λ (supposing $\lambda = 0$ if $P(0) \neq 0$), then we have the following finite product factorization[3]

(4) $$P(z) = Cz^\lambda \prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right)$$

where $C \in \mathbb{C}\backslash\{0\}$ is a constant, so that a polynomial, except a constant factor, may be determined by its zeros. For transcendental entire functions, this last property is much more articulated respect to the polynomials: indeed, whilst the indeterminacy for polynomials is given by a constant C, for transcendental entire functions it is larger and related to the presence of an exponential factor which is need to be added to warrant the convergence of infinite product development. A great part of history of the approach and resolution of this last problem is the history of entire function factorization.

Following (Vivanti 1928, Sections 135-141), the rise of the first formulation of the entire function factorization theorem by K. Weierstrass in 1876 (see (Weierstrass 1876)), was mainly motivated by the purpose to give a solution to the latter formal problem, concerning the convergence of the infinite product development of a transcendental entire function $f(z)$ having an infinite number of zeros, namely $z = 0$, with multiplicity λ, and $z_1, \dots, z_n, \dots$ such that $0 < |z_j| \le |z_{j+1}|$, $z_j \neq z_{j+1}$ $j = 1, 2, \dots$, trying to extend the case of a finite number of zeros $z_1, \dots, z_n$, in which such a factorization is given by

(5) $$f(z) = e^{g(z)} z^\lambda \prod_{j=1}^{n} \left(1 - \frac{z}{z_j}\right),$$

to the case of infinite zeros, reasoning, by analogy, as follows. The set of infinite zeros z_j is a countable set having only one accumulation point, that at infinite. Therefore, for every infinite increasing natural number sequence $\{\rho_i\}_{i\in\mathbb{N}}$, it will be always possible to arrange the zeros z_j according to their modulus in such a manner to have the following non-decreasing sequence $|z_1| \le |z_2| \le \cdots$ with $\lim_{n\to\infty}|z_n| = \infty$. In such a case, if one wants, by analogy, to extend (5) as follows

(6) $$f(z) = e^{g(z)} z^\lambda \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right),$$

then it will not be possible to fully avoid divergence's problems inherent to the related infinite product. The first hint towards a possible overcoming of these difficulties, was suggested to Weierstrass (see (Weierstrass 1856a)) by looking at the form of the inverse of the *Euler integral of the second kind* – that is to say, the *gamma function* – and given by

[3] It is noteworthy the historical fact pointed out by Giuseppe Bagnera (1927, Chapter III, Section 12, Number 73) according to which already Cauchy himself had considered first forms of infinite product developments, after Euler work. The author himself, then, quotes too Betti.

(7) $$\frac{1}{\Gamma(z)} = z\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)\left(\frac{j}{1+j}\right)^{z} = z\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)e^{-z\log\frac{j+1}{j}},$$

from which he descried the possible utility of the exponential factors there involved to, as say, force the convergence of the infinite product of the second last equality; these his ideas concretized only in 1876 with the explicit formulation of his celebrated theorem on entire function factorization.

Weierstrass (1856a) attributes, however, the infinite product expansion (7) to Gauss, but some next historical studies attribute to Euler this formula, that he gave in the famous 1748 *Introductio in Analysin Infinitorum*. Nevertheless, Ullrich (1989, Section 3.5) says that the real motivation to this Weierstrass' results about entire function factorization were mainly due to attempts to characterize the factorization of quotients of meromorphic functions on the basis of their zero sets, rather than to solve the above problem. Furthermore, Ullrich (1989, Section 3.5) observes too that other mathematicians dealt with questions concerning entire function factorization methods, amongst whom Enrico Betti and Bernard Riemann in whose important 1861 *sommersemester* lectures on analytic functions he argued upon the construction of particular complex functions with simple zeros, even if, all in all, he didn't give, according to Ullrich, nothing more what Euler done about gamma function through 1729 to his celebrated 1748 treatise on infinitesimal analysis[4]. Instead, Laugwitz (2008, Chapter 1, Section 1.1.6) states that Riemann's work on meromorphic functions was ahead of the Weierstrass' one, having been carried out with originality and simplicity. To this point, for our purposes, it would be of a certain importance to deepen the possible relationships between Riemann and Weierstrass: for instance, in this regards, Laugwitz (2008, Chapter 1, Section 1.1.5) says that Riemann was aware of the Weierstrass' works until 1856-57, in connection with the composition of his own paper on Abelian functions. Again according to (Laugwitz 2008, Chapter 1, Section 1.1.6), one of the key themes of Riemann's work on complex function theory is the determination of a function from its singularities which, in turn, implies the approach of another problem, the one concerning the determination of a function from its zeros. In this regards, Riemann limited himself to consider the question to determine a function with infinitely many zeroes whose only point of accumulation is ∞. What he is after is the product representation later named after Weierstrass. Riemann uses a special case to explain the general procedure. He does it in such a way that by following his direction one could immediately give a proof of the Weierstrass product theorem. Therefore, it would be hoped a deeper study of these 1861 Riemann's lectures to historically clarify this last question which is inside the wider historical framework concerning the work of Riemann in complex function theory.

4. On the history of gamma function: a first flash out

To this point, it doesn't seem irrelevant to further highlight, although in a sketchily manner, some of the main aspects of the history of gamma function. To this end, we follow the as many notable work of Reinhold Remmert (1998) which, besides to mainly be an important textbook on some advanced complex analysis topics, it is also a very valuable historical source on the subject, which seems to remember the style of the above mentioned Vivanti's textbook whose German edition, on the other hand, has always been a constant reference point in drawing up the Remmert's textbook itself. Following (Remmert 1998, Chapter 2), the early origins of gamma function should be search into the attempts to extend the function $n!$ to real arguments. Euler was the first one to approach this problem since 1729, giving a first expression of this function, in a letter to Goldbach (see (Whittaker & Watson 1927, Chapter XII, Section 12.1)), providing a first infinite product expression of this new function, but only for real values. Gauss, who did not know Euler work, taking also into account Newton's work on interpolation (see (Schering 1881, Sections XI and

[4] Following (Lunts 1950), (Markuševič 1988, Capitolo VII) and references therein, Lobačevskij too, since 1830s, made some notable studies on gamma function which pre-empted times.

XII)), in 1811 considered as well complex values, denoting such a new function with Π, while it was Legendre, in 1814, to introduce a unified notation both for Euler and Gauss functions, denoting these last with $\Gamma(z)$ and speaking, since then, of the gamma function. Afterwards, in 1854, Weierstrass began to consider an Euler infinite product expansion of the function $1/\Gamma(z)$, that he denoted with $Fc(z)$ and is given by $1/\Gamma(z) = ze^{\gamma z}\prod_{j=1}^{\infty}\left(1+\frac{z}{j}\right)e^{-\frac{z}{j}}$, where γ is the Euler-Mascheroni constant[5], from which he recognized, for the first time, the importance of the use of exponential factors as infinite product convergence-producing elements. However, according to (Whittaker & Watson 1927, Chapter XII, Section 12.1), the formula (7) had already been obtained by F.W. Newman (1848) starting from Euler's expression of gamma function given by (7).

5. On the history of entire function factorization theorem, II

Following (Vivanti 1928, Section 135-141), (Remmert 1998, Chapter 3) and, above all, (Bottazzini & Gray 2013, Section 6.7), Weierstrass extended the product (5) in such a manner to try to avoid divergence problems with the *ad hoc* introduction, into the product expansion, of certain forcing convergence factors. This attempt was successfully attended, since 1874, as a solution to a particular question – the one which may be roughly summarized as the attempt to build up an entire transcendental function with prescribed zeros – which arose within the general Weierstrass' intent to solve the wider problem to find a representation for a single-valued function as a quotient of two convergent power series. To be precise, he reached, amongst other things, to the following main result:

> *«Given a countable set of non-zero complex points $z_1, z_2, \dots$, such that $0 < |z_1| \leq |z_2| \leq \cdots$ with $lim_{n\to\infty}|z_n| = \infty$, then it is possible to find, in infinite manners, a non-decreasing sequence of natural numbers $p_1, p_2, \dots$ such that the series $\sum_{j=1}^{\infty}|z/z_j|^{p_j+1}$ be convergent for every finite value of z, in such a manner that the most general entire function which is zero, with their own multiplicity, in the points $z_1, z_2, \dots$, and has a zero of order λ in the origin, is given by*[6]

(8) $$f(z) = e^{g(z)}z^{\lambda}\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)E_j(z),$$

> *where $E_j(z) = (1-z)\left(\sum_{i=1}^{j} z^i/i\right)$ for $j \geq 1$ and $E_0(z) = 1-z$, $g(z)$ being an arbitrary entire function, and the infinite product is absolutely convergent for each finite value of $|z|$».*

The factors $E_j(z)$ will be later called *Weierstrass' factors*, whilst the numbers p_j will be called *convergence exponents*. The sequence $\{E_j(z)\}_{j\in\mathbb{N}_0}$ plays a very fundamental role in the Weierstrass' theorem: from the equation

(9) $$1-z = \exp(\log(1-z)) = \exp\left(-\sum_{i\geq 1}\frac{z^i}{i}\right),$$

Weierstrass obtained the formula $E_j(z) = \exp\left(-\sum_{i>j} z^j/j\right)$ in proving convergence properties which, on the other hand, would have been easier obtained by means of the following estimates

[5] See (Pepe 2012) for a contextual brief history of the Euler-Mascheroni constant.

[6] Historically, in relation to (8), the function $f(z)$ was usually denoted, d'après Weierstrass, by $G(z)$, whilst $z^{\lambda}\prod_{j=1}^{\infty}\left(1-\frac{z}{z_j}\right)E_j(z)$ was named *canonical (or primitive) function* – see (Sansone 1972, Chapter IV, Section 3), where there are too many interesting historical notes.

$$(10)\qquad \left|E_j(z)-1\right| \le |z|^{j+1},\ \forall j \in \mathbb{N}_0,\ \forall z \in \mathbb{C},\ |z| \le 1$$

that have been proved only later. Amongst the first ones to have made this, seems there being L. Fejér (see (Hille 1959, Section 8.7)), but the argument appears as early as 1903 in a paper of Luciano Orlando[7] (1903) which starts from Weierstrass' theorem as treated by Borel's monograph on entire functions. As has already been said above, Weierstrass was led to develop his theory by the chief objective to establish the general expression for all analytic functions meromorphic in $\mathbb{C}$ excepts at finitely many points, reaching the scope only in 1876 (see (Weierstrass 1876a)), after a series of previous futile attempts, with notable results concerning the class of transcendental entire functions. But what was new and sensational in Weierstrass' construction was the introduction and application of the so-called *convergence-producing factors* (or *primary factors* or *Weierstrass' factors*) which have no influence on the behavior and distribution of the zeros. In the Weierstrass' necrology, Poincaré (1899, 6.) said that Weierstrass' major contribution to the development of function theory has just been the discovery of primary factors. Also Hermite was, in a certain sense, astonished and intrigued from the introduction of this new Weierstrass' notion of prime factor, which he considered of capital importance in analysis and making later notable studies in this direction; he also suggested to Èmile Picard a French translation of the original 1876 Weierstrass' work, so opening a French research trend on this area. *En passant*, we also point out on the fact that, from the notion of prime factor and from the convergence of the infinite product $\prod_{j\in\mathbb{N}} E_j({}^{z}/{}_{a_j})$, representing an entire transcendental function vanishing, in a prescribed way, in each a_j, Hilbert drawn inspiration to formulate his valuable algebraic notion of prime ideal[8].

6. On the history of entire function factorization theorem, III

Following (Pincherle 1922, Chapter IX, Section 137), (Vivanti 1928, Section 136), (Burckel 1979, Chapter XI), (Remmert 1998, Chapters 3 and 6), (Ullrich 1989, Section 3.5) and (Bottazzini & Gray 2013, Sections 5.11.5 and 6.7), since the late 1850s, Enrico Betti had reached to notable results, about convergence properties of infinite products of the type (6), very near to the Weierstrass' ones related to the resolution of a fundamental problem of entire function theory, the so-called *Weierstrass' problem*[9] (see (Pincherle 1922, Chapter IX, Section 137)). Betti exposed these outcomes in his celebrated advanced analysis 1859-60 Pisa lectures *La teorica delle funzioni ellittiche* (see (Betti 1903-1913, Tomo I, XXII)), published into the Tomes III and IV of the *Annali di matematica pura ed applicata, Serie I*, after having published, into the Tomo II of these last, an Italian translation of the celebrated 1851 Riemann's inaugural dissertation on complex function theory, which can be considered as an introduction to his lectures on elliptic functions. Indeed, in the latter, Betti before places an *Introduzione* on the general principles on complex functions, essentially based on these 1859 Riemann lectures. From the point 3. onwards of this introduction, Betti argues on entire functions, their finite and infinite zeroes (there called roots), as well as on

[7] Luciano Orlando (1887-1915) was an Italian mathematician prematurely died in the First World War – see the very brief obituary (Marcolongo 1918). His supervisors were G. Bagnera and R. Marcolongo who led him to make algebraic, integral equation theory and mathematical physics researches.

[8] Usually, the notion of prime ideal of commutative algebra, with related operations, would want to be stemmed from the factorization of natural numbers.

[9] According to (Forsyth 1918, Chapter V, Section 50) (see also (Bottazzini & Gray 2013, Section 4.2.3.2)), in relation to the infinite product expression of an entire transcendental function, prior to 1876 Weierstrass' paper, attention should be also paid to a previous 1845 work of A. Cayley on doubly periodic elliptic functions. Furthermore, according to (Tannery & Molk 1893, Section 85), already in some previous 1847 works of G. Eisenstein on elliptic functions, some notable problems having to do with the construction of analytic functions with prescribed zeros as a quotient of entire functions with the involvement of certain transcendental entire functions of exponential type (similar to the *Weierstrass problem* as historically related to meromorphic functions), had been considered. See also certain function's quotients stemmed from the developments of certain determinants given in (Gordan 1874).

possible quotients between them. In particular, taking into account (Briot & Bouquet 1859), he reached to the consideration of infinite products of the type $\prod_{\rho}\left(1-\frac{s}{\rho}\right)$, where ρ are the zeros of an entire function, hence to the introduction of a factor of the type e^{w}, where w is an arbitrary entire function, to make convergent this infinite product. Furthermore, Betti dealt with these types of infinite products beginning to consider infinite product representations of the following particular function $es(z) = z\prod_{m=1}^{\infty}\left(\frac{m}{m+1}\right)^{z}\left(1+\frac{z}{m}\right)$, which satisfies some functional equations and verifies the relation $\Gamma(z) = 1/es(z)$. Therefore, as Weierstrass too will do later, Betti started from the consideration of the infinite product expansion of the inverse of gamma function, in studying the factorization of entire functions. Afterwards, Betti proved some theorems which can be considered as particular cases of Weierstrass' result, concluding affirming that

> *«Da questi teoremi si deduce che le funzioni intere potranno decomporsi in un numero infinito di fattori di primo grado ed esponenziali, e qui comparisce una prima divisione delle funzioni intere. Quelle che hanno gl'indici delle radici in linea retta, e quelle che le hanno disposte comunque nel piano; le prime, che sono espresse da un prodotto semplicemente infinito, le chiameremo di prima classe, le seconde, che sono espresse da un prodotto doppiamente infinito, le diremo di seconda classe. Le funzioni di prima classe si dividono anch'esse in due specie, la prima, che comprende quelle che hanno gl'indici delle radici disposti simmetricamente rispetto a un punto, e che possono esprimersi per un prodotto infinito di fattori di primo grado, le altre, che hanno gl'indici delle radici disposti comunque sopra la retta, le quali si decomporranno in fattori di primo grado ed esponenziali. Ogni funzione intera di prima classe della prima specie potrà decomporsi nel prodotto di più funzioni intere della stessa classe di seconda specie, e data una funzione della seconda specie se ne potrà sempre trovare un'altra che moltiplicata per la medesima dia per prodotto una funzione della prima specie. Le funzioni di seconda classe si dividono anch'esse in due specie; la prima comprenderà quelle che hanno gl'indici delle radici disposti egualmente nei quattro angoli di due assi ortogonali, in modo che facendo una rotazione intorno all'origine di un quarto di circolo, gl'indici di tutte le radici vengano a sovrapporsi, le quali funzioni possono esprimersi per un prodotto doppiamente infinito di fattori di primo grado; la seconda comprenderà quelle che hanno gl'indici disposti comunque, e si decompongono in un prodotto doppiamente infinito di fattori di primo grado e di fattori esponenziali. Data una funzione della seconda specie se ne potrà sempre trovare un'altra che moltiplicata per quella dia una funzione della prima specie».*

Then, Betti goes on with entire function treatment in the *Parte Prima* of his lessons on elliptic functions, followed by the *Parte Seconda* devoted to quotients of functions, mentioning twice Weierstrass in relation either to (Weierstrass 1856a) and to (Weierstrass 1856b). Therefore, Betti work on entire function factorization was very forerunner of the Weierstrass' one *a fortiori* for what will now be said. Indeed, following Francesco Cecioni comments to some works of Ulisse Dini (1953-59, Volume II), it turns out that Betti's work could easily reach, only with very slight modifications, the same generality and abstraction of Weierstrass' one, as Dini explicitly proved in (Dini 1881); furthermore, Dini proved too that Betti's work could be able to give a particular case, given in the years 1876-77, of the general Gösta Mittag-Leffler theorem – see (Mittag-Leffler 1884), (Vivanti 1928, Section 145) and (Bottazzini & Gray 2013, Section 6.7.6) – independently of what Weierstrass himself was doing in the same period, in regards to these arguments. Cecioni says that this Dini's work had already been worked out since 1880, whilst the Weierstrass' theorem was published in 1876 – see (Weierstrass 1876). Much before, namely in 1860, Betti had proved, as we have already said, a particular but important case of this theorem, albeit he didn't go beyond, because the results achieved by him were enough to his pragmatic scopes concerning Abelian and elliptic functions[10]; nevertheless, as again Pincherle (1922, Chapter IX, Section 135) claimed later, the Weierstrass' method will be essentially the same of the Betti's one with slight modifications. In the years 1876-77, also G. Mittag-Leffler proved a particular case of a theorem that himself will give later, to be precise in 1884, after a long series of previous works in which he gradually, through particular cases, reached to the general form of his theorem as we know it nowadays. In the meanwhile, Weierstrass reconsidered Mittag-Leffler's works in the early 1880s, in relation to his previous ones, whilst Casorati (1880-82) had too some interesting ideas similar to the Mittag-Leffler ones. Almost at the same time, amongst others, Ernst Schering (1881), Charles Hermite (1881), Émile Picard (1881), Felice Casorati (1882), Ulisse Dini (1881), Paolo Cazzaniga[11] (1882) and Claude Guichard (1884) achieved notable results about the general problem to build up a complex function with prescribed singularities, although related to a generality degree less than the Mittag-Leffler results. Thus, the history of Mittag-Leffler theorem makes too its awesome appearance within the general history of meromorphic functions, a part of which may be retraced in the same Mittag-Leffler 1884 paper in which, amongst others, also this work of Ulisse Dini is quoted. However, both Schering (1881, Section XVI) and Casorati (1880-82, p. 269, footnote [(***)]), in discussing the above mentioned Mittag-Leffler results, quote Betti's work about Weierstrass' theorem; in particular, the former speaks of *Betti's convergence factors* and the latter states that

> *«Anche il sig. Dini, nella sua Nota sopra citata, dimostra questo teorema, riducendo lo studio del prodotto infinito a quello della serie dei logaritmi dei fattori; riduzione di cui s'era già valso felicemente, per il caso di distribuzione degli zeri a distanze non mai minori di una quantità fissa d, il sig. Betti nella Introduzione della sua Monografia delle funzioni ellittiche (Annali di Matematica, Tomo III, Roma, 1860), dove precede assai più oltre di Gauss nella via che mena al teorema del sig. Weierstrass».*

Therefore, from Mittag-Leffler works onward, together to all those works made by other mathematicians amongst whom Dini, Schering, Casorati, Hermite, Picard, Cazzaniga, Guichard and Weierstrass himself, starts the theory of entire transcendental functions whose early history surely would need larger space besides this. In any case, with Mittag-Leffler, we have the most general theorems for the construction, by infinite products, of an entire function with prescribed singularities. To be precise, following (Gonchar et al. 1997, Part I, Introduction) and (Vivanti 1901, Section 215), the works by Weierstrass, Mittag-Leffler and Picard, dating back to 1870s, marked the beginning of systematic studies of the theory of entire and meromorphic functions. The Weierstrass and Mittag-Leffler theorems gave a general description of the structure of entire and meromorphic functions, while the representation of entire functions as an infinite product à la Weierstrass, served as the basis for studying properties of entire and meromorphic functions. The Picard theorem initiated the theory of value distribution of meromorphic functions, while J.L.W.V. Jensen works, at the late 1890s, were of a great importance for the further developments of the theory of entire and meromorphic functions which started, in the same period, to gradually become

[10] In this regards, Salvatore Pincherle (1899, Chapter IX, Section 175) said that Betti solved the Weierstrass' problem in a quite general case.

[11] Some sources refer of Paolo Cazzaniga, whereas others refer of Paolo Gazzaniga. Very likely, they are the same person, that is to say, Paolo Gazzanica (1853-1930), an Italian mathematician graduated from Pavia University in 1878 under the supervision of F. Casorati. In the years 1878-1883, he was interim assistant professor at Pavia, then he spent a period of study in Germany under Weierstrass and Kronecker. Afterwards, from 1888, he became professor at the high school Tito Livio in Padua, teaching as well in the local University. He was too one of the most influential teachers of Tullio Levi-Civita during his high school studies. His researches, mainly concerned with applied algebra and number theory. Paolo Gazzaniga has to be distinguished from Tito Camillo Cazzaniga (1872-1900), a prematurely died Italian mathematician, graduated from Pavia University in 1896, whose researches concerned matrix theory and analytic functions according to the trend of Ernesto Pascal during his teaching in Pavia. Both Tito Cazzaniga and Paolo Gazzaniga are quoted in (Vivanti 1901), but not in (Vivanti 1928).

a separate scientific discipline after the pioneering works of Laguerre, Hadamard, Poincaré, Lindelöff and Borel, until the Rolf Nevanlinna work of early 1900s.

7. On the history of entire function factorization theorem, IV

As has already been recalled in (Iurato2013), the Weierstrass' entire function factorization theorem has had further remarkable applications in many other pure and applied mathematical contexts. In this place, we wish to point out on another possible interesting historical connection there not mentioned. To be precise, following (Markuševič 1967, Volume II, Chapters 8 and 9), (Burckel 1979, Chapter VII) and (Remmert 1998, Chapter 4), a very similar problem to that considered by Weierstrass is the one considered in (Markuševič 1967, Volume II, Chapter 8, Theorem 8.5) where, roughly, a bounded analytic function with prescribed zeros is constructed by means of certain infinite products introduced by Wilhelm Blaschke (1915) in relation to questions related to Giuseppe Vitali convergence theorem for sequences of holomorphic functions, and defined upon those complex numbers assigned as given zeros of the function that has to be determined. We shall return later on such aspects concerning *Blaschke products*, which form a special class of Weierstrass' products, when we will go on in deepening the history of entire function theory; due to this, to our historical purposes, we would want to try to analyze whether the previous Weierstrass' work on entire function factorization theorems have played a certain role in Blaschke's work.

Note. Recently, a truly masterful, updated and complete history of complex function theory has been published, namely the 2013 Bottazzini and Gray book, written by two of the most leading historians of mathematics. We have constantly followed this treatise, which may be considered as the most systematic and organic historical treatment of this argument published so far. We will refer to (Bottazzini & Gray 2013) for any further in-depth study on the subject.

Acknowledgements. My *grātiās agō* to Professor Umberto Bottazzini of the University of Milan as well as to Professor Mauro Nacinovich of the University of Rome ''Tor Vergata'', for having promptly and kindly supplied me useful bibliographical comments and hints about my research project.

References

Ahlfors, L.V. (1979), *Complex Analysis. An Introduction to the Theory of Analytic Function of One Complex Variable,* Third Edition, New York: McGraw-Hill, Inc.

Bagnera, G. (1927), *Lezioni sopra la Teoria delle Funzioni Analitiche*, lezioni litografate raccolte dal Dott. Giovanni Ricci, Roma: Stabilimento Tipo-litografico Attilio Sampaolesi.

Betti, E. (1903-1913), *Opere Matematiche*, pubblicate per cura della R. Accademia de' Lincei, in due tomi, Milano-Roma: Ulrico Hoepli Editore-Librajo della Real Casa e della R. Accademia dei Lincei.

Blaschke, W. (1915), Eine Erweiterung des Satzes von Vitali über Folgen analytischer Funktionen, *Berichte über die Verhandlungen der Königlich-Sächsischen Gesellschaft der Wissenschaften zu Leipzig, Mathematisch-Physikalische Klasse*, 67: 194-200.

Bottazzini, U. (2003), Complex Function Theory, 1780-1900, in: Jahnke, H.N. (Ed.) (2003), *A History of Analysis*, Providence (RI): American Mathematical Society and London Mathematical Society, pp. 213-260.

Bottazzini, U. and Gray, J. (2013), *Hidden Harmony – Geometric Fantasies. The Rise of Complex Function Theory*, New York (NY): Springer Science + Business Media, Inc.

Briot, M. and Bouquet, M. (1859), *Théorie des functions doublement périodiques et, en particulier, des fonctions elliptiques*, Paris: Mallet-Bachelier, Imprimeur-Libraire.

Burckel, R.B. (1979), *An Introduction to Classical Complex Analysis*, Volume I, New York: Academic Press, Inc.

Casorati, F. (1880-82), Aggiunte a recenti lavori dei sig. Weierstrass e Mittag-Leffler sulle funzioni di una variabile complessa, *Annali di Matematica Pura ed Applicata*, 10 (1): 261-278.

Cazzaniga, P. (1880-82), Espressione di funzioni intere che in posti dati arbitrariamente prendono valori prestabiliti, *Annali di Matematica Pura ed Applicata*, 10 (1): 279-290.

Dedekind, R. (1876), Bernhard Riemann's Lebenslauf, in 1876 Riemann's *Werke*.

Dini, U. (1881), Alcuni teoremi sulle funzioni di una variabile complessa, in: *In Memoriam Dominici Chelini. Collectanea Mathematica nunc primum edita cura et studio L. Cremona et E. Beltrami*, Milano: Ulrico Hoepli Editore-Librajo della Real Casa, pp. 258-276.

Dini, U. (1953-59), *Opere*, a cura dell'Unione Matematica Italiana e con il contributo del Consiglio Nazionale delle Ricerche, 5 Voll., Roma: Edizioni Cremonese (già Casa Editrice Perrella).

Edwards, H.M. (1974), *Riemann's Zeta Function*, New York: Academic Press, Inc.

Forsyth, A.R. (1918), *Theory of Functions of a Complex Variable*, Third Edition, Cambridge (UK): Cambridge University Press.

Gonchar, A.A., Havin, V.P. and Nikolski, N.K. (Eds.) (1997), *Complex Analysis I: Entire and Meromorphic Functions. Polyanalytic Functions and Their Generalizations*, Encyclopaedia of Mathematical Sciences, Volume 85, Berlin and Heidelberg: Springer-Verlag.

Gordan, P. (1874), Über den grössten gemeinsame Factor, *Mathematische Annalen*, 7 (1): 433-448.

Guichard, C. (1884), Sur les fonctions entières, *Annales scientifiques de l'École Normale Supérieure, 3^e^ Série*, 1: 427-432.

Hille, E. (1959), *Analytic Function Theory*, Volume I, New York: Chelsea Publishing Company.

Hermite, C. (1881), Sur quelques points de la théorie des fonctions, *Bulletin des sciences mathématiques et astronomiques*, *2^e^ Série*, 5 (1): 312-320.

Iurato, G. (2013), *On some historical aspects of Riemann Zeta Function, 1*, preprint No. hal-00828587 version 1, available at HAL archives-ouvertes.

Janovitz, A. and Mercanti, F. (2008), *Sull'apporto evolutivo dei matematici ebrei mantovani nella nascente nazione italiana*, Teramo: Monografie di Eiris.

Laugwitz, D. (2008), *Bernhard Riemann 1826-1866. Turning Points in the Conception of Mathematics*, Boston: Birkhäuser.

Lunts, G.L. (1950), The analytic work of N.I. Lobachevskii, *Uspekhi Matematicheskikh Nauk*, 5 1(35): 187-195.

Marcolongo, R. (1918-1919), Necrologio di Luciano Orlando, *Rendiconti del Seminario Matematico della Facoltà di Scienze della R. Università di Roma*, 5, Numero Speciale: 11-15.

Markuševič, A.I. (1965-67), *Theory of functions of a complex variable*, I, II and III volume, Englewood Cliffs (NJ): Prentice-Hall, Inc.

Markuševič, A.I. (1988), *Teoria delle funzioni analitiche*, Mosca-Roma: Edizioni Mir-Editori Riuniti.

Maz'ya, V. and Shaposhnikova, T. (1998), *Jacques Hadamard. A Universal Mathematician*, Providence (RI): American Mathematical Society and London Mathematical Society.

Mittag-Leffler, G. (1884), Sur la représentation analytique des fonctions monogènes uniformes d'une variable indépendante, *Acta Mathematica*, 4 (1): 1-79.

Newman, F.W. (1848), On Γa especially when a is negative, *The Cambridge and Dublin Mathematical Journal*, 3: 57-60.

Orlando, L. (1903), Sullo sviluppo della funzione $(1-z)e^{\left(z+\frac{z^2}{2}+\cdots+\frac{z^{p-1}}{p-1}\right)}$, *Giornale di Matematiche di Battaglini per il progresso degli studi nelle università italiane*, Volume XLI, 40° della 2ª Serie, pp. 377-378.

Pepe, L. (2012), Mascheroni and Gamma, in: Burenkov, V.I., Goldman, M.L., Laneev, E.B. and Stepanov V.D. (Eds.), *Peoples' Friendship University of Russia. Progress in Analysis. Proceedings* *of the 8th Congress of the International Society for Analysis, its applications and computation, 22-27 August 2011*, 2 Vols., Moscow: Peoples' Friendship University of Russia Publishers, pp. 23-36.

Picard, É. (1881), Sur la décomposition en facteurs primaires des fonctions uniformes ayant une ligne de points singuliers essentiels, *Comptes Rendus de l'Académie des Sciences de Paris*, 92: 690-692.

Pierpont, J. (1914), *Function of a complex variable*, Boston (MA): Ginn and Company Publisher.

Pincherle, S. (1899-1900), *Lezioni sulla teoria delle funzioni analitiche tenute nella R. Università di Bologna*, Bologna: Litografia Sauer e Barigazzi.

Picherle, S. (1922), *Gli elementi della teoria delle funzioni analitiche*, Parte prima, Bologna: Nicola Zanichelli editore.

Poincaré, H.J. (1899), L'oeuvre mathématique de Weierstrass, *Acta Mathematica*, 22 (1): 1-18.

Remmert, R. (1998), *Classical Topics in Complex Function Theory*, New York: Springer-Verlag New York, Inc.

Sansone, G. (1972), *Lezioni sulla teoria delle funzioni di una variabile complessa*, Volume I, quarta edizione, Padova: CEDAM – Casa Editrice Dott. Antonio Milani.

Schering, E. (1881), Das Anschliessen einer Function an algebraische Functionen in unendlich vielen Stellen, *Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen*, Band 27: 3-64.

Tannery, J. and Molk, J. (1893), *Éléments de la théorie des fonctions elliptiques*, Tome I, Paris: Gauthier-Villars et Fils, Imprimeurs-Libraires de l'École Polytechnique, du Bureau des Longitudes.

Torelli, G. (1901), *Sulla totalità dei numeri primi fino ad un limite assegnato*, Memoria estratta dagli Atti della R. Accademia delle Scienze Fisiche e Matematiche di Napoli, Volume XI, Serie 2a, N. 1, Napoli: Tipografia della Reale Accademia delle Scienze Fisiche e Matematiche, diretta da E. De Rubertis fu Michele.

Ullrich, P. (1989), Weierstraß Vorlesung zur ''Einleitung in die Theorie der analytischen Funktionen'', *Archive for History of Exact Sciences*, 40 (2): 143-172.

Vivanti, G. (1901), *Teoria delle funzioni analitiche*, Milano: Ulrico Hoepli Editore-Librajo della Real Casa (with a 1928 revised and enlarged second edition entitled *Elementi della teoria delle funzioni analitiche e delle funzioni trascendenti intere*, published by the same publishing house).

Weierstrass, K. (1856a), Über die Theorie der analytischen Facultäten, *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 51: 1-60.

Weierstrass, K. (1856b), Theorie der Abel'schen Functionen, *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, 52: 285-380.

Weierstrass, K. (1876), Zur Theorie der eindeutigen analytischen Funktionen, *Mathematische Abhandlungen der Akademie der Wissenschaften zu Berlin*, S. 11-60.

Whittaker, E.T. and Watson, G.N. (1927), *A Course in Modern Analysis. An Introduction to the General Theory of Infinite Processes and of Analytic Functions; with an Account of the Principal Transcendental Functions,* Fourth Edition, Cambridge (UK): Cambridge University Press.