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Viktoria Heu, Frank Loray

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| Viktoria Heu, Frank Loray. Flat rank 2 vector bundles on genus 2 curves. 2014. hal-00927061v1

**HAL Id: hal-00927061**

**<https://hal.science/hal-00927061v1>**

Preprint submitted on 10 Jan 2014 (v1), last revised 15 Jul 2015 (v2)

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# FLAT RANK 2 VECTOR BUNDLES ON GENUS 2 CURVES

VIKTORIA HEU AND FRANK LORAY

ABSTRACT. We study the moduli space of trace-free irreducible rank 2 connections over a curve of genus 2 and the forgetful map towards the (non-separated) moduli space of underlying vector bundles (including unstable bundles). We draw the geometric picture of the latter and compute a natural Lagrangian rational section of the forgetful map. As a particularity of the genus 2 case, such connections are invariant under the hyperelliptic involution : they descend as rank 2 logarithmic connections over the Riemann sphere. We establish explicit links between the well-known moduli space of the underlying parabolic bundles with the classical approaches by Narasimhan-Ramanan, Tyurin and Bertram. By the hyperelliptic approach, we recover a Poincaré family on a degree 2 cover of the Narasimhan-Ramanan moduli space, due to Bolognesi. Moreover, we compare the explicit equations of the Kummer surface and the Hitchin map for each point of view, allowing us to explain a certain number of geometric phenomena in the considered moduli spaces.

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1991 *Mathematics Subject Classification*. Primary 14H60; Secondary 34Mxx, 32G34, 14Q10.

*Key words and phrases*. Vector Bundles, Moduli Spaces, Parabolic Connections, Higgs Bundles, Kummer Surface.

The first author is partially supported by ANR-13-JS01-0002-0.

The second author is supported by CNRS.

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## INTRODUCTION

Let  $X$  be a smooth projective curve of genus 2 over  $\mathbb{C}$ . A rank 2 holomorphic connection on  $X$  is the data  $(E, \nabla)$  of a rank 2 vector bundle  $E \rightarrow X$  together with a  $\mathbb{C}$ -linear map  $\nabla : E \rightarrow E \otimes \Omega_X^1$  satisfying the Leibniz rule. The trace  $\text{tr}(\nabla)$  defines a holomorphic connection on  $\det(E)$ ; we say that  $(E, \nabla)$  is trace-free (or a  $\mathfrak{sl}_2$ -connection) when  $(\det(E), \text{tr}(\nabla))$  is the trivial connection  $(\mathcal{O}_X, dz)$ . From the analytic point of view,  $(E, \nabla)$  is determined (up to bundle isomorphism) by its monodromy representation, *i.e.* an element of  $\text{Hom}(\pi_1(X), \text{SL}_2)/\text{PGL}_2$  (up to conjugacy). The goal of this paper is a deeper understanding of the moduli space  $\mathfrak{Con}(X)$  of those connections and in particular the forgetful map  $(E, \nabla) \mapsto E$  towards the moduli space  $\mathfrak{Bun}(X)$  of flat vector bundles. Over an open set of the base, the map  $\text{bun} : \mathfrak{Con}(X) \rightarrow \mathfrak{Bun}(X)$  is known to be an affine  $\mathbb{A}^3$ -bundle. The former moduli space may be constructed by Geometric Invariant Theory (see [37, 25, 26]) and we get a quasi-projective variety  $\text{Con}^{ss}(X)$  whose stable locus  $\text{Con}^s(X)$  is open, smooth and parametrizes equivalence classes of irreducible connections. There, several equivalence classes of reducible connections may be identified to the same point in the strictly semi-stable locus. Since our initial motivation is to understand the phase portrait of the isomonodromic flow for connections, we can take reducible connections apart and deal with by another method.

The moduli space of bundles, even if we restrict to those bundles admitting an irreducible connection, is however non Hausdorff as a topological space, due to the fact that some unstable bundles arise in this way. We can start with the classical moduli space  $\text{Bun}^{ss}(X)$  of semi-stable bundles constructed by Narasimhan-Ramanan (see [35]), but we have to investigate how to complete this picture with missing flat unstable bundles.

**Hyperelliptic descent.** The main tool of our study, elaborated in section 2, directly follows from the hyperellipticity property of such objects. Denote by  $\iota : X \rightarrow X$  the hyperelliptic involution, by  $\pi : X \rightarrow \mathbb{P}^1$  the quotient map and by  $\underline{W}$  the critical divisor on  $\mathbb{P}^1$  (projection of the 6 Weierstrass points). We can think of  $\mathbb{P}^1 = X/\iota$  as an orbifold quotient (see [39]) and any representation  $\rho \in \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2)$  of the orbifold fundamental group, *i.e.* with 2-torsion around points of  $\underline{W}$ , can be lifted on  $X$  to define an element  $\pi^*\rho$  in  $\text{Hom}(\pi_1(X), \text{SL}_2)$ . As a particularity of the genus 2 case, both moduli spaces of representations have the same dimension 6 and one can check

that the map  $\text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) \rightarrow \text{Hom}(\pi_1(X), \text{SL}_2)$  is dominant: any irreducible  $\text{SL}_2$ -representation of the fundamental group of  $X$  is in the image, invariant under the hyperelliptic involution  $\iota$  and can be pushed down to  $X/\iota$ .

From the point of view of connections, this means that every irreducible connection  $(E, \nabla)$  on  $X$  is invariant by the hyperelliptic involution  $\iota : X \rightarrow X$ . By pushing forward  $(E, \nabla)$  to the quotient  $X/\iota \simeq \mathbb{P}^1$ , we get a rank 4 logarithmic connection that splits into the direct sum  $\pi_*(E, \nabla) = (\underline{E}_1, \underline{\nabla}_1) \oplus (\underline{E}_2, \underline{\nabla}_2)$  of two rank 2 connections. Precisely, each  $\underline{E}_i$  has degree  $-3$  and  $\underline{\nabla}_i : \underline{E}_i \rightarrow \underline{E}_i \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$  is logarithmic with residual eigenvalues  $0$  and  $\frac{1}{2}$  at each pole. Conversely,  $\pi^*(\underline{E}_i, \underline{\nabla}_i)$  is a logarithmic connection on  $X$  with only apparent singular points: residual eigenvalues are now  $0$  and  $1$  at each pole, *i.e.* at each Weierstrass point of the curve. After performing a birational bundle modification (an elementary transformation over each of the 6 Weierstrass points) one can make it into a holomorphic and trace-free connection on  $X$ : we recover the initial connection  $(E, \nabla)$ . After deleting some reducible locus, we deduce a  $(2 : 1)$  map

$$\Phi : \mathfrak{Con}(X/\iota) \longrightarrow \mathfrak{Con}(X)$$

where  $\mathfrak{Con}(X/\iota)$  denotes the moduli space of logarithmic connections like above. Moduli spaces of logarithmic connections on  $\mathbb{P}^1$  have been widely studied by many authors.

One can associate to a connection  $(\underline{E}, \underline{\nabla}) \in \mathfrak{Con}(X/\iota)$  a parabolique structure  $\underline{p}$  on  $\underline{E}$  consisting of the data of the residual eigenspace  $p_j \subset \underline{E}|_{w_j}$  associated to the  $\frac{1}{2}$ -eigenvalue for each pole  $w_j$  in the support of  $\underline{W}$ . Denote by  $\mathfrak{Bun}(X/\iota)$  the moduli space of such parabolic bundles  $(\underline{E}, \underline{p})$ , *i.e.* defined by an irreducible connection  $(\underline{E}, \underline{\nabla}) \in \mathfrak{Con}(X/\iota)$ . In fact, the descending procedure described above can be already constructed at the level of bundles (see [7]) and we can construct a  $(2 : 1)$  map  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$  making the following diagram commutative:

$$\begin{array}{ccc} \mathfrak{Con}(X/\iota) & \xrightarrow[\Phi]{2:1} & \mathfrak{Con}(X) \\ \text{bun} \downarrow & & \text{bun} \downarrow \\ \mathfrak{Bun}(X/\iota) & \xrightarrow[\phi]{2:1} & \mathfrak{Bun}(X) \end{array}$$

Vertical arrows are locally trivial affine  $\mathbb{A}^3$ -bundles in restriction to a large open set of the bases.

**Narasimhan-Ramanan moduli space.** Having this picture at hand, we study in section 3 the structure of  $\mathfrak{Bun}(X)$ , partly surveying Narasimhan-Ramanan's classical work [35]. They construct a quotient map

$$NR : \text{Bun}^{ss}(X) \rightarrow \mathbb{P}_{NR}^3 := |2\Theta|$$

defined on the open set  $\text{Bun}^{ss}(X) \subset \mathfrak{Bun}(X)$  of semi-stable bundles onto the 3-dimensional linear system generated by twice the  $\Theta$ -divisor on  $\text{Pic}^1(X)$ . This map is one-to-one in restriction to the open set  $\text{Bun}^s(X)$  of stable bundles; it however identifies some strictly semi-stable bundles, as usually does GIT theory to get a Hausdorff quotient. Precisely, the Kummer surface  $\text{Kum}(X) = \text{Jac}(X)/_{\pm 1}$  naturally parametrizes the set of decomposable semi-stable bundles, and the classifying map  $NR$  provides an embedding  $\text{Kum}(X) \hookrightarrow \mathbb{P}_{NR}^3$  as a quartic surface with 16 nodes. The open set of stable bundles is therefore parametrized by the complement  $\mathbb{P}_{NR}^3 \setminus \text{Kum}(X)$ . Over a smooth point of  $\text{Kum}(X)$ , the fiber of  $NR$  consists in 3 isomorphism classes of semi-stable bundles, namely a decomposable one  $L_0 \oplus L_0^{-1}$  and the two non trivial extensions between

$L_0$  and  $L_0^{-1}$ ; the latter ones, however, only carry reducible connections, so that we might delete them from our definition of  $\mathfrak{Bun}(X)$  while focusing on irreducible connections. Over each singular point of  $\text{Kum}(X)$ , the fiber of  $NR$  consists in a decomposable bundle  $E_\tau$  (a twist of the trivial bundle by a 2-torsion point  $\tau$  of  $\text{Jac}(X)$ ) and the (rational) one-parameter family of non trivial extensions of  $\tau$  by itself. These latter ones we call (twists of) unipotent bundles; each of them is infinitesimally close to  $E_\tau$  in  $\mathfrak{Bun}(X)$ . To complete this classical picture, we have to add flat unstable bundles: by Weil's criterion, they are exactly those unique non-trivial extensions  $\kappa \rightarrow E_\kappa \rightarrow \kappa^{-1}$  where  $\kappa \in \text{Pic}^1(X)$  runs over the 16 theta-characteristics  $\kappa^2 = K_X$ . We call them Gunning bundles in reference to [22]: those connections defining a projective  $\text{PGL}_2$ -structure on  $X$  (an oper in the sense of [5]) are defined on these very special bundles  $E_\kappa$ , including the uniformization equation for  $X$ . These bundles occur as non Hausdorff points of  $\mathfrak{Bun}(X)$ : the bundles infinitesimally close to  $E_\kappa$  are precisely those extensions  $\kappa^{-1} \rightarrow E \rightarrow \kappa$ , which are sent onto a plane  $\Pi_\kappa \subset \mathbb{P}_{NR}^3$  by the classifying map. We call them Gunning planes: they are precisely the 16 planes involved in the classical (16, 6)-configuration of Kummer surfaces (see [24, 20]). As far as we know, these planes have had no modular interpretation so far.

For each of these special bundles, we describe the set of connections, and the quotient of the irreducible ones by the automorphism group. This is resumed in table 1; columns list for each type of bundle the projective part of the bundle automorphism group, the affine space of connections and lastly the moduli space of irreducible connections up to bundle automorphism. The 16-order group of 2-torsion points of  $\text{Jac}(X)$  is naturally acting on  $\mathfrak{Bun}(X)$  by tensor product, preserving each type of bundle.

bundle type	$E$	$\mathbb{P}\text{Aut}(E)$	connections	moduli
stable	$E$	1	$\mathbb{A}^3$	$\mathbb{A}^3$
decomposable	$L_0 \oplus L_0^{-1}$	$\mathbb{G}_m$	$\mathbb{A}^4$	$\mathbb{C}^2 \times \mathbb{C}^*$
affine	$L_0 \rightarrow E \rightarrow L_0^{-1}$	$\text{Aff}(\mathbb{C})$	$\mathbb{A}^5$	$\emptyset$
trivial+twists	$E_0, E_\tau$	$\text{PGL}_2(\mathbb{C})$	$\mathbb{A}^6$	$\mathbb{C}_{(a,b,c)}^3 \setminus \{b^2 = 4ac\}$
unipotent+twists	$\tau \rightarrow E \rightarrow \tau$	$\mathbb{G}_a$	$\mathbb{A}^4$	$\mathbb{C}^2 \times \mathbb{C}^*$
Gunning	$\kappa \rightarrow E_\kappa \rightarrow \kappa^{-1}$	$H^0(X, \Omega_X^1)$	$\mathbb{A}^5$	$\mathbb{A}^3$

TABLE 1. Automorphisms and moduli spaces of irreducible connections on special bundles

We supplement this geometric study with explicit computations of Narasimhan-Ramanan coordinates, together with the equation of  $\text{Kum}(X)$ , as well as the 16-order symmetry group. These computations are done for the genus 2 curve defined by an affine equation  $y^2 = x(x-1)(x-r)(x-s)(x-t)$  as functions of the free parameters  $(r, s, t)$ .

**The branching cover**  $\phi : \mathfrak{Bun}(X/\iota) \xrightarrow{2:1} \mathfrak{Bun}(X)$ . We provide a full description of this map in section 5 together with a complete dictionary between special bundles  $E$  listed above and special parabolic bundles  $(\underline{E}, \underline{p})$  in  $\mathfrak{Bun}(X/\iota)$ . The map  $\phi$  is a double cover (once we have taken the affine bundles off  $\mathfrak{Bun}(X)$ ) branching over the locus of decomposable bundles, including the trivial bundle and its 15 twists. The 16 latter bundles lift as 16 decomposable parabolic bundles. If we restrict to the complement of these very special bundles, we can follow the previous work of [2, 29]: the moduli space  $\mathfrak{Bun}^u(X/\iota)$  of indecomposable bundles can be constructed by patching together GIT quotients  $\text{Bun}_\mu^{ss}(X/\iota)$  of  $\mu$ -semi-stable parabolic bundles for a finite number of weights

$\mu \in [0, 1]^6$ . These moduli spaces are smooth projective manifolds (where semistable objects are actually stable) and they are patched together along strict Zariski open sets, giving  $\mathfrak{Bun}^u(X/\iota)$  the structure of a non Hausdorff scheme. In the present work, we mainly study a one-parameter family of weights, namely the diagonal family  $\mu = (\mu, \mu, \mu, \mu, \mu, \mu)$ . For  $\mu = \frac{1}{2}$ , the restriction map  $\phi : \text{Bun}_{NR}^{\text{ss}}(X/\iota) \rightarrow \mathbb{P}_{NR}^3$  is exactly the 2-fold cover of  $\mathbb{P}_{NR}^3$  ramifying over the Kummer surface  $\text{Kum}(X)$ ; it is singular for this special value. We pay more attention to the chart given by any  $\frac{1}{6} < \mu < \frac{1}{4}$  which is a 3-dimensional projective space, that we will denote  $\mathbb{P}_B^3$ : it is naturally isomorphic to a certain space of extensions studied by Bertram and Bolognesi [6, 11, 12]. The restriction map  $\phi : \mathbb{P}_B^3 \dashrightarrow \mathbb{P}_{NR}^3$  is rational and also related to the classical geometry of Kummer surfaces. Precisely, there is a natural embedding  $X/\iota \hookrightarrow \mathbb{P}_B^3$  as a twisted cubic and  $\phi|_{\mathbb{P}_B^3}$  is defined by the linear system of quadrics passing through the 6 conic points of  $X/\iota$ . The Galois involution  $\Upsilon : \mathfrak{Bun}(X/\iota) \xrightarrow{\sim} \mathfrak{Bun}(X/\iota)$  of  $\phi$  is defined by elementary transformations:  $\Upsilon = \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+$ . After restriction to the chart  $\mathbb{P}_B^3$ , it is known as Geiser involution (see Dolgachev [14]); its decomposition as sequence of blow-up and contraction directly follows from the study of wall-crossing phenomena when weights vary inside  $\frac{1}{6} < \mu < \frac{5}{6}$ . In this picture, unipotent bundles come from those parabolic bundles parametrized by the cubic  $X/\iota$ , and twisted unipotent, from those 15 lines passing through 2 among 6 points. Also Gunning planes with even theta-characteristic come from those 20 planes passing through 3 among 6 points, while odd Gunning planes come from the 6 conic points of  $X/\iota$ , that are indeterminacy points for  $\phi$ . Finally, the Kummer surface lifts as the dual Weddle surface (another quartic birational model of  $\text{Kum}(X)$ ).

**Anticanonical subbundles.** In order to establish our above dictionary, we study in section 4 the space of homomorphisms  $\mathcal{O}_X(-K_X) \rightarrow E$  for each type of bundle  $E$ . This is a 2-dimensional vector space for a generic vector bundle  $E$  defining a 1-parameter family of subbundles. This has been used by Tyurin to construct a parametrization of an open set of  $\mathfrak{Bun}(X)$ . Only two of these anti-canonical subbundles are invariant under the hyperelliptic involution. Once we have chosen one of them, one can associate the parabolic structure  $\mathbf{p}$  directed by this line bundle over the set  $W$  of Weierstrass points. After applying a negative elementary transformation to the parabolic bundle  $(E, \mathbf{p})$ , we precisely get  $\pi^*(\underline{E}, \mathbf{p})$  for one of the two preimages  $\phi^{-1}(E, \mathbf{p})$ ; the choice of the invariant anticanonical subbundle (or  $\mathbf{p}$ ) is a descent data. This allows us to link our moduli space  $\mathfrak{Bun}(X/\iota)$  with the space of  $\iota$ -invariant extensions  $-K_X \rightarrow E \rightarrow K_X$  studied by Bertram and Bolognesi: their moduli space coincides with our chart  $\mathbb{P}_B^3$ .

**Universal bundle and Hitchin fibration.** There is no universal bundle for  $\mathbb{P}_{NR}^3$ , but there is one for the 2-fold cover  $\mathbb{P}_B^3$ . We actually provide an explicit universal connection for  $\mathfrak{Con}(X/\iota)$  as well as a universal Higgs bundle for  $\mathfrak{Higgs}(X/\iota)$  in section 6. This allows us to explicitly compute the Hitchin Hamiltonians for the Hitchin system on  $\mathfrak{Higgs}(X/\iota)$ . Using the natural identification with the cotangent bundle  $T^*\mathfrak{Bun}(X/\iota)$  together with the double cover  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$ , we derive the explicit Hitchin Hamiltonians for  $\mathfrak{Higgs}(X)$  in a very direct way.

**Tyurin parameters.** In sections 4 and 6, we investigate the Tyurin point of view. Anticanonical morphisms provide, for a generic bundle  $E$ , a birational morphism  $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X) \rightarrow E$ , or after tensoring by  $\mathcal{O}_X(K_X)$ , a birational and minimal trivialisation  $E_0 \rightarrow E$ . Precisely, this birational bundle map consists in 4 elementary transformations for a parabolic structure on  $E_0$  supported by a divisor belonging to

the linear system  $|2K_X|$ . The moduli space of such parabolic structures is a birational model for  $\mathfrak{Bun}(X)$  (from which we easily deduce the rationality of this moduli space). We provide explicit change of coordinates between the Tyurin parameters and the other previous parameters. We also describe connections on  $E$  when pulled-back to  $E_0$  (connections with 4 apparent singular points). These computations allow us to construct an explicit rational section  $\mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$  which is regular over the stable open subset of  $\mathfrak{Bun}(X)$  and is, moreover, Lagrangian. In other words, over the stable open set, the Lagrangian fiber-bundle  $\mathfrak{Con}(X) \rightarrow \mathfrak{Bun}(X)$  is isomorphic to the cotangent bundle  $T^*\mathfrak{Bun}(X)$  (i.e.  $T^*P_{NR}^3$ ) as symplectic manifolds.

## 1. PRELIMINARIES ON CONNECTIONS

**1.1. Logarithmic connections.** Let  $X$  be a smooth projective curve over  $\mathbb{C}$  and  $E \rightarrow X$  be a rank  $r$  vector bundle. Let  $D$  be a reduced effective divisor on  $X$ . A *logarithmic connection* on  $E$  with polar divisor  $D$  is a  $\mathbb{C}$ -linear map

$$\nabla : E \rightarrow E \otimes \Omega_X^1(D)$$

satisfying the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for any local section  $s$  of  $E$  and function  $f$  on  $X$ . Locally, for a trivialization of  $E$ , the connection writes  $\nabla = d_X + A$  where  $d_X : \mathcal{O}_X \rightarrow \Omega_X^1$  is the differential operator on  $X$  and  $A$  is a  $r \times r$  matrix with coefficients in  $\Omega_X^1(D)$ , thus 1-forms having at most simple poles located along  $D$ . The *true polar divisor*, i.e. the singular set of such a logarithmic connection  $\nabla$  is a subset of  $D$ . Depending on the context, we may assume them to be equal. At each pole  $x_0 \in D$ , the residual matrix intrinsically defines an endomorphism of the fiber  $E_{x_0}$  that we denote  $\text{Res}_{x_0} \nabla$ . *Residual eigenvalues* and *residual eigenspaces* in  $E_{x_0}$  hence are well-defined.

**1.2. Twists and trace.** As before, let  $E$  be a rank  $r$  vector bundle endowed with a logarithmic connection  $\nabla$  on a curve  $X$ . The connection  $\nabla$  induces a logarithmic connection  $\text{tr}(\nabla)$  on the determinant line bundle  $\det(E)$  over  $X$  with

$$\text{Res}_{x_0} \text{tr}(\nabla) = \text{tr}(\text{Res}_{x_0} \nabla)$$

for each  $x_0 \in D$ . By the residue theorem, the sum of residues of a global meromorphic 1-form on  $X$  is zero. We thereby obtain *Fuchs' relation*:

$$(1) \quad \deg(E) + \sum_{x_0 \in D} \text{tr}(\text{Res}_{x_0} \nabla) = 0.$$

We can define the *twist* of the connection  $(E, \nabla)$  by a rank 1 meromorphic connection  $(L, \zeta)$  as the rank  $r$  connection  $(E', \nabla')$  with

$$(E', \nabla') = (E, \nabla) \otimes (L, \zeta) := (E \otimes L, \nabla \otimes \text{id}_L + \text{id}_E \otimes \zeta).$$

We have

$$\det(E') = \det(E) \otimes L^{\otimes r} \quad \text{and} \quad \text{tr}(\nabla') = \text{tr}(\nabla) \otimes \zeta^{\otimes r}.$$

When  $L \rightarrow X$  is a line bundle such that  $L^{\otimes r} \simeq \mathcal{O}_X$ , then there is a unique (holomorphic) connection  $\nabla_L$  on  $L$  such that the connection  $\nabla_L^{\otimes r}$  is the trivial connection on  $L^{\otimes r} \simeq \mathcal{O}_X$ . The twist by such a  $r$ -torsion connection has no effect on the trace: modulo isomorphism, we have  $\det(E') = \det(E)$  and  $\text{tr}(\nabla') = \text{tr}(\nabla)$ .

**1.3. Projective connections and Riccati foliations.** From now on, let us assume the rank to be  $r = 2$ . After projectivizing the bundle  $E$ , we get a  $\mathbb{P}^1$ -bundle  $\mathbb{P}E$  over  $X$  whose total space is a ruled surface  $S$ . Since  $\nabla$  is  $\mathbb{C}$ -linear, it defines a projective connection  $\mathbb{P}\nabla$  on  $\mathbb{P}E$  and the graphs of horizontal sections define a foliation by curves  $\mathcal{F}$  on the ruled surface  $S$ . The foliation  $\mathcal{F}$  is transversal to a generic member of the ruling  $S \rightarrow X$  and is thus a Riccati foliation (see [13], chapter 4). If the connection locally writes

$$\nabla : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto d \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

then in the corresponding trivialization  $(z_1 : z_2) = (1 : z)$  of the ruling, the foliation is defined by the (pfaffian) Riccati equation

$$dz - \beta z^2 + (\delta - \alpha)z + \gamma = 0.$$

Tangencies between  $\mathcal{F}$  and the ruling are concentrated on fibers over the (true) polar divisor  $D$  of  $\nabla$ . These singular fibers are totally  $\mathcal{F}$ -invariant. According to the number of residual eigendirections of  $\nabla$ , the restriction of  $\mathcal{F}$  to such a fibre is the union of a leaf and 1 or 2 points.

Any two connections  $(E, \nabla)$  and  $(E', \nabla')$  on  $X$  define the same Riccati foliation if, and only if,  $(E', \nabla') = (E, \nabla) \otimes (L, \zeta)$  for a rank 1 connection  $(L, \zeta)$ . Conversely, a Riccati foliation  $(S, \mathcal{F})$  is always the projectivization of a connection  $(E, \nabla)$ : once we have chosen a lifting  $E$  of  $S$  and a rank 1 connection  $\zeta$  on  $\det(E)$ , there is a unique connection  $\nabla$  on  $E$  such that  $\text{trace}(E) = \zeta$  and  $\mathbb{P}\nabla = \mathcal{F}$ .

**1.4. Parabolic structures.** A *parabolic structure* on  $E$  supported by  $D = \{x_1, \dots, x_n\} \subset X$  is the data  $\mathbf{p} = (p_1, \dots, p_n)$  of a 1-dimensional subspace  $p_i \in E_{x_i}$  for each  $x_i \in D$ . A *parabolic connection* is the data  $(E, \nabla, \mathbf{p})$  of a logarithmic connection  $(E, \nabla)$  with polar divisor  $D$  and a parabolic structure  $\mathbf{p}$  supported by  $D$  such that, at each pole  $x_i \in D$ , the parabolic direction  $p_i$  is an eigendirection of the residual endomorphism  $\text{Res}_{x_i} \nabla$ . For the corresponding Riccati foliation,  $\mathbf{p}$  is the data, on the ruled surface  $S$ , of a singular point of the foliation  $\mathcal{F}$  for each fiber over  $D$ .

**1.5. Elementary transformations.** Let  $(E, p)$  be a parabolic bundle on  $X$  supported by a single point  $x_0 \in X$ . Consider the vector bundle  $E^-$  defined by the subsheaf of those sections of  $E$  directed by  $p$  at  $x_0$ . A natural parabolic direction on  $E^-$  is defined by those sections of  $E$  which are vanishing at  $x_0$  (and thus belong to  $E^-$ ). If  $x$  is a local coordinate at  $x_0$  and  $E$  is generated by  $\langle e_1, e_2 \rangle$  with  $p$  directed by  $e_1$ , then  $E^-$  is locally generated by  $\langle e_1, x e_2 \rangle$  and we define  $p^-$  to be  $e_2' = x e_2$ . By identifying the sections of  $E$  and  $E^-$  outside  $x_0$ , we obtain a natural birational morphism (see also [30])

$$\text{elm}_{x_0}^- : E \dashrightarrow E^-.$$

In a similar way, we define the parabolic bundle  $(E^+, p^+)$  by the sheaf of those meromorphic sections of  $E$  having (at most) a single pole at  $x_0$ , whose residual part is directed by  $p$ . The parabolic  $p^+$  is directed by the holomorphic sections of  $E$  at  $x_0$ . In other words,  $E^+$  is generated by  $\langle \frac{1}{x} e_1, e_2 \rangle$  and  $p^+$  defined by  $e_2$ . The natural morphism

$$\text{elm}_{x_0}^+ : E \dashrightarrow E^+$$

is now regular, but fails to be an isomorphism at  $x_0$ .

These *elementary transformations* satisfy the following properties:

- $\det(E^\pm) = \det(E) \otimes \mathcal{O}_X(\pm[x_0])$ ,

- $\text{elm}_{x_0}^+ \circ \text{elm}_{x_0}^- = \text{id}_{(E,p)}$  and  $\text{elm}_{x_0}^- \circ \text{elm}_{x_0}^+ = \text{id}_{(E,p)}$ ,
- $\text{elm}_{x_0}^+ = \mathcal{O}_X([x_0]) \otimes \text{elm}_{x_0}^-$ .

In particular, positive and negative elementary transformations coincide for a projective parabolic bundle  $(\mathbb{P}E, p)$ . It consists, for the ruled surface  $S$ , in composing the blowing-up of  $p$  with the contraction of the strict transform of the fiber [19]. This latter contraction gives the new parabolic  $p^\pm$ . Elementary transformations on projective parabolic bundles are clearly involutive.

More generally, given a parabolic bundle  $(E, \mathbf{p})$  with support  $D$ , we define the elementary transformations  $\text{elm}_D^\pm$  as the composition of the (commuting) single elementary transformations over all points of  $D$ . We define  $\text{elm}_{D_0}^\pm$  for any subdivisor  $D_0 \subset D$  in the obvious way.

Given a parabolic connection  $(E, \nabla, \mathbf{p})$  with support  $D$ , the elementary transformations  $\text{elm}_D^\pm$  yield new parabolic connections  $(E^\pm, \nabla^\pm, \mathbf{p}^\pm)$ . In fact, the compatibility condition between  $\mathbf{p}$  and the residual eigenspaces of  $\nabla$  insures that  $\nabla^\pm$  is still logarithmic. The monodromy is obviously left unchanged, but the residual eigenvalues are shifted as follows: if  $\lambda_1$  and  $\lambda_2$  denote the residual eigenvalues of  $\nabla$  at  $x_0$ , with  $p$  contained in the  $\lambda_1$ -eigenspace, then

- $\nabla^+$  has eigenvalues  $(\lambda_1^+, \lambda_2^+) := (\lambda_1 - 1, \lambda_2)$ ,
- $\nabla^-$  has eigenvalues  $(\lambda_1^-, \lambda_2^-) := (\lambda_1, \lambda_2 + 1)$ ,

and  $p^\pm$  is now contained in the  $\lambda_2^\pm$ -eigenspace.

Finally, if the parabolic connections  $(E, \nabla, \mathbf{p})$  and  $(\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}})$  are isomorphic, then one can easily check that  $(E^\pm, \nabla^\pm, \mathbf{p}^\pm)$  and  $(\tilde{E}^\pm, \tilde{\nabla}^\pm, \tilde{\mathbf{p}}^\pm)$  are also isomorphic. This will allow to define elementary transformations  $\text{elm}_D^\pm$  on moduli spaces of parabolic connections.

**1.6. Stability and moduli spaces.** Given a collection  $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$  of weights  $\mu_i \in [0, 1]$  attached to  $p_i$ , we define the *parabolic degree* with respect to  $\boldsymbol{\mu}$  of a line subbundle  $L \hookrightarrow E$  as

$$\text{deg}_{\boldsymbol{\mu}}^{\text{par}}(L) := \text{deg}(L) + \sum_{p_i \in L} \mu_i$$

(where the summation is taken over those parabolics  $p_i$  which are directed by  $L$ ). Setting

$$\text{deg}_{\boldsymbol{\mu}}^{\text{par}}(E) := \text{deg}(E) + \sum_{i=1}^n \mu_i$$

(where the summation is taken over all parabolics), we define the *stability index* of  $L$  by

$$\text{ind}_{\boldsymbol{\mu}}(L) := \text{deg}_{\boldsymbol{\mu}}^{\text{par}}(E) - 2 \text{deg}_{\boldsymbol{\mu}}^{\text{par}}(L).$$

The parabolic bundle  $(E, \mathbf{p})$  is called *semi-stable* (resp. *stable*) with respect to  $\boldsymbol{\mu}$  if

$$\text{ind}_{\boldsymbol{\mu}}(L) \geq 0 \quad (\text{resp. } > 0) \quad \text{for each line subbundle } L \subset E.$$

For vanishing weights  $\mu_1 = \dots = \mu_n = 0$ , we get the usual definition of (semi-)stability of vector bundles. Semi-stable parabolic bundles admit a coarse moduli space  $\text{Bun}_{\boldsymbol{\mu}}^{ss}$  which is a normal projective variety; the stable locus  $\text{Bun}_{\boldsymbol{\mu}}^s$  is smooth (see [33]).

Note that tensoring by a line bundle does not affect the stability index. In fact, if  $S$  denotes again the ruled surface defined by  $\mathbb{P}E$ , line bundles  $L \hookrightarrow E$  are in one to one correspondence with sections  $\sigma : X \rightarrow S$ , and for vanishing weights,  $\text{ind}_{\boldsymbol{\mu}}(L)$  is precisely

the self-intersection number of the curve  $C := \sigma(X) \subset S$  (see also [31]). For general weights, we have

$$\mathrm{ind}_{\boldsymbol{\mu}}(L) = \#(C \cdot C) + \sum_{p_i \notin C} \mu_i - \sum_{p_i \in C} \mu_i.$$

For weighted parabolic bundles  $(E, \mathbf{p}, \boldsymbol{\mu})$ , it is natural to extend the definition of elementary transformations as follows. Given a subdivisor  $D_0 \subset D$ , define

$$\mathrm{elm}_{D_0}^{\pm} : (E, \mathbf{p}, \boldsymbol{\mu}) \dashrightarrow (E', \mathbf{p}', \boldsymbol{\mu}')$$

by setting

$$\mu'_i = \begin{cases} 1 - \mu_i & \text{if } p_i \in D_0, \\ \mu_i & \text{if } p_i \notin D_0. \end{cases}$$

When  $L' \hookrightarrow E'$  denotes the strict transform of  $L$ , we can easily check that

$$\mathrm{ind}_{\boldsymbol{\mu}'}(L') = \mathrm{ind}_{\boldsymbol{\mu}}(L).$$

Therefore,  $\mathrm{elm}_{D_0}^{\pm}$  acts as an isomorphism between the moduli spaces  $\mathrm{Bun}_{\boldsymbol{\mu}}^{ss}$  and  $\mathrm{Bun}_{\boldsymbol{\mu}'}^{ss}$  (resp.  $\mathrm{Bun}_{\boldsymbol{\mu}}^s$  and  $\mathrm{Bun}_{\boldsymbol{\mu}'}^s$ ). A parabolic connection  $(E, \nabla, \mathbf{p})$  is said to be *semi-stable* (resp. *stable*) with respect to  $\boldsymbol{\mu}$  if

$$\mathrm{ind}_{\boldsymbol{\mu}}(L) \geq 0 \quad (\text{resp. } > 0) \quad \text{for all } \nabla\text{-invariant line subbundles } L \subset E.$$

In particular, irreducible connections are stable for any weight  $\boldsymbol{\mu} \in [0, 1]^n$ . Semi-stable parabolic connections admit a coarse moduli space  $\mathrm{Con}_{\boldsymbol{\mu}}^{ss}$  which is a normal quasi-projective variety; the stable locus  $\mathrm{Con}_{\boldsymbol{\mu}}^s$  is smooth (see [37]).

## 2. HYPERELLIPTIC CORRESPONDENCE

Let  $X$  be the smooth complex projective curve given in an affine chart of  $\mathbb{P}^1 \times \mathbb{P}^1$  by

$$y^2 = x(x-1)(x-r)(x-s)(x-t).$$

Denote its hyperelliptic involution, defined in the above chart by  $(x, y) \mapsto (x, -y)$ , by  $\iota : X \rightarrow X$  and denote its hyperelliptic cover, defined in the above chart by  $(x, y) \mapsto x$ , by  $\pi : X \rightarrow \mathbb{P}^1$ . Denote by  $\underline{W} = \{0, 1, r, s, t, \infty\}$  the critical divisor on  $\mathbb{P}^1$  and by  $W = \{w_0, w_1, w_r, w_s, w_t, w_\infty\}$  the *Weierstrass divisor* on  $X$ , *i.e.* the branching divisor with respect to  $\pi$ . Note that we make no difference in notation between a reduced effective divisor and its support.

Consider a rank 2 vector bundle  $\underline{E} \rightarrow \mathbb{P}^1$  of degree  $-3$ , endowed with a logarithmic connection  $\underline{\nabla} : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$  having residual eigenvalues 0 and  $\frac{1}{2}$  at each pole. We fix the parabolic structure  $\underline{\mathbf{p}}$  attached to the  $\frac{1}{2}$ -eigenspaces over  $\underline{W}$ . After lifting the parabolic connection  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  via  $\pi : X \rightarrow \mathbb{P}^1$ , we get a parabolic connection on  $X$

$$\left( \tilde{E} \rightarrow X, \tilde{\nabla}, \tilde{\mathbf{p}} \right) = \pi^* \left( \underline{E} \rightarrow \mathbb{P}^1, \underline{\nabla}, \underline{\mathbf{p}} \right).$$

We have  $\det(\tilde{E}) \simeq \mathcal{O}_X(-3K_X)$  and the residual eigenvalues of the connection  $\tilde{\nabla} : \tilde{E} \rightarrow \tilde{E} \otimes \Omega_X^1(W)$  are 0 and 1 at each pole, with parabolic structure  $\tilde{\mathbf{p}}$  directed by the 1-eigenspaces. After applying elementary transformations directed by  $\tilde{\mathbf{p}}$ , we get a new parabolic connection:

$$\mathrm{elm}_W^{\pm} : \left( \tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}} \right) \dashrightarrow (E, \nabla, \mathbf{p})$$

which is now holomorphic and trace-free. Let us fix homogenous weights  $\underline{\mu}_i = \frac{1}{2}$  for the original connection  $(\underline{E}, \underline{\nabla}, \underline{p})$ . We then get weights  $\tilde{\mu}_i = 1$  for the lift  $(\tilde{E}, \tilde{\nabla}, \tilde{p})$  and finally  $\mu_i = 0$  for the holomorphic connection  $(E, \nabla, p)$  so that we can omit  $\mu$  and  $p$ .

Recall from the introduction that we denote by  $\mathfrak{Con}(X/\iota)$  the moduli space of logarithmic rank 2 connections on  $\mathbb{P}^1$  with residual eigenvalues 0 and  $\frac{1}{2}$  at each pole in  $\underline{W}$ , and we denote by  $\mathfrak{Con}(X)$  the moduli space of trace-free holomorphic rank 2 connections on  $X$ . Since to every element  $(\underline{E}, \underline{\nabla})$  of  $\mathfrak{Con}(X/\iota)$ , the parabolic structure  $\underline{p}$  is intrinsically defined as above, we have just defined a map

$$\Phi : \begin{cases} \mathfrak{Con}(X/\iota) & \rightarrow \mathfrak{Con}(X) \\ (\underline{E}, \underline{\nabla}, \underline{p}) & \mapsto (E, \nabla). \end{cases}$$

Roughly counting dimensions, we see that both spaces of connections have same dimension 6 up to bundle isomorphisms. We may expect to obtain most of all holomorphic and trace-free rank 2 connections on  $X$  by this construction. This turns out to be true and will be proved along this section. In particular, any *irreducible* holomorphic and trace-free rank 2 connection  $(E, \nabla)$  on  $X$  can be obtained like above. Note that the stability of  $E$  is a sufficient condition for the irreducibility of  $\nabla$ .

**2.1. Topological considerations.** By the Riemann-Hilbert correspondence, the two moduli spaces of connections considered above are in one-to-one correspondence with moduli spaces of representations. Let us start with  $\mathfrak{Con}(X)$  which is easier. The monodromy of a trace-free holomorphic rank 2 connection  $(E, \nabla)$  on  $X$  gives rise to a monodromy representation, namely a homomorphism  $\rho : \pi_1(X, w) \rightarrow \mathrm{SL}_2$ . In fact, this depends on the choice of a basis on the fiber  $E_w$ . Another choice will give the conjugate representation  $M\rho M^{-1}$  for some  $M \in \mathrm{SL}_2$ . The class  $[\rho] \in \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2) / \mathrm{PGL}_2$  however is well-defined by  $(E, \nabla)$ . Conversely, the monodromy  $[\rho]$  characterizes the connection  $(E, \nabla)$  on  $X$  modulo isomorphism, which yields a bijective correspondence

$$RH : \mathfrak{Con}(X) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2) / \mathrm{PGL}_2$$

which turns out to be complex analytic where it makes sense, *i.e.* on the smooth part. Yet this map is highly transcendental, since we have to integrate a differential equation to compute the monodromy. Note that the space of representations only depends on the topology of  $X$ , not on the complex and algebraic structure.

In a similar way, parabolic connections in  $\mathfrak{Con}(X/\iota)$  are in one-to-one correspondence with representations  $\rho : \pi_1^{\mathrm{orb}}(X/\iota) \rightarrow \mathrm{GL}_2$  of the orbifold fundamental group (killing squares of simple loops around punctures, see the proof below). Thinking of  $\mathbb{P}^1 = X/\iota$  as the orbifold quotient of  $X$  by the hyperelliptic involution, these representations can also be seen as representations

$$\rho : \pi_1(\mathbb{P}^1 \setminus \underline{W}, x) \rightarrow \mathrm{GL}_2(\mathbb{C})$$

with 2-torsion monodromy around the punctures, having eigenvalues 1 and  $-1$ .

If  $x = \pi(w)$ , the branching cover  $\pi : X \rightarrow X/\iota$  induces a monomorphism

$$\pi_* : \pi_1(X, w) \hookrightarrow \pi_1^{\mathrm{orb}}(X/\iota, x),$$

whose image consists of words of even length in the alphabet of a system of simple generators of  $\pi_1^{\mathrm{orb}}(X/\iota, x)$ . This allows to associate, to any representation  $\rho : \pi_1^{\mathrm{orb}}(X/\iota, x) \rightarrow \mathrm{GL}_2$  as above, a representation  $\rho \circ \pi_* : \pi_1(X, w) \rightarrow \mathrm{SL}_2$ . We have thereby defined a map

$\Phi^{\text{top}}$  between corresponding representation spaces, which makes the following diagram commutative

$$\begin{array}{ccc} \mathfrak{Con}(X/\iota) & \xrightarrow[\sim]{RH} & \text{Hom}(\pi_1^{\text{orb}}(X/\iota, x), \text{GL}_2) / \text{PGL}_2 \\ \downarrow \Phi & & \downarrow \Phi^{\text{top}} \\ \mathfrak{Con}(X) & \xrightarrow[\sim]{RH} & \text{Hom}(\pi_1(X, w), \text{SL}_2) / \text{PGL}_2. \end{array}$$

We now want to describe the map  $\Phi^{\text{top}}$ . The quotient  $\pi_1^{\text{orb}}(X/\iota, x) / \pi_*(\pi_1(X, w)) \simeq \mathbb{Z}_2$  acts (by conjugacy) as outer automorphisms of  $\pi_1(X, w)$ . It coincides with the outer action of the hyperelliptic involution  $\iota$ . Since the hyperelliptic involution is an outer automorphism, it acts non-trivially on  $\text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2$ .

**Theorem 2.1.** *Given a representation  $[\rho] \in \text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2$ , the following properties are equivalent:*

- (a)  $[\rho]$  is either irreducible or abelian;
- (b)  $[\rho]$  is  $\iota$ -invariant;
- (c)  $[\rho]$  is in the image of  $\Phi^{\text{top}}$ .

In this case,  $[\rho]$  has 1 or 2 preimages under  $\Phi^{\text{top}}$  depending on whether it is diagonal or not.

*Proof.* We start making explicit the monomorphism  $\pi_*$  and the involution  $\iota$ . Let  $x \in \mathbb{P}^1 \setminus \underline{W}$  and  $w \in X$  one of the two preimages. Choose simple loops around the punctures to generate the orbifold fundamental group of  $\mathbb{P}^1 \setminus \underline{W}$  with the standard representation

$$\pi_1^{\text{orb}}(X/\iota, x) = \left\langle \gamma_0, \gamma_1, \gamma_r, \gamma_s, \gamma_t, \gamma_\infty \mid \begin{array}{l} \gamma_0^2 = \gamma_1^2 = \gamma_r^2 = \gamma_s^2 = \gamma_t^2 = \gamma_\infty^2 = 1 \\ \text{and } \gamma_0\gamma_1\gamma_r\gamma_s\gamma_t\gamma_\infty = 1 \end{array} \right\rangle.$$

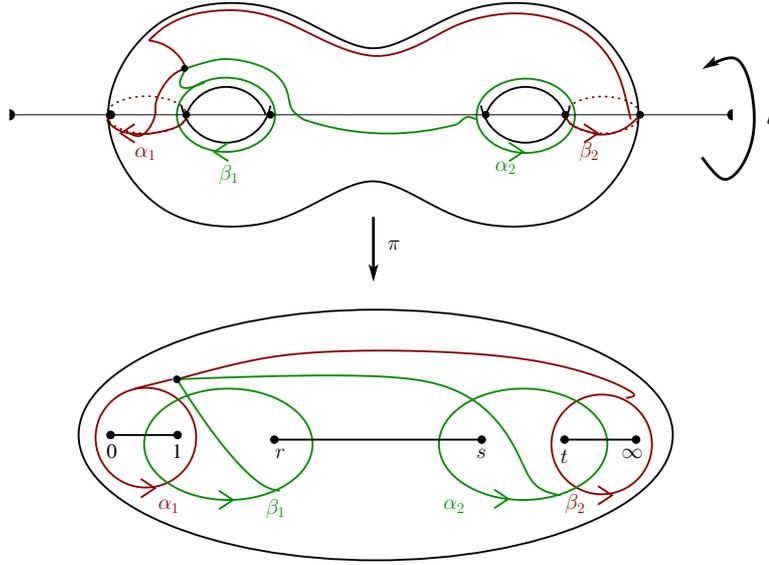
Even words in these generators can be lifted as loops based in  $w$  on  $X$ , generating the ordinary fundamental group of  $X$ . Using the relations, we see that  $\pi_1(X, w)$  is actually generated by the following pairs

$$\begin{cases} \alpha_1 := \gamma_0\gamma_1 \\ \beta_1 := \gamma_r\gamma_1 \end{cases} \quad \begin{cases} \alpha_2 := \gamma_s\gamma_t \\ \beta_2 := \gamma_\infty\gamma_t \end{cases}$$

and they provide the standard presentation

$$(2) \quad \pi_1(X, w) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] = 1 \rangle,$$

where  $[\alpha_i, \beta_i] = \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$  denotes the commutator. In other words, the monomorphism  $\pi_*$  is defined by  $\alpha_1 \mapsto \gamma_0\gamma_1$  et cetera.



After moving the base point  $w \rightarrow w_i$  to a Weierstrass point, the involution  $\iota$  acts as an involutive automorphism of  $\pi_1(X, w_i)$ : it coincides with the outer automorphism given by  $\gamma_i$ -conjugacy. For instance, for  $i = 1$ , we get

$$\begin{cases} \alpha_1 \mapsto \alpha_1^{-1} \\ \beta_1 \mapsto \beta_1^{-1} \end{cases} \quad \begin{cases} \alpha_2 \mapsto \gamma \alpha_2^{-1} \gamma^{-1} \\ \beta_2 \mapsto \gamma \beta_2^{-1} \gamma^{-1} \end{cases} \quad \text{with} \quad \gamma = \beta_1^{-1} \alpha_1^{-1} \beta_2 \alpha_2.$$

Let us now prove (a) $\Leftrightarrow$ (b). That irreducible representations are  $\iota$ -invariant already appears in the last section of [18]. Let us recall the argument given there. There is a natural surjective map

$$\Psi : \text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2 \longrightarrow \text{Hom}(\pi_1(X), \text{SL}_2) // \text{PGL}_2 =: \chi$$

to the GIT quotient  $\chi$ , usually called character variety, which is an affine variety. The singular locus is the image of reducible representations. There can be many different classes  $[\rho]$  over each singular point. The smooth locus of  $\chi$  however is the geometric quotient of irreducible representations, which are called stable points in this context. The above map  $\Psi$  is injective over this open subset. The involution  $\iota$  acts on  $\chi$  as a polynomial automorphism and we want to prove that the action is trivial. First note that the canonical fuchsian representation given by the uniformisation  $\mathbb{H} \rightarrow X$  must be invariant by the hyperelliptic involution  $\iota : X \rightarrow X$ . The corresponding point in  $\chi$  therefore is fixed by  $\iota$ . On the other hand, the definition of  $\chi$  only depends on the topology of  $X$  and, considering all possible complex structures on  $X$ , we now get a large set of fixed points  $\chi^{\text{fuchsian}} \subset \chi$ . Those fuchsian representations actually form an open subset of  $\text{Hom}(\pi_1(X), \text{SL}_2\mathbb{R}) // \text{SL}_2\mathbb{R}$ , and thus a Zariski dense subset of  $\chi$ . It follows that the action of  $\iota$  is trivial on the whole space  $\chi$ . By injectivity of  $\Psi$ , any irreducible representation is  $\iota$ -invariant.

In other words, if an irreducible representation  $\rho$  is defined by matrices  $A_i, B_i \in \text{SL}_2$ ,  $i = 1, 2$  with  $[A_1, B_1] \cdot [A_2, B_2] = I_2$ , then there exists  $M \in \text{GL}_2$  satisfying:

$$(3) \quad \begin{cases} M^{-1}A_1M = A_1^{-1} \\ M^{-1}B_1M = B_1^{-1} \end{cases} \quad \begin{cases} M^{-1}A_2M = CA_2^{-1}C^{-1} \\ M^{-1}B_2M = CB_2^{-1}C^{-1} \end{cases} \quad \text{with} \quad C = B_1^{-1}A_1^{-1}B_2A_2.$$

Since the action of  $\iota$  is involutive,  $M^2$  commutes with  $\rho$  and is thus a scalar matrix. The matrix  $M$  has two opposite eigenvalues which can be normalized to  $\pm 1$  after replacing  $M$  by a scalar multiple. There are exactly two such normalizations, namely  $M$  and  $-M$ .

It remains to check what happens for reducible representations. In the strict reducible case, there is a unique common eigenvector for all matrices  $A_1, B_1, A_2, B_2$ ; the representation  $\rho$  restricts to it as a representation  $\pi_1(X) \rightarrow \mathbb{C}^*$  which must be  $\iota$ -invariant. This (abelian) representation must therefore degenerate into  $\{\pm 1\}$ . It follows that any reducible  $\iota$ -invariant representation is abelian. For abelian representations though, the action of  $\iota$  is simply given by

$$A_i \mapsto A_i^{-1} \quad \text{and} \quad B_i \mapsto B_i^{-1} \quad \text{for} \quad i = 1, 2.$$

But then, up to conjugacy:

- either  $A_1, B_1, A_2, B_2$  are diagonal and one can choose  $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ ,
- or  $A_1, B_1, A_2, B_2$  are upper triangular with eigenvalues  $\pm 1$  (projectively unipotent) and  $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  works.

Let us now prove (b)  $\Leftrightarrow$  (c). Given a representation  $[\rho] \in \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) / \text{PGL}_2$ , its image under  $\Phi^{\text{top}}$  is  $\iota$ -invariant, *i.e.* the action of  $\iota$  coincides in this case with the conjugacy by  $\rho(\gamma_1) \in \text{GL}_2$ . Conversely, let  $[\rho] \in \text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2$  be  $\iota$ -invariant, *i.e.*  $\iota^*\rho = M^{-1} \cdot \rho \cdot M$  for some  $M \in \text{GL}_2$  as in (3). From the cases discussed above, we know that  $M$  can be chosen with eigenvalues  $\pm 1$ . Then setting

$$\begin{cases} M_0 := A_1 M \\ M_1 := M \\ M_r := B_1 M \end{cases} \quad \begin{cases} M_s := B_2^{-1} A_1 B_1 M \\ M_t := A_1 B_1 M A_2 B_2 \\ M_\infty := A_1 B_1 M A_2 \end{cases}$$

we get a preimage of  $[\rho]$ . The preimage depends only of the choice of  $M$ . Any other choice writes  $M' := CM$  with  $C$  commuting with  $\rho$ . In the general case, *i.e.* when  $\rho$  is irreducible, we get two preimages given by  $M$  and  $-M$ . However, when  $\rho$  is diagonal, we get only one preimage, because the anti-diagonal matrices  $M$  and  $-M$  are conjugated by a diagonal matrix (commuting with  $\rho$ ).  $\square$

**Corollary 2.2.** *The Galois involution of the double cover  $\Phi^{\text{top}}$  is given by*

$$\left\{ \begin{array}{ccc} \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) / \text{PGL}_2 & \longrightarrow & \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) / \text{PGL}_2 \\ [\rho] & \mapsto & [-\rho] \end{array} \right\}.$$

So far, Theorem 2.1 provides an analytic description of the map  $\Phi$ : although  $\Phi^{\text{top}}$  is a polynomial branching cover, the Riemann-Hilbert correspondence is only analytic. In the next section, we will follow a more direct approach providing algebraic informations about  $\Phi$ . However, note that we can already deduce the following:

**Corollary 2.3.** *An irreducible trace-free holomorphic connection  $(E, \nabla)$  on  $X$  is invariant under the hyperelliptic involution: there exists a bundle isomorphism  $h : E \rightarrow \iota^*E$  conjugating  $\nabla$  with  $\iota^*\nabla$ . We can moreover assume  $h \circ \iota^*h = \text{id}_E$  and  $h$  is unique up to a sign.*

**Remark 2.4.** *Note that  $h$  acts as  $-\text{id}$  on the determinant line bundle  $\det(E) = \det(\iota^*E) \simeq \mathcal{O}$ .*

Each Weierstrass point  $w \in X$  is fixed by  $\iota$  and the restriction of  $h$  to the fibre  $E_w = \iota^*E_w$  is an automorphism with simple eigenvalues  $\pm 1$ .

**2.2. A direct algebraic approach.** Let  $(E, \nabla)$  be a holomorphic trace-free rank 2 connection on  $X$ . As in corollary 2.3, let  $h$  be a  $\nabla$ -invariant lift to the vector bundle  $E$  of the action of  $\iota$  on  $X$ . Following [7] and [8], we can naturally associate a parabolic logarithmic connection  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}}, \underline{\boldsymbol{\mu}})$  on  $\mathbb{P}^1$  with polar divisor  $\underline{W}$ . Let us briefly recall this construction. The isomorphism  $h$  induces a non-trivial involutive automorphism on the rank 4 bundle  $\pi_*E$  on  $\mathbb{P}^1$ . The spectrum of such an automorphism is  $\{-1, +1\}$  with respective multiplicities 2, which yields a splitting  $\pi_*E = \underline{E} \oplus \underline{E}'$  with  $\underline{E}$  denoting the  $h$ -invariant subbundle.

In local coordinates, the automorphism  $h$  acts on  $\pi_*E$  in the following way. If  $U \subset X$  is a sufficiently small open set outside of the critical points, we have  $\Gamma(\pi(U), \pi_*E) = \Gamma(U, E) \oplus \Gamma(\iota(U), E)$  and  $h$  permutes both factors. Locally at a Weierstrass point with local coordinate  $y$ , one can choose sections  $e_1$  and  $e_2$  generating  $E$  such that  $h(e_1) = e_1$  and  $h(e_2) = -e_2$  (recall that  $h$  has eigenvalues  $\pm 1$  in restriction to the Weierstrass fiber). On the corresponding open set of  $\mathbb{P}^1$ , the bundle  $\pi_*E$  is generated by  $\langle e_1, e_2, ye_1, ye_2 \rangle$ , and we see that  $\langle e_1, ye_2 \rangle$  spans the  $h$ -invariant subspace. Since the connection  $\nabla$  on  $E$  is  $h$ -invariant, we can choose the sections  $e_1$  and  $e_2$  above to be horizontal for  $\nabla$ . Then considering the basis  $\underline{e}_1 = e_1$  and  $\underline{e}_2 = ye_2$  of  $\underline{E}$ , we get

$$\underline{\nabla}\underline{e}_1 = \nabla e_1 = 0 \quad \text{and} \quad \underline{\nabla}\underline{e}_2 = \nabla ye_2 = dy \otimes e_2 = \frac{dy}{y} \otimes \underline{e}_2 = \frac{1}{2} \frac{dx}{x} \otimes \underline{e}_2$$

so that  $\underline{\nabla}$  is logarithmic with eigenvalues 0 and  $\frac{1}{2}$ . To each pole in  $\underline{W}$ , we associate the parabolic  $\underline{p}_i$  directed by the eigenspace with eigenvalue  $\frac{1}{2}$ , with the natural parabolic weight  $\underline{\mu}_i = \frac{1}{2}$ .

However, since we consider the rank 2 case, this general construction can also be viewed in the following way. Denote by  $\mathbf{p}$  the parabolic structure on  $E$  directed by the  $h$ -invariant directions over  $W = \{w_0, w_1, w_r, w_s, w_t, w_\infty\}$  and associate the trivial homogenous weight  $\boldsymbol{\mu} = 0$ . In the coordinates above, the basis  $(\underline{e}_1, \underline{e}_2)$  generates the vector bundle  $E$  after one negative elementary transformation in that direction. Now the hyperelliptic involution acts trivially on the parabolic logarithmic connection on  $X$  defined by

$$\left( \tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}}, \tilde{\boldsymbol{\mu}} \right) := \text{elm}_{\bar{W}}(E, \nabla, \mathbf{p}, \boldsymbol{\mu})$$

and we have

$$\left( \tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}}, \tilde{\boldsymbol{\mu}} \right) = \pi^* \left( \underline{E}, \underline{\nabla}, \underline{\mathbf{p}}, \underline{\boldsymbol{\mu}} \right).$$

**2.3. Galois involution and symmetry group.** With the notations above, let  $(\underline{E}', \underline{\nabla}')$  be the connection on  $\mathbb{P}^1$  we obtain for the other possibility of a lift of the hyperelliptic involution on  $(E \rightarrow X, \nabla)$ , namely for  $h' = -h$ . It is straightforward to check that the map from  $(\underline{E}', \underline{\nabla}', \underline{\mathbf{p}}')$  to  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  and vice-versa is obtained by the elementary transformations  $\text{elm}_W^+$  over  $\mathbb{P}^1$ , followed by the tensor product with a certain logarithmic rank 1 connection  $\sqrt{d \log(W)}$  over  $\mathbb{P}^1$  defined below.

$$\begin{array}{ccccc}
 (\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}}) & \xleftarrow{\text{elm}_W^-} & (E, \nabla, \mathbf{p}) & \xleftarrow{h} & (E, \nabla) & \xrightarrow{-h} & (E, \nabla, \mathbf{p}') & \xrightarrow{\text{elm}_W^-} & (\tilde{E}', \tilde{\nabla}', \tilde{\mathbf{p}}') \\
 \uparrow \pi^* & & & & \downarrow \pi_* & & & & \uparrow \pi^* \\
 (\underline{E}, \underline{\nabla}, \underline{\mathbf{p}}) & \xleftarrow{\text{eig}_{\frac{1}{2}}} & (\underline{E}, \underline{\nabla}) & \hookrightarrow & (\underline{E}, \underline{\nabla}) \oplus (\underline{E}', \underline{\nabla}') & \longleftarrow & (\underline{E}', \underline{\nabla}') & \xrightarrow{\text{eig}_{\frac{1}{2}}} & (\underline{E}', \underline{\nabla}', \underline{\mathbf{p}}') \\
 & & & & \text{\scriptsize } \sqrt{d \log(W)} \otimes \text{elm}_W^+ & & & & 
 \end{array}$$

There is a unique rank 1 logarithmic connection  $(L, \zeta)$  on  $\mathbb{P}^1$  having polar divisor  $\underline{W}$  and eigenvalues 1; note that  $L = \mathcal{O}_{\mathbb{P}^1}(-6)$ . We denote by  $d \log(\underline{W})$  this connection and by  $\sqrt{d \log(\underline{W})}$  its unique square root. In a similar way, define  $\sqrt{d \log(D)}$  for any even order subdivisor  $D \subset \underline{W}$ .

The Galois involution of our map  $\Phi : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X)$  is therefore given by

$$\sqrt{d \log(\underline{W})} \otimes \text{elm}_W^+ : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X/\iota).$$

There is a 16-order group of symmetries on  $\mathfrak{Bun}(X)$  (resp.  $\mathfrak{Con}(X)$ ) consisting of twists with 2-torsion line bundles (resp. rank 1 connections). It can be lifted as a 32-order group of symmetries on  $\mathfrak{Bun}(X/\iota)$  (resp.  $\mathfrak{Con}(X/\iota)$ ), namely those transformations  $\sqrt{d \log(D)} \otimes \text{elm}_D^+$  with  $D \subset \underline{W}$  even. For instance, if  $D = w_i + w_j$ , then its action on  $\mathfrak{Con}(X/\iota)$  corresponds via  $\Phi$  to the twist by the 2-torsion connection on  $\mathcal{O}_X(w_i + w_j - K_X)$ . In particular, it permutes the two parabolics (of  $\mathbf{p}$  and  $\mathbf{p}'$ ) over  $w_i$  and  $w_j$ .

### 3. FLAT VECTOR BUNDLES OVER $X$

In this section, we provide a description of the space of trace-free holomorphic connections on a given flat rank 2 vector bundle  $E$  over the genus 2 curve  $X$ . We review the classical construction of the moduli space of these bundles due to Narasimhan and Ramanan, where the classical geometry of Kummer surfaces arises.

**3.1. Flatness criterion.** Recall the well-known flatness criterion for vector bundles over curves [42, 1].

**Theorem 3.1** (Weil). *A holomorphic vector bundle on a compact Riemann surface is flat, i.e. it admits a holomorphic connection, if and only if it is the direct sum of indecomposable bundles of degree 0.*

In our case of rank 2 vector bundles  $E$  over a genus 2 curve  $X$  with trivial determinant bundle  $\det(E) = \mathcal{O}_X$ , Weil's criterion demands that either  $E$  is indecomposable, or it is the direct sum of degree 0 line bundles. We get the following list of flat bundles:

- stable bundles (forming a Zariski-open subset of the moduli space),
- decomposable bundles of the form  $E = L \oplus L^{-1}$  where  $L \in \text{Jac}(X)$  is a degree 0 line bundle,
- strictly semi-stable indecomposable bundles,
- Gunning bundles.

We recall that a *Gunning bundle* over  $X$  is an unstable indecomposable rank 2 vector bundle with trivial determinant bundle. There are precisely 16 such bundles: for

each of the 16 line bundles  $L \in \text{Pic}^1(X)$  such that  $L^{\otimes 2} = \mathcal{O}(K_X)$  there is a unique indecomposable extension  $0 \rightarrow L \rightarrow E \rightarrow L^{-1} \rightarrow 0$  of  $L$  by  $L^{-1}$ .

Given a flat bundle  $E$ , and a  $\mathfrak{sl}_2$ -connection  $\nabla$  on  $E$ , any other  $\mathfrak{sl}_2$ -connection writes

$$\nabla' = \nabla + \theta$$

where  $\theta \in \text{Hom}(\mathfrak{sl}(E) \otimes \Omega_X^1)$  is a Higgs field. Here,  $\mathfrak{sl}(E)$  denotes the vector bundle whose sections are trace-free endomorphisms of  $E$ . On the other hand, by the Riemann-Roch Theorem and Serre Duality we have

$$(4) \quad h^0(\mathfrak{sl}(E) \otimes \Omega_X^1) = 3 \cdot \text{genus}(X) - 3 + h^0(\mathfrak{sl}(E))$$

( $\mathfrak{sl}(E)$  is self-dual). Since there is no natural choice for the initial connection  $\nabla$ , the set of connections on  $E$  is an affine space. We will see in the following that for generic bundles we have  $h^0(\mathfrak{sl}(E)) = 0$  and the *moduli space* of  $\mathfrak{sl}_2$ -connections on  $E$  is  $\mathbb{A}^3$  in this case. There are, however, flat bundles with non-trivial automorphisms for which the *moduli space* of  $\mathfrak{sl}_2$ -connections will be a quotient of some  $\mathbb{A}^n$  by the automorphism group, yet the dimension of this quotient is always 3, as suggested by (4).

**3.2. Semi-stable bundles and the Narasimhan-Ramanan theorem.** Two semi-stable vector bundles of same rank and degree over a curve are called *S-equivalent*, if the graded bundles associated to Jordan-Hölder filtrations of these bundles are isomorphic. In our case, *i.e.* rank 2 and determinant-free bundles, we get that

- two stable bundles are *S-equivalent* if and only if they are isomorphic;
- two strictly semi-stable bundles are *S-equivalent* if and only if there is a line bundle  $L \in \text{Jac}(X)$  such that each of the two bundles is an extension either of  $L$  by  $L^{-1}$  or of  $L^{-1}$  by  $L$ .

To a semi-stable bundle  $E$ , we associate (following [35]) the set

$$C_E = \{L \in \text{Pic}^1(X) \mid h^0(X, E \otimes L) > 0\}.$$

Equivalently,  $L \in C_E$  if and only if there is a non-trivial homomorphism  $L^{-1} \rightarrow E$ . For stable bundles, such a homomorphism is an embedding and  $C_E$  then parametrizes line subbundles of maximal degree.

Narasimhan and Ramanan proved that this set  $C_E$  is the support of a uniquely defined effective divisor  $D_E$  on  $\text{Pic}^1(X)$  linearly equivalent to  $2\Theta$ , where

$$\Theta = \{[p] \mid p \in X\} \subset \text{Pic}^1(X)$$

is the subset of effective divisors, naturally parametrized by the curve  $X$  itself. Moreover, for strictly semi-stable bundles, the divisor  $D_E$  only depends on the Jordan-Hölder filtration, *i.e.* on the *S-equivalence* class of  $E$ . We thus get a map

$$NR : \mathcal{M}_{NR} \rightarrow \mathbb{P}(H^0(\text{Pic}^1(X), \mathcal{O}(2\Theta)))$$

from the moduli space of *S-equivalence* classes to the linear system  $|2\Theta|$  on  $\text{Pic}^1(X)$ .

**Theorem 3.2** (Narasimhan-Ramanan). *The map  $NR$  defined above is an isomorphism. Let  $\pi : \mathcal{E} \rightarrow T$  be an algebraic family of semi-stable rank 2 vector bundles with trivial determinant over  $X$ . Then the map  $\phi : T \rightarrow \mathcal{M}_{NR}$  associating to  $t \in T$  the *S-equivalence* class of  $E_t = \pi^{-1}(t)$  is a morphism.*

In particular, the moduli space of stable bundles naturally identifies with an open subset of  $\mathcal{M}_{NR} \simeq \mathbb{P}^3$ . A stable bundle has no non-trivial automorphism: we have  $\text{Aut}(E) = \mathbb{C}^*$  acting by scalar multiplication in the fibres (see [21], thm 29). Therefore,

the moduli space of holomorphic connections  $\nabla : E \rightarrow E \otimes \Omega_X^1$  on a given stable bundle  $E$  is an  $\mathbb{A}^3$ -affine space. Note that all holomorphic connections on a stable bundle are irreducible.

We now list special flat bundles and explain how they arise in the Narasimhan-Ramanan moduli space.

**3.3. Semi-stable decomposable bundles.** Let  $E = L_0 \oplus L_0^{-1}$  with  $L_0 \in \text{Jac}(X)$ . Given  $L \in \text{Pic}^1(X)$ , non-trivial sections of  $E \otimes L$  come from non-trivial sections of  $L_0 \otimes L$  or  $L_0^{-1} \otimes L$ . We promptly deduce that

$$D_E = L_0 \cdot \Theta + L_0^{-1} \cdot \Theta$$

where  $L_0 \cdot \Theta$  denotes the translation of  $\Theta$  by  $L_0$  for the group law on  $\text{Pic}(X)$ . A special case occurs for the 16 torsion points  $L_0^2 = \mathcal{O}_X$  for which  $L_0 = L_0^{-1}$  and hence  $D_E = 2(L_0 \cdot \Theta)$  is not reduced.

The moduli space of semi-stable decomposable bundles naturally identifies with the *Kummer variety*

$$\text{Kum}(X) := \text{Jac}(X) / \iota,$$

the quotient of the Jacobian  $\text{Jac}(X)$  by the involution  $\iota : L_0 \mapsto \iota^* L_0 = L_0^{-1}$ . The Narasimhan-Ramanan classifying map provides a natural embedding

$$NR : \text{Kum}(X) := \text{Jac}(X) / \iota \hookrightarrow \mathcal{M}_{NR}$$

and the image is a quartic surface in  $\mathcal{M}_{NR} \simeq \mathbb{P}^3$ . The moduli space of stable bundles identifies with the complement of this surface. The 16 torsion points  $L_0^2 = \mathcal{O}_X$  of the Jacobian are precisely the fixed points of the involution  $\iota$  and yield 16 conic singularities on  $\text{Kum}(X)$ .

**3.3.1. The 2-dimensional family of decomposable bundles.** When  $L_0^2 \neq \mathcal{O}_X$ , the corresponding rank 2 bundle  $E = L_0 \oplus L_0^{-1}$  lies on the smooth part of  $\text{Kum}(X)$ . Non-trivial automorphisms come from the independent action of  $\mathbb{G}_m$  on the two factors: we get a  $\mathbb{G}_m$ -action on  $\mathbb{P}(E)$ .

Given a connection on  $L_0$ , we easily deduce a totally reducible connection  $\nabla_0$  on  $E$  (preserving both factors). Any other connection will differ from  $\nabla_0$  by a Higgs bundle:  $\nabla = \nabla_0 + \theta$  where  $\theta : E \rightarrow E \otimes \Omega_X^1$  is  $\mathcal{O}_X$ -linear and may be represented in the matrix way

$$\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \begin{cases} \alpha : L_0 \rightarrow L_0 \otimes \Omega_X^1, \\ \beta : L_0^{-1} \rightarrow L_0 \otimes \Omega_X^1, \\ \gamma : L_0 \rightarrow L_0^{-1} \otimes \Omega_X^1. \end{cases}$$

Under our assumption that  $L_0^2 \neq \mathcal{O}_X$ , our space of connections is parametrized by  $\mathbb{C}_\alpha^2 \times \mathbb{C}_\beta^1 \times \mathbb{C}_\gamma^1$ . Since  $E$  has no degree 0 subbundle distinct from  $L_0$  and  $L_0^{-1}$ , reducible connections on  $E$  are precisely those for which one of the two factors is invariant, *i.e.*  $\beta = 0$  or  $\gamma = 0$ . The  $\mathbb{G}_m$ -action is trivial on  $\alpha$  but not on the two other coefficients: the quotient  $\mathbb{C}_\beta^1 \times \mathbb{C}_\gamma^1 / \mathbb{G}_m$  is  $\mathbb{C}^*$  after deleting reducible connections (for which  $\beta = 0$  or  $\gamma = 0$ ). *The moduli space of irreducible connections on  $E$  is thus given by  $\mathbb{C}^2 \times \mathbb{C}^*$ .*

The involution  $\iota$  preserves those connections that are irreducible or totally reducible. *The moduli space of  $\iota$ -invariant connections is  $\mathbb{C}^2 \times \mathbb{C}$ .*

**3.3.2. The trivial bundle and its 15 twists.** All 16 special decomposable bundles are equivalent to the trivial one after twisting by a convenient line bundle. Let us study the case  $E = \mathcal{O}_X \oplus \mathcal{O}_X$  which admits the trivial connection  $\nabla_0 = d$ . Any other connection is obtained by adding a Higgs bundle of the matrix form

$$\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \alpha, \beta, \gamma \in H^0(X, \Omega_X^1)$$

(here, a trivialization of  $E$  is chosen). Our space of connections is parametrized by  $\mathbb{C}_\alpha^2 \times \mathbb{C}_\beta^2 \times \mathbb{C}_\gamma^2$  but now the group acting is  $\text{Aut}(\mathbb{P}E) = \text{PGL}_2$ .

Since the trivial connection is  $\text{PGL}_2$ -invariant, the data of a connection  $\nabla = d + \theta$  is equivalent to the data of the Higgs field  $\theta$  itself. Moreover, the determinant map

$$\det : H^0(\mathcal{E}nd(E) \otimes \Omega_X^1) \rightarrow H^0(\Omega_X^1 \otimes \Omega_X^1) ; \theta \mapsto -(\alpha \otimes \alpha + \beta \otimes \gamma)$$

is invariant under the  $\text{PGL}_2$ -action. Through this map, we claim the following.

**Proposition 3.3.** *The moduli space of irreducible trace-free connections on the trivial bundle of rank 2 over  $X$  coincides with the open set in  $H^0(X, \Omega_X^1 \otimes \Omega_X^1)$  of those quadratic differentials that are not the square  $\omega \otimes \omega$  of a holomorphic 1-form  $\omega$ .*

*Proof.* Note that in our usual coordinates on  $X$ , we have

$$H^0(X, \Omega_X^1) = \text{Vect}_{\mathbb{C}} \left( \frac{dx}{y}, x \frac{dx}{y} \right).$$

The eigendirections of  $\theta$  define a curve  $C$  on the total space  $X \times \mathbb{P}^1$  of the projectivized bundle (for eigendirections to make sense, we have to compose  $\theta$  by local isomorphisms  $E \otimes \Omega_X^1 \rightarrow E$ ; the resulting curve  $C$  does not depend on this choice). In a concrete way, for each vector  $v \in E$ , we compute the determinant  $v \wedge \theta(v)$ . Under trivializing coordinates  $(1 : z) \in \mathbb{P}^1$  we find that  $C$  is defined by

$$C : -\gamma + 2z\alpha + z^2\beta = 0.$$

It follows that  $C$  has bidegree  $(2, 2)$  in  $X \times \mathbb{P}^1$  (*i.e.* with respect to the variables  $(y, z)$ ) and is invariant by the hyperelliptic involution  $\iota$ . Hence it defines a bidegree  $(1, 2)$  curve  $\underline{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$  (*i.e.* with respect to the variables  $(x, z)$ ). It is easy to check that  $C$  is reducible if and only if  $\nabla$  is reducible. In the irreducible case, the curve  $C$  defines a  $(2 : 1)$ -map  $\mathbb{P}_z^1 \rightarrow \mathbb{P}_x^1$  whose Galois involution may be normalized to  $z \mapsto -z$  under the  $\text{PGL}_2$ -action. After this normalization, we get that  $\alpha = 0$  and the involution  $\iota$  lifts as  $(x, y, z) \mapsto (x, -y, -z)$ . In particular,  $z = 0$  and  $z = \infty$  are the two  $\iota$ -invariant subbundles. This normalization is unique up to the dihedral group  $\mathbb{D}_\infty$  (preserving  $z \in \{0, \infty\}$ ). Clearly, the determinant  $-\beta \otimes \gamma$  is invariant and determines  $\nabla$  up to this action since, in genus 2, any quadratic form splits as a product  $\det(\theta) = -\beta \otimes \gamma$ . Finally, one can easily check that the following properties are equivalent:

- $\nabla$  (or  $\theta$ ) is reducible,
- the  $(1, 2)$  curve  $\underline{C}$  splits,
- the determinant  $\det(\theta)$  viewed on  $\mathbb{P}_x^1$  has a double zero,
- the determinant  $\det(\theta)$  (viewed on  $X$ ) is a square.

□

It may be of interest to pursue the discussion of the proof above in the reducible case. In this case,  $C$  is reducible and has a bidegree  $(0, 1)$ -factor which is  $\nabla$ -invariant.

We can normalize

$$\theta = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}.$$

The gauge freedom is given by the group of upper-triangular matrices and we are led to the following cases

- (1)  $\alpha \neq 0$  and  $\beta$  is not proportional to  $\alpha$  (in particular  $\neq 0$ ); the monodromy is *affine* but *non-abelian* and the curve  $\underline{C}$  splits as a union of irreducible bidegree  $(0, 1)$  and  $(1, 1)$  curves.
- (2)  $\alpha \neq 0$  and  $\beta$  is proportional to  $\alpha$ : we can normalize  $\beta = 0$ ; the monodromy is *diagonal* and the curve  $\underline{C}$  splits as a union of two bidegree  $(0, 1)$  curves and one  $(1, 0)$  curve located at the vanishing point of  $\alpha$ .
- (3)  $\alpha = 0$  and  $\beta \neq 0$ ; the monodromy is *unipotent* but non-trivial and the curve  $\underline{C}$  splits as a union of a bidegree  $(0, 1)$  curve with multiplicity 2 and a bidegree  $(1, 0)$  curve located at the vanishing point of  $\beta$ .
- (4)  $\alpha = 0$  and  $\beta = 0$  and we get the *trivial* connection (the curve  $\underline{C}$  has vanishing equation and is not defined).

The determinant map  $\det$  defined above takes values in the set of quadratic differentials over  $X$ . Those are of the form

$$\nu = \frac{\nu_0 + \nu_1 x + \nu_2 x^2}{x(x-1)(x-r)(x-s)(x-t)} dx \otimes dx.$$

It is a square, say  $\det(\theta) = -\alpha \otimes \alpha$ , if and only if  $\nu_1^2 = 4\nu_0\nu_2$ . In this case,  $\alpha$  is uniquely defined up to a sign. It follows that a fiber  $\det^{-1}(\nu)$  of the determinant map above is

- a unique irreducible connection (up to  $\mathrm{PGL}_2$ -isomorphism) if  $\nu_1^2 \neq 4\nu_0\nu_2$ ;
- the union of two reducible connections of type (1) (upper and lower triangular once  $\alpha$  is fixed) and a reducible connection of type (2) over a smooth point of the cone  $\nu_1^2 = 4\nu_0\nu_2$ ;
- the union of the trivial connection (4) and a 1-parameter family of reducible connections of type (3) over the singular point  $\nu_0 = \nu_1 = \nu_2 = 0$ .

The moduli space of connections on the trivial bundle thus is not separated. Yet it is possible to define a separated moduli space of connections by excluding reducible connections of type (1) and (3) for example: consider the (real-algebraic) family of connections of the form

$$d + \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

such that

$$|\beta| = |\gamma|,$$

where  $|\cdot|$  is defined as

$$\left| \lambda_0 \frac{dx}{y} + \lambda_1 x \frac{dx}{y} \right| = \sqrt{|\lambda_0|^2 + |\lambda_1|^2}.$$

This family defines a moduli space of connections on the trivial bundle which can be identified via the determinant map with

$$H^0(\Omega_X^1 \otimes \Omega_X^1) \simeq \mathbb{C}^3.$$

**3.4. Semi-stable indecomposable bundles.** In this case, the bundle is a non-trivial extension  $0 \rightarrow L_0 \rightarrow E \rightarrow L_0^{-1} \rightarrow 0$  for some  $L_0 \in \text{Jac}(X)$ . It is identified with the corresponding trivial extension by the Narasimhan-Ramanan classifying map. For fixed  $L_0$ , the moduli space of those extensions is isomorphic to  $\mathbb{P}H^1(X, L_0^2)$  which, by Serre duality, identifies with  $\mathbb{P}H^0(X, L_0^{-2} \otimes \Omega_X^1)$ . Again, the discussion splits into two cases.

**3.4.1. Affine bundles.** When  $L_0^2 \neq \mathcal{O}_X$ , then  $\mathbb{P}H^0(X, L_0^{-2} \otimes \Omega_X^1)$  reduces to a single point: there is only one non-trivial extension up to isomorphism. For such a bundle  $E$ , a curious phenomenon occurs: *all connections on  $E$  are reducible, none of them is totally reducible.* Indeed,  $L_0$  is the unique subbundle of degree 0, but is not invariant by the hyperelliptic involution. Therefore, the vector bundle  $E$  itself is not  $\iota$ -invariant. This implies that the monodromy of a connection  $\nabla$  on  $E$  can be neither irreducible, nor totally reducible. We don't want to consider further this kind of bundle. Note that this phenomenon does not occur for higher genus (see [23], Prop. (3.3), p.70).

**3.4.2. The 1-dimensional family of unipotent bundles and its 15 twists.** When  $L_0^2 = \mathcal{O}_X$ , the moduli space of non-trivial extensions  $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$  is parametrized by  $\mathbb{P}H^1(X, \mathcal{O}_X) \simeq \mathbb{P}H^0(X, \Omega_X^1) \simeq \mathbb{P}^1$ ; we call *unipotent bundles* such bundles  $E$ . Following [32], the automorphism group of  $E$  is  $\text{Aut}(E) = \mathbb{G}_m \times \mathbb{G}_a$ . The action of  $\mathbb{G}_a$  is faithful in restriction to each fiber  $E_w$ , unipotent and fixing the unique subbundle  $\mathcal{O}_X \subset E$ ; the action of  $\mathbb{G}_m$  is scalar as usual.

For a convenient open covering  $(U_i)$  of  $X$ , the bundle  $E$  is defined by a matrix cocycle of the form

$$M_{ij} = \begin{pmatrix} 1 & b_{ij} \\ 0 & 1 \end{pmatrix}$$

where  $(b_{ij}) \in H^1(X, \mathcal{O}_X)$  is a non trivial scalar cocycle. Moreover, from the short exact sequence

$$0 \rightarrow H^0(X, \Omega_X^1) \rightarrow H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0,$$

$(b_{ij})$  may be lifted to  $H^1(X, \mathbb{C})$ , so that  $E$  is flat: the local connections  $d_X$  over  $U_i$  glue together to form a global connection (non-canonical)  $\nabla_0$  with unipotent monodromy. Conversely, if a connection  $(E, \nabla)$  has unipotent monodromy, defined by say

$$A_1 = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

(with respect to the standard basis (2)), then  $E$  is either the trivial bundle, or a unipotent bundle; in fact, we are in the former case if, and only if,  $(a_1, b_1, a_2, b_2)$  is the period data of a holomorphic 1-form on  $X$ .

**Proposition 3.4.** *Let  $\nabla_0$  be a unipotent connection on  $E$  like above. Then the general connection on  $E$  can be described as*

$$\nabla = \nabla_0 + \lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3 + \lambda_4\theta_4$$

with  $(\lambda_i) \in \mathbb{C}^4$  so that the  $\mathbb{G}_a$ -action is given by

$$\left( c, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \right) \longrightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 - c\lambda_1 \\ \lambda_3 + 2c\lambda_2 - c^2\lambda_1 \\ \lambda_4 \end{pmatrix}$$

Moreover, reducible (resp. unipotent) connections are given by  $\lambda_1 = 0$  (resp.  $\lambda_1 = \lambda_2 = 0$ ). The moduli space of irreducible connections on  $E$  identifies with  $\mathbb{C}^* \times \mathbb{C}^2$ .

*Proof.* A general trace-free connection on  $E$  is defined by a collection

$$d + \theta_i \quad \text{where} \quad \theta_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & -\alpha_i \end{pmatrix}$$

are matrices of holomorphic 1-forms on  $U_i$  satisfying the compatibility condition

$$\theta_j = M_{ij}^{-1} \theta_i M_{ij} + M_{ij}^{-1} dM_{ij}$$

on  $U_i \cap U_j$  or, equivalently,

$$(5) \quad \begin{cases} \alpha_i - \alpha_j & = b_{ij} \gamma_i \\ \beta_i - \beta_j & = -2b_{ij} \alpha_i + b_{ij}^2 \gamma_i \\ \gamma_i - \gamma_j & = 0 \end{cases}$$

When  $\alpha_i = \gamma_i = 0$ , we precisely obtain all those connections with unipotent monodromy on  $E$ ; the second equation (5) then tells us that  $(\beta_i)$  defines a global holomorphic 1-form  $\beta \in H^0(X, \Omega_X^1)$ .

When  $\gamma_i = 0$ , we get all reducible connections on  $E$ . The first equation (5) tells us that  $(\alpha_i)$  defines a global holomorphic 1-form  $\alpha \in H^0(X, \Omega_X^1)$ . To solve the second equation (5), we need that the image under Serre duality

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X) \times H^0(X, \Omega_X^1) & \rightarrow & H^1(X, \Omega_X^1) \xrightarrow{\sim} \mathbb{C} \\ (b_{ij}, \alpha) & \mapsto & (b_{ij} \alpha) \end{array}$$

is the zero cocycle. In other words,  $\alpha$  must belong to the orthogonal  $(b_{ij})^\perp$  (with respect to Serre duality). In this case, we can solve  $(\beta_i)$ , and the solution is unique up to addition by a global holomorphic 1-form  $\beta$ .

Irreducible connections occur for  $\gamma \neq 0$  (note that the third equation (5) states that  $(\gamma_i)$  is a global 1-form). Then, the first equation (5) imposes that  $\gamma \in (b_{ij})^\perp$  (the orthogonal for Serre duality). Then, the collection  $(\alpha_i)$  solving the cocycle  $(b_{ij} \gamma)$  is unique up to the addition of a global holomorphic 1-form  $\alpha \in H^0(X, \Omega_X^1)$ . Finally, to solve the second equation in  $(\beta_i)$ , we have to insure that the cocycle

$$(-2b_{ij} \alpha_i + b_{ij}^2 \gamma) \in H^1(X, \Omega_X^1)$$

is zero, which can be achieved by conveniently using the freedom  $\alpha$  when solving the first equation. Precisely, if we add some global 1-form  $\alpha$  to the collection  $(\alpha_i)$ , then we translate the previous cocycle by  $(-2b_{ij} \alpha)$ ; for a convenient choice of  $\alpha$  (or  $(\alpha_i)$ ), the cocycle becomes trivial. Note that we still have the freedom to add any 1-form  $\alpha$  belonging to the orthogonal  $(b_{ij})^\perp$ . Finally, we can find a solution  $(\beta_i)$  which is unique up to addition by a global holomorphic 1-form  $\beta \in H^0(X, \Omega_X^1)$ .

Finally, given an irreducible connection as above, defined by  $(\alpha_i)$ ,  $(\beta_i)$  and  $\gamma \neq 0$ , and given a global holomorphic 1-form  $\beta \notin (b_{ij})^\perp$ , it follows from above case-by-case discussion that any connection  $\nabla$  on  $E$  takes the form

$$d_X + \lambda_1 \begin{pmatrix} \alpha_i & \beta_i \\ \gamma & -\alpha_i \end{pmatrix} + \lambda_2 \begin{pmatrix} \gamma & -2\alpha_i \\ 0 & -\gamma \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

over charts  $U_i$ , for convenient scalars  $\lambda_i$ . Unipotent bundle automorphisms are given in these charts by a constant matrix  $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$ , with  $c \in \mathbb{C}$  not depending on  $U_i$ , and it is straightforward to check that the action on  $\lambda_i$  is the one of the statement.  $\square$

**3.5. Unstable and indecomposable: the 6 + 10 Gunning bundles.** There are 16 theta characteristics, i.e. square-roots of  $\Omega_X^1 = \mathcal{O}_X(K_X)$ . They split into

- 6 odd theta characteristics  $\mathcal{O}_X([w_i])$ ,  $i = 0, 1, r, s, t, \infty$ ;
- 10 even theta characteristics  $\mathcal{O}_X([w_i] + [w_j] - [w_\infty])$ ,  $i \neq j \neq \infty$ .

Given a theta characteristic  $\kappa$ , there is a unique non-trivial extension  $0 \rightarrow \kappa \rightarrow E_\kappa \rightarrow \kappa^{-1} \rightarrow 0$  up to isomorphism, which is called the *Gunning bundle*  $E_\kappa$  associated to  $\kappa$ . We talk about *even or odd* Gunning bundle depending on the nature of  $\kappa$ . We have  $\text{Aut}(E_\kappa) \simeq \mathbb{G}_m \ltimes H^0(X, \Omega_X^1)$  (see [32]); the group  $H^0(X, \Omega_X^1)$  is acting by unipotent bundle automorphisms on fibers  $E|_w$ , fixing the subbundle  $\kappa$ .

The Narasimhan-Ramanan modular map is not well-defined on those bundles, but is in the closure: there are deformations of  $E$  such that all neighbouring bundles are stable. Precisely, the bundle  $E$  is arbitrarily close to semi-stable extensions  $0 \rightarrow \kappa^{-1} \rightarrow E' \rightarrow \kappa \rightarrow 0$  (see Proposition 3.5). Those latter ones are such that the corresponding divisor  $D_{E'} \sim 2\Theta$  (see section 3.2) passes through the point  $\kappa$  on  $\text{Pic}^1(X)$ . They define a 2-plane in  $\mathcal{M}_{NR}$  which we will call *Gunning plane* and denote it by  $\Pi_\kappa$ .

The intersection of  $\Pi_\kappa$  with the Kummer surface is easily described as

$$\Pi_\kappa \cap \text{Kum}(X) = \{L_0 \oplus L_0^{-1} \mid L_0 \in \kappa^{-1} \cdot \Theta\}.$$

In fact, these 16 planes  $\Pi_\kappa$  are well-known; each of them is tangent to the Kummer surface along a conic passing through 6 of the 16 nodes. The above description gives a natural parametrization of the hyperelliptic cover of this marked conic by the curve  $X$  itself (via the  $\Theta$  divisor). Precisely, for each  $\Pi_\kappa$ , the 6 corresponding nodes are those parametrized by the 2-torsion points  $\kappa^{-1} \otimes \mathcal{O}([w_i])$  where  $w_i$  runs over the six Weierstrass points. Conversely, through each node pass 6 of the 16 planes. This so-called (16, 6) configuration is classical (see [24, 20]) as well as the interpretation in terms of the moduli space of vector bundles (see [35, 11]). However, the interpretation of  $\Pi_\kappa$  in terms of accessible points for Gunning bundles seems to be new so far.

**Proposition 3.5.** *Given two extensions*

$$0 \rightarrow L \rightarrow E_0 \rightarrow L' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow L' \rightarrow E'_0 \rightarrow L \rightarrow 0$$

*of the same (but permuted) line bundles, there are two deformations  $E_t$  and  $E'_t$  of these bundles (parametrized by  $\mathbb{A}^1$ ) such that  $E_t \simeq E'_t$  for  $t \neq 0$ .*

*Proof.* The vector bundles  $E_0$  and  $E'_0$  are respectively defined by a cocycle of the form

$$(6) \quad \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & d_{ij} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{ij} & 0 \\ c_{ij} & d_{ij} \end{pmatrix}$$

for a convenient open covering  $(U_i)$  of  $X$ . Here,  $(a_{ij})$  and  $(d_{ij})$  are cocycles respectively defining  $L$  and  $L'$ . We claim that this can be succeeded with only two Zariski open sets  $X = U_1 \cup U_2$  so that we can neglect the cocycle condition. Before proving this claim, let us show how to conclude the proof. Consider the deformations  $E_t$  and  $E'_t$  respectively defined by

$$\begin{pmatrix} a_{ij} & b_{ij} \\ tc_{ij} & d_{ij} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{ij} & tb_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}.$$

They define the same vector bundle up to isomorphism for  $t \neq 0$  since these cocycles are conjugated by the automorphism of  $L \oplus L'$  defined in the matrix way by  $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$ . On the other hand, we clearly have  $E_t \rightarrow E_0$  and  $E'_t \rightarrow E'_0$  when  $t \rightarrow 0$ . For a general open

covering, these matrices fail to satisfy the cocycle condition  $A_{ij}A_{jk}A_{ki} = I$ ; this is why we need to work with only two open sets.

Although it might be standard, let us prove the claim. Up to tensoring by a very ample line bundle  $\tilde{L} = \mathcal{O}_X(\tilde{D})$ , we may assume that  $L, L', E_0$  and  $E'_0$  are all generated by global holomorphic sections. Choose one such section  $s_1$  for  $L$ ; it is then easy to construct another section  $s_2$  such that the corresponding (effective) divisors  $D_1$  and  $D_2$  have disjoint support. Indeed, given any non-zero section  $s_2$ , for any common zero with  $s_1$  one can find some section  $s$  non-vanishing at that point: one can then perturbate  $s_2 := s_2 + \epsilon \cdot s$ . This means that  $L$  may be trivialized on each open set  $U_i = X \setminus \text{supp}(D_i)$ ,  $i = 1, 2$ , and therefore defined with respect to this covering by a single cocycle  $a_{12}$ . In a similar way, we can construct sections  $\sigma_1$  and  $\sigma_2$  of  $E_0$  such that the two sections  $s_i \wedge \sigma_i$  of  $\det(E_0)$  have disjoint zeroes. In other words, possibly by deleting more points in the open sets  $U_i$ , the vector bundle  $E_0$  can simultaneously be trivialized on each of these open sets, and is therefore defined by a cocycle of the above form. To deal simultaneously with  $L'$  and  $E'_0$ , the easiest way is to consider the zero set of sections  $s_i \wedge \sigma_i \wedge s'_i \wedge \sigma'_i$  of  $\det(E_0 \oplus E'_0)$ . Finally, the same manipulation can be done with sections  $\tilde{s}_i$  of the ample bundle  $\tilde{L}$ : considering the zeros of sections  $s_i \wedge \sigma_i \wedge s'_i \wedge \sigma'_i \wedge \tilde{s}_i$  of  $\det(E_0 \oplus E'_0 \oplus \tilde{L})$  we can assume that the sections  $\sigma_i, s_i, s'_i$  and  $\tilde{s}_i$  have no common zeroes for  $i = 1, 2$ . Tensoring with  $\tilde{L}^{\otimes -1}$  we have constructed bases  $(s_i, \sigma_i)$  (resp.  $(\sigma'_i, s'_i)$ ) of  $E|_{U_i}$  (resp.  $E'|_{U_i}$ ) with  $i = 1, 2$  such that the corresponding cocycles of  $E$  and  $E'$  respectively are of the form (6).  $\square$

A connection  $\nabla$  on  $E$  necessarily satisfies Griffiths transversality with respect to the flag  $0 \subset \kappa \subset E_\kappa$  and defines an "oper" (see [5]). Following [22], the data of  $\nabla$  up to automorphism of  $E$  is equivalent to the data of a projective structure on  $X$ . Moreover, any two projective structures differ on  $X$  by a quadratic differential: once a projective structure has been chosen, the moduli space identifies with  $H^0(X, \mathcal{O}_X(2K_X))$ . However, there is no natural choice of "origin", *i.e.* there is no canonical projective structure on  $X$  from an algebraic point of view (see [28]). *The moduli space of (irreducible) connections on  $E_\kappa$  is therefore an  $\mathbb{A}^3$ -affine space.*

**3.6. Computation of a system of coordinates.** For all computations, the curve  $X$  is the smooth compactification of the affine complex curve defined by

$$X : y^2 = x(x-1)(x-r)(x-s)(x-t)$$

where  $0, 1, r, s, t \in \mathbb{C}$  are pair-wise distinct; we denote by  $\infty$  the point at infinity.

Let us first calculate a basis of  $H^0(\text{Pic}^1(X), \mathcal{O}_X(2\Theta))$  in order to introduce explicit projective coordinates on the three-dimensional projective space

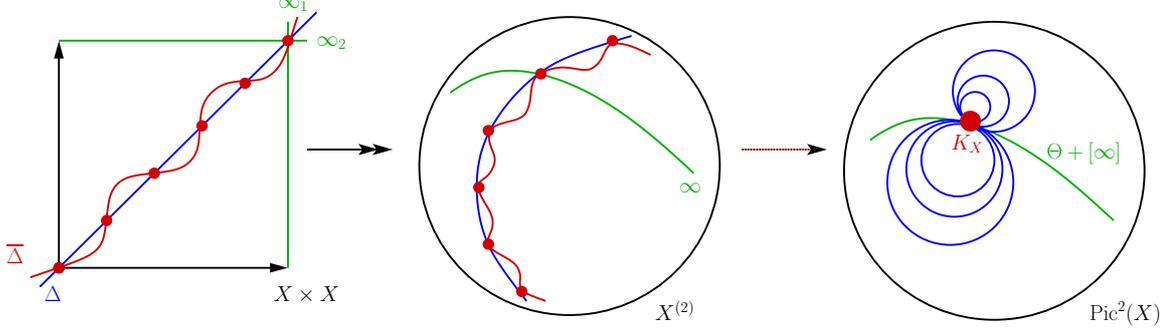
$$\mathbb{P}_{NR}^3 := \mathbf{P}H^0(\text{Pic}^1(X), \mathcal{O}_X(2\Theta)).$$

Since  $\text{Pic}^1(X)$  is birationally equivalent to the symmetric product  $X^{(2)}$ , rational functions on  $\text{Pic}^1(X)$  can be conveniently expressed as symmetric rational functions on  $X \times X$ .

$$\begin{array}{ccc} X \times X & \longrightarrow & X^{(2)} \xrightarrow{\phi^{(2)}} \text{Pic}^2(X) \xrightarrow{\sim} \text{Pic}^1(X) \\ & & \{P, Q\} \longmapsto [P] + [Q] \end{array}$$

The pull-back of the divisor  $\Theta \subset \text{Pic}^1(X)$  (resp.  $\Theta + [\infty] \subset \text{Pic}^2(X)$ ) to  $X \times X$  is  $\overline{\Delta} + \infty_1 + \infty_2$ , where

- $\overline{\Delta}$  is the anti-diagonal  $\{(P, Q) \in X \times X \mid Q = \iota(P)\}$ ,
- $\infty_1$  is the divisor  $\{\infty\} \times X$  and
- $\infty_2$  the divisor  $X \times \{\infty\}$ .



The pull-back to  $X \times X$  of  $2\Theta$ , viewed as a divisor on  $\text{Pic}^1(X)$  is then

$$2\overline{\Delta} + 2\infty_1 + 2\infty_2.$$

**Lemma 3.6.** *Let  $(P_1, P_2) = ((x_1, y_1), (x_2, y_2))$  be coordinates of  $X \times X$ . Then*

$$H^0(X \times X, \mathcal{O}_X^{sym}(2\overline{\Delta} + 2\infty_1 + 2\infty_2)) = \text{Vect}_{\mathbf{C}}(1, \text{Sum}, \text{Prod}, \text{Diag})$$

with

$$(7) \quad \begin{aligned} 1 &: (P_1, P_2) \mapsto 1 \\ \text{Sum} &: (P_1, P_2) \mapsto x_1 + x_2 \\ \text{Prod} &: (P_1, P_2) \mapsto x_1 x_2, \\ \text{Diag} &: (P_1, P_2) \mapsto \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 - (x_1 + x_2)^3 + (1 + \sigma_1)(x_1 + x_2)^2 + \\ &\quad + x_1 x_2 (x_1 + x_2) - (\sigma_1 + \sigma_2)(x_1 + x_2) \end{aligned}$$

where  $\sigma_1, \sigma_2$  and  $\sigma_3$  are the following constants:  $\sigma_1 = r + s + t, \sigma_2 = rs + st + tr, \sigma_3 = rst$ .

*Proof.* We have  $h^0(X \times X, \mathcal{O}_X^{sym}(2\overline{\Delta} + 2\infty_1 + 2\infty_2)) = h^0(\text{Pic}^1(X), \mathcal{O}(2\Theta)) = 4$  (see [35] or [34]). The function *Diag* can be rewritten as

$$(8) \quad \text{Diag} = \frac{1}{(x_1 - x_2)^2} \cdot \left[ -2y_1 y_2 - 2(1 + \sigma_1)x_1^2 x_2^2 - (\sigma_2 + \sigma_3)(x_1^2 + x_2^2) + (x_1 + x_2) \cdot (x_1^2 x_2^2 + (\sigma_1 + \sigma_2)x_1 x_2 + \sigma_3) \right]$$

The expression of *Diag* in (7) shows that it has no poles off the anti-diagonal and the infinity (and in particular no poles on the diagonal). From the expression (8) follows easily that *Diag* has polar divisor  $2\overline{\Delta} + 2\infty_1 + 2\infty_2$ . Indeed, if  $u_1$  is the local parameter for  $X_1$  near  $\infty_1$  defined by  $x_1 = \frac{1}{u_1}$ , then the principal part of the generating functions is given by

$$1, \quad \text{Sum} = \frac{1}{u_1^2} + x_2, \quad \text{Prod} = \frac{x_2}{u_1^2} \quad \text{and} \quad \text{Diag} \sim \frac{x_2^2}{u_1^2} - \frac{y_2}{u_1} + \dots$$

As a section of  $H^0(\text{Pic}^1(X), \mathcal{O}_X(2\Theta))$ , the function 1 vanishes twice along  $\Theta$  while the other ones do not vanish identically.  $\square$

In the sequel, denote by  $(v_0 : v_1 : v_2 : v_3)$  the projective coordinate on  $\mathbb{P}_{NR}^3$  representing the function

$$v_0 + v_1 \cdot \text{Sum} + v_2 \cdot \text{Prod} + v_3 \cdot \text{Diag}.$$

In order to compute the strictly semi-stable locus, namely the Kummer surface embedded in  $\mathcal{M}_{NR}$ , it is enough to consider the image in  $\mathbb{P}_{NR}^3$  of decomposable semi-stable bundles. Let  $L = \mathcal{O}_X([\underline{P}_1] + [\underline{P}_2] - [\infty]) \in \text{Pic}^1(X)$  be a line bundle such that  $L^2 \neq \mathcal{O}(K_X)$  and denote by  $\tilde{L}$  the associated degree 0 bundle  $\tilde{L} = \mathcal{O}_X([\underline{P}_1] + [\underline{P}_2] - 2[\infty])$ . Let us now calculate the explicit coordinates of the corresponding Narasimhan-Ramanan divisor  $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$  on  $\text{Pic}^1(X)$ , which is linearly equivalent to the divisor  $2\Theta$  (see section 3.3). The first component  $\tilde{L} \cdot \Theta$  is parametrized by

$$X \rightarrow \text{Pic}^1(X) ; Q \mapsto [\underline{P}_1] + [\underline{P}_2] + [Q] - 2[\infty].$$

Setting  $[\underline{P}_1] + [\underline{P}_2] + [Q] - 2[\infty] \sim [\underline{P}_1] + [\underline{P}_2] - [\infty]$ , we get that  $[\underline{P}_1] + [\underline{P}_2] + [Q]$  belongs to the linear system  $[\underline{P}_1] + [\underline{P}_2] + [\infty]$ . This latter one is generated by the two functions 1 and  $f(P) := \frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2}$  on the curve. Therefore,  $[\underline{P}_1] + [\underline{P}_2] - [\infty] \in \tilde{L} \cdot \Theta$  (the support of) if, and only if,  $f(\underline{P}_1) = f(\underline{P}_2)$ ; this gives the following equation for  $P_1 = (x_1, y_1)$  and  $P_2 = (x_2, y_2)$ :

$$\frac{y_1 + y_1}{x_1 - x_1} - \frac{y_1 + y_2}{x_1 - x_2} = \frac{y_2 + y_1}{x_2 - x_1} - \frac{y_2 + y_2}{x_2 - x_2}.$$

The equation for the other component  $\tilde{L}^{-1} \cdot \Theta$  is deduced by changing signs  $y_i \rightarrow -y_i$  for  $i = 1, 2$ . Taking into account the two equations, we get an equation for  $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$ :

$$\left( \frac{y_1 + y_1}{x_1 - x_1} - \frac{y_1 + y_2}{x_1 - x_2} - \frac{y_2 + y_1}{x_2 - x_1} + \frac{y_2 + y_2}{x_2 - x_2} \right) \left( \frac{-y_1 + y_1}{x_1 - x_1} - \frac{-y_1 + y_2}{x_1 - x_2} - \frac{-y_2 + y_1}{x_2 - x_1} + \frac{-y_2 + y_2}{x_2 - x_2} \right) = 0$$

which, after reduction, writes

$$(9) \quad -\text{Diag}(\underline{P}_1, \underline{P}_2) \cdot 1 + \text{Prod}(\underline{P}_1, \underline{P}_2) \cdot \text{Sum} - \text{Sum}(\underline{P}_1, \underline{P}_2) \cdot \text{Prod} + 1 \cdot \text{Diag} = 0$$

**Remark 3.7.** *The symmetric form of this equation is due to the fact that for any vector bundle  $E \in \mathcal{M}_{NR}$  and any line bundle  $L \in \text{Pic}^1(X)$  such that  $h^0(X, E \otimes L) > 0$ , the divisor  $D_E$  associated to  $E$  and the divisor  $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$  associated to  $\tilde{L} \oplus \tilde{L}^{-1}$  intersect precisely in  $L$  and  $\iota(L)$  on  $\text{Pic}^1(X)$ .*

Hence, according to equation (9), the Kummer embedding

$$\begin{aligned} \text{Jac}(X) &\rightarrow \text{Kum}(X) \subset \mathbb{P}_{NR}^3 \\ \mathcal{O}_X([\underline{P}_1] + [\underline{P}_2] - 2[\infty]) &\mapsto (v_0 : v_1 : v_2 : v_3) \end{aligned}$$

is explicitly given by

$$(10) \quad (v_0 : v_1 : v_2 : v_3) = (-\text{Diag}(\underline{P}_1, \underline{P}_2) : \text{Prod}(\underline{P}_1, \underline{P}_2) : -\text{Sum}(\underline{P}_1 : \underline{P}_2) : 1)$$

One can now eliminate parameters  $\underline{P}_1$  and  $\underline{P}_2$  from (10) as follows: express  $y_1 y_2$  in terms of functions  $\underline{x}_1 + \underline{x}_2$  and  $\underline{x}_1 \underline{x}_2$  and variable  $v_0/v_3$ , so that the square can be replaced by

$$\left( \frac{y_1 y_2}{x_1 x_2} \right)^2 = \prod_{w=0,1,r,s,t} (w^2 - (\underline{x}_1 + \underline{x}_2)w + \underline{x}_1 \underline{x}_2);$$

then replace  $\underline{x}_1 \underline{x}_2$  and  $\underline{x}_1 + \underline{x}_2$  by  $v_1/v_3$  and  $-v_2/v_3$  respectively. We get

$$\begin{aligned} \text{Kum}(X) : \\ 0 = & (v_0 v_2 - v_1^2)^2 \cdot 1 \\ & -2 [[(\sigma_1 + \sigma_2)v_1 + (\sigma_2 + \sigma_3)v_2](v_0 v_2 - v_1^2) \\ & + 2(v_0 + \sigma_1 v_1)(v_0 + v_1)v_1 + 2(\sigma_2 v_1 + \sigma_3 v_2)(v_1 + v_2)v_1] \cdot v_3 \\ & -2\sigma_3(v_0 v_2 - v_1^2) + [[(\sigma_1 + \sigma_2)^2 v_1 + (\sigma_2 + \sigma_3)^2 v_2] (v_1 + v_2) \\ & - (\sigma_1 + \sigma_3)^2 v_1 v_2 + 4[(\sigma_2 + \sigma_3)v_0 - \sigma_3 v_2]v_1] \cdot v_3^2 \\ & -2\sigma_3 [(\sigma_1 + \sigma_2)v_1 - (\sigma_2 + \sigma_3)v_2] \cdot v_3^3 \\ & + \sigma_3^2 \cdot v_3^4. \end{aligned}$$

Here, we see that  $v_3 = 0$  is a (Gunning-) plane tangent to  $\text{Kum}(X)$  along a conic.

Following formula (10), we can compute the locus of the trivial bundle  $E_0$  and its 15 twists  $E_\tau := E_0 \otimes \mathcal{O}_X(\tau)$ , where  $\tau = [w_i] - [w_j]$  with  $i \neq j$ .

$E_\tau$	$(v_0 : v_1 : v_2 : v_3)$
$E_{[w_0]-[w_1]}$	$(rs + st + rt : 0 : -1 : 1)$
$E_{[w_0]-[w_r]}$	$(r(st + s + t) : 0 : -r : 1)$
$E_{[w_0]-[w_s]}$	$(s(rt + r + t) : 0 : -s : 1)$
$E_{[w_0]-[w_t]}$	$(t(rs + r + s) : 0 : -t : 1)$
$E_{[w_1]-[w_\infty]}$	$(1 : 0 : 0 : 0)$
$E_{[w_1]-[w_r]}$	$(1 : -1 : 1 : 0)$
$E_{[w_1]-[w_s]}$	$(r^2 : -r : 1 : 0)$
$E_{[w_1]-[w_t]}$	$(s^2 : -s : 1 : 0)$
$E_{[w_r]-[w_\infty]}$	$(1 : 0 : 0 : 0)$
$E_{[w_r]-[w_s]}$	$((1 + r)st : r : -1 - r : 1)$
$E_{[w_r]-[w_t]}$	$((1 + s)rt : s : -1 - s : 1)$
$E_{[w_s]-[w_\infty]}$	$(1 : 0 : 0 : 0)$
$E_{[w_s]-[w_r]}$	$((1 + t)rs : t : -1 - t : 1)$
$E_{[w_s]-[w_t]}$	$((r + s)t : rs : -r - s : 1)$
$E_{[w_t]-[w_\infty]}$	$(1 : 0 : 0 : 0)$
$E_{[w_t]-[w_r]}$	$((r + t)s : rt : -r - t : 1)$
$E_{[w_t]-[w_s]}$	$((s + t)r : st : -s - t : 1)$

The Gunning planes  $\Pi_\kappa$  are the planes passing through 6 of these 16 singular points. Precisely, the odd Gunning plane with  $\kappa = [w_i]$  is passing through all  $E_\tau$  with  $\tau = [w_i] - [w_j]$  (including the trivial bundle  $E_0$  for  $i = j$ ); for an even Gunning plane with  $\kappa = [w_i] + [w_j] - [w_k] \sim [w_l] + [w_m] - [w_n]$ , where  $\{i, j, k, l, m, n\} = \{0, 1, r, s, t, \infty\}$ , we get

$$\left. \begin{aligned} & E_{[w_i]-[w_j]}, E_{[w_j]-[w_k]}, E_{[w_i]-[w_k]} \\ & E_{[w_l]-[w_m]}, E_{[w_m]-[w_n]}, E_{[w_l]-[w_n]} \end{aligned} \right\} \in \Pi_{[w_i]+[w_j]-[w_k]} = \Pi_{[w_l]+[w_m]-[w_n]}.$$

In particular, we can derive explicit equations, for instance:

$\Pi_{[w_0]}$	$v_1 = 0$
$\Pi_{[w_1]}$	$v_1 + v_2 + v_3 = 0$
$\Pi_{[w_\infty]}$	$v_3 = 0$
$\Pi_{[w_0]+[w_1]-[w_\infty]}$	$v_0 + v_1 = (rs + st + rt)v_3$

We can also compute the 16-order linear group given by twisting the general bundle  $E$  by a 2-torsion line bundle  $\mathcal{O}_X(\tau)$ ,  $\tau = [w_i] - [w_j]$ , by looking at the induced permutation on Kummer's singular points. For instance, we get

$$(v_0 : v_1 : v_2 : v_3) \xrightarrow{\otimes E_{[w_0]-[w_\infty]}} ((\sigma_2 + \sigma_3)v_1 + \sigma_3 v_2 : \sigma_3 v_3 : v_0 - (\sigma_2 + \sigma_3)v_3 : v_1)$$

$$(v_0 : v_1 : v_2 : v_3) \xrightarrow{\otimes E_{[w_1]-[w_\infty]}} (v_0 : v_1 : v_2 : v_3) \cdot \begin{pmatrix} 1 & \sigma_1 + \sigma_3 & \sigma_2 & 0 \\ -1 & -1 & 0 & \sigma_2 \\ 1 & 0 & -1 & -(\sigma_1 + \sigma_3) \\ 0 & 1 & 1 & 1 \end{pmatrix}^T$$

One can find in [24, 20] classical equations for Kummer surfaces which are nicer than the above one, but no more rational in  $(r, s, t)$ . For instance, we can choose  $E_0$ ,  $E_{[w_0]-[w_1]}$ ,  $E_{[w_1]-[w_\infty]}$  and  $E_{[w_0]-[w_\infty]}$  as a projective frame so that Gunning bundles  $\Pi_{[w_0]}$ ,  $\Pi_{[w_1]}$ ,  $\Pi_{[w_\infty]}$  and  $\Pi_{[w_0]+[w_1]-[w_\infty]}$  become coordinate hyperplanes. The Kummer equation therefore becomes quadratic in each coordinate. However, to reach the nice form given in §54 (page 83) of [24], we must choose square roots  $\lambda^2 = rst$  and  $\mu^2 = (r-1)(s-1)(t-1)$ . Then, setting

$$(u_0 : u_1 : u_2 : u_3) = (\lambda\mu(v_0 + v_1 - (rs + st + rt)v_3) : \lambda v_1 : \mu(v_1 + v_2 + v_3) : v_3),$$

we get the new equation

$$\begin{aligned} \text{Kum}(X) : \\ 0 = & (\lambda\mu v_1 v_2 + \lambda\mu^2 v_0 v_2 + \lambda^2 \mu v_0 v_1) \cdot 1 \\ & + 2 [\lambda^3 \mu^3 v_0 (v_1^2 - v_2^2) + \lambda^3 \mu^2 v_1 (v_2^2 - v_0^2) + \lambda^2 \mu^3 v_2 (v_0^2 - v_1^2) \\ & \quad + \lambda^2 \mu^2 (2 - \sigma_1 - \sigma_2 + 2\sigma_3) u_0 u_1 u_2] \cdot v_3 \\ & (\lambda\mu v_0^2 + \lambda\mu^2 v_1^2 + \lambda^2 \mu v_2^2 - 2\lambda^3 \mu^3 v_1 v_2 - 2\lambda^3 \mu^2 v_0 v_2 - 2\lambda^2 \mu^3 v_0 v_1) \cdot v_3^2. \end{aligned}$$

In these coordinates, the translations computed above simply become:

$$\begin{aligned} (u_0 : u_1 : u_2 : u_3) & \xrightarrow{\otimes E_{[w_0]-[w_\infty]}} (u_2 : u_3 : u_0 : u_1) \\ (u_0 : u_1 : u_2 : u_3) & \xrightarrow{\otimes E_{[w_1]-[w_\infty]}} (u_1 : -u_0 : -u_3 : u_2). \end{aligned}$$

Another classical presentation of the Kummer surface consists in normalizing the finite translation group. For the coordinates

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \sigma_2 + \sigma_3 & \sqrt{\sigma_3} & \sigma_2 + \sigma_3 & -\sqrt{\sigma_3} \\ \sqrt{\sigma_3} & 0 & -\sqrt{\sigma_3} & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b & 0 & 0 \\ c & d & 0 & 0 \\ 0 & 0 & 1 & \tilde{b} \\ 0 & 0 & \gamma & \tilde{d} \end{pmatrix} \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

with

$$\begin{aligned} a &= -\frac{r(r+\gamma)-\sqrt{\sigma_3}}{\sqrt{r(r-1)(r-s)(r-t)}} \\ b &= \frac{1+\gamma-\sqrt{\sigma_3}}{\sqrt{-1+\sigma_1-\sigma_2+\sigma_3}} & \tilde{b} &= \frac{r(s+t+\gamma)-(r+1+\gamma)\sqrt{\sigma_3}}{\sqrt{r(r-1)(r-s)(r-t)}} \\ c &= \frac{(r^2+\sqrt{\sigma_3})\gamma+r(\sigma_2-\sigma_1+2(r-st))}{\sqrt{r(r-1)(r-s)(r-t)}} & \gamma &= \text{root of } \Gamma(X) \\ d &= -\frac{\sigma_1-\sigma_2+(\sqrt{\sigma_3}+1)\gamma}{\sqrt{-1+\sigma_1-\sigma_2+\sigma_3}} & \tilde{d} &= \frac{r(2r-(\sigma_1+\sigma_2)-(s+t)\gamma)+(\sigma_1-\sigma_2+2r(s+t)+(r+1)\gamma)\sqrt{\sigma_3}}{\sqrt{r(r-1)(r-s)(r-t)}} \end{aligned}$$

where

$$\begin{aligned}
\Gamma(X) = & \left[ s(r+t) - (s+1)\sqrt{\sigma_3} \right] \cdot X^4 \\
& + 2 \left[ s(\sigma_2 + r + t + 3rt) - (\sigma_1 + s(r+t+3))\sqrt{\sigma_3} \right] \cdot X^3 \\
& + 6 \left[ s^2(r+t+rt) + rst(2+\sigma_1) - (2\sigma_2 + rt(s-1) + s(s+1))\sqrt{\sigma_3} \right] \cdot X^2 \\
& + 2 \left[ rs(s+r(-1+2s) + s^2(1-r)) + st(-t+s(1+2t) + s^2(1-t)) \right. \\
& \quad \left. + (3(\sigma_1 + \sigma_2) + 2 + 5s + 2s^2 + rt)\sigma_3 \right. \\
& \quad \left. - [st(-st+2t+3s) + sr(-rs+2r+3s) + rt(r+t) \right. \\
& \quad \left. + 2\sigma_2 + s\sigma_1 - t^2 - r^2 + 3s^2 + \sigma_3(8 + \sigma_1 + s)]\sqrt{\sigma_3} \right] \cdot X \\
& + 4\sigma_3^2 + \sigma_3(\sigma_1 + 3s + 4s^2 + (s^2 + rt)(r+t) + 4\sigma_2) - (r^3 + t^3)(s-1)^2 s \\
& \quad + (r+t)s^3 + [(r-t)^2 - 1]s^3 - ((r-t)^2 + 1 + 4(r+t) + 2rt)s^2 \\
& \quad - (r^2 + t^2 + r^2t^2)s + (t-r)^2 - 4\sigma_3(\sigma_1 + 1) - r^2t^2 \Big] \sqrt{\sigma_3} \cdot 1
\end{aligned}$$

we get the following very nice equation of the Kummer surface (see §53 page 80-81 of [24])

$$(11) \quad \begin{aligned} & (t_0^2 + t_1^2 + t_2^2 + t_3^2) + 2D(t_0t_1t_2t_3) \\ & + A(t_0^2t_3^2 + t_1^2t_2^2) + B(t_1^2t_3^2 + t_0^2t_2^2) + C(t_2^2t_3^2 + t_0^2t_1^2) = 0 \end{aligned}$$

where coefficients  $A, B, C, D$  depend on  $(r, s, t)$  (in an algebraic way) and satisfy the following relation

$$4 - A^2 - B^2 - C^2 + ABC + D^2 = 0.$$

Here the 16-order translation group is generated by double-transpositions of variables and double changes of signs:

$\tau$	$(t_0 : t_1 : t_2 : t_3) \otimes E_\tau$
0	$(t_0 : t_1 : t_2 : t_3)$
$[w_0] - [w_\infty]$	$(t_0 : t_1 : -t_2 : -t_3)$
$[w_1] - [w_\infty]$	$(t_3 : -t_2 : t_1 : -t_0)$
$[w_r] - [w_\infty]$	$(t_2 : -t_3 : t_0 : -t_1)$
$[w_s] - [w_\infty]$	$(t_2 : t_3 : -t_0 : -t_1)$
$[w_t] - [w_\infty]$	$(t_3 : t_2 : t_1 : t_0)$

The five  $t$ -polynomials occurring in the Kummer equation (11) are fundamental invariants for the action of the translation group and define a natural map  $\mathbb{P}_{NR}^3 \rightarrow \mathbb{P}^4$  whose image is a quartic hypersurface (see [15], Proposition 10.2.7).

**Corollary 3.8.** *The quartic in  $\mathbb{P}^4$  defined by the natural map  $\mathbb{P}_{NR}^3 \rightarrow \mathbb{P}^4$  is a coarse moduli space of  $S$ -equivalence classes of semi-stable  $\mathbb{P}^1$ -bundles over  $X$ .*

**Remark 3.9.** *Recall that a  $\mathbb{P}^1$ -bundle  $P$  over  $X$  is called semi-stable if  $\#(s, s) \geq 0$  for every section  $s : X \rightarrow P$ . If  $E$  is a rank 2 vector bundle over  $X$  such that  $\mathbb{P}(E) = P$ , then the (semi-)stability of  $P$  is equivalent to the semi-stability of  $E$  [4].*

*Proof.* Let  $T$  be a smooth parameter space and  $\mathcal{P} \rightarrow X \times T$  a family of  $\mathbb{P}^1$ -bundles over  $X$ . Denote by  $\pi_T$  the projection  $X \times T \rightarrow T$ . The  $\mathbb{P}^1$ -bundle  $\mathcal{P}$  lifts to a rank 2 bundle  $\mathcal{E} \rightarrow X \times T$  such that  $\det(\mathcal{E}) = \pi_T^* \mathcal{O}_X$  and  $\mathbb{P}(\mathcal{P}) = \mathcal{E}$ . This vector bundle is unique up to tensor product with  $\pi_T^*(L)$  where  $L$  is a 2-torsion line bundle on  $X$ . According to theorem 3.2, the classification map  $T \rightarrow \mathcal{M}_{NR}$  then is a morphism as is its composition

with the natural map  $\mathbb{P}_{NR}^3 \rightarrow \mathbb{P}^4$ . The resulting morphism  $T \rightarrow \mathbb{P}^4$  no longer depends on the choice of  $\mathcal{E}$ .  $\square$

#### 4. ANTI-CANONICAL SUBBUNDLES

Before describing the moduli space  $\mathfrak{Bun}(X/\iota)$  and the 2-fold ramified cover  $\mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$  in detail, let us give another interpretation and recall the classical approaches of Tyurin [41] and Bertram [6], as well as related works of Bolognesi [11, 12].

Let  $E$  be a flat vector bundle with trivial determinant bundle on  $X$ . Given an irreducible connection  $\nabla$  on  $E$ , corollary 2.3 provides a lift  $h : E \rightarrow \iota^*E$  of the hyperelliptic involution  $\iota : X \rightarrow X$  whose action on the Weierstrass fibers is non-trivial, with two distinct eigenvalues  $\pm 1$ . We want to understand which are the subbundles  $\mathcal{O}(-K_X) \hookrightarrow E$  and how  $h$  acts on this set. In section 4.1, we will prove that a generic  $E \in \mathfrak{Bun}(X)$  carries a 1-parameter family of such subbundles, only two of them being  $h$ -invariant:

- $L^+ \subset E$  on which  $h$  acts as  $\text{id}_{L^+}$ ,
- $L^- \subset E$  on which  $h$  acts as  $-\text{id}_{L^-}$ .

The two parabolic structures  $\mathbf{p}$  and  $\mathbf{p}'$  discussed in sections 2.2 and 2.3 are therefore respectively directed (over the Weierstrass points) by  $L^+$  and  $L^-$ .

By our main construction (section 2), we can reinterpret  $\mathfrak{Bun}(X/\iota)$  as the moduli space of hyperelliptic parabolic bundles  $(E, \mathbf{p})$  together with the forgetful map  $\mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X); (E, \mathbf{p}) \mapsto E$ . Another point of view arises from the moduli space of hyperelliptic flags  $(E, L)$  with  $E \supset L \simeq \mathcal{O}(-K_X)$ : Bertram considered in [6] the projective space of non-trivial extensions

$$0 \longrightarrow \mathcal{O}(-K_X) \longrightarrow E \longrightarrow \mathcal{O}(K_X) \longrightarrow 0$$

on which the hyperelliptic involution acts naturally. The invariant hyperplane, the set of hyperelliptic extensions, is a  $\mathbb{P}^3$ , naturally birational to  $\mathfrak{Bun}(X/\iota)$ . We will see that this  $\mathbb{P}^3$  naturally identifies with  $\text{Bun}_\mu^{ss}(X/\iota)$  for  $\frac{1}{6} < \mu < \frac{1}{4}$  and we thereby recover the nice description of Kumar and Bolognesi in [27, 11, 12]. Moreover, we explain the approach of Tyurin and deduce a parametrization of an open chart of  $\mathfrak{Con}(X)$  (a finite cover).

**4.1. Tyurin subbundles.** Let  $(E, \nabla)$  be an irreducible trace-free connection over  $X$ , and let  $h : E \rightarrow \iota^*E$  be the lift of the hyperelliptic involution  $\iota : X \rightarrow X$  given by corollary 2.3. Recall that  $h$  acts non-trivially with two distinct eigenvalues on each Weierstrass fiber  $E|_w$ . The involution  $\iota$  acts linearly on  $\mathcal{O}(-K_X)$  and therefore  $h$  acts on  $\text{Hom}(\mathcal{O}(-K_X), E)$ . Since it is involutive, this action induces a splitting  $\text{Hom}(\mathcal{O}(-K_X), E) = H^+ \oplus H^-$  into eigenspaces (relative to  $\pm 1$  eigenvalues). We call *Tyurin subbundle* of  $E$  the subbundles generated by non-trivial elements  $\varphi \in \text{Hom}(\mathcal{O}(-K_X), E)$ .

**Proposition 4.1.** *Let  $E$  and  $h$  be as above. The space of morphisms  $\text{Hom}(\mathcal{O}(-K_X), E)$  is 2-dimensional except in the following cases*

- $E$  is either unipotent, or an odd Gunning bundle, and then the dimension is 3,
- $E$  is the trivial bundle, and then the dimension is 4.

*If  $E$  is not an even Gunning bundle, the images of these morphisms span the vector bundle  $E$  at a generic point. The two eigenspaces  $H^+$  and  $H^-$  then have positive dimension; they correspond to morphisms into two distinct  $h$ -invariant subbundles,  $L^+$  and  $L^-$ . There are no other  $h$ -invariant Tyurin subbundles.*

**Remark 4.2.** *As we shall see in section 4.1.5, in the case of even Gunning bundles, the eigenspaces  $H^+$  and  $H^-$  still have positive dimension, but the associated  $h$ -invariant subbundles  $L^+$  and  $L^-$  are equal.*

*Proof.* First we have  $\mathrm{Hom}(\mathcal{O}(-K_X), E) \simeq H^0(E \otimes \mathcal{O}(K_X))$  and by the Riemann-Roch formula  $h^0(E \otimes \mathcal{O}(K_X)) - h^0(E) = 2$ . Here, we use Serre duality and the fact that  $E$  is selfdual (because  $\mathrm{rank}(E) = 2$  and  $\det(E) = \mathcal{O}_X$ ). We promptly deduce that  $h^0(E \otimes \mathcal{O}(K_X)) \geq 2$  and  $> 2$  if and only if  $E$  has non-zero sections or, equivalently, if it contains a subbundle  $L$  of the form  $L = \mathcal{O}_X$ ,  $\mathcal{O}_X([p])$  or  $\deg(L) > 1$ . Because of flatness (see section 3), the only possibilities are actually  $L = \mathcal{O}_X$  or  $\mathcal{O}_X([w])$  for some Weierstrass point  $w \in X$ .

When the image of a 2-dimensional subspace of  $\mathrm{Hom}(\mathcal{O}(-K_X), E)$  is degenerate, *i.e.* contained in a strict subbundle  $L \subset E$ , then  $h^0(L \otimes \mathcal{O}(K_X)) = 2$  which implies  $L = \mathcal{O}_X$  or  $L = \kappa$ , a theta characteristic. Yet in the cases when  $L$  is trivial or an odd theta characteristic, we have  $\dim(\mathrm{Hom}(\mathcal{O}(-K_X), E)) > 2 = \dim(\mathrm{Hom}(\mathcal{O}(-K_X), L))$  and thus not all morphisms take values into  $L$ : we get enough freedom to span  $E$  at a generic point.

Now, given two morphisms  $\varphi_i : \mathcal{O}(-K_X) \rightarrow E$  for  $i = 1, 2$ , taking value into two different subbundles  $L_i \subset E$ ,  $L_1 \neq L_2$ , we get a morphism  $\varphi_1 \oplus \varphi_2 : \mathcal{O}(-K_X) \oplus \mathcal{O}(-K_X) \rightarrow E$  whose image spans the vector bundle  $E$  at all fibers but those corresponding to the (effective) zero divisor of  $\varphi_1 \wedge \varphi_2 : \mathcal{O}(-2K_X) \rightarrow \mathcal{O}$ . Such a divisor takes the form  $[P_1] + [\iota(P_1)] + [P_2] + [\iota(P_2)]$  for some  $P_1, P_2 \in X$ . We thus get an isomorphism between the 2-dimensional vector space  $\mathrm{Vect}_{\mathbb{C}}(\varphi_1, \varphi_2) \subset \mathrm{Hom}(\mathcal{O}(-K_X), E)$  and the fiber of  $E$  over each point of  $X \setminus \{P_1, \iota(P_1), P_2, \iota(P_2)\}$ . In particular, over a Weierstrass point  $w \neq P_1, P_2$ , we have  $E|_w \simeq \mathrm{Vect}_{\mathbb{C}}(\varphi_1, \varphi_2)$  and since the action  $h$  on  $\mathrm{Hom}(\mathcal{O}(-K_X), E)$  is non-trivial, neither  $H^+$  nor  $H^-$  is reduced to  $\{0\}$ . Moreover,  $\varphi_1$  and  $\varphi_2$  cannot belong to a common eigenspace of the action of  $h$  on  $\mathrm{Hom}(\mathcal{O}(-K_X), E)$ . In other words, any two morphisms belonging to the same eigenspace  $H^{\pm}$  take image in the same subbundle, say  $L^{\pm}$ .

Let now  $L$  be a Tyurin subbundle distinct from  $L^+$  and  $L^-$ :  $L$  is generated by  $\varphi = \varphi_1 + \varphi_2$  for some  $\varphi_1 \in H^+$  and  $\varphi_2 \in H^-$ . Again, there is a Weierstrass point  $w$  where  $\varphi_1 \wedge \varphi_2$  does not vanish: the action of  $h$  is homothetic on the  $\varphi_i$  with opposite eigenvalues and cannot fix the direction  $\mathbb{C} \cdot \varphi(w)$ . Thus  $L$  is not  $h$ -invariant.  $\square$

Mind that a Tyurin subbundle  $L \subset E$  may be *degenerate*, *i.e.*  $L \not\cong \mathcal{O}(-K_X)$ . This so happens when the corresponding morphism  $\varphi \in \mathrm{Hom}(\mathcal{O}(-K_X), E)$  is not injective. Note that if the line bundles  $L^{\pm}$  are non-degenerate, they define the parabolic structures  $\mathfrak{p}^{\pm}$ . As we shall see, any flat vector bundle  $E$  has degenerate Tyurin subbundles; some of them can be  $h$ -invariant, even in the stable case.

In the following paragraphs, we will study the Tyurin subbundles for each type of bundle, following the list of section 3.

**4.1.1. Stable bundles.** When  $E$  is stable, any holomorphic connection is irreducible. Since the only bundle automorphisms of  $E$  are homothecies, the same bundle isomorphism  $h : E \rightarrow \iota^*E$  works for all connections and it therefore only depends on the bundle (up to a sign). The two  $h$ -invariant Tyurin bundles  $L^+$  and  $L^-$  depend (up to permutation) only on  $E$ .

Consider two elements  $\varphi^+, \varphi^- \in \mathrm{Hom}(\mathcal{O}(-K_X), E)$  generating  $L^+$  and  $L^-$  (at a generic point) and consider the divisor  $\mathrm{div}(\varphi^+ \wedge \varphi^-) = [P] + [\iota(P)] + [Q] + [\iota(Q)]$ . This divisor  $D_E^T \in |2K_X|$  is an invariant of the bundle, we call it the *Tyurin divisor*. Let

bundle type	degenerate invariant Tyurin subbundles	# $\left\{ \begin{array}{l} \text{parabolic} \\ \text{structures } \mathbf{p}^\pm \\ \text{(up to autom.)} \\ \text{determined by } L^\pm \end{array} \right\}$
stable off Gunning planes generic on $\Pi_{[w_i]}$ stable on $\Pi_{[w_i]} \cap \Pi_{[w_j]}$	$\emptyset$ $L^+ = \mathcal{O}_X(-[w_i])$ $L^+ = \mathcal{O}_X(-[w_i]), L^- = \mathcal{O}_X(-[w_j])$	2 out of 2 1 out of 2 0 out of 2
generic decomposable $L_0 \oplus L_0$ with $L^2 = \mathcal{O}_X$	$\emptyset$ $L^+ = L_0, L^- = L_0$	1 out of 1 1 out of 1
generic unipotent special unipotent twists of unipotent	$L^+ = \mathcal{O}_X$ $L^+ = \mathcal{O}_X, L^- = \mathcal{O}(-[w_i])$ $L^+ = \mathcal{O}_X([w_i] - [w_j])$	2 out of 2 1 out of 2 1 out of 2
even Gunning odd Gunning	$L^+ = L^- = \mathcal{O}_X(\kappa)$ $L^+ = \mathcal{O}_X(\kappa)$	2 out of 2 2 out of 2

TABLE 2. Summary of the Tyurin subbundles for the different types of bundles. Recall that non-degenerate invariant subbundles are by definition isomorphic to  $\mathcal{O}_X(-K_X)$ .

$D_E \in |2\Theta|$  be the divisor on  $\text{Pic}^1(X)$  defined by Narasimhan-Ramanan (see section 3.2).

**Proposition 4.3.** *Let  $E$  be stable. Then the divisor  $D_E^T$  is the intersection between the divisor  $D_E$  and the natural embedding  $X \rightarrow \Theta; P \mapsto [P]$  on  $\text{Pic}^1(X)$ :*

$$D_E^T = D_E \cdot \Theta.$$

For each point  $P$  of the support of  $D_E^T$ , there is exactly one subbundle  $L_P \equiv \mathcal{O}_X(-[P])$  of  $E$ . These are precisely the degenerate Tyurin subbundles. Such a degenerate Tyurin subbundle  $L_P$  is  $h$ -invariant if, and only if,  $P = w$  is a Weierstrass point. This so happens precisely when  $E$  lies on the odd Gunning plane  $\Pi_{[w]}$ .

*Proof.* First note that  $D_E^T = \text{div}(\varphi_1 \wedge \varphi_2)$  for any basis  $(\varphi_1, \varphi_2)$  of  $\text{Hom}(\mathcal{O}(-K_X), E)$ . A point  $P \in X$  belongs to the support of  $D_E^T$  if and only if  $\iota(P)$  does. This is equivalent to the fact that  $\varphi^+$  and  $\varphi^-$  are colinear at  $\iota(P)$ . Equivalently, there is a morphism  $\varphi_P \in \text{Hom}(\mathcal{O}(-K_X), E)$  which vanishes at  $\iota(P)$  (and can be completed to a basis with  $\varphi^+$  or  $\varphi^-$ ). By stability of the vector bundle  $E$ , the morphism  $\varphi_P$  cannot vanish elsewhere. Denote by  $L_P$  the line subbundle corresponding to  $\varphi_P$ . Finally, we have  $P \in D_E^T$  if and only if there is a line subbundle  $L_P$  of  $E$  such that  $L_P \simeq \mathcal{O}([\iota(P)] - K_X) = \mathcal{O}(-[P])$ . On the other hand,  $P$  belongs to the support of  $D_E \cdot \Theta$  if and only if there is a line subbundle  $L_P \simeq \mathcal{O}(-[P])$  of  $E$ . Since these divisors are generically reduced, we can conclude by continuity that  $D_E^T = D_E \cdot \Theta$ .

Now suppose  $E$  has two line subbundles  $L_P$ . A linear combination of the two corresponding homomorphisms in  $\text{Hom}(\mathcal{O}(-K_X), E)$  then would have a double zero at  $P$ , which is impossible by stability of  $E$ . So for each point  $P$  in the support of  $D_E^T$ , we get a unique subbundle  $L_P \simeq \mathcal{O}_X(-[P])$  and there are no other degenerate Tyurin subbundles.

Finally, note that the finite set of (at most 4) degenerate Tyurin subbundles must be  $h$ -invariant. Thus such a bundle  $L_P$  is invariant if, and only if,  $P$  is  $\iota$ -invariant.  $\square$

**Corollary 4.4.** *When  $E$  is stable and outside of odd Gunning planes  $\Pi_{[w_i]}$ , there are exactly two  $h$ -invariant subbundles  $L^+, L^- \simeq \mathcal{O}(-K_X)$  in  $E$  that are invariant under the hyperelliptic involution  $h$ . The two parabolic structures  $\mathbf{p}$  and  $\mathbf{p}'$  defined in sections 2.2 and 2.3 are directed by these two subbundles.*

Another important consequence of the proposition above is the Tyurin parametrization of the moduli space of stable bundles which relies on the following

**Corollary 4.5.** *When  $E$  is stable and the Tyurin divisor  $D_E^T = [P] + [\iota(P)] + [Q] + [\iota(Q)]$  is reduced (4 distinct points), then the natural map*

$$\varphi^+ \oplus \varphi^- : \mathcal{O}(-K_X) \oplus \mathcal{O}(-K_X) \rightarrow E$$

*is a positive elementary transformation for the parabolic structure defined over  $D_E^T = [P] + [\iota(P)] + [Q] + [\iota(Q)]$  by generators of  $L_P, L_{\iota(P)}, L_Q$  and  $L_{\iota(Q)}$ .*

We thus get a full set of invariants for generic bundles by considering the Tyurin divisor  $D_E^T \in |2K_X|$  and taking into account the cross-ratio of degenerate Tyurin subbundles.

**Remark 4.6.** *When  $E$  belongs to an odd Gunning plane  $\Pi_{[w]}$ , then one of the two  $h$ -invariant Tyurin subbundles is degenerate, say  $L^- = \mathcal{O}_X(-[w])$ , and fails to determine the parabolic structure  $\mathbf{p}^-$  over the Weierstrass point  $w$ . When  $E \in \Pi_{[w_i]} \cap \Pi_{[w_j]}$ , then the two  $h$ -invariant Tyurin subbundles are degenerate and neither  $\mathbf{p}^+$ , nor  $\mathbf{p}^-$  are determined by these bundles.*

**4.1.2. Generic decomposable bundles.** Let  $E = L_0 \oplus L_0^{-1}$ , where  $L_0 = \mathcal{O}([P] + [Q] - K_X)$  is not 2-torsion:  $L_0^2 \neq \mathcal{O}_X$ . There is (up to scalar multiple) a unique morphism  $\varphi : \mathcal{O}(-K_X) \rightarrow L_0$  (resp.  $\varphi' : \mathcal{O}(-K_X) \rightarrow L_0^{-1}$ ) vanishing at  $[P] + [Q]$  (resp.  $[\iota(P)] + [\iota(Q)]$ ). They generate all Tyurin subbundles and they are the only degenerate ones. Clearly, neither  $L_0$  nor  $L_0^{-1}$  is invariant. The projective part  $\mathbb{G}_m$  of the automorphism group  $\text{Aut}(E)$  fixes both  $L_0$  and  $L_0^{-1}$  and acts transitively on the remaining part of the family. Any involution  $h$  interchanges  $L_0$  and  $L_0^{-1}$  while it fixes two generic members  $L^+$  and  $L^-$  of the family. The parabolic structures are directed by these two bundles. Another choice of lift  $h' = g \circ h \circ g^{-1}$ ,  $g \in \text{Aut}(E)$ , just translates the two subbundles  $L^\pm$  by  $g$ . Finally, *up to automorphism, there is a unique invariant Tyurin bundle, and thus a unique parabolic structure.*

**4.1.3. The trival bundle and its 15 twists.** When  $E$  is the trival bundle,  $\text{Hom}(\mathcal{O}(-K_X), E)$  is 4-dimensional and generated by 2-dimensional subspaces  $\text{Hom}(\mathcal{O}(-K_X), \mathcal{O}_X)$  for two distinct embeddings  $\mathcal{O}_X \hookrightarrow E$ . We get a 3-dimensional family of Tyurin subbundles, that contains the 1-parameter family of degenerate ones formed by all embeddings  $\mathcal{O}_X \hookrightarrow E$ . Given any irreducible connection, the corresponding lift  $h$  fixes only two subbundles, two degenerate ones (see section 3.3.2). The two parabolic structures are directed by these two embeddings  $\mathcal{O}_X \hookrightarrow E$ . Therefore, *up to automorphism, there is exactly one parabolic structure on the trival vector bundle.*

When  $E = L_0 \oplus L_0$  with  $L_0 = \mathcal{O}([w_i] + [w_j] - K_X)$ ,  $i \neq j$ , then  $\text{Hom}(\mathcal{O}(-K_X), E)$  is 2-dimensional and all Tyurin subbundles are degenerate in this case: they form the 1-parameter family of subbundles  $L_0 \hookrightarrow E$ . Still  $\text{Aut}(E) = \text{GL}_2(\mathbb{C})$  acts transitively on them. Remind that the two parabolic structures can be deduced from the previous case just by permuting the two parabolics over  $w_i$  and  $w_j$  (see section 2.3). Each parabolic structure is thus distributed on two embeddings  $L_0 \hookrightarrow E$ ; since  $\text{Aut}(E)$  acts 2-transitively on them, *there is a unique parabolic structure up to automorphisms.*

4.1.4. *Unipotent bundles and their 15 twists.* Let  $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$  be a non-trivial extension. Here the space of morphisms  $\text{Hom}(\mathcal{O}(-K_X), E)$  has dimension 3 and the subbundle  $\mathcal{O}_X \subset E$  is responsible for this extra dimension:  $\text{Hom}(\mathcal{O}(-K_X), \mathcal{O}_X)$  has dimension 2. There are many lifts  $h$  of the hyperelliptic involution  $\iota$  since there are non-trivial automorphisms on  $E$ : any other lift is, up to a sign, given by  $g \circ h \circ g^{-1}$  for some  $g \in \text{Aut}(E)$ . But once  $h$  is fixed, we can apply Proposition 4.1 and get that there are exactly two  $h$ -invariant Tyurin subbundles  $L^\pm$ , one of them is the unique embedding  $\mathcal{O}_X \hookrightarrow E$ ; maybe replacing  $h$  by  $-h$ , we may assume  $L^+ = \mathcal{O}_X$ .

Let  $\varphi^+$  be a non-zero element of  $\text{Hom}(\mathcal{O}(-K_X), L^+)$ , vanishing at say  $[P] + [\iota(P)]$ ; mind that we can choose  $P$  arbitrarily on  $X$ . Let  $\varphi^-$  be a section of  $\text{Hom}(\mathcal{O}(-K_X), L^-)$  (unique up to a constant) and consider the divisor defined by zeroes of  $\varphi^+ \wedge \varphi^-$ : as an element of the linear system  $|2K_X|$ , it takes the form  $[P] + [\iota(P)] + [Q] + [\iota(Q)]$  including the vanishing divisor of  $\varphi^+$ . Clearly,  $[Q] + [\iota(Q)]$  is an invariant of the bundle (while  $[P] + [\iota(P)]$  can be chosen arbitrarily by switching to another  $\varphi^+$ ).

**Proposition 4.7.** *The divisor  $[Q] + [\iota(Q)]$  characterizes the extension  $E$ : we thus get a natural identification between the space  $\mathbb{P}\text{Hom}(\mathcal{O}(K_X))^\vee$  parametrizing extensions and  $\mathbb{P}\text{Hom}(\mathcal{O}(K_X))$  parametrizing those divisors  $[Q] + [\iota(Q)]$ .*

*The bundle  $L^-$  is degenerate if, and only if,  $[Q] + [\iota(Q)] = 2[w_i]$  where  $w_i$  is a Weierstrass point. In this case,  $L^- = \mathcal{O}_X(-[w_i])$  (and  $\varphi^-$  vanishes at  $w_i$ ).*

*Proof.* The morphism  $\varphi^-$  defines a natural morphism

$$\text{id}|_{L^+} \oplus \varphi^- : \mathcal{O}_X \oplus \mathcal{O}(-K_X) \rightarrow E$$

whose determinantal map vanishes at  $[Q] + [\iota(Q)]$ . When  $Q \neq \iota(Q)$ , this is a positive elementary transformation for a parabolic structure defined over  $[Q] + [\iota(Q)]$  neither directed by  $\mathcal{O}_X$ , nor by  $\mathcal{O}(-K_X)$  (otherwise  $E$  would be decomposable). One can easily check that, up to automorphism of the bundle  $\mathcal{O}_X \oplus \mathcal{O}(-K_X)$ , there is a unique parabolic structure over  $[Q] + [\iota(Q)]$ , and  $E$  is well determined by this divisor. This provides a natural identification as stated, outside of the 6 special bundles for which  $Q = \iota(Q) = w_i$ ; it extends by continuity at those points.

Since  $E$  is semi-stable and indecomposable, we have  $\deg(L^-) < 0$ . In the degenerate case, the only possibility is that  $\varphi^-$  has a single zero, at say  $Q$ , and  $L^- = \mathcal{O}_X(-[\iota(Q)])$ . But  $L^-$  being  $h$ -invariant,  $Q = \iota(Q)$  has to be a Weierstrass point,  $w_i$  say. On the other hand, when  $Q = w_i$ , choosing  $P \neq w_i$  we get two sections  $\varphi^+$  and  $\varphi^-$  colinear at  $Q$ ; if  $\varphi^-$  does not vanish, then a linear combination will vanish, producing some  $\mathcal{O}_X(-[\iota(Q)]) \subset E$  which is invariant. Yet  $L^\pm$  are the only  $h$ -invariant Tyurin subbundles, so we obtain a contradiction.  $\square$

The two hyperelliptic parabolic structures associated to  $h$  are directed by these two bundles, except for the 6 special extensions  $E$  for which  $L^-$  is degenerate. The  $h'$ -invariant Tyurin subbundles for  $h' = g \circ h \circ g^{-1}$  are  $L^+$  and  $g(L^-)$  ( $\text{Aut}(E)$  is fixing the subbundle  $L^+ \simeq \mathcal{O}_X$  and the  $\mathbb{G}_a$ -part is moving all other directions in each fiber). Therefore, *there are exactly two hyperelliptic parabolic structures on  $E$  up to automorphism.*

**Remark 4.8.** *In the geometric picture, the 1-parameter family of extensions  $(E_t)_{t \in \mathbb{P}^1}$  of the trivial line bundle can be seen as the tangent cone to the Kummer surface after blowing up the singular point corresponding to the trivial bundle. The strict transform of the Gunning plane  $\Pi_{[w_i]}$  then intersects this  $\mathbb{P}^1$  in a unique point which is the bundle satisfying  $L^- \simeq \mathcal{O}(-[w_i])$  as above.*

Let  $L_0 = \mathcal{O}([w_i] + [w_j] - K_X)$  be a non-trivial 2-torsion point of  $\text{Pic}^0(X)$ ,  $i \neq j$ , and consider a non-trivial extension  $0 \rightarrow L_0 \rightarrow E \rightarrow L_0 \rightarrow 0$ . This time,  $\text{Hom}(\mathcal{O}(-K_X), E)$  has dimension 2 and generates a 1-parameter family of Tyurin subbundles. One of them is  $L_0$ , the only one having degree 0. It is degenerate and must be invariant, say  $L^+$ . The group  $\text{Aut}(E)$  is acting transitively on the remaining part of the family and, like for unipotent bundles,  $h$  fixes one of them, say  $L^-$ . The intersection  $L^+ \cap L^-$  has to be  $[w_i] + [w_j]$  and  $L^-$  is therefore non-degenerate, characterizing the parabolic structure.

4.1.5. *The 6 + 10 Gunning bundles.* Let  $\kappa \in \text{Pic}^1(X)$  be a theta characteristic and  $E_\kappa$  be the associated Gunning bundle. The subbundle  $\kappa \subset E_\kappa$  is the unique one having degree  $> -1$ ; it is a degenerate  $h$ -invariant Tyurin subbundle.

When  $\kappa$  is an even theta characteristic  $\kappa = [w_i] + [w_j] + [w_k] - K_X$ , we have

$$\dim(\text{Hom}(\mathcal{O}_X(-K_X), \kappa)) = \dim(\text{Hom}(\mathcal{O}_X(-K_X), E_\kappa)) = 2$$

and all morphisms  $\varphi : \mathcal{O}_X(-K_X) \rightarrow E_\kappa$  factor through the subbundle  $\kappa \subset E_\kappa$ : there is a unique Tyurin bundle in this case. Through the identification  $\text{Hom}(\mathcal{O}(-K_X), \kappa) \simeq H^0(X, \mathcal{O}_X([w_i] + [w_j] + [w_k]))$ , the space of morphisms is generated by the two sections

$$1, \frac{(x - x_l)(x - x_m)(x - x_n)}{y} \in H^0(X, \mathcal{O}([w_i] + [w_j] + [w_k])),$$

where  $\{i, j, k, l, m, n\} = \{1, \dots, 6\}$  and  $w_i = (x_i, 0) \in X$ . The hyperelliptic involution acts as  $\text{id}$  on the first one and  $-\text{id}$  on the second one. There are two types of hyperelliptic parabolic structures on  $E_\kappa$ :

- parabolics corresponding to  $w_i, w_j$  and  $w_k$  lying on  $\kappa \hookrightarrow E_\kappa$ , the others outside;
- parabolics corresponding to  $w_l, w_m$  and  $w_n$  lying on  $\kappa \hookrightarrow E_\kappa$ , the others outside.

This implies that *up to automorphism, there are exactly two parabolic structures on a Gunning bundle  $E_\kappa$  with even theta characteristic.*

Let us now consider the case where  $\kappa$  is an odd theta characteristic  $\kappa = \mathcal{O}_X([w_i])$ . The  $h$ -invariant Tyurin subbundles  $L^+$  and  $L^-$  are distinct and one of them is the maximal subbundle of  $E_\kappa$ , say  $L^+ = \kappa$ , which is the only degenerate Tyurin subbundle of  $E_\kappa$ . Note that in particular, the parabolic  $\mathfrak{p}_i^-$  is directed by  $L^+|_{w_i} = L^-|_{w_i}$  and  $\mathfrak{p}_i^+$  is elsewhere. Since  $\text{Aut}(E_\kappa)$  fixes  $L^+$  and acts transitively on the set of line subbundles of the form  $\mathcal{O}(-K_X)$ , *there are, up to automorphism, exactly two parabolic structures on a Gunning bundle  $E_\kappa$  with odd theta characteristic.*

4.2. **Extensions of the canonical bundle.** Here, we recall some results obtained by Bertram in [6], completed in the genus 2 case by Bolognesi in [11, 12] (see also [27]).

The space of non trivial extensions  $0 \rightarrow \mathcal{O}(-K_X) \rightarrow E \rightarrow \mathcal{O}(K_X) \rightarrow 0$  is  $\mathbb{P}H^1(-2K_X)$  which identifies, by Serre duality, to  $\mathbb{P}H^0(3K_X)^\vee$ . This space naturally parametrizes the moduli space of those pairs  $(E, L)$  where  $L \subset E$  is a non-degenerate Tyurin bundle. The hyperelliptic involution  $\iota$  acts naturally on  $H^0(3K_X)$  and thus on its dual: the invariant subspace is an hyperplane  $\mathbb{P}_B^3 \subset \mathbb{P}H^0(3K_X)^\vee \simeq \mathbb{P}^4$  that naturally parametrizes those pairs  $(E, L)$  that are invariant under the involution. As we have seen in section 4.1, most stable bundles  $E$  admit exactly two invariant and non-degenerate Tyurin subbundles and most decomposable bundles  $E$  admit only one. This suggests that  $\mathbb{P}_B^3$  is a birational model for the 2-fold cover of  $\mathbb{P}_{NR}^3$  ramified over the Kummer surface.

A cubic differential  $\omega \in H^0(3K_X)$  writes  $\omega = (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4y) \left(\frac{dx}{y}\right)^{\otimes 3}$  uniquely so that the coefficients  $a_i$  provide a full set of coordinates. Let  $(b_0 : b_1 : b_2 : b_3 : b_4)$

be dual homogeneous coordinates for  $\mathbb{P}_B^4 := \mathbb{P}H^0(3K_X)^\vee$ . We have the following description (see introductions of [6, 27] and §5 of [11])

The locus of unstable bundles is given by the natural embedding of the curve  $X$ :

$$X \hookrightarrow \mathbb{P}_B^4; (x, y) \mapsto (1 : x : x^2 : x^3 : y).$$

The locus of strictly semi-stable bundles is given by the quartic hypersurface  $\text{Wed} \subset \mathbb{P}_B^4$  spanned by the 2-secant lines of  $X$ . The natural action of the hyperelliptic involution  $\iota : X \rightarrow X$  on cubic differentials induces an involution on  $\mathbb{P}_B^4$  that fixes the hyperplane  $\mathbb{P}_B^3 = \{b_4 = 0\}$  and the point  $(0 : 0 : 0 : 0 : 1)$ .

The Narasimhan-Ramanan moduli map

$$\mathbb{P}_B^4 \dashrightarrow \mathbb{P}_{NR}^3$$

is given by the full linear system of quadrics that contain  $X$ ; it restricts to  $\mathbb{P}_B^3$  as the full linear system of quadrics (of  $\mathbb{P}_B^3$ ) that contain the six points  $X \cap \mathbb{P}_B^3$ . After blowing-up the locus  $X$  of unstable bundles, we get a morphism

$$\tilde{\mathbb{P}}_B^4 \rightarrow \mathbb{P}_{NR}^3$$

namely a conic bundle; its restriction to the strict transform  $\tilde{\mathbb{P}}_B^3$  of  $\mathbb{P}_B^3$  is generically  $2 : 1$ , ramifying over the Kummer surface  $\text{Kum} \subset \mathbb{P}_{NR}^3$ . The quartic hypersurface  $\text{Wed}$  restricts to  $\mathbb{P}_B^3$  as the (dual) Weddle surface; it is sent onto the Kummer surface.

There is a Poincaré vector bundle  $\mathcal{E} \rightarrow X \times \mathbb{P}_B^4$  realizing the classifying map above. Hence by restriction, there is a Poincaré bundle  $\mathcal{E} \rightarrow X \times \mathbb{P}_B^3$  on the double cover  $\mathbb{P}_B^3$  of  $\mathbb{P}_{NR}^3$ . The projectivized Poincaré bundle  $\mathbf{P}(\mathcal{E}) \rightarrow X \times \mathbb{P}_B^3$  defines a conic bundle  $\mathcal{C} \rightarrow X \times \mathbb{P}_{NR}^3$  over the quotient  $\mathbb{P}_{NR}^3$ . For each vector bundle  $E \in \mathbb{P}_{NR}^3$ , the fibre  $\mathcal{C}_E$  of the conic bundle represents the family of Tyurin-subbundles of  $E$ . Yet the conic bundle  $\mathcal{C}$  is not a projectivized vector bundle over  $\mathbb{P}_{NR}^3$ , not even up to birational equivalency, because a Poincaré bundle over a Zariski-open set of  $\mathbb{P}_{NR}^3$  does not exist [36].

**4.3. Tyurin parametrization.** Let  $E$  be a flat rank two vector bundle with trivial determinant bundle over  $X$ . It follows from Corollary 4.5 that, when  $E$  is stable and off the odd Gunning planes, then  $E$  can be deduced from  $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$  by applying 4 positive elementary transformations, namely over the Tyurin divisor  $D_E^T$ . In fact, if we allow non reduced divisors, then this remains true for all flat bundles except even Gunning bundles. Indeed, it follows from Proposition 4.1 that we have a non degenerate map

$$\varphi^+ \oplus \varphi^- : \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X) \rightarrow E$$

by selecting  $\varphi^+$  and  $\varphi^-$  generating  $H^+$  and  $H^-$  respectively; non degenerate means that the image spans the generic fiber. Comparing the degree of both vector bundles, we promptly deduce that this map decomposes into 4 successive positive elementary transformations, possibly over non distinct points. This so happens when the divisor  $D_E^T \in |2K_X|$  is non reduced.

Conversely, let us consider a divisor, say reduced for simplicity:

$$D = [\underline{P}_1] + [\iota(\underline{P}_1)] + [\underline{P}_2] + [\iota(\underline{P}_2)] \in |2K_X|,$$

and consider also a parabolic structure  $\mathbf{q}$  over  $D$  on the trivial bundle  $E_0 \rightarrow X$ : given  $e_1$  and  $e_2$  two independant sections of  $E_0$ , the parabolic structure is defined by

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) \in (\mathbb{P}^1)^4$$

where  $e_1 + \lambda_{\underline{P}_i} e_2$  generates the parabolic direction over  $\underline{P}_i$ , and similarly for  $\iota(\underline{P}_i)$ . From this data, one can associate a vector bundle with trivial determinant  $E$  by

$$\mathcal{O}(-K_X) \otimes \text{elm}_D^+(E_0, \mathbf{q}) \rightarrow E.$$

**Proposition 4.9.** *The list of rank 2 vector bundles of trivial determinant over  $X$  that can be obtained from the trivial bundle by  $\mathcal{O}(-K_X) \otimes \text{elm}_D^+(\mathcal{O}_X \oplus \mathcal{O}_X, \mathbf{q})$  is the following:*

- (1) *all stable bundles:*
  - (a) *stable bundles off the Gunning planes with odd theta characteristic,*
  - (b)\* *stable bundles on Gunning planes  $\Pi_w$  with odd theta characteristic ,*
- (2) *all semi-stable bundles:*
  - (a) *decomposable bundles  $E = L \oplus L^{-1}$  where  $L$  is not of 2-torsion,*
  - (b) *the trivial bundle,*
  - (c) *generic unipotent bundles,*
  - (d) *affine bundles, possibly (\*),*
  - (e)\* *twists of the trivial bundle,*
  - (f)\* *the 6 special unipotent bundles,*
  - (g)\* *twists of the unipotent bundles,*
- (3) *some unstable bundles:*
  - (a) *decomposable bundles  $E = L \oplus L^{-1}$  where  $L = \mathcal{O}([\underline{P}_1])$  or  $L = \mathcal{O}(K_X)$ ,*
  - (b)\* *decomposable bundles  $E = L \oplus L^{-1}$  where  $L = \mathcal{O}([w])$ ,*
  - (c)\* *odd Gunning bundles.*

*For the bundles marked with (\*) we need to use multiple elementary transformations in some points. Neither even Gunning bundles nor general decomposable unstable bundles can be obtained.*

*Proof.* This proposition is mainly a *résumé* of 4, where we detail the reduced or non-reduced nature of the Tyurin divisor.

- (1) The Tyurin divisor for a stable bundle is reduced if and only if it does not ly on an odd Gunning plane.
- (2) (a) For  $\lambda_{\underline{P}_1} = \lambda_{\underline{P}_2} = 0$  and  $\lambda_{\iota(\underline{P}_1)} = \lambda_{\iota(\underline{P}_2)} = \infty$  we get  $E = L \oplus L^{-1}$  with  $L = \mathcal{O}([\underline{P}_1] + [\underline{P}_2] - K_X)$ .
- (b) Take  $\lambda_{\underline{P}_1} = \lambda_{\iota(\underline{P}_1)} = 0$  and  $\lambda_{\underline{P}_2} = \lambda_{\iota(\underline{P}_2)} = \infty$ .
- (c) We have seen that  $E \otimes \mathcal{O}(K_X)$  can be obtained from the bundle  $\mathcal{O} \oplus \mathcal{O}(K_X)$  by positive elementary transformations in  $\underline{P}_2$  and  $\iota(\underline{P}_2)$ . Moreover,  $\mathcal{O} \oplus \mathcal{O}(K_X)$  can be obtained from the trivial bundle by a two elementary transformations over  $\underline{P}_1$  and  $\iota(\underline{P}_1)$  (for an arbitrary  $\underline{P}_1$ ) on the same trivial subbundle. The Tyurin divisor can thus be chosen reduced if and only if  $\underline{P}_2$  is not a Weierstrass point.
- (d) The space of Tyurin subbundles of an affine bundle is generated by two distinct Tyurin subbundles.
- (e)\* For  $\lambda_{\underline{P}_1} = \lambda_{\underline{P}_2} = 0$  and  $\lambda_{\iota(\underline{P}_1)} = \lambda_{\iota(\underline{P}_2)} = \infty$  for  $\underline{P}_1 = \iota(\underline{P}_1) = w_i$  and  $\underline{P}_2 = \iota(\underline{P}_2) = w_j$  we obtain  $L \oplus L$  with  $L = \mathcal{O}([w_i] + [w_j] - K_X)$ .
- (g)\* If  $E$  is a non trivial extension of  $L = \mathcal{O}([w_i] + [w_j])$  with  $i \neq j$ , then  $E$  is obtained by two double elementary transformations in  $w_i$  and  $w_j$  on a trivial subbundle of the trivial bundle.
- (3) (a) Take  $\lambda_{\underline{P}_2} = \lambda_{\iota(\underline{P}_2)} = \lambda_{\iota(\underline{P}_1)} = 0$ .
- (b)\* Like (a) with  $\underline{P}_1 = w$ .

(c)\* If the theta characteristic is  $\kappa = [w]$ , we need to perform a double elementary transformation in  $w$ .

Even Gunning bundles cannot be obtained. Otherwise two distinct trivial subbundles of the trivial bundle would generate two distinct Tyurin subbundles on an even Gunning bundle. Reasoning on the possible preimages of the destabilizing subbundle, it is straightforward to check that the above mentioned decomposable bundles are the only possible ones.  $\square$

As a consequence, the moduli space  $\mathcal{M}_{NR}$  is birational to the moduli space of parabolic structures over  $D$  on  $E_0$ , when  $D$  runs over the linear system  $|2K_X|$ . Let us be more precise. Consider the parameter space

$$(\underline{P}_1, \underline{P}_2, \lambda) \in X \times X \times \mathbb{P}^1$$

and associate to each such data, the parabolic structure defined on the vector bundle  $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$  by

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) := \left( \lambda, -\lambda, \frac{1}{\lambda}, -\frac{1}{\lambda} \right).$$

Equivalently, one can view the parabolic structure as the collection of points

$$(\underline{P}_1, \lambda), \quad (\iota(\underline{P}_1), -\lambda), \quad \left( \underline{P}_2, \frac{1}{\lambda} \right) \quad \text{and} \quad \left( \iota(\underline{P}_2), -\frac{1}{\lambda} \right)$$

on the total space  $X \times \mathbb{P}^1$  of the projectivized  $\mathbb{P}^1$ -bundle  $\mathbb{P}(\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X))$ . The natural rational map  $X \times X \times \mathbb{P}^1 \dashrightarrow \mathbb{P}_{NR}^3$  is not birational however, since for a given bundle over  $X$  there are several possibilities to choose  $\underline{P}_1$ ,  $\underline{P}_2$  and  $\lambda$ . One can first independently permute  $\underline{P}_1 \leftrightarrow \iota(\underline{P}_1)$ ,  $\underline{P}_2 \leftrightarrow \iota(\underline{P}_2)$  and  $\underline{P}_1 \leftrightarrow \underline{P}_2$ : this generates a order 8 group of permutations. Moreover, once  $\underline{P}_1$  and  $\underline{P}_2$  have been chosen to parametrize the linear system  $|2K_X|$ , there is still a freedom in the choice of  $\lambda$ : our choice of normalization, characterized by

$$\lambda_{\underline{P}_1} + \lambda_{\iota(\underline{P}_1)} = \lambda_{\underline{P}_2} + \lambda_{\iota(\underline{P}_2)} = 0 \quad \text{and} \quad \lambda_{\underline{P}_1} \cdot \lambda_{\underline{P}_2} = 1,$$

is invariant under the Klein 4 group  $\langle z \mapsto -z, z \mapsto \frac{1}{z} \rangle$  acting on the projective variable  $e_1 + ze_2$ . The transformation group taking into account all this freedom is generated by the following 4 transformations

$$\begin{aligned} (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) &\longrightarrow (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) \\ ((\underline{P}_1, \underline{P}_2, \lambda), ((x, y), z)) &\begin{cases} \xrightarrow{\sigma_{12}} & ((\underline{P}_2, \underline{P}_1, \frac{1}{\lambda}), ((x, y), z)) \\ \xrightarrow{\sigma_\iota} & ((\iota(\underline{P}_1), \iota(\underline{P}_2), -\lambda), ((x, y), z)) \\ \xrightarrow{\sigma_{iz}} & ((\underline{P}_1, \iota(\underline{P}_2), i\lambda), ((x, y), iz)) \\ \xrightarrow{\sigma_{1/z}} & ((\underline{P}_1, \underline{P}_2, \frac{1}{\lambda}), ((x, y), \frac{1}{z})) \end{cases} \end{aligned}$$

(here,  $i = \sqrt{-1}$ ). In fact, our choice of normalization for  $(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)})$  may not be the most naive one, which would have consisted to fix 3 of them to 0, 1 and  $\infty$ ; but our choice has the advantage that the transformation

$$\begin{aligned} (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) &\longrightarrow (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) \\ ((\underline{P}_1, \underline{P}_2, \lambda), ((x, y), z)) &\longmapsto ((\underline{P}_1, \underline{P}_2, \lambda), ((x, -y), -z)) \end{aligned}$$

preserves the parabolic structure, and corresponds to the projectivized hyperelliptic involution  $h : E \rightarrow \iota^*E$ . In particular, the subbundles  $z = 0$  and  $z = \infty$  generated

respectively by  $e_1$  and  $e_2$  precisely correspond to the two  $\iota$ -invariant Tyurin subbundles of  $E$ .

The 32-order group  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}, \sigma_{1/z} \rangle$  acts faithfully on the parameter space  $\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1$ . Setting  $\underline{P}_1 = (\underline{x}_1, \underline{y}_1)$  and  $\underline{P}_2 = (\underline{x}_2, \underline{y}_2)$ , the field of rational invariant functions is generated by

$$\underline{s} := \underline{x}_1 + \underline{x}_2, \quad \underline{p} := \underline{x}_1 \underline{x}_2 \quad \text{and} \quad \lambda := \left( \lambda^2 + \frac{1}{\lambda^2} \right) \underline{y}_1 \underline{y}_2$$

so that a quotient map (up to birational equivalence) is given by

$$\begin{array}{ccc} \underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1 & \xrightarrow{(32:1)} & \mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \\ \left( (\underline{x}_1, \underline{y}_1), (\underline{x}_2, \underline{y}_2), \lambda \right) & \mapsto & \left( (1 : -\underline{x}_1 - \underline{x}_2 : \underline{x}_1 \underline{x}_2), \left( \lambda^2 + \frac{1}{\lambda^2} \right) \underline{y}_1 \underline{y}_2 \right) \end{array}$$

Here,  $\mathbb{P}_D^2 = |2K_X|$  is just the linear system parametrizing those divisors  $D_E^T$ . This quotient is our sharp Tyurin configuration space, and we get a natural birational map

$$\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \dashrightarrow \mathbb{P}_{NR}^3$$

which can be explicitly described as follows.

**Proposition 4.10.** *The natural classifying map  $\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \dashrightarrow \mathbb{P}_{NR}^3$  writes*

$$\begin{aligned} (\underline{s}, \underline{p}, \lambda) &\mapsto (v_0 : v_1 : v_2 : v_3) \\ &= \left( \frac{\lambda - \underline{s}\underline{p}^2 + 2(1 + \sigma_1)\underline{p}^2 - (\sigma_1 + \sigma_2)\underline{s}\underline{p} + (\sigma_2 + \sigma_3)(\underline{s}^2 - 2\underline{p}) - \sigma_3\underline{s}}{\underline{s}^2 - 4\underline{p}} : \underline{p} : -\underline{s} : 1 \right). \end{aligned}$$

Before proving it, let us make some observations. First, the fibration  $\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_D^2$  is sent onto the pencil of lines of  $\mathbb{P}_{NR}^3$  passing through the trivial bundle  $E_0 : (1 : 0 : 0 : 0)$ . In fact, the surface  $\{\lambda = \infty\}$  in Tyurin parameter space, corresponding to  $\lambda = 0$  or  $\infty$ , is the locus of the trivial bundle. Also, the surface defined by  $\lambda = \{1, -1, i, -i\}$  corresponds to generic decomposable flat bundles and is sent onto the Kummer surface; we note that it is also defined by  $\lambda^2 = 4(y_1 y_2)^2$  which, after expansion, writes

$$\begin{aligned} \lambda^2 &= \underline{p}(\underline{p} - \underline{s} + 1) \cdot \left( \underline{p}^3 - \sigma_1 \underline{p}^2 \underline{s} + \sigma_2 \underline{p} \underline{s}^2 - \sigma_3 \underline{s}^3 + (\sigma_1^2 - 2\sigma_2) \underline{p}^2 + (3\sigma_3 - \sigma_1 \sigma_2) \underline{p} \underline{s} \right. \\ &\quad \left. + \sigma_1 \sigma_3 \underline{s}^2 + (\sigma_2^2 - 2\sigma_1 \sigma_3) \underline{p} - \sigma_2 \sigma_3 \underline{s} + \sigma_3^2 \right) \end{aligned}$$

which allow us to retrieve the equation of  $\text{Kum}(X) \subset \mathbb{P}_{NR}^3$ .

*Proof.* The Tyurin divisor  $D_E^T$  has equation  $x^2 - \underline{s}x + \underline{p}$ . Following Proposition 4.3, the divisor  $D_E^T$  is also defined by the restriction of Narasimhan-Ramanan divisor  $D_E$  on the embedded curve  $X \xrightarrow{\sim} \Theta \subset \text{Pic}^1(X)$ . The latter one has equation  $v_0 + v_1 \cdot \text{Sum} + v_2 \cdot \text{Prod} + v_3 \cdot \text{Diag} = 0$ . From notations of section 3.6, we can compute its restriction to the lift  $\infty_1$  of the divisor  $\Theta$  on  $X \times X$ ; it is parametrized as  $X \xrightarrow{\sim} \infty_1; P \mapsto (\infty, P)$ . The generating functions  $1, \text{Sum}, \text{Prod}, \text{Diag}$  restrict to  $\infty_1$  as (see proof of Lemma 3.6)

$$1|_{\infty_1} \equiv 0, \quad \text{Sum}|_{\infty_1} \equiv 1, \quad \text{Prod}|_{\infty_1} = x \quad \text{and} \quad \text{Diag}|_{\infty_1} = x^2$$

so that equation for  $D_E^T$  is also given by  $v_1 + v_2 \cdot x + v_3 \cdot x^2 = 0$ . This already fixes the last 3 terms of the map.

One would like to conclude as follows. The map being birational, it is in restriction to a generic  $\mathbb{P}^1$ -fiber of  $\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_D^2$ . Moreover, it is affine since  $\lambda = \infty$  has to be sent to the point  $E_0 = (1 : 0 : 0 : 0)$ . Finally, the 2-section  $\lambda^2 = 4(\underline{y}_1 \underline{y}_2)^2$  has to be sent onto

the Kummer surface (intersecting a generic line through  $E_0$  twice outside of  $E_0$ ). Yet this fixes the map only up to an involution.

Let us restart in a more direct way. Assume we are given  $(\underline{P}_1, \underline{P}_2, \lambda)$  and the associated parabolic structure on  $(E_0, \mathbf{q}) \rightarrow (X, D_E^T)$ ; then, we want to compute the Narasimhan-Ramanan divisor  $D_E \subset \text{Pic}^1(X)$  for the corresponding vector bundle  $E$  obtained after 4 elementary transformations. Given a degree 3 line bundle  $L_0$ , we can look at holomorphic sections  $s_0 : X \rightarrow E_0 \otimes L_0$ ; it is straightforward to check that a section  $s_1 e_1 + s_2 e_2$  taking value in Tyurin parabolic directions over  $D_E^T$  will produce, after elementary transformations, a holomorphic section of  $E \otimes L_0(\mathbb{K}_X - D_E^T)$ , showing that  $L_0(\mathbb{K}_X - D_E^T) = L_0(-\mathbb{K}_X) \in D_E$ . Since sections of  $L_0 = \mathcal{O}_X([P_1] + [P_2] + [\infty])$  are generated by  $\langle 1, \frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2} \rangle$ , up to automorphisms of  $E_0$ , we can assume  $s_1 = 1$  and  $s_2 = f := \frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2}$ . Therefore, computing the cross-ratio, we get

$$\gamma := \frac{\lambda_{\underline{P}_2} - \lambda_{\underline{P}_1}}{\lambda_{\iota(\underline{P}_1)} - \lambda_{\underline{P}_1}} : \frac{\lambda_{\underline{P}_2} - \lambda_{\iota(\underline{P}_2)}}{\lambda_{\iota(\underline{P}_1)} - \lambda_{\iota(\underline{P}_2)}} = \frac{f(\underline{P}_2) - f(\underline{P}_1)}{f(\iota(\underline{P}_1)) - f(\underline{P}_1)} : \frac{f(\underline{P}_2) - f(\iota(\underline{P}_2))}{f(\iota(\underline{P}_1)) - f(\iota(\underline{P}_2))}$$

which, after reduction, gives

$$\frac{4y_1 y_2 \gamma}{(x_1 - x_2)^2} = (-\text{Diag}(\underline{P}_1, \underline{P}_2) \cdot 1 + \text{Prod}(\underline{P}_1, \underline{P}_2) \cdot \text{Sum} - \text{Sum}(\underline{P}_1, \underline{P}_2) \cdot \text{Prod} + \text{Diag})$$

with notations of section 3.6. On the other hand, from Tyurin parameters, we get

$$\gamma = -\frac{(1 - \lambda^2)^2}{4\lambda^2}$$

hence the result.  $\square$

The total space  $(X_1 \times X_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1)$  is equipped with the 4 rational sections

$$(\underline{P}_1, \lambda), (\iota(\underline{P}_1), -\lambda), \left(\underline{P}_2, \frac{1}{\lambda}\right), \left(\iota(\underline{P}_2), -\frac{1}{\lambda}\right) : (X_1 \times X_2 \times \mathbb{P}_\lambda^1) \rightarrow (X \times \mathbb{P}_z^1)$$

which are globally invariant under the action of  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}, \sigma_{1/z} \rangle$ . The quotient provides a projective Poincaré bundle, namely a (non trivial)  $\mathbb{P}^1$ -bundle over  $(\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1) \times X$  (actually, over an open set of the parameters) equipped with a universal parabolic structure. After positive elementary transformation, we get a universal  $\mathbb{P}^1$ -bundle over an open subset of  $\mathbb{P}_{NR}^3$ . However, we cannot lift the construction to a vector bundle because the action of  $\langle z \mapsto -z, z \mapsto \frac{1}{z} \rangle$  (induced by  $\langle \sigma_{iz}^2, \sigma_{1/z} \rangle$ ) does not lift to a linear  $\text{GL}_2$ -action (indeed,  $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$  and  $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  do not commute). This is the reason why there is no Poincaré bundle for  $\mathbb{P}_{NR}^3$ , but only a projective version of it. The ambiguity is killed-out if we do not take  $\sigma_{1/z}$  into account, meaning that we choose one of the two  $h$ -invariants Tyurin subbundles: we then retrieve the Poincaré bundle defined by extensions in section 4.2, which here is explicitly given as follows. Consider the vector bundle

$$\tilde{\mathcal{E}} = p^*(\mathcal{O}_X(\mathbb{K}_X)) \otimes \text{elm}_{\delta_1, \delta_2, \delta_3, \delta_4}^+(\!(X_1 \times X_2 \times \mathbb{P}^1) \times (X \times \mathbb{C}^2)\!)$$

over  $(X_1 \times X_2 \times \mathbb{P}^1) \times X$ , where  $\delta_1 : p = p_1$ ,  $\delta_2 : p = \iota(p_1)$ ,  $\delta_3 : p = p_2$ ,  $\delta_4 : p = \iota(p_2)$  if  $p$  denotes the projection from  $X_1 \times X_2 \times \mathbb{P}^1 \times X$  to  $X$  and  $p_i$  the projection to  $X_i$ ; and the parabolic structure over these divisors is given respectively by

$$(\underline{P}_1, \underline{P}_2, \lambda, \underline{P}_1, \begin{pmatrix} \lambda \\ 1 \end{pmatrix}), (\underline{P}_1, \underline{P}_2, \lambda, \iota(\underline{P}_1), \begin{pmatrix} -\lambda \\ 1 \end{pmatrix}), (\underline{P}_1, \underline{P}_2, \lambda, \underline{P}_2, \begin{pmatrix} 1 \\ \lambda \end{pmatrix}), (\underline{P}_1, \underline{P}_2, \lambda, \iota(\underline{P}_2), \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}).$$

This vector bundle is clearly invariant for the action

$$(\underline{P}_1, \underline{P}_2, \lambda, P, Z) \begin{cases} \xrightarrow{\sigma_{12}} & (\underline{P}_2, \underline{P}_1, \frac{1}{\lambda}, P, Z) \\ \xrightarrow{\sigma_\iota} & (\iota(\underline{P}_1), \iota(\underline{P}_2), -\lambda, P, Z) \\ \xrightarrow{\sigma_{iz}} & \left( (\underline{P}_1, \iota(\underline{P}_2), i\lambda, P, \begin{pmatrix} \sqrt{i} & 0 \\ 0 & \frac{1}{\sqrt{i}} \end{pmatrix} Z) \right), \end{cases}$$

i.e.  $\tilde{\mathcal{E}} \simeq \sigma^* \tilde{\mathcal{E}}$  for each  $\sigma \in \langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle$ . The quotient (in the sense of [7]) thus defines a universal vector bundle  $\mathcal{E} \rightarrow X \times B$  with trivial determinant bundle parametrized by the 2-cover  $B = (X_1 \times X_2 \times \mathbb{P}^1) / \langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle = \mathbb{P}_D^2 \times \mathbb{P}_\lambda^1$  of  $\mathcal{M}_{NR}$ , defined over an open set.

## 5. FLAT PARABOLIC VECTOR BUNDLES OVER THE QUOTIENT $X/\iota$

Consider the data  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$  where

- $\underline{E}$  is a rank 2 vector bundle over  $\mathbb{P}^1$ ,
- $\underline{\nabla} : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$  is a rank 2 logarithmic connection on  $\underline{E}$  with polar divisor  $\underline{W} = [0] + [1] + [r] + [s] + [t] + [\infty]$  and residual eigenvalues 0 and  $\frac{1}{2}$  over each pole,
- $\underline{\mathbf{p}} = (\underline{p}_0, \underline{p}_1, \underline{p}_r, \underline{p}_s, \underline{p}_t, \underline{p}_\infty)$  the quasi-parabolic structure defined by all  $\frac{1}{2}$ -eigendirections over  $x = 0, 1, r, s, t, \infty$ .

Via the Riemann-Hilbert correspondance, an equivalent data is the monodromy representation  $\pi_1(\mathbb{P}^1 \setminus \{0, 1, r, s, t, \infty\}) \rightarrow \mathrm{GL}_2$  with local monodromy  $\sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  at the punctures. We denote by  $\mathfrak{Con}(X/\iota)$  the coarse moduli space of such parabolic connections  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$ . We note that the parabolic structure  $\underline{\mathbf{p}}$  is determined by the connection  $(\underline{E}, \underline{\nabla})$  so that we may just ignore it; however, it plays a crucial role in the bundle map.

**5.1. Flatness criterion.** We denote by  $\mathfrak{Bun}(X/\iota)$  the coarse moduli space of those parabolic bundles  $(\underline{E}, \underline{\mathbf{p}})$  subjacent to some irreducible parabolic connection  $(\underline{E}, \underline{\nabla}, \underline{\mathbf{p}})$ . We note that, from Fuchs relations, we get that

$$\deg(E) = -3 \quad \text{for any } (\underline{E}, \underline{\mathbf{p}}) \in \mathfrak{Bun}(X/\iota).$$

Following [10, 2], we have the complete characterization of flat parabolic bundles:

**Proposition 5.1.** *Given a parabolic bundle  $(\underline{E}, \underline{\mathbf{p}})$  like above, there exists a connection  $\underline{\nabla}$  compatible with the parabolic structure like above if and only if  $\deg(\underline{E}) = -3$  and*

- either  $(\underline{E}, \underline{\mathbf{p}})$  is indecomposable,
- or  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  with 2 parabolics directed by  $\mathcal{O}_{\mathbb{P}^1}(-1)$ , the 4 other ones by  $\mathcal{O}_{\mathbb{P}^1}(-2)$ ,
- or  $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  with all parabolics directed by  $\mathcal{O}_{\mathbb{P}^1}(-3)$ .

Moreover, in each case, one can choose  $\underline{\nabla}$  irreducible.

*Proof.* We refer to the proof of Proposition 3 in [2] to show that indecomposable parabolic bundles are flat: this part of their proof does not use genericity of eigenvalues. In the decomposable case,  $\underline{E} = L_1 \oplus L_2$  and parabolics are distributed along  $L_1$  and  $L_2$  giving a decomposition  $\underline{W} = D_1 + D_2$ . If it exists, a connection  $\underline{\nabla}$  writes in matrix form

$$\underline{\nabla} = \begin{pmatrix} \nabla_1 & \theta_{1,2} \\ \theta_{2,1} & \nabla_2 \end{pmatrix}$$

where

- $\nabla_i : L_i \rightarrow L_i \otimes \Omega_{\mathbb{P}^1}^1(D_i)$  is a logarithmic connection with eigenvalues  $\frac{1}{2}$  for  $i = 1, 2$ ;
- $\theta_{i,j} : L_j \rightarrow L_i \otimes \Omega_{\mathbb{P}^1}^1(D_i)$  is a morphism for  $i \neq j$ .

Fuchs relation for  $\underline{E}$  gives  $\deg(\underline{E}) = -3$ , and for  $\nabla_i$ , gives

$$-2 \deg(L_i) = \text{number of parabolics lying on } L_i.$$

It follows that the only flat decomposable parabolic bundles are those listed in the statement. Now we note that connections  $\nabla_i$  exist and are uniquely determined by above conditions. Setting  $\theta_{i,j} = 0$ , we get a (totally reducible) parabolic connection  $\nabla$  on  $(E, \mathbf{p})$ . In all cases,  $\theta_{i,j}$  are morphisms  $\mathcal{O}_{\mathbb{P}^1}(n) \rightarrow \mathcal{O}_{\mathbb{P}^1}(n+1)$  for some  $n$  and live in a 2-dimensional vector space. We claim that

- $\underline{\nabla}$  is reducible if, and only if, one of the  $\theta_{i,j} = 0$ ,
- $\underline{\nabla}$  is totally reducible if, and only if, all  $\theta_{i,j} = 0$ .

Indeed, if a line bundle  $L \hookrightarrow E$  is  $\underline{\nabla}$ -invariant, then Fuchs relation for  $\underline{\nabla}|_L$  gives the following possible cases:

- $\deg(L) = -3$  and  $L$  passes through all parabolics;
- $\deg(L) = -2$  and  $L$  passes through 4 parabolics;
- $\deg(L) = -1$  and  $L$  passes through 2 parabolics;
- $\deg(L) = 0$  and  $L$  passes through no parabolics.

This forces  $L$  to be one of factors of the decomposable cases above. For instance, when  $\deg(L) = -3$ , either  $L \hookrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  and must coincide with the second factor (since both must contain all parabolics), or  $L \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  but then  $L$  intersects the first factor at only one point and thus cannot share the 2 parabolics.  $\square$

It follows from Proposition 5.1 above that the only flat decomposable parabolic bundles are

- $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  with 2 parabolics directed by  $\mathcal{O}_{\mathbb{P}^1}(-1)$ , the 4 other ones by  $\mathcal{O}_{\mathbb{P}^1}(-2)$ , and
- $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  with all parabolics directed by  $\mathcal{O}_{\mathbb{P}^1}(-3)$ .

For each such bundle  $(\underline{E}, \mathbf{p})$ , the space of connections is  $\mathbb{C}_{\theta_{1,2}}^2 \times \mathbb{C}_{\theta_{2,1}}^2$  (the  $\theta_{i,j}$  are those defined in the proof of Proposition 5.1) where  $\{0\} \times \mathbb{C}^2$  and  $\mathbb{C}^2 \times \{0\}$  stand for reducible connections and  $\{0\} \times \{0\}$  for the unique totally reducible one. The automorphism group of  $(\underline{E}, \mathbf{p})$  is  $\mathbb{C}^*$  acting as follows:

$$\mathbb{C}^* \times \mathbb{C}^4 \rightarrow \mathbb{C}^4 ; (\lambda, (a_0, a_1, b_0, b_1)) \mapsto (\lambda a_0, \lambda a_1, \lambda^{-1} b_0, \lambda^{-1} b_1).$$

The GIT quotient is the affine threefold  $xy = zw$  where  $x = a_0 a_1$ ,  $y = b_0 b_1$ ,  $z = a_0 b_1$  and  $w = a_1 b_0$ ; the singular point  $x = y = z = w = 0$  stands for reducible connections.

**5.2. How special bundles on  $X$  occur as special bundles on  $X/\iota$ .** Let us recall the construction of the map  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$  (see sections 2.2 and 2.3). Given a flat parabolic bundle  $(\underline{E}, \mathbf{p})$  in  $\mathfrak{Bun}(X/\iota)$ , we lift it up to the curve  $X$  as  $\pi^*(\underline{E}, \mathbf{p}) = (\tilde{E}, \tilde{\mathbf{p}})$ , then apply elementary transformations  $(E, \mathbf{p}) := \text{elm}_W^+(\tilde{E}, \tilde{\mathbf{p}})$  over the Weierstrass points and get a determinant-free vector bundle  $E$  over  $X$ , an element of  $\mathfrak{Bun}(X)$ . Conversely, given a generic bundle  $E$  on  $X$ , say stable and off the Gunning planes, then it has exactly two  $\iota$ -invariant anti-canonical subbundles  $\mathcal{O}_X(-K_X) \hookrightarrow E$  (see Corollary 4.4); consider the parabolic structure  $\mathbf{p}$  directed by one of them  $L \subset E$ . Then after applying elementary transformations over the Weierstrass points  $(\tilde{E}, \tilde{\mathbf{p}}) := \text{elm}_W^-(E, \mathbf{p})$ , we get the lift of a unique parabolic bundle  $(\underline{E}, \mathbf{p})$  on  $X/\iota$ ; precisely,  $\tilde{E} = \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-2K_X)$

and  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ . The two anti-canonical subbundles  $L, L' \subset E$ , being  $\iota$ -invariant, descend as two subbundles of  $(\underline{E}, \underline{p})$ ; one easily checks that they are the destabilizing bundle  $\underline{L} = \mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \times \mathcal{O}_{\mathbb{P}^1}(-2)$  and the unique  $\underline{L}' \simeq \mathcal{O}_{\mathbb{P}^1}(-4) \subset \underline{E}$  passing through all parabolics  $\underline{p}$ .

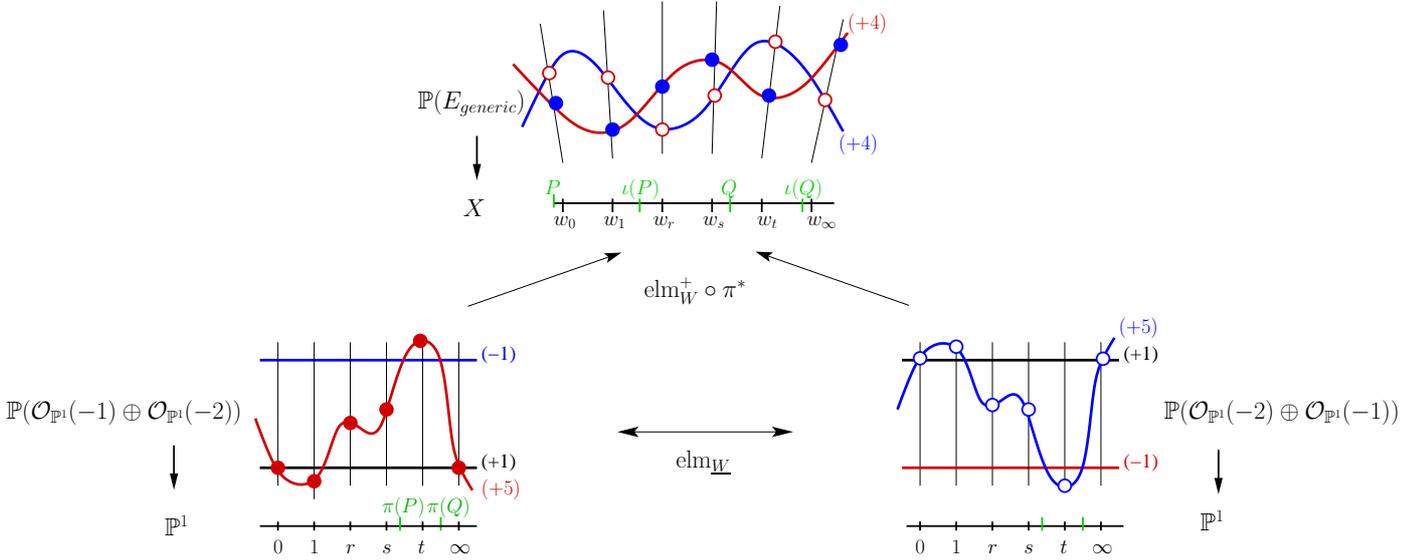


FIGURE 1. A generic stable bundle on  $X$

In figure 1, we can see the total space (ruled surfaces) of the parabolic bundles associated to  $E$ , and its two preimages  $\underline{E}$  and  $\underline{E}'$  in  $\mathfrak{Bun}(X/\iota)$ . The anti-canonical subbundles  $L$  and  $L'$  of  $E$ , and the corresponding subbundles of  $\underline{E}$  and  $\underline{E}'$ , are the blue and red curves (sections) on the ruled surfaces. We can see the self-intersection of the curves in each case. Parabolics are just points in Weierstrass fibers; those corresponding to  $\underline{p}$  and  $\underline{p}'$  (directed by the blue curve  $L$  up-side) are the red ones. The intersection of the two curves determines (in each ruled surface) the Tyurin divisor  $D_E^T$ . The Galois involution of  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$  permutes the roles of  $L$  and  $L'$ ; down-side, the elementary transformation permutes the role of the two curves.

We now list the parabolic bundles of  $\mathfrak{Bun}(X/\iota)$  giving rise to special bundles of  $\mathfrak{Bun}(X)$  and illustrate on pictures the corresponding configurations of curves and points on the ruled surfaces.

5.2.1. *Generic decomposable bundles.* Let  $E = L_0 \oplus L_0^{-1}$ , where  $L_0 = \mathcal{O}([P] + [Q] - K_X)$  is not 2-torsion:  $L_0^2 \neq \mathcal{O}_X$ . Assume also, for simplicity, that neither  $P$ , nor  $Q$  is a Weierstrass point. Recall (see section 4.1.2) that, up to automorphism, there is a unique parabolic structure  $\underline{p}$  which is directed by any embedding  $\mathcal{O}_X(-K_X) \hookrightarrow E$ . On the projective bundle  $\mathbb{P}(E)$ , there are two sections  $\sigma_0, \sigma_\infty : X \rightarrow \mathbb{P}(E)$  coming from the two factors  $L_0$  and  $L_0^{-1}$  respectively, both having 0 self-intersection and permuted by the involution  $\iota : X \rightarrow X$ . On the other hand, the anticanonical embedding defines a section  $\sigma : X \rightarrow \mathbb{P}(E)$  intersecting  $\sigma_0$  at  $[P] + [Q]$  and  $\sigma_\infty$  at  $[\iota(P)] + [\iota(Q)]$ . One can view  $\mathbb{P}(E)$  as the fiber-wise compactification of  $\mathcal{O}_X([P] + [Q] - [\iota(P)] - [\iota(Q)])$  with  $\sigma_0$  as the zero section and  $\sigma_\infty$  as the compactifying section; then  $\sigma$  is a rational section with divisor  $[P] + [Q] - [\iota(P)] - [\iota(Q)]$ .

For the corresponding parabolic bundle  $(\underline{E}, \underline{p})$ , the anticanonical embedding descends as the destabilizing subbundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ . On the other hand,  $\sigma_0$  and  $\sigma_\infty$ , being permuted by the involution  $\iota$ , descend as a 2-section  $\Gamma \subset \mathbb{P}(\underline{E})$ , thus intersecting a generic member of the ruling twice. Moreover,  $\Gamma$  intersects twice the section  $\sigma_{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}(\underline{E})$  defined by the destabilizing bundle  $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E}$ , namely at  $\pi(P)$  and  $\pi(Q)$  (where  $\pi : X \rightarrow \mathbb{P}^1 = X/\iota$  is the hyperelliptic projection). The restriction of the ruling projection  $\mathbb{P}(\underline{E}) \rightarrow \mathbb{P}^1$  to the curve  $\Gamma$ :

$$\Gamma \rightarrow \mathbb{P}^1 (= X/\iota)$$

is a 2 : 1-cover branching precisely over the branching divisor  $\underline{W}$  of  $\pi : X \rightarrow \mathbb{P}^1$  (orbifold points of  $X/\iota$ ). The parabolic structure  $\underline{p}$  is precisely located at the double point of  $\Gamma \subset \mathbb{P}(\underline{E})$  over  $\underline{W}$ .

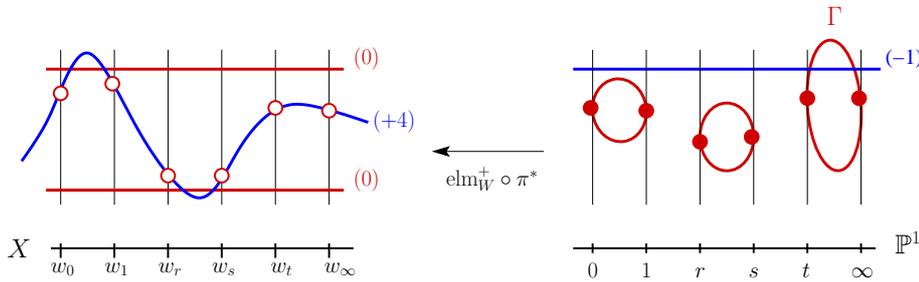


FIGURE 2. A generic decomposable bundle on  $X$

Conversely, a parabolic structure  $\underline{p}$  on  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  gives rise to a decomposable bundle  $E$  if, and only if, there is a smooth curve  $\Gamma \subset \mathbb{P}(\underline{E})$  belonging to the linear system defined by  $|2[\sigma_{-1}] + 2[f]|$  (with  $f$  any fiber of the ruling and  $\sigma_{-1}$  the negative section as before) such that  $\Gamma$  passes through all 6 parabolic points  $\underline{p}$  and is moreover vertical at these points (i.e. tangent to the ruling).

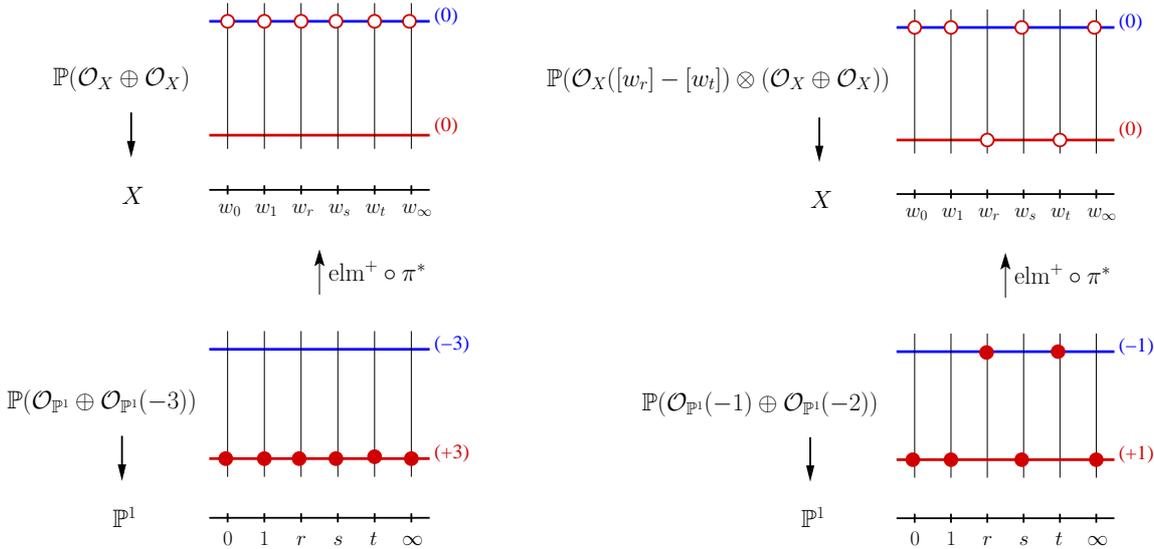


FIGURE 3. The trivial bundle over  $X$  and one of its twists

5.2.2. *The trivial bundle and its 15 twists.* Up to automorphism, the trivial bundle  $E_0 = \mathcal{O}_X \oplus \mathcal{O}_X$  has a unique parabolic structure  $\mathbf{p}$ , which is directed by any  $\mathcal{O}_X \hookrightarrow E_0$ . Descending to  $\mathbb{P}^1$ , we get the decomposable bundle  $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  with parabolic structure  $\underline{\mathbf{p}}$  directed by any  $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}_0$ . Note that  $(\underline{E}_0, \underline{\mathbf{p}})$  is a fixed point of  $\text{elm}_W$ . Similarly,  $\overline{E}_\tau = \tau \otimes E_0$  with  $\tau = \mathcal{O}_X([w_i] - [w_j])$  a 2-torsion line bundle, comes from the decomposable parabolic bundle  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  having parabolics  $\underline{p}_i$  and  $\underline{p}_j$  lying on the first factor, the other ones on the second.

These 16 parabolic bundles are exactly those flat decomposable bundles listed in Proposition 5.1.

5.2.3. *The unipotent family and its 15 twists.* A generic non trivial extension  $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$  has two hyperelliptic parabolic structures:

- $\mathbf{p}$  directed by some embedding  $\mathcal{O}_X(-K_X) \hookrightarrow E$  (unique up to bundle automorphism);
- $\mathbf{p}'$  directed by the destabilizing bundle  $\mathcal{O}_X \hookrightarrow E$ .

They respectively descend to elements of

- $\underline{\Delta} = \{(\underline{E}, \underline{\mathbf{p}}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{\mathbf{p}} \subset \mathcal{O}_{\mathbb{P}^1}(-3) \subset \underline{E}\}$ ;
- $\underline{\Delta}' = \{(\underline{E}', \underline{\mathbf{p}}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3) \text{ and } \underline{\mathbf{p}}' \subset \mathcal{O}_{\mathbb{P}^1}(-4) \subset \underline{E}'\}$ .

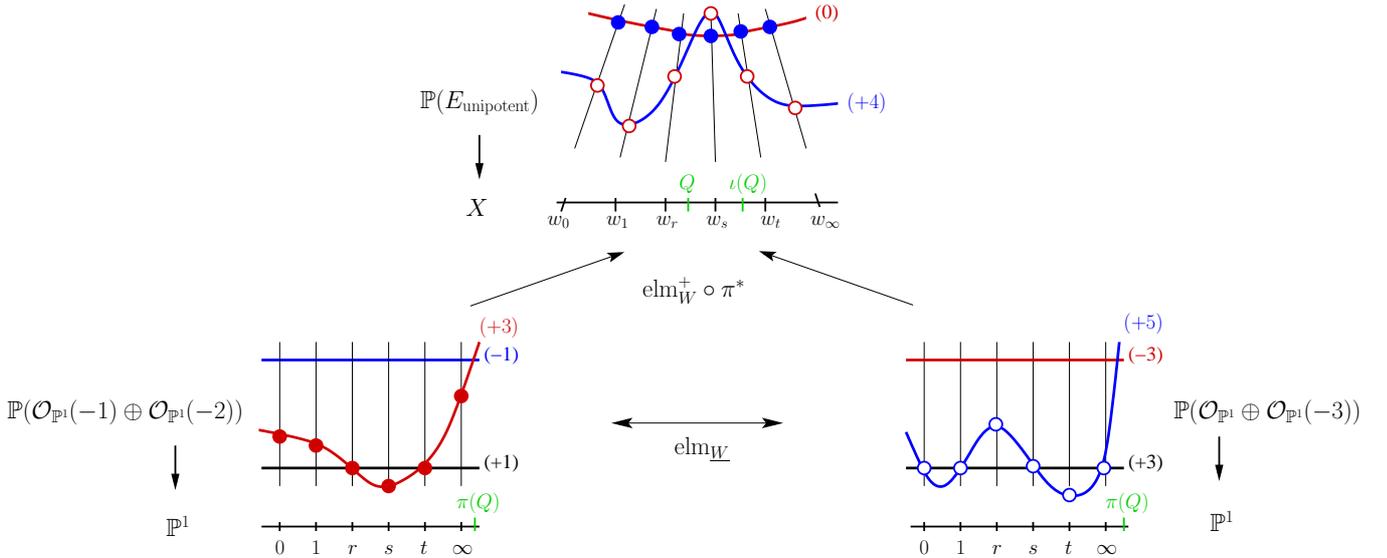


FIGURE 4. A unipotent bundle over  $X$

Denote by  $\Delta$  the 1-parameter family of the corresponding unipotent bundles in  $\mathfrak{Bun}(X)$  and by  $\underline{\Delta}$  and  $\underline{\Delta}'$  its respective preimages on  $\mathfrak{Bun}(X/\iota)$ . Both of these families are naturally parametrized by our base  $X/\iota$ : the extension class of  $E \in \Delta$  is characterized by the intersection locus of the two special subbundles  $\mathcal{O}_X(-K_X), \mathcal{O}_X \hookrightarrow E$ , an element of  $|\mathcal{O}_X(K_X)| \simeq |\mathcal{O}_{\mathbb{P}^1}(1)|$ . Similarly, the intersection locus of subbundles  $\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$  and  $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-4) \hookrightarrow \underline{E}'$  both define an element of  $|\mathcal{O}_{\mathbb{P}^1}(1)|$ . This unambiguously defines isomorphisms  $\Delta \simeq \underline{\Delta} \simeq \underline{\Delta}'$ , the latter one being induced by  $\mathcal{O}(-3) \otimes \text{elm}_W^+$ . Remind (see [29]) that, despite the point-wise identification just mentioned, any point of  $\underline{\Delta}$  is arbitrary close to any point of  $\underline{\Delta}'$  in the sense that they can

be simultaneously approximated by some deformation of stable parabolic bundles. This will give rise to a flop phenomenon when we will compare certain semi-stable projective charts.

The study of non-trivial extensions  $0 \rightarrow \tau \rightarrow E \rightarrow \tau \rightarrow 0$  where  $\tau = \mathcal{O}_X([w_i] - [w_j])$  is a 2-torsion line bundle, can be deduced from the study of the corresponding unipotent bundles  $\tau \otimes E$  by applying  $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \text{elm}_{[w_i]+[w_j]}^+$  on  $\underline{E}$  or, equivalently, by interchanging on  $\tau \otimes E$  the parabolic directions  $p_i$  and  $p_j$  with  $p'_i$  and  $p'_j$  respectively. We get a 1-parameter family  $\Delta_{i,j}$  naturally parametrized by  $X/\iota$ . There are two hyperelliptic parabolic structures for such a bundle  $E$ :

- $\underline{p}$  with parabolics  $p_i$  and  $p_j$  on  $\mathcal{O}_X \hookrightarrow E$  and the others outside;
- $\underline{p}'$  with parabolics  $p_i$  and  $p_j$  outside  $\mathcal{O}_X \hookrightarrow E$  and the others on it.

They respectively descend as elements of

- $\underline{\Delta}_{i,j} = \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2), \forall k \neq i, j\}$ ;
- $\underline{\Delta}'_{i,j} = \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}'_i, \underline{p}'_j \subset \mathcal{O}_{\mathbb{P}^1}(-1)\}$ .

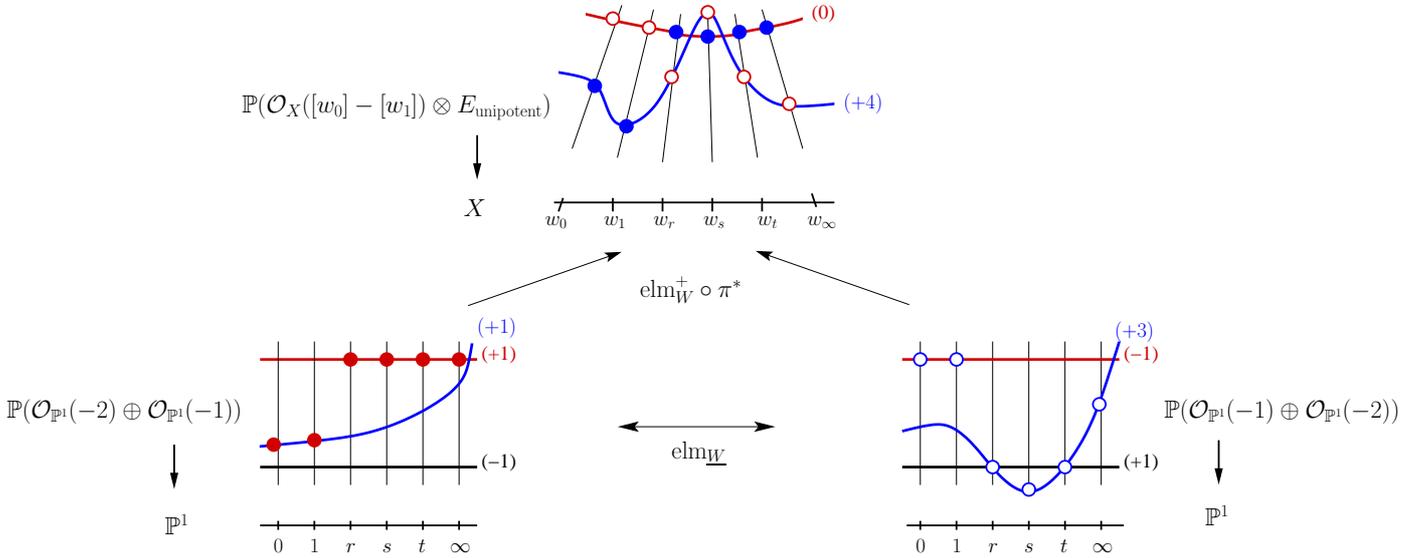


FIGURE 5. Twist of a unipotent bundle over  $X$

Again,  $\mathcal{O}(-3) \otimes \text{elm}_W^+$  point-wise permutes  $\underline{\Delta}_{i,j}$  and  $\underline{\Delta}'_{i,j}$ .

5.2.4. *The 6 + 10 Gunning bundles and Gunning planes.* We now list how arises the unique non trivial extension  $0 \rightarrow \mathcal{O}_X(\kappa) \rightarrow E \rightarrow \mathcal{O}_X(-\kappa) \rightarrow 0$  where  $\kappa$  runs over the 16 theta characteristics  $\kappa^2 = K_X$ .

**Six odd theta characteristics.** For odd theta characteristics  $\kappa = [w_i]$  lying along the divisor  $\Theta$ , the two hyperelliptic parabolic structures are:

- $\underline{p}$  with parabolic  $p_i$  on  $\mathcal{O}_X(\kappa) \hookrightarrow E$  and the others outside;
- $\underline{p}'$  with parabolic  $p_i$  outside  $\mathcal{O}_X(\kappa) \hookrightarrow E$  and the others lying on it.

They respectively descend as

- $Q_i : \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2), \forall k \neq i$ ;
- $Q'_i : \underline{E}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3) \text{ and } \underline{p}'_i \subset \mathcal{O}_{\mathbb{P}^1}$ .

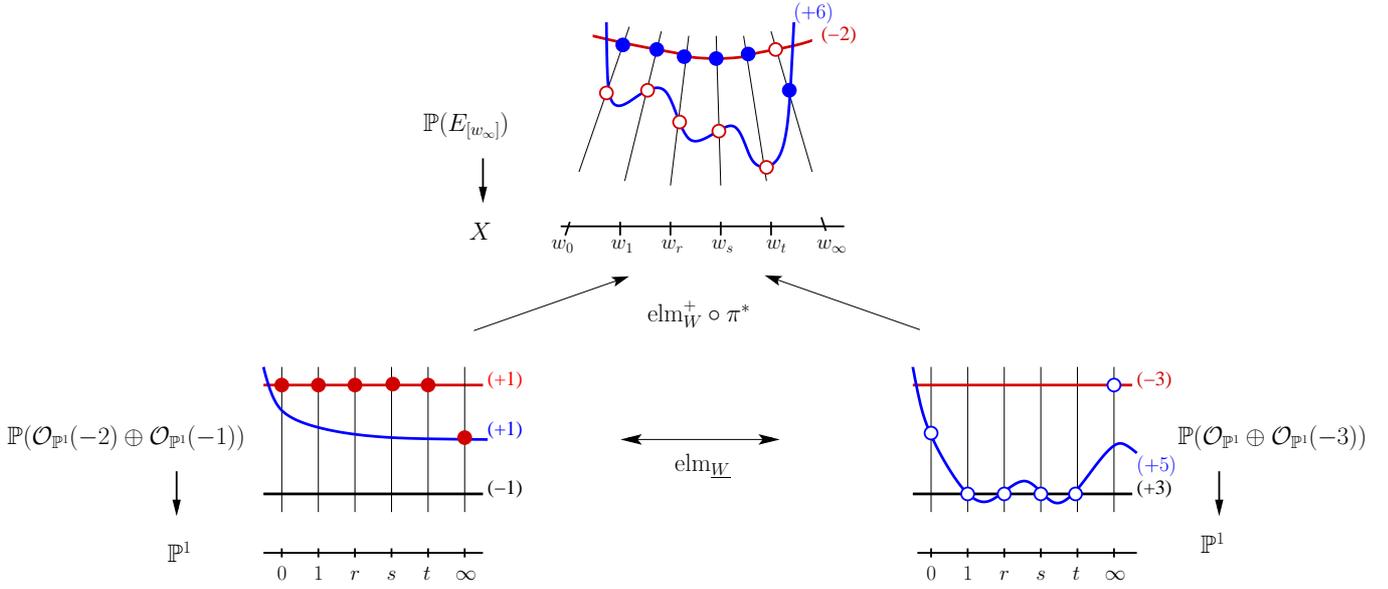


FIGURE 6. An odd Gunning bundle over  $X$

By the same way, the Gunning plane  $\Pi_\kappa$  descend as

- $\underline{\Pi}_i = \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_i \subset \mathcal{O}_{\mathbb{P}^1}(-1)\};$
- $\underline{\Pi}'_i = \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}'_k \subset \mathcal{O}_{\mathbb{P}^1}(-2), \forall k \neq i\}.$

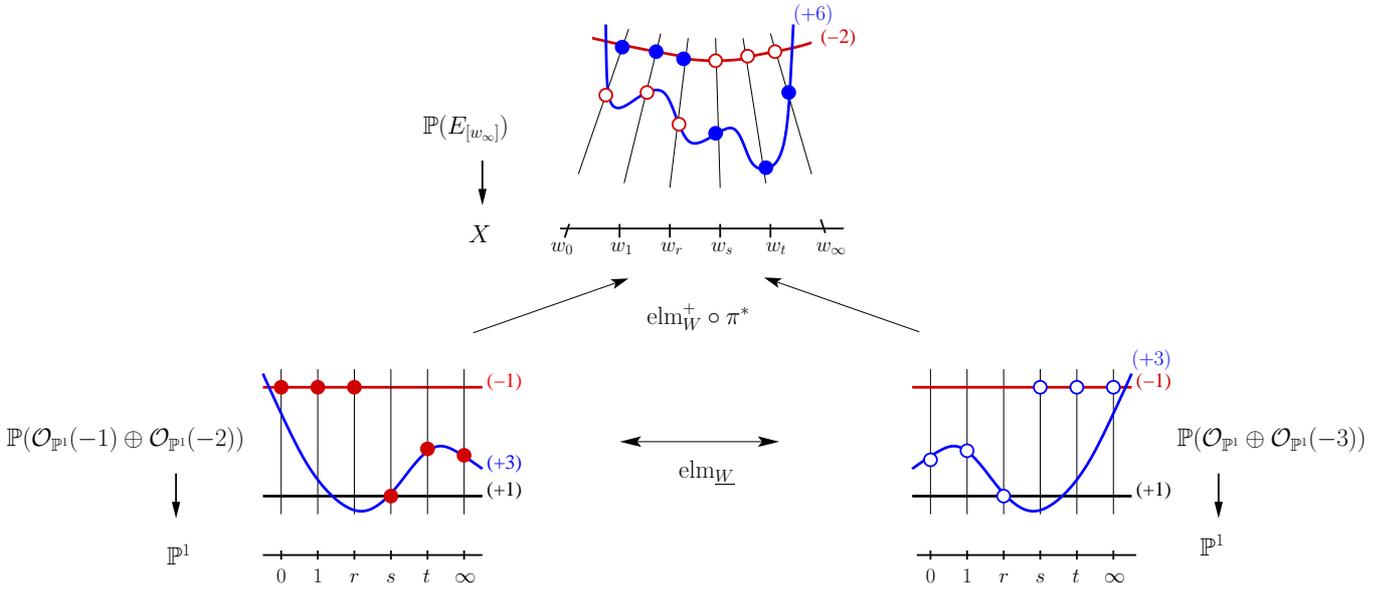


FIGURE 7. An even Gunning bundle over  $X$

**Ten even theta characteristics.** Somehow different is the case of even theta characteristics  $\kappa = [w_i] + [w_j] - [w_k]$ . Denote by  $\underline{W} = \{i, j, k\} \cup \{l, m, n\}$ . The two hyperelliptic parabolic structures are:

- $\mathbf{p}$  with parabolics  $p_i, p_j$  and  $p_k$  lying on  $\mathcal{O}_X(\kappa) \hookrightarrow E$  and the others outside;
- $\mathbf{p}'$  with parabolics  $p_i, p_j$  and  $p_k$  outside  $\mathcal{O}_X(\kappa) \hookrightarrow E$  and the others lying on it.

They respectively descend as elements of

- $Q_{i,j,k} : \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and  $\underline{p}_i, \underline{p}_j, \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-1)$ ;
- $Q_{l,m,n} : \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and  $\underline{p}'_i, \underline{p}'_j, \underline{p}'_k \subset \mathcal{O}_{\mathbb{P}^1}(-1)$ .

The corresponding Gunning planes descend to

- $\underline{\Pi}_{i,j,k} = \left\{ (\underline{E}, \mathbf{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_i, \underline{p}_j, \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-3) \subset \underline{E} \right\}$ ;
- $\underline{\Pi}_{l,m,n} = \left\{ (\underline{E}', \mathbf{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}'_i, \underline{p}'_j, \underline{p}'_k \subset \mathcal{O}_{\mathbb{P}^1}(-3) \subset \underline{E}' \right\}$ .

**5.3. Semi-stable bundles and projective charts.** The coarse moduli space  $\mathfrak{Bun}^u(X/\iota)$  of rank 2 indecomposable parabolic bundles  $(\underline{E}, \mathbf{p})$  over  $\mathbb{P}^1 = X/\iota$  is studied in [2, 29]. From the previous section,  $\mathfrak{Bun}(X/\iota) \setminus \mathfrak{Bun}^u(X/\iota)$  only consists of 16 bundles, that correspond to the trivial bundle and its 15 twists on  $X$  (see section 5.2.2). It turns out that a parabolic bundle  $(\underline{E}, \mathbf{p})$  is indecomposable if, and only if, it is stable for a good choice of weights  $\boldsymbol{\mu} = (\mu_0, \mu_1, \mu_r, \mu_s, \mu_t, \mu_\infty) \in [0, 1]^6$  (see [29]). One can thus cover the moduli space  $\mathfrak{Bun}^u(X/\iota)$  by projective charts  $\text{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota)$  for a finite collection of weights, giving  $\mathfrak{Bun}^u(X/\iota)$  a structure of non separated scheme. By the way,  $\mathfrak{Bun}^u(X/\iota)$  can be covered by charts isomorphic to  $(\mathbb{P}^1)^3$  (see [2]) or  $\mathbb{P}^3$  (see [29]). We will mainly consider two charts.

**5.3.1. The chart  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ .** One of them (see [2] and [29] section 3.4) is given by weights of the form

$$\mu_0 = \mu_1 = \mu_\infty = \frac{1}{2} \quad \text{and} \quad \mu_r = \mu_s = \mu_t = 0$$

and is isomorphic to  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ . Precisely,  $\boldsymbol{\mu}$ -stable bundles  $(\underline{E}, \mathbf{p})$  are given by  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  with  $\underline{p}_0, \underline{p}_1, \underline{p}_\infty$  outside of  $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$  and not all lying on the same  $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$ . One can choose  $\mathcal{O}_{\mathbb{P}^1}(-2)$  containing at least  $\underline{p}_0$  and  $\underline{p}_\infty$  say, and then choose meromorphic sections  $e_1$  and  $e_2$  of  $\mathcal{O}_{\mathbb{P}^1}(-1)$  and  $\mathcal{O}_{\mathbb{P}^1}(-2)$  (whose divisor is supported at  $x = \infty$ ) such that the parabolic structure is normalized to

$$\underline{p}_i = \lambda_i e_1 + e_2 \quad \text{with} \quad (\lambda_0, \lambda_1, \lambda_\infty) = (0, 1, 0) \quad \text{and} \quad (\lambda_r, \lambda_s, \lambda_t) = (R, S, T) \in \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1.$$

To compare to the point of view of [2], note that

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes \text{elm}_\infty^+(\underline{E}, \mathbf{p}) = (\underline{E}'_0, \mathbf{p}')$$

is the trivial bundle  $\underline{E}'_0 = \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}$  equipped with a parabolic structure having  $\underline{p}'_0, \underline{p}'_1$  and  $\underline{p}'_\infty$  pairwise distinct (with respect to the trivialization of the bundle). From this chart, we can compute the two-fold cover  $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$

**Proposition 5.2.** *The classifying map  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{NR}^3$  is explicitly given by  $(R, S, T) \mapsto (v_0 : v_1 : v_2 : v_3)$  where*

$$\begin{aligned} v_0 &= s^2 t^2 (r^2 - 1)(s - t)R - r^2 t^2 (s^2 - 1)(r - t)S + s^2 r^2 (t^2 - 1)(r - s)T + \\ &\quad + t^2 (t - 1)(r^2 - s^2)RS - s^2 (s - 1)(r^2 - t^2)RT + r^2 (r - 1)(s^2 - t^2)ST \\ v_1 &= rst [((r - 1)(s - t)R - (s - 1)(r - t)S + (t - 1)(r - s)T + \\ &\quad + (t - 1)(r - s)RS - (s - 1)(r - t)RT + (r - 1)(s - t)ST] \\ v_2 &= -st(r^2 - 1)(s - t)R + rt(s^2 - 1)(r - t)S - rs(t^2 - 1)(r - s)T - \\ &\quad - t(t - 1)(r^2 - s^2)RS + s(s - 1)(r^2 - t^2)RT - r(r - 1)(s^2 - t^2)ST \\ v_3 &= st(r - 1)(s - t)R - rt(s - 1)(r - t)S + sr(t - 1)(r - s)T + \\ &\quad + t(t - 1)(r - s)RS - s(s - 1)(r - t)RT + r(r - 1)(s - t)ST \end{aligned}$$

*This map is generically  $(2 : 1)$  with indeterminacy points*

$$(R, S, T) = (0, 0, 0), \quad (1, 1, 1), \quad (\infty, \infty, \infty) \quad \text{and} \quad (r, s, t).$$

*The Galois involution  $(R, S, T) \mapsto (\tilde{R}, \tilde{S}, \tilde{T})$  of this covering map is given by*

$$\begin{aligned} \tilde{R} &= \lambda(R, S, T) \cdot \frac{(s-t)+(t-1)S-(s-1)T}{-t(s-1)S+s(t-1)T+(s-t)ST} \\ \tilde{S} &= \lambda(R, S, T) \cdot \frac{(r-t)+(t-1)R-(r-1)T}{-t(r-1)R+r(t-1)T+(r-t)RT} \\ \tilde{T} &= \lambda(R, S, T) \cdot \frac{(r-s)+(s-1)R-(r-1)S}{-s(r-1)R+r(s-1)S+(r-s)RS} \end{aligned}$$

$$\text{where} \quad \lambda(R, S, T) = \frac{t(r-s)RS-s(r-t)RT+r(s-t)ST}{(s-t)R-(r-t)S+(r-s)T}.$$

*The ramification locus is over the Kummer surface; its lift on  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  is given by the equation*

$$\begin{aligned} &((s - t)R + (t - r)S + (r - s)T)RST \\ &+ t((r - 1)S - (s - 1)R)RS + r((s - 1)T - (t - 1)S)ST + s((t - 1)R - (r - 1)T)RT \\ &- t(r - s)RS - r(s - t)ST - s(t - r)RT = 0. \end{aligned}$$

*Proof.* For computations, we work with the parabolic bundle

$$(\underline{E}'_0, \underline{\mathbf{p}}') := \mathcal{O}_{\mathbb{P}^1}(1) \otimes \text{elm}_{\infty}^+(\underline{E}, \underline{\mathbf{p}})$$

where  $\underline{E}'_0 = \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}$  is the trivial bundle, generated by sections  $e'_1$  and  $e'_2$ , and  $\underline{\mathbf{p}}'$  is the parabolic structure defined by

$$\underline{p}'_i = \lambda_i e'_1 + e'_2 \quad \text{with} \quad (\lambda_0, \lambda_1, \lambda_{\infty}) = (0, 1, \infty) \quad \text{and} \quad (\lambda_r, \lambda_s, \lambda_t) = (R, S, T) \in \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1.$$

Let now  $E$  be the vector bundle over  $X$  obtained by

$$E := \text{elm}_W^+(\pi^*(\underline{E}, \underline{\mathbf{p}})) = \text{elm}_W^+(\pi^*(\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \text{elm}_{[\infty]}^-(\underline{E}'_0, \underline{\mathbf{p}}')));$$

this can be rewritten as

$$E := \mathcal{O}_X(-3[w_{\infty}]) \otimes \text{elm}_W^+(\pi^*(\underline{E}'_0, \underline{\mathbf{p}}')) = \mathcal{O}_X(-3[w_{\infty}]) \otimes \text{elm}_W^+(E_0, \pi^* \underline{\mathbf{p}}')$$

where  $E_0$  is the trivial vector bundle on  $X$ .

In order to calculate the classifying map, we need to make the Narasimhan-Ramanan divisor  $D_E$  explicit in our coordinates. We may assume that  $E$  is generic (*i.e.* stable), so that  $D_E$  precisely describes the 1-parameter family of degree  $-1$  line bundles  $L \subset E$ . After applying  $\mathcal{O}_X(-3[\infty]) \otimes \text{elm}_W^+$ , we get the family of degree  $-4$  subbundles  $L' \subset E_0$  (the trivial bundle over  $X$ ) passing through all 6 parabolics  $\underline{\mathbf{p}}'$ . Precisely, if

$L = \mathcal{O}_X([w_\infty] - [P_1] - [P_2])$ , then  $L' = \mathcal{O}_X(-3[w_\infty]) \otimes L = \mathcal{O}_X(-2[w_\infty] - [P_1] - [P_2])$ . In other words, the Narasimhan-Ramanan divisor  $D_E \subset \text{Pic}^1(X)$  is directly given by the 1-parameter family of points  $\{P_1, P_2\}$  such that there is a line subbundle  $L = \mathcal{O}_X(-[P_1] - [P_2] - 2[\infty]) \hookrightarrow E_0$  coinciding with the parabolic structure over  $W$ . Let  $\sigma = (\sigma_1, \sigma_2) : X \rightarrow \mathbf{C}^2$  be a meromorphic section of  $L$  with divisor  $-[P_1] - [P_2] - 2[\infty]$  with  $P_i = (x_i, y_i) \in X$ ,  $i = 1, 2$ :

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \alpha + \beta x + \gamma \left( \frac{y-y_1}{x-x_1} - \frac{y-y_2}{x-x_2} \right) \\ \delta + \varepsilon x + \varphi \left( \frac{y-y_1}{x-x_1} - \frac{y-y_2}{x-x_2} \right) \end{pmatrix}.$$

After normalizing  $\alpha = 1$ , there is a unique choice of  $\beta, \gamma, \delta, \varepsilon, \varphi \in \mathbf{C}$  such that

$$\sigma(0, 0) \parallel \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma(1, 0) \parallel \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \sigma(r, 0) \parallel \begin{pmatrix} R \\ 1 \end{pmatrix}, \quad \sigma(s, 0) \parallel \begin{pmatrix} S \\ 1 \end{pmatrix}, \quad \sigma(\infty, \infty) \parallel \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The condition  $\sigma(t, 0) \parallel \begin{pmatrix} T \\ 1 \end{pmatrix}$  depends now only on the choice of  $\{P_1, P_2\}$  and writes (after convenient reduction)

$$v_0 \cdot 1 + v_1 \cdot \text{Sum}(P_1, P_2) + v_2 \cdot \text{Prod}(P_1, P_2) + v_3 \cdot \text{Diag}(P_1, P_2) = 0$$

with  $v_i$  as given in the proposition.

One can easily deduce that a generic point  $(v_0 : v_1 : v_2 : v_3) \in \mathcal{M}_{NR}$  has precisely two preimages in  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  :

$$\begin{aligned} R &= \frac{r(t-1)(v_0+rv_1-r(s+t+st)v_3)T}{t(r-1)(v_0+tv_1-t(r+s+rs)v_3)-(r-t)(v_0+v_1-\sigma_2v_3)T} \\ S &= \frac{s(t-1)(v_0+sv_1-s(r+t+rt)v_3)T}{t(s-1)(v_0+tv_1-t(r+s+rs)v_3)-(s-t)(v_0+v_1-\sigma_2v_3)T}, \end{aligned}$$

where  $T$  is any solution of  $aT^2 + bT + ct^2 = 0$  with

$$\begin{aligned} a &= (v_1 + v_2t + v_3t^2)(v_0 + v_1 - \sigma_2v_3) \\ b &= -(1+t)(v_0v_2 + v_1^2 + tv_1v_3) - 2(v_0v_1 + tv_0v_3 + tv_1v_2) \\ &\quad + \sigma_2(tv_1 + v_2 + tv_3)v_3 + (r+s+rs)(v_1 + t^2v_2 + t^2v_3)v_3 \\ c &= (v_1 + v_2 + v_3)(v_0 + tv_1 - t(r+s+rs)v_3). \end{aligned}$$

The discriminant of this polynomial leads again to our equation of the Kummer surface in the coordinates  $(v_0 : v_1 : v_2 : v_3)$  given in section 3.6. We can easily calculate the Galois involution of the classifying map  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{NR}^3$ . Its fixed points provide the equation in coordinates  $(R, S, T)$  of the lift of the Kummer surface.  $\square$

5.3.2. *The chart  $\mathbb{P}_b^3$ .* The other chart (namely the main chart  $\mathbb{P}_b^3$  of [29]) is defined by democratic weights

$$\frac{1}{6} < \mu_0 = \mu_1 = \mu_r = \mu_s = \mu_t = \mu_\infty < \frac{1}{4}$$

and corresponds to the moduli space of the indecomposable parabolic structures on  $\underline{E} := \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  having no parabolic directed by  $\mathcal{O}_{\mathbb{P}^1}(-1)$ . Parabolic bundles belonging to this chart are exactly those given by extensions

$$0 \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-1), \emptyset) \rightarrow (\underline{E}, \underline{p}) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-2), \underline{W}) \rightarrow 0$$

*i.e.* defined by points of  $\mathbb{P}H^1(\mathbb{P}^1, \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(\underline{W}), \mathcal{O}_{\mathbb{P}^1}(-1)))$ , which by Serre duality, identifies to  $\mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^\vee$ . After lifting them on  $X \rightarrow \mathbb{P}^1$ , applying elementary transformations and forgetting the parabolic structure, we precisely get those extensions

$$0 \rightarrow \mathcal{O}(-K_X) \rightarrow E \rightarrow \mathcal{O}(K_X) \rightarrow 0$$

i.e. by those points of  $\mathbb{P}_B^4 = \mathbb{P}H^0(X, \mathcal{O}_X(3K_X))^\vee$ , that are  $\iota$ -invariant. Thus, the projective chart  $\mathbb{P}_b^3$  of [29] naturally identifies with that one  $\mathbb{P}_B^3$  introduced by Bertram (see section 4.2). From this point of view, we have natural projective coordinates  $\mathbf{b} = (b_0 : b_1 : b_2 : b_3)$ , dual to the coordinates of  $\iota$ -invariant cubic forms  $(a_0 + a_1x + a_2x^2 + a_3x^3) \left(\frac{dx}{y}\right)^{\otimes 3}$ ; after computation, we get

**Proposition 5.3.** *The natural birational map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  is given by*

$$(b_0 : b_1 : b_2 : b_3) \mapsto \begin{cases} R &= r \frac{b_3 - (s+t+1)b_2 + (st+s+t)b_1 - stb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \\ S &= s \frac{b_3 - (r+t+1)b_2 + (rt+r+t)b_1 - rtb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \\ T &= t \frac{b_3 - (r+s+1)b_2 + (rs+r+s)b_1 - rsb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \end{cases}$$

This will be proved in section 6.2, using Higgs fields. Combination with Proposition 5.2 yields

**Corollary 5.4.** *The natural map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_{NR}^3$  is given by*

$$(b_0 : b_1 : b_2 : b_3) \mapsto \begin{cases} v_0 &= b_2 b_3 - (1 + \sigma_1) b_2^2 + (\sigma_1 + \sigma_2) b_1 b_2 - (\sigma_2 + \sigma_3) b_0 b_2 + \sigma_3 b_0 b_1 \\ v_1 &= b_2^2 - b_1 b_3 \\ v_2 &= b_0 b_3 - b_1 b_2 \\ v_3 &= b_1^2 - b_0 b_2 \end{cases}$$

Moreover, the (dual) Weddle surface, i.e. the lift to  $\mathbb{P}_B^3$  of the Kummer equation, writes

$$\begin{aligned} &(-b_0 b_2 b_3^2 + b_1^2 b_3^2 + b_1 b_2^2 b_3 - b_2^4) + (1 + \sigma_1)(b_0 b_2^2 b_3 - 2b_1^2 b_2 b_3 + b_1 b_2^3) + (\sigma_1 + \sigma_2)(-b_0 b_2^3 + b_1^3 b_3) \\ &+ (\sigma_2 + \sigma_3)(-b_0 b_1^2 b_3 + 2b_0 b_1 b_2^2 - b_1^3 b_2) + \sigma_3(b_0^2 b_1 b_3 - b_0^2 b_2^2 - b_0 b_1^2 b_2 + b_1^4) = 0. \end{aligned}$$

This Corollary has to be compared to section 4.2. Indeed, the components of the map  $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_{NR}^3$  exactly correspond to the restriction to  $\mathbb{P}_B^3$  of the natural quadratic forms on  $\mathbb{P}_B^4$  vanishing along the embedding

$$X \hookrightarrow \mathbb{P}_B^4 ; (x, y) \mapsto (b_0 : b_1 : b_2 : b_3 : b_4) = (1 : x : x^2 : x^3 : y).$$

Indeed, the first one is the restriction of

$$b_4^2 - (b_2 b_3 - (1 + \sigma_1) b_2^2 + (\sigma_1 + \sigma_2) b_1 b_2 - (\sigma_2 + \sigma_3) b_0 b_2 + \sigma_3 b_0 b_1)$$

which vanishes along  $X \hookrightarrow \mathbb{P}_B^4$  from

$$y^2 = x(x-1)(x-r)(x-s)(x-t) = x^5 - (1 + \sigma_1)x^4 + (\sigma_1 + \sigma_2)x^3 - (\sigma_2 + \sigma_3)x^2 + \sigma_3 x.$$

The other 3 quadratic forms just come from the following equalities on  $X$

$$b_0 b_2 = b_1^2 = x^2, \quad b_0 b_3 = b_1 b_2 = x^3 \quad \text{and} \quad b_1 b_3 = b_2^2 = x^4.$$

It is quite surprising that the most natural basis both appearing from Bertram point of view, and Narasimhan-Ramanan point of view, are so compatible. They provide the same system of coordinate on  $\mathbb{P}_{NR}^3$  which is however not considered in the classical theory of Kummer surfaces (see [24, 20]).

5.3.3. *Special bundles in the chart  $\mathbb{P}_b^3$ .* Here is the list of those special parabolic bundles of section 5.2 that are semi-stable for  $\frac{1}{6} < \mu_0 = \mu_1 = \mu_r = \mu_s = \mu_t = \mu_\infty < \frac{1}{4}$  and how they occur as special points in the chart  $\mathbb{P}_b^3$ .

**Proposition 5.5.** *The only special bundles occurring (as semi-stable bundles) in  $\text{Bun}_\mu^{ss}(X/\iota) = \mathbb{P}_b^3$  are the generic bundles of the following families*

- **Unipotent bundles  $\underline{\Delta}$ :** *this 1-parameter family corresponds to the twisted cubic parametrized by*

$$X/\iota \rightarrow \mathbb{P}_b^3; x \mapsto (1 : x : x^2 : x^3).$$

- **Odd Gunning bundles  $Q_i$ :** *they are the 6 special points of the previous embedding  $X/\iota \rightarrow \mathbb{P}_b^3$ , namely  $Q_i$  is the image of the Weierstrass point  $w_i$ .*
- **Twisted unipotent bundles  $\underline{\Delta}_{i,j}$ :** *lines of  $\mathbb{P}_b^3$  passing through  $Q_i$  and  $Q_j$ .*
- **Even Gunning planes  $\underline{\Pi}_{i,j,k}$ :** *planes of  $\mathbb{P}_b^3$  passing through  $Q_i, Q_j$  and  $Q_k$ .*
- **Odd Gunning planes  $\underline{\Pi}'_i$ :** *the quadric surface of  $\mathbb{P}_b^3$  with a conic singular point at  $Q_i$  that contains the 5 lines  $\underline{\Delta}_{i,j}$  and the cubic  $\underline{\Delta}$ .*

*Proof.* It is easy to check which special parabolic bundles are semi-stable or not. For instance, the trivial bundle  $E_0$  descends as the vector bundle  $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  equipped with the decomposable parabolic structure  $\underline{p}$  directed by  $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}_0$  (see 5.2.2); then  $\mathcal{O}_{\mathbb{P}^1}$  is destabilizing.

Once this has been done, for each family occurring in  $\mathbb{P}_b^3$ , we already know from section 5.2 where they are sent on  $\mathbb{P}_{NR}^3$ , we know the corresponding explicit equations from section 3.6 and we can deduce equations on  $\mathbb{P}_b^3$  by using explicit formula from Corollary 5.4.  $\square$

**Remark 5.6.** *Actually, we have only dealt with generic bundles of each type so far. Indeed, only an open set of the family  $\underline{\delta}$  of unipotent bundles occurs in  $\text{Bun}_\mu^{ss}(X/\iota) = \mathbb{P}_b^3$ , namely the complement of Weierstrass points, since they are replaced by Gunning bundles  $Q_i$ .*

The preimage of the Kummer surface  $\text{Kum}(X)$  in the chart  $\mathbb{P}_b^3$  is nothing but the dual Weddle surface  $\text{Wed}(X)$ , another birational model of  $\text{Kum}(X)$ : it is also a quartic surface, but with only 6 nodes (see [24, 20]). Precisely, the 16 singular points of  $\text{Kum}(X)$  are blown-up and replaced by the lines  $\underline{\Delta}_{i,j}$ ; the 6 Gunning planes  $\Pi_i$  are contracted onto the points  $Q_i$ , giving rise to new conic points. In particular, all 16 quasi-unipotent families  $\underline{\Delta}$  and  $\underline{\Delta}_{i,j}$  are contained in  $\text{Wed}$ .

Actually, the map  $\phi : \mathbb{P}_b^3 \dashrightarrow \mathbb{P}_{NR}^3$  is defined by the linear system of quadrics passing through the 6 points  $Q_i$ ; indeed, for a general plane  $\Pi \in \mathbb{P}_{NR}^3$ ,  $\phi^*\Pi$  must intersect each contracted  $\Pi_i$ . We thus recover the quadric system in [14], §4.6. Those  $\Pi$  tangent to  $\text{Kum}(X)$  have a singular lift  $\underline{\Pi}$ ; when  $\Pi$  runs over the tangent planes of  $\text{Kum}(X)$ , the singular point of  $\underline{\Pi}$  runs over the Weddle surface.

**Remark 5.7.** *The complement of the (dual) Weddle surface covers the open set of stable bundles in  $\mathbb{P}_{NR}^3$*

$$\mathbb{P}_b^3 \setminus \text{Wed}(X) \xrightarrow{\phi} \mathbb{P}_{NR}^3 \setminus \text{Kum}(X).$$

*However, this is not a covering since over odd Gunning planes, only  $\underline{\Pi}'_i$  occurs in  $\mathbb{P}_b^3$ .*

**5.4. Moving weights and wall-crossing phenomena.** For a generic weight  $\mu$ , semi-stable bundles are automatically stable; in this case, the moduli space  $\text{Bun}_\mu^{ss}(X/\iota)$  is projective, smooth and a geometric quotient. The special weights  $\mu$ , for which some bundles are strictly semi-stable, form a finite collection of affine planes in the weight-space  $[0, 1]^6 \ni \mu$  called *walls*. They cut-out  $[0, 1]^6$  into finitely many *chambers*: the connected components of the complement of walls. Along walls, the moduli space is no more a geometric quotient, but a categorical quotient, identifying some semi-stable bundles together to get a (Hausdorff) projective manifold, which might be singular in this case; outside of the semi-stable locus,  $\text{Bun}_\mu^{ss}(X/\iota)$  is still smooth and a geometric quotient. The moduli space  $\text{Bun}_\mu^{ss}(X/\iota)$  is locally constant in a given chamber; if not empty, it has the right dimension 3 and contains as an open set the geometric quotient of those bundles  $(\underline{E}, \underline{p})$  with  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  and parabolics  $\underline{p}$  in general position:

- no parabolic on  $\mathcal{O}_{\mathbb{P}^1}(-1)$ ,
- no 3 parabolics on the same  $\mathcal{O}_{\mathbb{P}^1}(-2)$ ,
- no 5 parabolics on the same  $\mathcal{O}_{\mathbb{P}^1}(-3)$ .

Between any two (non empty !) moduli spaces we get a natural birational map

$$\text{can} : \text{Bun}_\mu^{ss}(X/\iota) \xrightarrow{\sim} \text{Bun}_{\mu'}^{ss}(X/\iota)$$

arising from the identification of these generic bundles occurring in both of them. The indeterminacy locus comes from those special parabolic bundles that are stable for  $\mu$  but not for  $\mu'$  and *vice-versa*; this happens each time we cross a wall. The moduli space  $\mathfrak{Bun}^u(X/\iota)$  of indecomposable bundles can be covered by a finite collection of such moduli spaces, by choosing one  $\mu$  in each non empty chamber; therefore,  $\mathfrak{Bun}^u(X/\iota)$  can be constructed by patching together these moduli spaces by means of canonical maps along the open set of common bundles. This gives  $\mathfrak{Bun}^u(X/\iota)$  a structure of smooth non separated scheme. However, in our case, we have also decomposable flat bundles that are not taken into account in this picture. For instance the preimage  $(\underline{E}_0, \underline{p}^0) := \phi^{-1}(E_0)$  of the trivial bundle on  $X$  (see section 5.2.2), being decomposable, can only arise as a singular point in semi-stable projective charts  $\text{Bun}_\mu^{ss}(X/\iota)$ . Indeed, if the bundle  $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$  equipped with the decomposable parabolic structure  $\underline{p}^0$  directed by  $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}_0$  is semi-stable for some choice of weights  $\mu$ , then all other parabolic structures  $\underline{p}$  on  $\underline{E}_0$  with no parabolics directed by  $\mathcal{O}_{\mathbb{P}^1} \subset \underline{E}_0$  are also semi-stable and infinitesimally close to  $\underline{p}^0$ ; they are represented by the same point in the Hausdorff quotient  $\text{Bun}_\mu^{ss}(X/\iota)$ . One can check that this point is necessarily singular.

Instead of being exhaustive, let us consider in this section the family of moduli spaces  $\text{Bun}_\mu^{ss}(X/\iota)$  with weights  $\mu = (\mu, \mu, \mu, \mu, \mu, \mu)$ , for  $\mu \in [0, 1]$ . One can easily check which family of special bundle is semi-stable, depending on the choice of  $\mu$ ; this is resumed in the following table.

$\mu$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	1
unipotent bundles (and twists)			$\Delta$		$\Delta_{ij}$		
			$\Delta'$		$\Delta'_{ij}$		
odd Gunning bundles and planes		$Q_i$		$\Pi_i$			
			$\Pi'_i$			$Q'_i$	
even Gunning planes			$\Pi_{ijk}$				

TABLE 3. Moving weights.

For  $\mu \in [0, \frac{1}{6}[$ . The moduli space  $\text{Bun}_{\mu}^{ss}(X/\iota)$  is empty since  $\mathcal{O}_{\mathbb{P}^1}(-1)$  is destabilizing the generic parabolic bundle (even if it carries no parabolic).

For  $\mu = \frac{1}{6}$ . The moduli space  $\text{Bun}_{\mu}^{ss}(X/\iota)$  reduces to a single point. Indeed, it also contains the (non flat) decomposable bundle  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  with all parabolics  $\underline{p}$  directed by  $\mathcal{O}_{\mathbb{P}^1}(-2)$ . But the generic parabolic bundle is infinitesimally close to this decomposable bundle so that they have to be identified in the Hausdorff quotient  $\text{Bun}_{\mu}^{ss}(X/\iota)$ .

For  $\mu \in ]\frac{1}{6}, \frac{1}{4}[$ . Here, we recover our chart  $\mathbb{P}_{\mathbf{b}}^3 := \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota)$  with special families  $\Delta$ ,  $\Delta_{ij}$ ,  $Q_i$ ,  $\Pi'_i$  and  $\Pi_{ijk}$ . The natural map  $\phi : \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota) \dashrightarrow \mathbb{P}_{NR}^3$  has indeterminacy points at all 6 points  $Q_i$ .

For  $\mu = \frac{1}{4}$ . Now, odd Gunning planes  $\Pi_i$  become semi-stable, but infinitesimally close to the corresponding point  $Q_i$ , so that they are identified in the quotient  $\text{Bun}_{\mu}^{ss}(X/\iota)$ . Therefore, the moduli space is still the same  $\mathbb{P}_{\mathbf{b}}^3$  but no more a geometric quotient.

For  $\mu \in ]\frac{1}{4}, \frac{1}{2}[$ . Odd Gunning bundles  $Q_i$  are no more semi-stable and are replaced by the corresponding Gunning planes  $\Pi_i$ . The natural map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota)$$

is the blow-up of  $\mathbb{P}_{\mathbf{b}}^3$  at all 6 points  $Q_i$ , and the exceptional divisors represent the corresponding planes  $\Pi_i$ . The natural map  $\phi : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \mathbb{P}_{NR}^3$  is a morphism.

For  $\mu = \frac{1}{2}$ . the trivial bundle and its 15 twists become semi-stable (and just for this special value of  $\mu$ ). In particular, unipotent families are identified with these bundles in the moduli space, which has effect to contract the strict transforms of lines  $\Delta_{ij}$  and the rational curve  $\Delta$  to 16 singular points of  $\text{Bun}_{\mu}^{ss}(X/\iota)$ . This moduli space is exactly the double cover of  $\mathbb{P}_{NR}^3$  ramified along  $\text{Kum}(X)$ , therefore singular with conic points over each singular point of  $\text{Kum}(X)$ . The natural map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$$

is a minimal resolution.

For  $\mu \in ]\frac{1}{2}, \frac{3}{4}[$ . The families  $\Delta$  and  $\Delta_{ij}$  are no more semi-stable, and replaced by the families  $\Delta'$  and  $\Delta'_{ij}$ . But mind that the canonical map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \dashrightarrow \text{Bun}_{\frac{1}{2}, \frac{3}{4}}^{ss}(X/\iota)$$

is not biregular: there is a flop phenomenon around each of the 16 above rational curves. Precisely, after blowing-up the 16 curves, we exactly get the resolution  $\widehat{\text{Bun}}_{\frac{1}{2}}^{ss}(X/\iota)$  of the previous moduli space by blowing-up the 16 conic points. Then, exceptional divisors are  $\simeq \mathbb{P}^1 \times \mathbb{P}^1$  and we can contract them back to rational curves by using the other ruling; this is the way the map  $\text{can}$  is constructed here. In particular, we get is another minimal resolution of  $\text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$ .

For  $\mu \in ]\frac{3}{4}, \frac{5}{6}[$ . Here, we finally contract the strict transforms of  $\Pi'_i$  to the points  $Q_i$ .

**5.5. Galois and Geiser involutions.** The Galois involution of the ramified cover  $\phi : \mathfrak{Bun}(X/\iota) \xrightarrow{2:1} \mathfrak{Bun}(X)$

$$\Upsilon := \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\mathbb{W}}^+ : \mathfrak{Bun}(X/\iota) \xrightarrow{\sim} \mathfrak{Bun}(X/\iota)$$

induces isomorphisms between moduli spaces

$$\Upsilon : \mathfrak{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota) \xrightarrow{\sim} \mathfrak{Bun}_{\boldsymbol{\mu}'}^{ss}(X/\iota)$$

where  $\boldsymbol{\mu}'$  is defined by  $\mu'_i = \frac{1}{2} - \mu_i$  for all  $i$ . In particular, it underlines the symmetry of our special family of moduli spaces around  $\mu = \frac{1}{2}$  (see section 5.4): the Galois involution induces a biregular involution of  $\mathfrak{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$ , as well as isomorphisms

$$\mathfrak{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \xleftarrow{\sim} \mathfrak{Bun}_{\frac{1}{2}, \frac{3}{4}}^{ss}(X/\iota) \quad \text{and} \quad \mathfrak{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota) \xleftarrow{\sim} \mathfrak{Bun}_{\frac{3}{4}, \frac{5}{6}}^{ss}(X/\iota).$$

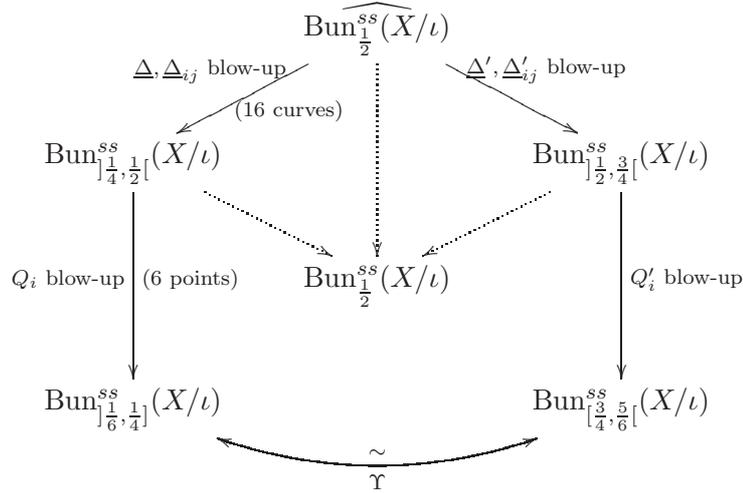
Considering now the composition

$$\mathfrak{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota) \xrightarrow{\text{can}} \mathfrak{Bun}_{\frac{3}{4}, \frac{5}{6}}^{ss}(X/\iota) \xrightarrow{\Upsilon} \mathfrak{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota),$$

we get the (birational) Galois involution of the map  $\phi : \mathbb{P}_{\mathbf{b}}^3 \dashrightarrow \mathbb{P}_{NR}^3$  described in Corollary 5.4. This is known as the Geiser involution (see [14], §4.6); it is a degree 7 birational map. The combination of all wall-crossing phenomena described in section 5.4, when  $\mu$  is varying from  $\frac{1}{6}$  to  $\frac{5}{6}$ , provides a complete decomposition of this map:

- first blow-up 6 points (those  $Q_i$  along the embedding  $X/\iota \xrightarrow{\sim} \underline{\Delta} \subset \mathbb{P}_{\mathbf{b}}^3$ ),
- flop 16 rational curves (those strict transforms of the twisted cubic  $\underline{\Delta}$  and all lines  $\underline{\Delta}_{ij}$ ),
- contract 6 planes (namely strict transforms of  $\Pi'_i$  onto  $Q_i$ ),
- then compose by the unique isomorphism sending  $Q'_i \rightarrow Q_i$ .

This is resumed in the following diagramm.



**Remark 5.8.** *Even Gunning bundles  $Q_{ijk}$  are semi-stable if, and only if,  $\mu = 1$ . This is why they do not appear in our family of moduli spaces. However, for some other choices of weights  $\boldsymbol{\mu}$ , they appear as stable points, and therefore smooth points of some projective charts.*

## 6. HIGGS BUNDLES AND CONNECTIONS

A *Higgs bundle* on a Riemann surface  $X$  is a vector bundle  $E \rightarrow X$  endowed with a *Higgs field*, i.e. an  $\mathcal{O}_X$ -linear morphism

$$\Theta : E \rightarrow E \otimes \Omega_X^1(D),$$

where  $D$  is an effective divisor. If  $D$  is reduced, then  $\Theta$  is called *logarithmic* and for any  $x \in D$ , the residual morphism  $\text{Res}_x(\Theta) \in \text{End}(E_x)$  is well-defined. As usual, we will only consider the case where  $E$  is a rank 2 vector bundle with trivial determinant bundle and  $\Theta$  is trace-free. By definition, a holomorphic ( $D = \emptyset$ ) and trace-free Higgs-field on  $E$  is an element of  $H^0(X, \mathfrak{sl}_2(E) \otimes \Omega_X^1)$ , which, by Serre duality, is isomorphic to  $H^1(X, \mathfrak{sl}_2(E))^\vee$ . On the other hand, stable bundles are simple (there are no non-scalar automorphism); for such bundles  $E$ , the vector space  $H^1(X, \mathfrak{sl}_2(E))$  is precisely the tangent space in  $E$  of our moduli space  $\mathfrak{Bun}(X)$  of flat vector bundles over  $X$ . Therefore, *in restriction to the open set of stable bundles* the moduli space  $\mathfrak{Higgs}(X)$  of Higgs bundles identifies in a natural way to

$$\mathfrak{Higgs}(X) := T^*\mathfrak{Bun}(X).$$

Just as naturally, we can define

$$\mathfrak{Higgs}(X/\iota) := T^*\mathfrak{Bun}(X/\iota),$$

but we need to clarify its meaning. Let  $(\underline{E}, \underline{\mathbf{p}})$  be a parabolic bundle in  $\mathfrak{Bun}(X/\iota)$ . Then  $T_{(\underline{E}, \underline{\mathbf{p}})}^*\mathfrak{Bun}(X/\iota) = H^0(X, \mathfrak{sl}_2(\underline{E}, \underline{\mathbf{p}}) \otimes \Omega_{\mathbb{P}^1}^1)$ , where  $\mathfrak{sl}_2(\underline{E}, \underline{\mathbf{p}})$  denotes the space of trace-free endomorphisms of  $\underline{E}$  leaving  $\underline{\mathbf{p}}$  invariant. Now consider the image of the natural embedding

$$H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}, \underline{\mathbf{p}}) \otimes \Omega_{\mathbb{P}^1}^1) \hookrightarrow H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})).$$

Via the (meromorphic) gauge transformation

$$\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+ \in H^0(\mathbb{P}^1, \text{SL}_2(\underline{E} \otimes \mathcal{O}_{\mathbb{P}^1}(\underline{W}), \underline{\mathbf{p}}))$$

it corresponds precisely to those logarithmic Higgs fields  $\underline{\Theta}$  in  $H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))$  that have *apparent singularities* in  $\underline{\mathbf{p}}$  over  $\underline{W}$ : the residual matrices are congruent to  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$  and  $\underline{\mathbf{p}}$  corresponds to their eigenvectors. We shall denote this set of apparent logarithmic Higgs fields on  $\underline{E}$  by

$$H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{\mathbf{p}}} \simeq H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}, \underline{\mathbf{p}}) \otimes \Omega_{\mathbb{P}^1}^1).$$

On the other hand, if we see  $\mathfrak{Bun}(X/\iota)$  as a space of bundles  $E$  over  $X$  with a lift  $h$  of the hyperelliptic involution, then the space of  $h$ -invariant Higgs fields on  $E$  also naturally identifies to the cotangent space  $T_{(E, h)}^*\mathfrak{Bun}(X/\iota)$ . Indeed, let  $(\underline{E}, \underline{\mathbf{p}})$  be a parabolic bundle in  $\mathfrak{Bun}(X/\iota)$  and consider the corresponding parabolic bundle  $(E, \mathbf{p}) = \text{elm}_W^+(\pi^*(\underline{E}, \underline{\mathbf{p}}))$  over  $X$  together with its unique isomorphism  $h : E \simeq \iota^*E$  such that  $\mathbf{p}$  corresponds to the +1-eigenspaces of  $h$ . Let  $\underline{\Theta}$  be a logarithmic Higgs field in

$$T_{(\underline{E}, \underline{\mathbf{p}})}^*\mathfrak{Bun}(X/\iota) \simeq H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{\mathbf{p}}}.$$

The corresponding Higgs bundle  $(E, \Theta) = \text{elm}_W^+(\pi^*(\underline{E}, \underline{\mathbf{p}}))$  then is  $h$ -invariant and holomorphic by construction.

Similarly to the case of connections, we obtain

$$\pi_*(E, \Theta) = \bigoplus_{i=1}^2 (E_i, \Theta_i),$$

where  $(\underline{E}_i, \Theta_i)$  are apparent logarithmic Higgs bundles on  $\mathbb{P}^1$  with  $D = \underline{W}$ . Note that if  $\nabla_1$  and  $\nabla_2$  are connections on the same vector bundle  $E \rightarrow X$ , then  $(E, \nabla_1 - \nabla_2)$  is a Higgs bundle. Hence the moduli space  $\mathbf{Con}(X)$  (resp.  $\mathbf{Con}(X/\iota)$ ) is an affine extension of  $\mathfrak{Higgs}(X)$  (resp.  $\mathfrak{Higgs}(X/\iota)$ ).

**6.1. A Poincaré family on the 2-fold cover  $\mathfrak{Bun}(X/\iota)$ .** Since we get a universal vector bundle on an open part of  $\mathfrak{Bun}(X/\iota)$  for our moduli problem (for instance over  $\mathbb{P}_B^3$ , see section 4.2), we can expect to find a universal family of Higgs bundles (resp. connections) there, which we will now construct over an open subset of the projective chart  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ , namely when  $(R, S, T) \in \mathbb{C}^3$  is finite.

For  $(i, z_i) = (r, R), (s, S), (t, T)$ , define the Higgs field  $\Theta_i$  given on a trivial chart  $(\mathbb{P}^1 \setminus \{\infty\}) \times \mathbb{C}^2$  of  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$  by

$$\Theta_i := \frac{dx}{x} \begin{pmatrix} 0 & 0 \\ 1 - z_i & 0 \end{pmatrix} + \frac{dx}{x-1} \begin{pmatrix} z_i & -z_i \\ z_i & -z_i \end{pmatrix} + \frac{dx}{x-i} \begin{pmatrix} -z_i & z_i^2 \\ -1 & z_i \end{pmatrix}$$

These parabolic Higgs fields are independant over  $\mathbb{C}$  (they do not share the same poles) and any other Higgs field  $\Theta$  on  $\underline{E}$  respecting the parabolic structure  $\underline{p}$  given by  $(R, S, T)$  is a linear combination of these  $\Theta_i$ :

$$\Theta = c_r \Theta_r + c_s \Theta_s + c_t \Theta_t \quad \text{for unique } c_r, c_s, c_t \in \mathbb{C}.$$

These generators are chosen such that the coefficient  $(2, 1)$  of  $\Theta_i$  vanishes at  $x = j$  and  $k$  where  $\{i, j, k\} = \{r, s, t\}$ . They are also very natural on our chart  $\mathbf{Bun}_\mu^{ss}(X/\iota) = \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  with  $\mu \in ]\frac{1}{6}, \frac{1}{4}[$ . Indeed, for our choice of chart and generators, we precisely get:

**Proposition 6.1.** *The differential 1-form  $dz_i$  on the affine chart  $(R, S, T) \in \mathbb{C}^3 \subset \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  identifies under Serre duality with the Higgs bundle  $\Theta_i \in H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}}$  for  $(i, z_i) = (r, R), (s, S), (t, T)$ .*

*Proof.* In an intrinsic way, the tangent space of the moduli space of parabolic bundles at a point  $(\underline{E}, \underline{p})$  is given by  $H^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}))$  where  $\mathfrak{sl}(\underline{E}, \underline{p})$  is the sheaf of trace-free endomorphisms of  $E$  over  $\mathbb{P}_x^1$  that preserve the parabolic structure. For instance the vector field  $\frac{\partial}{\partial R} \in T_{(R,S,T)} \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  can be represented by the two charts  $U_0 = \mathbb{P}_x^1 \setminus \{r\}$  and  $U_1$  an analytic disc surrounding  $x = r$  together with the cocycle

$$\phi_{0,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

on the punctured disc  $U_{0,1} = U_0 \cap U_1$ . Indeed, if we glue the restrictions  $(\underline{E}, \underline{p})|_{U_0}$  and  $(\underline{E}, \underline{p})|_{U_1}$  by the map

$$\exp(\zeta \phi_{0,1}) = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} : ((\underline{E}, \underline{p})|_{U_1})|_{U_{0,1}} \rightarrow (\underline{E}, \underline{p})|_{U_0},$$

we get the new parabolic bundle defined by  $\underline{p} = (0, 1, R + \zeta, S, T, \infty)$ , *i.e.* the point defined by the time- $\zeta$  map generated by the vector field  $\frac{\partial}{\partial R}$ . Let us now compute the perfect pairing

$$\langle \cdot, \cdot \rangle : H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}} \times H^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p})) \rightarrow H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \simeq \mathbb{C};$$

defining Serre duality in our coordinates. Given a Higgs field  $\Theta \in H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}}$ , the image in  $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1)$  is given by the cocycle

$$\langle \Theta, \phi_{0,1} \rangle = \text{trace}(\Theta \cdot \phi_{0,1})$$

on  $U_{0,1}$ , that is the  $(1,2)$ -coefficient of  $\Theta$  restricted to  $U_{0,1}$  (note that  $\Theta$  is holomorphic there). We fix an isomorphism  $H^1(\Omega_{\mathbb{P}^1}^1) \rightarrow \mathbb{C}$  as follows. Given a cocycle  $(U_{0,1}, \omega_{0,1}) \in H^1(\Omega_{\mathbb{P}^1}^1)$ , one can easily write  $\omega_{0,1} = \alpha_0 - \alpha_1$  for meromorphic 1-forms  $\alpha_i$  on  $U_i$ . Then  $\omega_{0,1}$  is trivial in  $H^1(\Omega_{\mathbb{P}^1}^1)$  if, and only if,  $\omega_{0,1} = \omega_0 - \omega_1$  for holomorphic 1-forms  $\omega_i$  on  $U_i$ , or, equivalently, if the principal part defined by  $(\alpha_i)_i$  is that of a global meromorphic 1-form  $(\alpha_i - \omega_i)_i$ . Since the obstruction is given precisely by the Residue Theorem, we are led to define

$$\text{Res} : H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \rightarrow \mathbb{C}$$

as the map which to a principal part  $(\alpha_i)_i$  representing the cocycle, associates the sum of residues. For instance,

$$\omega_{0,1} := \langle \Theta_r, \phi_{0,1} \rangle = (1-R) \frac{dx}{x} + R \frac{dx}{x-1} - \frac{dx}{x-r}$$

can be represented by the cocycle

$$\alpha_0 := 0 \quad \text{and} \quad \alpha_1 := -\omega_{0,1}$$

so that the principal part is just defined by  $\frac{dx}{x-r}$  at  $x=r$  and we get

$$\text{Res} \langle \Theta_r, \phi_{0,1} \rangle = 1$$

i.e.  $\langle \Theta_r, \frac{\partial}{\partial R} \rangle = 1$ . Similarly, we have

$$\left\langle \Theta_i, \frac{\partial}{\partial z_j} \right\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

□

**Corollary 6.2.** *The Liouville form on  $T^* \text{Bun}_{\mu}^{ss}(X/\iota)$  defines a holomorphic symplectic 2-form on the moduli space of Higgs bundles defined in the chart  $(R, S, T, c_r, c_s, c_t) \in \mathbb{C}^6$  by*

$$\omega = dR \wedge dc_r + dS \wedge dc_s + dT \wedge dc_t.$$

A connection on the parabolic bundle attached to a parameter  $(R, S, T)$  is given by

$$(12) \quad \begin{aligned} \nabla_0 := & d + \begin{pmatrix} 0 & 0 \\ 1 & \frac{1}{2} \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} -1 & \frac{3}{2} \\ -1 & \frac{3}{2} \end{pmatrix} \frac{dx}{x-1} \\ & + \frac{1}{2} \begin{pmatrix} 0 & R \\ 0 & 1 \end{pmatrix} \frac{dx}{x-r} + \frac{1}{2} \begin{pmatrix} 0 & S \\ 0 & 1 \end{pmatrix} \frac{dx}{x-s} + \frac{1}{2} \begin{pmatrix} 0 & T \\ 0 & 1 \end{pmatrix} \frac{dx}{x-t} \end{aligned}$$

and any other connection on this bundle writes uniquely as

$$\nabla = \nabla_0 + c_r \Theta_r + c_s \Theta_s + c_t \Theta_t.$$

This provides a universal family of parabolic connections on a large open subset of the moduli space.

**6.2. The apparent map on  $\mathfrak{Con}(X/\iota)$ .** Following [29], we will now recall the construction of the so-called *apparent map*, allowing us to prove Proposition 5.3. For a parabolic connection  $(\underline{E}, \underline{p}, \nabla)$  defined on the main vector bundle  $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ , we can associate a morphism

$$\nabla \mapsto \varphi_{\nabla} \in \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))$$

by composition of

$$\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E} \xrightarrow{\nabla} \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$$

where the last arrow is just the projection on the second factor.

**Remark 6.3.** *Geometrically, the zeroes of the apparent map (which is an element of  $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$ ) are the coordinates of the (three) tangencies between the destabilizing section  $\sigma_{-1}$  of  $\mathbb{P}(\underline{E})$  and the foliation on  $\mathbb{P}(\underline{E})$  defined by flat sections of  $\mathbb{P}(\nabla)$ . On the other hand, these are precisely the positions of the apparent singular points appearing when we derive the associate 2<sup>nd</sup> order fuchsian equation from the “cyclic vector”  $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow E$ .*

We can extend the definition of the apparent map to so-called  $\lambda$ -connections

$$\nabla = \lambda \cdot \nabla_0 + c_r \Theta_r + c_s \Theta_s + c_t \Theta_t, \quad (\lambda, c_r, c_s, c_t) \in \mathbb{C}^4,$$

including Higgs fields (for  $\lambda = 0$ ). There is a natural  $G_m$ -action by multiplication on the moduli space of  $\lambda$ -connections so that a generic element  $\nabla$ , with  $\lambda \neq 0$ , is equivalent to a unique connection (in the usual sense), namely  $\frac{1}{\lambda} \nabla$ . After projectivization, we thus obtain a natural compactification of the moduli space of connections on  $\underline{E}$  (an affine 3-space) by the moduli space of projective Higgs fields (i.e. up to  $G_m$ -action). In our coordinates, an element  $(\lambda : c_r : c_s : c_t) \in \mathbb{P}^3$  denotes either a connection (when  $\lambda \neq 0$ ) or a projective class of a Higgs field. It is proved in [29], Theorem 4.3, that the map  $\nabla \mapsto \mathbb{P}\varphi_{\nabla}$ , which is invariant under  $G_m$ -action, defines an isomorphism from the moduli space of  $\lambda$ -connections up to  $G_m$ -action onto  $\mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))$ . Moreover, we deduce a map

$$\text{Bun}_{\mu}^{ss}(X/\iota) \rightarrow \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\vee}$$

which to a parabolic bundle  $(\underline{E}, \underline{p})$  associates the image under  $\mathbb{P}\varphi$  of the hyperplane locus of Higgs bundles  $\lambda = 0$ . For  $\frac{1}{6} < \mu < \frac{1}{4}$ , this map is also an isomorphism.

On the other hand, looking at  $\text{Bun}_{\mu}^{ss}(X/\iota)$  as extensions (see section 5.3), we also get a natural isomorphism

$$\text{Bun}_{\mu}^{ss}(X/\iota) \xrightarrow{\sim} \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\vee}$$

It follows from [29], proof of Theorem 4.3, that these two maps coincide.

*Proof of Proposition 5.3.* For  $(R, S, T) \in \mathbb{C}^3$  finite, the corresponding parabolic bundle also belongs to  $\text{Bun}_{\mu}^{ss}(X/\iota)$  and we can use the apparent map to compute the corresponding point  $(b_0 : b_1 : b_2 : b_3) \in \mathbb{P}_b^3$ . Precisely, the apparent map  $\varphi_{\Theta_r}$  is given by the  $(2, 1)$ -coefficient of  $\Theta_r$

$$\mathbb{P}\varphi_{\Theta_r} = \frac{R-1}{R-r}(x-r)(x-s)(x-t) \in \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})) \simeq |\mathcal{O}_{\mathbb{P}^1}(3)|.$$

This provides a first equation

$$(R-r)b_0 - (\sigma_1 R - r(1+s+t))b_1 + (\sigma_2 R - r(s+t+st))b_2 - \sigma_3(R-1)b_3 = 0;$$

similar equations for  $\Theta_s$  and  $\Theta_t$  give the result.  $\square$

**6.3. A rational section of  $\mathfrak{C}\text{on}(X) \rightarrow \mathfrak{B}\text{un}(X)$ .** The rational section  $\nabla_0 : \mathfrak{B}\text{un}(X/\iota) \dashrightarrow \mathfrak{C}\text{on}(X/\iota)$  constructed in section 6.1 over the chart  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  is not invariant by the Galois involution of  $\Phi : \mathfrak{C}\text{on}(X/\iota) \xrightarrow{2:1} \mathfrak{C}\text{on}(X)$ , i.e. it defines a 2-section, but not a rational section  $\mathfrak{B}\text{un}(X/\iota) \dashrightarrow \mathfrak{C}\text{on}(X)$ . One can easily deduce a rational section by taking the barycenter (recall that  $\mathfrak{C}\text{on}(X) \rightarrow \mathfrak{B}\text{un}(X)$  is an affine bundle) but it is not the simplest one. Here, we start back from the Tyurin parametrization of bundles to construct such an explicit section.

Like in section 4, consider a generic data  $(\underline{P}_1, \underline{P}_2, \lambda) \in X \times X \times \mathbb{P}^1$  and associate the parabolic structure  $\tilde{\mathbf{p}}$  on  $\tilde{E} := \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$  defined over

$$D := [\underline{P}_1] + [\iota(\underline{P}_1)] + [\underline{P}_2] + [\iota(\underline{P}_2)] \in |2K_X|,$$

by

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) := \left( \lambda, -\lambda, \frac{1}{\lambda}, -\frac{1}{\lambda} \right)$$

(where  $\lambda_Q$  means the direction generated by  $\lambda_Q e_1 + e_2$  over  $Q$ , for fixed independant sections  $e_1, e_2$  over  $X \setminus \{\infty\}$ ). After 4 elementary transformations, we get a bundle  $E$  with trivial determinant. A holomorphic connection  $\nabla$  on  $E := \text{elm}_D^+(\tilde{E}, \tilde{\mathbf{p}})$  can be pulled-back to  $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$  and we get a parabolic logarithmic connection  $\tilde{\nabla}$  on this bundle with (apparent) singular points over  $D$ . In the basis  $\langle e_1, e_2 \rangle$ , we can write

$$\tilde{\nabla} : d + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where the trace is given by

$$\alpha + \delta = \frac{dx}{x-x_1} + \frac{dx}{x-x_2}$$

and the projective part takes the form (here  $z$  is the projective variable defined by  $ze_1 + e_2$ )

$$\mathbb{P}\tilde{\nabla} : dz - \gamma z^2 + (\alpha - \delta)z + \beta \quad \text{with} \quad \begin{cases} -\gamma & = \frac{A(x)}{(x-x_1)(x-x_2)} \frac{dx}{y} \\ \alpha - \delta & = \frac{by}{(x-x_1)(x-x_2)} \frac{dx}{y} \\ \beta & = \frac{C(x)}{(x-x_1)(x-x_2)} \frac{dx}{y} \end{cases}$$

where  $A, C$  are degree 3 polynomials in  $x$  and  $b \in \mathbb{C}$ . This is due to the fact that the connection has only simple poles over  $D$  and that it is invariant under the (normalized) lift of the hyperelliptic involution  $h : (x, y, z) \mapsto (x, -y, -z)$ . We note that  $e_1$  and  $e_2$  generate the two  $\iota$ -invariant Tyurin subbundles. Moreover, these coefficients  $\{(A, b, C)\}$  have to satisfy several additional conditions, namely the compatibility with the parabolic data, that eigenvalues are 0 and 1 (parabolic directed by 1) and the singularity is apparent, in the sense that it disappears after an elementary transformation in the parabolic. This gives 6 affine equations in the 9-dimensional space of coefficients  $\{(A, b, C)\}$ :

$$\begin{aligned} \text{parabolic data:} & \begin{cases} \lambda A(x_1) + by_1 + \frac{1}{\lambda} C(x_1) & = 0 \\ \frac{1}{\lambda} A(x_2) + by_2 + \lambda C(x_2) & = 0 \end{cases} \\ \text{eigenvalues:} & \begin{cases} 2\lambda A(x_1) + by_1 & = y_1(x_2 - x_1) \\ \frac{2}{\lambda} A(x_2) + by_2 & = y_2(x_1 - x_2) \end{cases} \\ \text{apparent:} & \begin{cases} 2y_1(\lambda A'(x_1) + \frac{1}{\lambda} C'(x_1)) + bF'(x_1) & = 0 \\ 2y_2(\frac{1}{\lambda} A'(x_2) + \lambda C'(x_2)) + bF'(x_2) & = 0 \end{cases} \end{aligned}$$

where  $F(x) = x(x-1)(x-r)(x-s)(x-t)$ . Viewing a Higgs field  $\tilde{\Theta} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$  as the difference of two connections, we get  $\alpha + \delta = 0$  and, for the projective part, the corresponding linearized equations (with 0 right-hand-side). Starting with a connection

$\tilde{\nabla}$  on  $\tilde{E} = \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$  as above, *via* 4 elementary transformations, we get a holomorphic  $\mathfrak{sl}_2$ -connection

$$(E, \nabla) = \text{elm}_D^+(\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}})$$

on  $X$  whose parabolic data  $\mathbf{p}$  is supported by the strict transform of the line bundle  $L_1 := \mathbb{C}\langle e_1 \rangle$  (we suppose  $\lambda \notin \{0, \infty\}$  by genericity). Pushing it down, we get a parabolic connection on  $X/\iota = \mathbb{P}_x^1$  for which  $L_1$  becomes the destabilizing subbundle  $\mathcal{O}_{\mathbb{P}^1}(-1)$ .

Selecting the  $\iota$ -invariant Tyurin subbundle  $L_1$ , we have a natural generically finite map

$$X \times X \times \mathbb{P}_\lambda^1 \xrightarrow{16:1} \mathbb{P}_B^3$$

with Galois group generated by  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle$  (see section 4). Then, the Galois involution of  $\mathbb{P}_B^3 \xrightarrow{2:1} \mathbb{P}_{NR}^3$  is induced by  $\sigma_{1/z}$  which is permuting  $e_1$  and  $e_2$ . We can thus compute the apparent map of a connection  $\tilde{\nabla}$  (or a Higgs field  $\tilde{\Theta}$ ) with respect to  $e_1$  and get that  $\varphi_{\tilde{\nabla}} = A(x)$ .

**Remark 6.4.** *The three zeroes of  $A(x)$  define six points on  $X$ , which are the coordinates of the tangencies between  $e_1$  and the foliation  $\mathbb{P}(\tilde{\nabla})$  on  $\mathbb{P}(\tilde{E})$ .*

Like in the proof of Proposition 5.3 (see section 6.2) we can use the apparent map for Higgs fields to compute the corresponding Bertram coordinates of  $\mathbb{P}_B^3$ . A straightforward computation yields:

**Proposition 6.5.** *The natural map  $X \times X \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_B^3$  is given by*

$$\begin{cases} b_0 &= \lambda y_2 - \frac{1}{\lambda} y_1 \\ b_1 &= \lambda x_1 y_2 - \frac{1}{\lambda} x_2 y_1 \\ b_2 &= \lambda x_1^2 y_2 - \frac{1}{\lambda} x_2^2 y_1 \\ b_3 &= \lambda x_1^3 y_2 - \frac{1}{\lambda} x_2^3 y_1 \end{cases}$$

It follows from [29] that a connection on a parabolic bundle belonging to the chart  $\mathbb{P}_B^3$  is determined by its apparent map. It is particularly easy to see this fact in above equations: after prescribing  $\varphi_{\tilde{\nabla}} = A(x) \in \mathbb{P}_A^3$  (up to homothecy), *i.e.* after prescribing the roots of  $A(x)$ , we get a unique solution  $(A, b, C)$  except when  $A(x)$  lies in the hyperplane of Higgs bundles defined by Proposition 6.5 above. In the latter case, there is a solution  $(A, b, C)$  as a Higgs field which is unique up to an homothecy. Note that the group  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle$  acts on connections (and Higgs fields) and the induced action on the coefficient  $A(x)$  is by homothecy. It follows that the corresponding point  $A(x) \in \mathbb{P}_A^3$  is invariant. The fourth involution  $\sigma_{1/z}$  however permutes  $A(x)$  and  $C(x)$  (and changes the sign). In order to construct a rational section  $\nabla_0 : \mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$ , we can consider connections for which  $A(x)$  and  $C(x)$  are homothetic to each other, *i.e.* define the same point in  $\mathbb{P}_A^3$ . A straightforward computation shows that there are exactly two possibilities:

$$\begin{aligned} \nabla^+ : b &= \frac{\lambda^2 + 1}{\lambda^2 - 1} (x_1 - x_2) \quad \text{and} \quad A(x) = C(x) = \\ & \frac{1}{2(x_1 - x_2)^2} \frac{\lambda}{\lambda^2 - 1} \left( (y_1 - y_2)(4x^3 - 6(x_1 + x_2)x^2 + 12x_1x_2x) - 6x_1x_2(x_2y_1 - x_1y_2) \right. \\ & \left. + 2(x_2^3y_1 - x_1^3y_2) - (x_1 - x_2)(x - x_1)(x - x_2) \left( y_1 \frac{F'(x_1)}{F(x_1)}(x - x_2) + y_2 \frac{F'(x_2)}{F(x_2)}(x - x_1) \right) \right) \end{aligned}$$

and

$$\begin{aligned} \nabla^- : b &= \frac{\lambda^2 - 1}{\lambda^2 + 1}(x_1 - x_2) \quad \text{and} \quad A(x) = -C(x) = \\ & \frac{1}{2(x_1 - x_2)^2} \frac{\lambda}{\lambda^2 + 1} \left( (y_1 + y_2)(4x^3 - 6(x_1 + x_2)x^2 + 12x_1x_2x) - 6x_1x_2(x_2y_1 + x_1y_2) \right. \\ & \left. + 2(x_2^3y_1 + x_1^3y_2) - (x_1 - x_2)(x - x_1)(x - x_2) \left( y_1 \frac{F'(x_1)}{F(x_1)}(x - x_2) - y_2 \frac{F'(x_2)}{F(x_2)}(x - x_1) \right) \right) \end{aligned}$$

This provides two ‘‘universal connections’’ over the parameter space  $X \times X \times \mathbb{P}_\lambda^1$  which are each invariant under  $\sigma_{1/z}$  and  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}^2 \rangle$ , but permuted by  $\sigma_{iz}$ . Taking the barycenter of these two connections for each parameter  $(\underline{P}_1, \underline{P}_2, \lambda)$  yields a fully invariant section

$$\nabla_0 := \frac{\nabla^+ + \nabla^-}{2}$$

whose coefficients are given by

$$\begin{aligned} \frac{b_0 y}{(x-x_1)(x-x_2)} &:= \frac{\lambda^4 + 1}{\lambda^4 - 1} \left( \frac{1}{x-x_1} - \frac{1}{x-x_2} \right) \\ \frac{A_0(x)}{(x-x_1)(x-x_2)} &:= \frac{\lambda}{\lambda^4 - 1} \left\{ \left( \frac{y_2}{x-x_2} - \frac{\lambda^2 y_1}{x-x_1} \right) + \frac{(\lambda^2 y_1 - y_2)(2x - x_1 - x_2)}{(x_1 - x_2)^2} \right. \\ & \quad \left. - \frac{\lambda^2 y_1 \frac{F'(x_1)}{F(x_1)}(x-x_2) + y_2 \frac{F'(x_2)}{F(x_2)}(x-x_1)}{2(x_1 - x_2)} \right\} \\ \frac{C_0(x)}{(x-x_1)(x-x_2)} &:= \frac{\lambda}{\lambda^4 - 1} \left\{ \left( \frac{\lambda^2 y_2}{x-x_2} - \frac{y_1}{x-x_1} \right) + \frac{(y_1 - \lambda^2 y_2)(2x - x_1 - x_2)}{(x_1 - x_2)^2} \right. \\ & \quad \left. - \frac{y_1 \frac{F'(x_1)}{F(x_1)}(x-x_2) + \lambda^2 y_2 \frac{F'(x_2)}{F(x_2)}(x-x_1)}{2(x_1 - x_2)} \right\}. \end{aligned}$$

**Proposition 6.6.** *The induced rational section*

$$\nabla_0 : \mathfrak{Bun}(X/\iota) \dashrightarrow \mathfrak{Con}(X)$$

*is Lagrangian, and moreover regular over the open set of stable bundles.*

*Proof.* This connection is well-defined provided that  $\lambda^4 \neq 1$  and  $x_2 \neq x_1$ . We get a universal connection for all stable bundles. Indeed, we first check that all stable bundles off odd Gunning planes are covered by the open subset where the connection  $\nabla_0$  is well-defined:

$$X \times X \times \mathbb{P}_\lambda^1 \setminus (\{\lambda^4 = 1\} \cup \{x_1 = x_2\}) \dashrightarrow \mathbb{P}_{NR}^3 \setminus (\text{Kum}(X) \cup \Pi_{w_0} \cup \dots \cup \Pi_{w_\infty}).$$

We thus get a rational section  $\nabla_0 : \mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$  which is holomorphic over stable bundles, off odd Gunning planes. We can check that it actually extends holomorphically along odd Gunning planes. It is sufficient to extend it outside intersections of odd Gunning planes since those form a codimension 2 subset. The Gunning plane  $\Pi_{w_0}$  comes from the indeterminacy locus  $\{w_0\} \times X \times \{0\}$  of the map  $X \times X \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_{NR}^3$ . Precisely, a generic element of  $\Pi_{w_0}$  is obtained as follows. We first renormalize  $z = w/\lambda$  so that parabolic directions become

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) = (\lambda^2, -\lambda^2, 1, -1)$$

and then make the first two parabolic tending to 0 while  $y_1 \rightarrow 0$  with some fixed slope  $\frac{\lambda^2}{y_1} = c$ . The limiting connection has now a double pole at  $w_0$ , which disappears after two elementary transformations.

Finally, that this section is Lagrangian directly follows from straightforward verification.  $\square$

**Remark 6.7.** *There are precisely two Higgs fields invariant under  $\sigma_{1/z}$ :*

$$(x - x_1)(\lambda^2 z^2 - 1) \frac{dx}{y} \quad \text{and} \quad (x - x_2)(z^2 - \lambda^2) \frac{dx}{y}.$$

*They are also permuted by  $\sigma_{iz}$  and invariant under  $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}^2 \rangle$ . We obtain a basis of the space of Higgs bundles by adding for example  $\nabla^+ - \nabla^-$ .*

**6.4. The Hitchin fibration on  $\mathfrak{Higgs}(X/\iota)$  and  $\mathfrak{Higgs}(X)$ .** On the moduli space of Higgs bundles on  $X$ , the Hitchin fibration is defined by the map

$$\text{Hitch} : \mathfrak{Higgs}(X) \rightarrow H^0(X, 2K_X) ; (E, \Theta) \mapsto \det(\Theta).$$

Viewing  $\mathfrak{Higgs}(X)$  as the total space of the cotangent bundle  $T^*\mathfrak{Bun}(X)$  (over the open set of stable bundles), the Liouville form defines a symplectic structure on  $\mathfrak{Higgs}(X)$ . The above map defines a completely integrable system on this space: writing a quadratic differential as  $(h_2 x^2 + h_1 x + h_0) \left(\frac{dx}{y}\right)^{\otimes 2}$ , the 3 components of Hitch

$$h_0, h_1, h_2 : \mathfrak{Higgs}(X) \rightarrow \mathbb{C}$$

are holomorphic functions commuting to each other for the Poisson structure. Moreover, fibers of the map Hitch are (open sets of) 3-dimensional abelian varieties. One can also associate to  $(E, \Theta)$  the spectral curve  $\text{spec}(\Theta)$  which is the double-section of the projectivized bundle  $\mathbb{P}E \rightarrow X$  defined by the eigendirections of  $\Theta$ . This curve  $\text{spec}(\Theta)$  is thus a two-fold ramified cover of  $X$ , ramifying at zeroes of the quadratic form  $\text{Hitch}(E, \Theta)$ ; the spectral curve is thus constant along Hitchin fibers and its Jacobian is the compactification of the fiber.

Viewing a Higgs field as the difference of two connections, we have seen that Higgs bundles are invariant under involution and descend, likely as connections, as parabolic Higgs fields on  $\mathbb{P}_x^1 = X/\iota$ . The induced map

$$\mathfrak{Higgs}(X/\iota) \xrightarrow{2:1} \mathfrak{Higgs}(X)$$

allows us to compute the Hitchin fibration easily. Note that, applying an elementary transformation to some Higgs bundle  $(E, \Theta)$  does not modify  $\det(\Theta)$  since an elementary transformation is just a birational bundle transformation, acting by conjugacy on  $\Theta$ . Therefore, to get Hitchin Hamiltonians on the chart  $(R, S, T, c_r, c_s, c_t)$ , we just have to compute

$$\det(c_r \Theta_r + c_s \Theta_s + c_t \Theta_t) = (h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)}.$$

A straightforward computation yields

$$h_0 = (c_r(R-1) + c_s(S-1) + c_t(T-1))(c_rst(R-r)R + c_srt(S-s)S + c_trs(T-t)T)$$

$$\begin{aligned} h_1 = & +c_r (c_r(s+t)(r+1) + c_s s(t+1) + c_t t(s+1)) R^2 - c_r^2 (t+s) R^3 \\ & +c_s (c_s(r+t)(s+1) + c_r r(t+1) + c_t t(r+1)) S^2 - c_s^2 (t+r) S^3 \\ & +c_t (c_t(r+s)(t+1) + c_r r(s+1) + c_s s(r+1)) T^2 - c_t^2 (r+s) T^3 \\ & -c_r c_s (t(R-1+S-1) + r(S-s) + s(R-r)) RS \\ & -c_r c_t (s(R-1+T-1) + r(T-t) + t(R-r)) RT \\ & -c_s c_t (r(S-1+T-1) + s(T-t) + t(S-s)) ST \\ & - (c_t t(r+s) + c_r r(s+t) + c_s s(r+t)) (c_r R + c_s S + c_t T) \end{aligned}$$

$$h_2 = (c_r(R-1)R + c_s(S-1)S + c_t(T-1)T) (c_r(R-r) + c_s(S-s) + c_t(T-t))$$

It is easy to check that these functions indeed Poisson-commute: for any  $f, g \in \{h_0, h_1, h_2\}$ , we have

$$\sum_{i=r,s,t} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i}$$

in Darboux notation  $(p_r, p_s, p_t, q_r, q_s, q_t) := (R, S, T, c_r, c_s, c_t)$ .

In proposition 5.3, we specified the birational map  $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_B^3$ , allowing us to express the Bertram coordinates  $(b_0 : b_1 : b_2 : b_3)$  as functions of  $(R, S, T)$ . Setting

$$c_r dR + c_s dS + c_t dT = \lambda_1 d \frac{b_1}{b_0} + \lambda_2 d \frac{b_2}{b_0} + \lambda_3 d \frac{b_3}{b_0}$$

allows us to express the coefficients  $c_r, c_s, c_t$  as functions of the Bertram coordinates as well. The Hitchin map in Bertram coordinates then writes

$$(h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)}$$

with

$$h_0 = \frac{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}{b_0^4} \cdot \begin{cases} -b_0 \sigma_3 \cdot [\lambda_1 b_0 b_{10} + \lambda_2 (b_0 b_{21} + b_1 b_{10}) + \lambda_3 (b_0 b_{32} + b_1 b_{21} + b_2 b_{10})] \\ +b_0 \sigma_2 \cdot [\lambda_1 b_1 b_{10} + \lambda_2 (b_2 b_{10} + b_1 b_{21}) + \lambda_3 (b_1 b_{32} + b_2 b_{21} + b_3 b_{10})] \\ -\sigma_1 \cdot [\lambda_1 b_1^2 b_{10} + \lambda_2 b_2 (b_1 b_{10} + b_0 b_{21}) \\ + \lambda_3 (b_0 b_2 b_{32} + b_0 b_3 b_{21} + b_1 b_3 b_{10})] \\ +1 \cdot [\lambda_1 (b_1^2 b_{21} + b_2 (b_1^2 - b_0 b_2)) + \lambda_2 b_2 (b_1 b_{21} + b_2 b_{10}) \\ + \lambda_3 b_3 (b_0 b_{32} + b_1 b_{21} + b_2 b_{10})] \end{cases}$$

$$\begin{aligned}
h_1 &= \frac{1}{b_0^4} \cdot \left\{ \begin{aligned}
& b_0 \sigma_3 \cdot [\lambda_2^2 b_0 (b_{21}^2 - b_2 b_{20}) + \lambda_3^2 (-b_0 b_2 (2b_3 - b_2) - b_1 b_3 (b_1 - 2b_0)) \\
& \quad - \lambda_1 \lambda_2 b_0 b_1 b_{10} - \lambda_1 \lambda_3 b_1^2 b_{10} \\
& \quad + \lambda_2 \lambda_3 (2b_0 b_1 b_2 + b_2 b_0 (b_0 - 2b_2) - b_0 b_3 (2b_1 - b_0) - b_2 b_{10}^2)] \\
& + b_0 \sigma_2 \cdot [\lambda_2^2 b_2 (b_{10}^2 + b_0 b_{20}) + \lambda_3^2 b_3 (b_0 (b_3 - 2b_2) + b_1 (2b_2 - b_1)) + \lambda_1 \lambda_2 b_1^2 b_{10} \\
& \quad + \lambda_1 \lambda_3 (b_1^2 (2b_2 - b_1) - b_0 b_2^2) + \lambda_2 \lambda_3 (b_1^2 b_{32} + 2b_2^2 b_{10} + 2b_0 b_3 b_{21})] \\
& + b_0 \sigma_1 \cdot [\lambda_2^2 b_2 (b_{21}^2 - b_2 b_{20}) + \lambda_3^2 b_3 (-b_2 (b_2 - 2b_1) - b_3 (2b_1 - b_0)) \\
& \quad + \lambda_1 \lambda_2 (b_0 b_2^2 - b_1^2 (2b_2 - b_1)) \\
& \quad + \lambda_1 \lambda_3 (b_0 b_2 (2b_3 - b_2) - b_1 (b_{21}^2 + b_1 (2b_3 - b_1))) \\
& \quad + \lambda_2 \lambda_3 (b_3 (b_1^2 - 2b_2 (2b_1 - b_0)) - b_2^2 (b_2 - 2b_1))] \\
& + 1 \cdot [\lambda_1^2 (b_1^2 (b_1^2 - 2b_0 b_2) + b_0^2 b_2^2) + \lambda_2^2 b_2^2 (b_{10}^2 + b_0 b_{20}) \\
& \quad + \lambda_3^2 b_3 (-b_0 (b_2^2 + 2b_1 b_3) + b_3 (b_1^2 + 2b_0 b_2)) \\
& \quad + \lambda_1 \lambda_2 b_2 (2b_1^2 - b_0 b_2) b_{10} + \lambda_1 \lambda_3 (-b_0 b_1 b_2^2 + 2b_1^2 b_3 b_{10} - b_0^2 b_3 (b_3 - 2b_2)) \\
& \quad + \lambda_2 \lambda_3 (b_0 b_2^2 (3b_3 - b_2) + 2b_1 b_2 b_3 (b_1 - 2b_0))]
\end{aligned} \right. \\
h_2 &= \frac{\lambda_3}{b_0^3} \cdot \left\{ \begin{aligned}
& \sigma_3 \cdot [-\lambda_1 b_0 b_1 b_{10} - b_0 \lambda_2 (b_1 b_{21} + b_2 b_{10}) - \lambda_3 b_0 (b_1 b_{32} + b_2 b_{21} + b_3 b_{10})] \\
& + \sigma_2 \cdot [\lambda_1 b_1^2 b_{10} + \lambda_2 b_2 (b_1 b_{10} + b_0 b_{21}) + \lambda_3 (b_0 b_2 b_{32} + b_0 b_3 b_{21} + b_1 b_3 b_{10})] \\
& - \sigma_1 \cdot [\lambda_1 (b_1^2 b_{21} + b_2 (b_1^2 - b_0 b_2)) + \lambda_2 b_2 (b_2 b_{10} + b_1 b_{21}) \\
& \quad + \lambda_3 b_3 (b_0 b_{32} + b_1 b_{21} + b_2 b_{10})] \\
& + 1 \cdot [\lambda_1 ((b_1^2 - b_0 b_2) (2b_3 - b_2) + b_1 b_2 b_{21}) + \lambda_2 (b_2^2 (b_2 - 2b_1) + b_3 (2b_1 b_2 - b_0 b_3)) \\
& \quad + \lambda_3 (b_3^2 (2b_1 - b_0) + b_2 b_3 (b_2 - 2b_1))]
\end{aligned} \right.
\end{aligned}$$

Here  $b_{ij}$  denotes  $b_i - b_j$ .

We can now push-down formulae onto  $X$  to give the explicit Hitchin Hamiltonians on  $\mathfrak{Higgs}(X) \simeq T^* \mathfrak{Bun}(X)$ . In order to do this, we consider the natural rational map  $\phi^* : T^* \mathbb{P}_{NR}^3 \dashrightarrow T^* \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$  induced by the explicit map  $\phi : \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{NR}^3$  of Proposition 5.2. Then, for a general section  $\mu_0 d(\frac{v_0}{v_3}) + \mu_1 d(\frac{v_1}{v_3}) + \mu_2 d(\frac{v_2}{v_3})$ , the Hitchin Hamiltonians are given, after straightforward computation, by

$$\begin{aligned}
h_0 &= \frac{1}{v_3^3} \cdot \left\{ \begin{aligned}
& \mu_0^2 \cdot [v_0^3 - (2\sigma_{23} v_0 + \sigma_3 v_1 - (\sigma_{12} \sigma_3 + \sigma_{23}^2) v_3) v_0 v_3 \\
& \quad + \sigma_3 (\sigma_{23} v_1 + \sigma_3 v_2 + v_3) v_3^2 - \sigma_{123} \sigma_2 v_3^3] \\
& + v_1 \mu_1^2 \cdot [v_0 v_1 + \sigma_3 v_2 v_3] \\
& + v_1 \mu_2^2 \cdot [v_0 v_3 + v_1 v_2 + (1 + \sigma_1) v_1 v_3 - \sigma_{23} v_3^2] \\
& + \mu_0 \mu_1 \cdot [2(v_0 - \sigma_{23} v_3) v_0 v_1 + (v_0 v_2 - (v_1 - \sigma_{12} v_3) v_1 - (\sigma_{23} v_2 + \sigma_3 v_3) v_3) \sigma_3 v_3] \\
& + \mu_0 \mu_2 \cdot [v_0^2 v_2 + v_0 v_1^2 - (\sigma_{12} v_1 + 2\sigma_{23} v_2 + \sigma_3 v_3) v_0 v_3 \\
& \quad - (\sigma_{23} v_1 + 3\sigma_3 v_2 + (\sigma_1 \sigma_3 - \sigma_{123} \sigma_2 + 2\sigma_3) v_3) v_1 v_3 + \sigma_{23} (\sigma_{23} v_2 + \sigma_3 v_3) v_3^2] \\
& + v_1 \mu_1 \mu_2 \cdot [v_0 v_2 + v_1^2 - (\sigma_{12} v_1 + \sigma_{23} v_2 - \sigma_3 v_3) v_3]
\end{aligned} \right. \\
h_1 &= \frac{1}{v_3^3} \cdot \left\{ \begin{aligned}
& \mu_0^2 \cdot [2v_0^2 v_1 - 2\sigma_{23} v_0 v_1 v_3 + \sigma_3 v_0 v_2 v_3 + \sigma_{12} \sigma_3 v_1 v_3^2 - \sigma_2 \sigma_{23} v_2 v_3^2 - \sigma_3^2 v_3^3] \\
& + v_1 \mu_1^2 \cdot [v_0 v_2 + v_1^2 - \sigma_{12} v_1 v_3 - \sigma_{23} v_2 v_3 + \sigma_3 v_3^2] \\
& + \mu_2^2 \cdot [v_0 v_2 v_3 - v_1^2 v_3 + 2v_1 v_2^2 + 2(1 + \sigma_1) v_1 v_2 v_3 + \sigma_{12} v_1 v_3^2 - \sigma_{23} v_2 v_3^2 - \sigma_3 v_3^3] \\
& + \mu_0 \mu_1 \cdot [v_0^2 v_2 - (\sigma_{12} v_1 + \sigma_{23} v_2 + \sigma_3 v_3) v_0 v_3 - \sigma_{23} v_1^2 v_3 + (\sigma_{123} \sigma_2 - (\sigma_1 + 2)\sigma_3) v_1 v_3^2 \\
& \quad + \sigma_{23} (\sigma_{23} v_2 + \sigma_3 v_3) v_3^2] \\
& + \mu_0 \mu_2 \cdot [v_0 (2v_0 v_3 + 4v_1 v_2 + 4(1 + \sigma_3) v_1 v_3 + \sigma_{12} v_2 v_3 - 2\sigma_{23} v_3^2) \\
& \quad + \sigma_{12} (v_1 - \sigma_{12} v_3) v_1 v_3 + 2\sigma_3 v_2^2 v_3 - (\sigma_{123} \sigma_2 - (\sigma_1 + 2)\sigma_3) v_2 v_3^2 + \sigma_{12} \sigma_3 v_3^3] \\
& + \mu_1 \mu_2 \cdot [v_0 v_2^2 + 3v_1^2 v_2 + 2((1 + \sigma_1) v_1 - \sigma_{23} v_3) v_1 v_3 - (\sigma_{12} v_1 + \sigma_{23} v_2 + \sigma_3 v_3) v_2 v_3]
\end{aligned} \right.
\end{aligned}$$

$$h_2 = \frac{1}{v_3^3} \cdot \begin{cases} v_1\mu_0^2 \cdot [v_0v_1 + \sigma_3v_2v_3] \\ +v_1\mu_1^2 \cdot [v_0v_3 + v_1v_2 + (1 + \sigma_1)v_1v_3 - \sigma_{23}v_3^2] \\ +v_1\mu_0\mu_1 \cdot [v_0v_2 + v_1^2 - \sigma_{12}v_1v_3 - \sigma_{23}v_2v_3 + \sigma_3v_3^2] \\ +\mu_0\mu_2 \cdot [-2v_0v_1v_3 + v_0v_2^2 + v_1^2v_2 - \sigma_{12}v_1v_2v_3 - \sigma_{23}^2v_3 - \sigma_3v_2v_3^2] \\ +\mu_1\mu_2 \cdot [v_0v_2v_3 - v_1^2v_3 + 2v_1v_2^2 + 2(1 + \sigma_1)v_1v_2v_3 + \sigma_{12}v_1v_3^2 - \sigma_{23}v_2v_3^2 - \sigma_3v_3^3] \end{cases}$$

where  $\sigma_{ij} = \sigma_i + \sigma_j$ ,  $ij = 12, 13, 23$ , and  $\sigma_{123} = \sigma_1 + \sigma_2 + \sigma_3$ .

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IRMA, 7 RUE RENÉ-DESCARTES, 67084 STRASBOURG CEDEX, FRANCE  
*E-mail address:* `heu@math.unistra.fr`

IRMAR, CAMPUS DE BEAULIEU, 35042 RENNES CEDEX, FRANCE  
*E-mail address:* `frank.loray@univ-rennes1.fr`