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Adding modular predicates^{*}

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Abstract. This paper is a contribution to the study of regular languages defined by fragments of first order or even monadic second order logic. More specifically, we consider the operation of enriching a given fragment by adding modular predicates. Our first result gives a simple algebraic counterpart to this operation in terms of semidirect products of varieties together with a combinatorial description based on elementary operations on languages. Now, a difficult question is to know whether the decidability of a given fragment is preserved under this enrichment. We first prove that this is always the case for so-called local varieties. The problem remains open in the nonlocal case but our main results also give several sufficient conditions to preserve decidability. We use these latter results to establish the decidability of three fragments of the first order logic with two variables.

1 Introduction

The *decision problem* for a given class of regular languages consists in deciding, given a regular language, whether or not it belongs to this class. Solving the decision problem for various fragments of monadic second order is a well-studied problem on regular languages [1–13]. Fragments of logic are usually defined in terms of their quantifier complexity (Σ_n -classes) or number of variables allowed in the formulae. Another possible parameter is to impose restrictions on the numerical predicates in the signature. A complete classification of the numerical predicates defining only regular languages was given by Péladeau [14] and Straubing [10]. There are essentially three basic groups of such predicates: the linear order, the local predicates LOC and the modular predicates MOD. Given a fragment $\mathcal{F}[\sigma]$ on the signature σ , the enrichment $\mathcal{F}[\sigma] \rightarrow \mathcal{F}[\sigma, \text{LOC}]$ has been widely studied [15, 9, 16]. For instance, Straubing [9] gave a nice algebraic interpretation of the enrichment $\mathcal{F}[<] \rightarrow \mathcal{F}[<, \text{LOC}]$ when \mathcal{F} is the fragment $\mathcal{B}\Sigma_n$ of Boolean combinations of Σ_n -formulae. The natural framework to state this kind of result is Eilenberg’s theory of varieties and can be roughly summarized as follows:

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1. In good cases (but not always) the enrichment by LOC corresponds to the operation $\mathbf{V} \rightarrow \mathbf{V} * \mathbf{LI}$ (the semidirect product by the variety \mathbf{LI} of locally trivial semigroups) on varieties.
2. If \mathbf{V} is a local variety, then \mathbf{V} is decidable if and only if $\mathbf{V} * \mathbf{LI}$ is decidable.
3. The nonlocal case requires advanced algebraic tools (notably derived categories) and is still the topic of intense research. Several important cases have been solved positively, but Ainger [17] exhibited an example of a decidable variety \mathbf{V} such that $\mathbf{V} * \mathbf{LI}$ is undecidable.

The aim of this paper is to establish similar results for the enrichment $\mathcal{F}[\sigma] \rightarrow \mathcal{F}[\sigma, \text{MOD}]$. Our first result (Theorem 2) states that the algebraic counterpart to this enrichment is another semidirect product, the operation $\mathbf{V} \rightarrow \mathbf{V} * \mathbf{MOD}$, where \mathbf{MOD} is the variety of cyclic stamps. Our second result (Theorem 5) shows that when \mathbf{V} is local, then \mathbf{V} is decidable if and only if $\mathbf{V} * \mathbf{MOD}$ is decidable. Finally, our main result (Theorem 6) deals with the nonlocal case. Surprisingly, its proof (Section 6) reduces to an instance of the *separation problem*. Figure 1, which can be found at the end of section 5, summarizes the consequences of our results for deciding various fragments of first-order logic.

2 Preliminaries

2.1 Words and logic

Let A be a finite alphabet and σ a relational signature. Given a word $u = a_0 \cdots a_{n-1}$ of length n , we associate to u the *relational structure* $M_u = \{[0, n-1], (\mathbf{a})_{a \in A}, (P^u)_{P \in \sigma}\}$, where P^u is the interpretation of the symbol P over the interval $[0, n-1]$ and $(\mathbf{a})_{a \in A}$ are disjoint monadic predicates given by the positions of the letters over the structure. For instance, if $u = aabbab$, then $\mathbf{a} = \{0, 1, 4\}$ and $\mathbf{b} = \{2, 3, 5\}$. Basic examples of predicates include the binary predicate $<$, interpreted as the usual order on integers. For each $k \geq 0$, we define the LOC_k predicates to be the unary predicates $x = \min + k$, which is true at the position k , the dual predicate $x = \max - k$ and the binary predicate $x = y + k$. The class LOC of local predicates is the union of all LOC_k . We also consider the modular predicate MOD_i^d , which holds at all positions equal to i modulo d , and the 0-ary predicate D_i^d which is true if the word length is equal to i modulo d . For $u = aabbab$, we have $\text{MOD}_0^2 = \{0, 2, 4\}$, and D_1^3 is *false* whereas D_0^3 is *true*. We denote by MOD_d the set of *modular* predicates modulo d . We define MOD as the union of all MOD_d .

Formulae are interpreted on words in the usual way (see [10]). For instance the formula $\exists x \exists y \exists z \mathbf{ax} \wedge \mathbf{by} \wedge \mathbf{az} \wedge (x < y) \wedge (y < z)$ defines the language $A^* a A^* b A^* a A^*$. Since a sentence defines a language, one can naturally associate a class of languages to a class of sentences.

In [4], Kufleitner and Lauser defined *fragments of logic* as sets of formulae closed under some syntactical substitutions. Here, we only require substitutions on an atomic level. Thus in this paper, a *fragment of logic* is a set of formulae closed under atomic substitutions.

If needed the alphabet will be specified. For instance $\mathcal{F}[\sigma](B^*)$ will denote the set of languages of B^* definable by a formula of the fragment \mathcal{F} on the signature σ .

2.2 Enriched words

We now fix a positive integer d and an alphabet A . Let \mathbb{Z}_d be the cyclic group of order d .

Definition 1 (Enriched alphabet). We call the set $A_d = A \times \mathbb{Z}_d$ the enriched alphabet of A , and we denote by $\pi_d : A_d^* \rightarrow A^*$ the projection defined by $\pi_d(a, i) = a$ for each $(a, i) \in A_d$. For example, the word $(a, 2)(b, 1)(b, 2)(a, 0)$ is an enriched word of $abba$ for $d = 3$. We say that $abba$ is the underlying word of $(a, 2)(b, 1)(b, 2)(a, 0)$.

Definition 2 (Well-formed words). A word $(a_0, i_0)(a_1, i_1) \cdots (a_n, i_n)$ of A_d^* is well-formed if for $0 \leq j \leq n$, $i_j = j \pmod{d}$. We denote by K_d the set of all well-formed words of A_d^* .

Let $\alpha_d : A^* \rightarrow A_d^*$ be the function defined for any word $u = a_0 a_1 \cdots a_n \in A^*$ by $\alpha_d(u) = (a_0, 0)(a_1, 1) \cdots (a_n, n \pmod{d})$. The word $\alpha_d(u)$ is called the well-formed word attached to u .

Note that the restriction of π_d to the set of well-formed words is one-to-one. For instance, the enriched word $(a, 0)(b, 1)(b, 2)(a, 0)$ is a well-formed word for $d = 3$. It is the unique well-formed word having the word $abba$ as underlying word. The following lemma is an easy consequence of this observation.

Lemma 1. Let L and L' be languages of A_d^* , then the following equalities hold

1. $\pi_d((L \cup L') \cap K_d) = \pi_d(L \cap K_d) \cup \pi_d(L' \cap K_d)$,
2. $\pi_d((L \cap L') \cap K_d) = \pi_d(L \cap K_d) \cap \pi_d(L' \cap K_d)$
3. $(\pi_d(L \cap K_d))^c = \pi_d(L^c \cap K_d)$.

2.3 Semigroups and recognizable languages

We refer to [15] for the standard definitions of semigroup theory. A *semigroup* is a set equipped with a binary associative operation, which we will denote multiplicatively. A *monoid* is a semigroup with a neutral element 1. Recall that a monoid M divides another monoid N if M is a quotient of a submonoid of N . This defines a partial order on finite monoids.

A *stamp* is a surjective monoid morphism from A^* onto a finite monoid. A language L is *recognized* by a finite monoid M if there exists a stamp $\varphi : A^* \rightarrow M$ and a subset P of M such that $L = \varphi^{-1}(P)$. A language is *recognizable* if it is recognized by a finite monoid. Kleene's theorem states that the set of recognizable languages is exactly the set of rational (or regular) languages. The *syntactic congruence* of a regular language L of A^* is the equivalence relation \sim_L defined by $u \sim_L v$ if and only if for all $w, w' \in A^*$,

$$wuw' \in L \Leftrightarrow wvw' \in L.$$

The monoid $M_L = A^*/\sim_L$ is the *syntactic monoid* of L and the morphism $\eta_L : A^* \rightarrow A^*/\sim_L$ its *syntactic stamp*.

2.4 Stable Semigroup, Stable Monoid, Stable Stamp

For a stamp $\varphi : A^* \rightarrow M$, the set $\varphi(A)$ is an element of the powerset monoid of M . As such it has an idempotent power. The *stability index* of a stamp is the least positive integer s such that $\varphi(A^s) = \varphi(A^{2s})$. This set forms a subsemigroup called the *stable semigroup* of φ . The set $\varphi((A^s)^*)$ is called the *stable monoid* of φ and the morphism from $(A^s)^*$ onto the stable monoid induced by φ is called the *stable stamp*. The stable monoid of a regular language is the stable monoid of its syntactic stamp.

2.5 Stamps and varieties

A (pseudo) variety of finite monoids is a class of finite monoids closed under division and finite products. According to Eilenberg [18], a *variety of languages* \mathcal{V} is a class of languages closed under finite union, finite intersection and complementation, left and right quotients and closed under inverse of monoid morphisms. This means that, for any monoid morphism $\varphi : A^* \rightarrow B^*$, $X \in \mathcal{V}(B^*)$ implies $\varphi^{-1}(X) \in \mathcal{V}(A^*)$. Furthermore Eilenberg [18] proved that there is a *natural* bijective correspondence between varieties of monoids and varieties of languages.

If the class of languages $\mathcal{F}[\sigma]$ is a variety of languages, a potential problem is that the classes $\mathcal{F}[\sigma, \text{LOC}]$ and $\mathcal{F}[\sigma, \text{MOD}]$ might not be closed under inverses of morphisms. Thus Eilenberg's varieties theory does not apply in this case. To overcome this difficulty, one needs the more general theory of \mathcal{C} -varieties introduced by Esik and Ito [19] and Straubing [11] and developed in [20]. We say that a morphism between finitely generated monoids is *length-preserving* if the image of each letter is a letter. Let \mathcal{C} be a class of morphisms between finitely generated free monoids closed under composition and containing the length-preserving morphisms. Examples include the morphisms between finitely generated free monoids (*all*), the *non-erasing* (*ne*) morphisms (morphisms for which the image of letters are non empty words) and the *length-multiplying* (*lm*) morphisms (morphisms for which there is an integer k such that the image of each letter is a word of size k).

Let us now recall the notion of a \mathcal{C} -variety of stamps. The *restricted product stamp* of two stamps $\eta_1 : A^* \rightarrow M_1$ and $\eta_2 : A^* \rightarrow M_2$ is the stamp η defined by $\eta(a) = (\eta_1(a), \eta_2(a))$. The image of η is a submonoid of $M_1 \times M_2$. A stamp $\varphi : A^* \rightarrow M$ \mathcal{C} -*divides* another stamp $\psi : B^* \rightarrow N$ if and only if there exists a pair (α, β) such that α is a \mathcal{C} -morphism from A^* to B^* , $\beta : N \rightarrow M$ is a partial surjective monoid morphism and $\varphi = \beta \circ \mu \circ \alpha$. The pair (α, β) is called an \mathcal{C} -*division*. Then a \mathcal{C} -variety of stamps is a class of stamps closed under \mathcal{C} -division and finite restricted products. Note that if \mathbf{V} is a variety of monoids, then the class of all stamps whose image is a monoid in \mathbf{V} forms a \mathcal{C} -variety of stamps, also denoted by \mathbf{V} . A \mathcal{C} -variety of languages is a class of languages closed under finite union, finite intersection and complementation, left and right quotients

$$\begin{array}{ccc}
A^* & \xrightarrow{\alpha} & B^* \\
\varphi \downarrow & & \downarrow \psi \\
M & \xleftarrow{\beta} & N
\end{array}$$

and closed under inverse of \mathcal{C} -morphisms. Eilenberg's varieties theorem can be extended to \mathcal{C} -varieties: there is a natural bijective correspondence between \mathcal{C} -varieties of stamps and \mathcal{C} -varieties of languages [11].

Finally, there is also a natural bijective correspondence between varieties of monoids and *all*-varieties. Therefore given a variety of monoids \mathbf{V} we will also denote by \mathbf{V} the corresponding *all*-variety of stamps.

Example 1. Given $d > 0$, let \mathbf{MOD}_d be the class of all stamps of the form $\pi_d : A^* \rightarrow \mathbb{Z}_d$ with $\pi_d(a) = \pi_d(b)$ for all letters a and b . Then \mathbf{MOD}_d is a *lm*-variety of stamps and the corresponding *lm*-variety of languages MOD_d is the *lm*-variety generated by the languages $\{(A^d)^* A^i \mid 0 \leq i < d\}$. The class $\mathbf{MOD} = \bigcup_{d>0} \mathbf{MOD}_d$ is also a *lm*-variety of stamps.

Example 2. Let \mathbf{DA} be the variety of monoids satisfying the equation $(xy)^\omega = (xy)^\omega x (xy)^\omega$ where ω is the idempotent power of the monoid. Alternatively \mathbf{DA} is the variety of monoids whose regular \mathcal{D} -classes are aperiodic semigroups. The corresponding variety of languages is the class of languages definable in $\mathbf{FO}^2[<]$, the two-variable first order logic [12]. When adding the local predicates we obtain a *ne*-variety of languages. The variety of stamps corresponding to $\mathbf{FO}^2[<, \text{LOC}]$ is \mathbf{LDA} , the class of stamps $\eta : A^* \rightarrow M$ such that for every idempotent e of the semigroup $\eta(A^+)$, the submonoid eMe is in \mathbf{DA} . For instance, the syntactic stamp of the language $(ab)^*$ is in \mathbf{LDA} but the syntactic stamp of the language $c^*(ce^*bc^*)^*$ is not.

3 Wreath product

3.1 Wreath Product Principle for MOD

The wreath product is an algebraic operation on monoids that specializes the semidirect product. This operation has been studied intensively in semigroup theory. The reader is referred to [21] for applications to languages. In logic, this operation often encodes the addition of some new predicates. In particular, for many cases, the $-*\mathbf{LI}$ operation corresponds to adding local predicates to a given signature. The rather technical definition of the wreath product is omitted (see Appendix¹). We will only use it through the following theorem, a consequence of the Wreath Product Principle for stamps presented in [1].

¹ The references to the Appendix are given for convenience of the referees. They are not part of the paper

Theorem 1 (Wreath Product Principle for MOD [1]). Let \mathbf{V} be a (ne)-variety, let \mathcal{V} be the corresponding (ne)-variety of languages, L a regular language of A^* and d a positive integer. Then the following properties are equivalent:

- (1) The language L is recognized by a stamp in $\mathbf{V} * \text{MOD}_d$,
- (2) The language L belongs to the lattice of languages generated by the languages of the form $(A^d)^* A^i$ for $i < d$ and of the form $\pi_d(L' \cap K_d)$ where $L' \in \mathcal{V}(A_d^*)$.

Furthermore, a language L is recognized by a stamp in $\mathbf{V} * \text{MOD}$ if and only if there exists $d > 0$ such that L is recognized by a stamp in $\mathbf{V} * \text{MOD}_d$.

The next theorem is the main result of this section.

Theorem 2. Let $\mathcal{F}[\sigma]$ be a fragment equivalent to a (ne)-variety \mathbf{V} , L a regular language and d a positive integer. Then the following properties are equivalent:

- (1) L is definable by a formula of $\mathcal{F}[\sigma, \text{MOD}_d]$,
- (2) η_L belongs to $\mathbf{V} * \text{MOD}_d$,
- (3) there exists some languages L_0, \dots, L_{d-1} of $\mathcal{V}(A_d^*)$ such that:

$$L = \bigcup_{i=0}^{d-1} ((A^d)^* A^i \cap \pi_d(L_i \cap K_d)) \quad (1)$$

Furthermore, a language L is definable in $\mathcal{F}[\sigma, \text{MOD}]$ if and only if L is recognized by a stamp in $\mathbf{V} * \text{MOD}$.

Proof. We only treat the case of a variety of monoids, since the proof for a ne-variety is the same. (3) implies (2) follows from Theorem 1.

(2) implies (3). Assume that η_L belongs to $\mathbf{V} * \text{MOD}$. By Theorem 1, we can suppose that L belongs to the lattice generated by languages of the form $(A^d)^* A^i$ for $i < d$ and $\pi_d(L' \cap K_d)$ with $L' \in \mathcal{V}(A_d^*)$. Recall that the lattice is distributive and the languages $(A^d)^* A^i$ form a partition of A^* . Therefore, there are languages H_0, \dots, H_{d-1} in the lattice of languages generated by $\pi_d(L' \cap K_d)$ with $L' \in \mathcal{V}(A_d^*)$ such that $L = \bigcup_{i=0}^{d-1} ((A^d)^* A^i \cap H_i)$. Thus, thanks to Lemma 1, for $0 \leq i < d$, there exists a language $L_i \in \mathcal{V}(A_d^*)$ such that $H_i = \pi_d(L_i \cap K_d)$.

For the equivalence between (1) and (3) we need an auxiliary result which gives a decomposition of the language defined by a formula into smaller pieces.

Lemma 2. Let $\mathcal{F}[\sigma, \text{MOD}]$ be a fragment of logic and φ a formula of $\mathcal{F}[\sigma, \text{MOD}_d]$. Then there exists d formulae ψ_i of $\mathcal{F}[\sigma, \text{MOD}_d]$ that do not contain any predicate D_j^d and such that $\varphi \equiv \bigvee_{i=0}^{d-1} (\psi_i \wedge D_i^d)$. Moreover, we have:

$$L(\varphi) = \bigcup_{i=0}^{d-1} ((A^d)^* A^i \cap L(\psi_i)).$$

The proof is omitted here. It relies on some elementary manipulations of formulae. We now conclude the proof of Theorem 2. Let φ be a formula of $\mathcal{F}[\sigma, \text{MOD}]$.

Then φ belongs to $\mathcal{F}[\sigma, \text{MOD}_d]$ for some $d > 0$. Using Lemma 2, we know it is sufficient to consider a formula φ without any length predicate. We transform it into a formula ψ by replacing every predicate $\text{MOD}_i^d(x)$ by $\bigvee_{a \in A} (\mathbf{a}, \mathbf{i})x$ and every predicate $\mathbf{a}x$ by $\bigvee_{0 \leq i < d} (\mathbf{a}, \mathbf{i})x$. The resulting formula ψ is in $\mathcal{F}[\sigma](A_d^*)$ and $L(\varphi) = \pi_d(L(\psi) \cap K_d)$. Conversely, we transform a formula ψ of $\mathcal{F}[\sigma](A_d^*)$ into a formula φ of $\mathcal{F}[\sigma, \text{MOD}_d]$ by replacing every predicate $(\mathbf{a}, \mathbf{i})x$ in ψ by $\mathbf{a}x \wedge \text{MOD}_i^d(x)$. We also get $L(\varphi) = \pi_d(L(\psi) \cap K_d)$. \square

The semidirect product does not necessarily preserve decidability [17]. The next sections will focus on some particular cases of semidirect products of varieties with **MOD** where decidability is preserved.

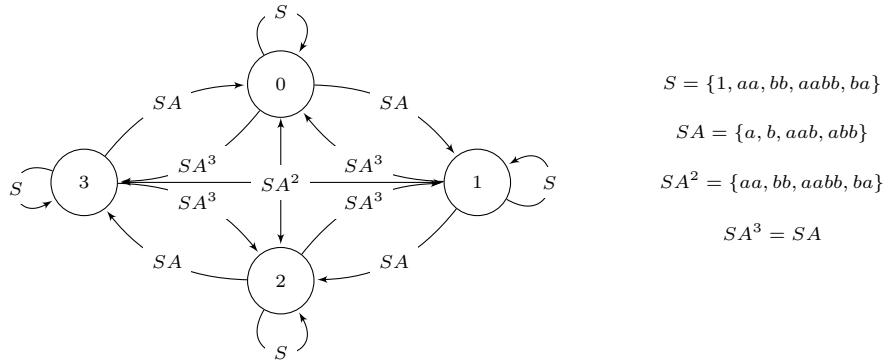
3.2 The Derived Category Theorem

A (small) *category* is a set of objects equipped with a set of arrows between any pair of objects, with a composition law for consecutive arrows. A *loop* is an arrow whose initial object is the same as its final object. The set of loops around a given object, equipped with the composition law, forms a monoid, called the *local monoid* of that object. We refer to Tilson [16] for complete definitions. Here we only consider finite categories, seen as a generalization of finite monoids, since a monoid can be viewed as a one-object category. Here we give the definition of the derived category for **MOD** which is an adaptation of the one introduced by Tilson [16] and specialized for **MOD** in [1].

Definition 3. Let $\varphi : A^* \rightarrow M$ be a stamp and d an integer. The d -derived category of φ , denoted $C_d(\varphi)$, is the category with \mathbb{Z}_d as set of objects, and the arrows from i to j are the elements of M such that there exists a word u such that $\varphi(u) = m$ and $i + |u| \equiv j \pmod{d}$. The d -derived category of a regular language L , denoted $C_d(L)$, is the category $C_d(\eta_L)$.

Lemma 3. Let d be a positive integer, and L be a regular language of stability index s . Then the local monoids of $C_d(L)$ are isomorphic to $\eta_L((A^d)^*)$. In particular, the local monoids of $C_s(L)$ are isomorphic to the stable monoid of L .

Example 3. The 4-derived category of the language $(aa)^*ab(bb)^*$ is given below. Let η be its syntactic morphism and S its stable monoid. Its stability index is 4.



We omit the definition of the division of categories [16] (see Appendix). The *global* of a variety \mathbf{V} , denoted by $g\mathbf{V}$, is the variety of all categories that divide a monoid in \mathbf{V} , seen as a one-object category. The derived category theorem was originally proved by Tilson [16] for varieties of monoids and semigroups. In [22], Chaudard extended this theorem to \mathcal{C} -varieties. Here we give the specialization to **MOD** of this latter generalization.

Theorem 3 (Derived Category's Theorem for MOD [22]). *Let \mathbf{V} be a (ne-)variety and L a regular language. A language L has its syntactic stamp in $\mathbf{V} * \mathbf{MOD}$ if and only if there exists $d > 0$ such that $C_d(L)$ is in $g\mathbf{V}$.*

4 The local case

For any (ne)-variety \mathbf{V} , we define \mathbf{QV} to be the *lm*-variety of stamps with a stable stamp in \mathbf{V} . Following Tilson [16], we denote by $\ell\mathbf{V}$ the variety of categories whose local monoids are all in \mathbf{V} . The next theorem makes explicit the link between \mathbf{QV} and $\ell\mathbf{V}$.

There is a similar definition of $\ell\mathbf{V}$ for a ne-variety [22]. This definition is too technical to be presented in this paper, but the link between \mathbf{QV} and $\ell\mathbf{V}$ presented in the next theorem holds for varieties and for ne-varieties.

Theorem 4. *Let \mathbf{V} be a (ne-)variety and L a regular language of A^* of stability index s . The following properties are equivalent:*

- (1) *L is recognized by a stamp in \mathbf{QV} ,*
- (2) *there exists an integer d such that $C_d(L)$ is in $\ell\mathbf{V}$,*
- (3) *$C_s(L)$ is in $\ell\mathbf{V}$.*

Proof.

(1) \rightarrow (3). If L is recognized by a stamp in \mathbf{QV} , then its syntactic stamp is also in \mathbf{QV} and its stable monoid is in \mathbf{V} . But, thanks to Lemma 3, the local monoids of $C_s(L)$ belong to \mathbf{V} , and thus $C_s(L)$ is in $\ell\mathbf{V}$.

(3) \rightarrow (2). Is obvious.

(2) \rightarrow (1). Suppose that $C_d(L)$ is in $\ell\mathbf{V}$. Then the local monoids of $C_d(L)$, which are isomorphic to $\eta_L((A^d)^*)$ by Lemma 3, belong to \mathbf{V} . Thus $\eta_L((A^{ds})^*)$, which is a submonoid of $\eta_L((A^d)^*)$, also belongs to \mathbf{V} . Finally, by definition of the stability index, the monoid $\eta_L((A^s)^*) = \eta_L((A^{ds})^*)$ is in \mathbf{V} and thus η_L is in \mathbf{QV} . \square

Observe that any monoid of \mathbf{V} , viewed as a one-object category, belongs to $\ell\mathbf{V}$. Therefore by definition of $g\mathbf{V}$, any category of $g\mathbf{V}$ divides a category of $\ell\mathbf{V}$, and thus $g\mathbf{V} \subseteq \ell\mathbf{V}$. A variety such that $g\mathbf{V} = \ell\mathbf{V}$ is said to be *local*. Combining Theorem 3 and Theorem 4 yields the following theorem.

Theorem 5. *Let \mathbf{V} be a (ne-)variety. Then $\mathbf{V} * \mathbf{MOD} \subseteq \mathbf{QV}$. If furthermore \mathbf{V} is local, then $\mathbf{V} * \mathbf{MOD} = \mathbf{QV}$.*

Since the stability index and the stable monoid of a given regular language are computable, one gets the following corollary.

Corollary 1. Let $\mathcal{F}[\sigma]$ be a fragment equivalent to a local (ne-)variety. Then $\mathcal{F}[\sigma]$ is decidable if and only if $\mathcal{F}[\sigma, \text{MOD}]$ is decidable.

Remark 1. The equality $\mathbf{V} * \text{MOD} = \mathbf{QV}$ does not always hold. A counterexample is the variety \mathbf{J} , which is known to be nonlocal. Chaubard, Pin and Straubing proved the decidability of $\mathbf{J} * \text{MOD}$ [1], using the characterization of $g\mathbf{J}$ given by Knast in [23]. Using this characterization, we can prove that the language $(aa)^*ab(bb)^*$, whose stable monoid is in \mathbf{J} does not satisfy Knast's equation, proving that $\mathbf{J} * \text{MOD} \subsetneq \mathbf{QJ}$ (see Example 3).

5 Main result

Theorem 2 gives a description of the languages definable in $\mathcal{F}[\sigma, \text{MOD}]$ which makes use of a parameter d . To derive an effective characterization from this result, two problems have to be solved. The first one consists in computing effectively this integer d , given the language L . We call it the *Delay problem* for **MOD** in reference to the Delay Theorem [9, 16] which solves a similar problem for the operation $\mathbf{V} \rightarrow \mathbf{V} * \mathbf{LI}$. The second problem is to find effectively the languages L_0, \dots, L_{d-1} occurring in Theorem 2 (1). Finding these languages can be reduced to the membership problem for $g\mathbf{V}$. In several situations, this is known to be decidable (but not always, see [17]). Local varieties, handled in the previous section, form a good example. We now state our main result, which gives a sufficient condition to solve the Delay Problem for **MOD** in the nonlocal case.

Theorem 6 (A partial Delay Theorem for MOD). Let \mathbf{V} be a (ne-)variety such that, for each alphabet A and any letter a of A , \mathbf{V} contains the syntactic stamps of the languages aA^* and A^*a . Then a stamp with stability index s belongs to $\mathbf{V} * \text{MOD}$ if and only if it belongs to $\mathbf{V} * \text{MOD}_s$.

The proof of this Theorem is given in Section 6. Let us first deduce several Corollaries from this result. It is known that $g\mathbf{V}$ is decidable if and only if $\mathbf{V} * \mathbf{LI}$ is decidable [17, 16]. By Theorems 3 and 6, we have:

Corollary 2. Let \mathbf{V} be a variety containing the syntactic stamps of the languages aA^* and A^*a . If $\mathbf{V} * \mathbf{LI}$ is decidable, then $\mathbf{V} * \text{MOD}$ is also decidable.

The global variety of any decidable variety containing the syntactic stamp of $(ab)^*$ is known to be decidable [16]. The following corollary, which is proved in Section 7, makes use of similar results compatible for both varieties and (ne)-varieties.

Corollary 3. Let \mathbf{V} be a (ne-)variety that contains the syntactic stamps of the languages aA^* , $(ab)^*$ and A^*a . Then \mathbf{V} is decidable if and only if $\mathbf{V} * \text{MOD}$ is decidable.

We summarized in the following table the consequences of our results for deciding various fragments of first-order logic.

	$\mathcal{B}\Sigma_1 = \mathbf{FO}_0^2$	\mathbf{FO}_k^2	\mathbf{FO}^2	\mathbf{FO}
[<]	J [8, 13]	V_k [6, 3]	DA [12]	A [7, 24]
[<, LOC]	J * LI [23]	V_k * LI [5]	LDA [12]	A [7, 24]
[<, MOD]	J * MOD [1]	V_k * MOD Corollary 2 New	QDA Theorem 5 or [2]	QA Theorem 5 or [10, 25]
[<, LOC, MOD]	J * LI * MOD Corollary 3 or [26]	V_k * LI * MOD Corollary 3 New	LDA * MOD Corollary 3 New	QA Theorem 5 or [10, 25]

Fig. 1.

As a consequence one can obtain (see Appendix) the following extension of the previous work of the authors [2].

Corollary 4. $\mathbf{FO}^2[<, \text{LOC}, \text{MOD}] = \mathbf{LDA} * \mathbf{MOD} = \mathbf{QLDA}$

6 Proof of the Delay Theorem for MOD

Surprisingly, our proof of Theorem 6 does not rely on the arguments presented in Section 3.2. Instead, we reduce the proof of the Delay Theorem for **MOD** to a particular instance of a problem known as the *separation problem*, which can be summarized as follows. Given a variety of languages \mathcal{V} , two disjoint regular languages L and L' are \mathcal{V} -separable if there exists a language R in \mathcal{V} such that $L \subseteq R$ and $R \cap L' = \emptyset$. More specifically, our proof relies on the following result.

Proposition 1. *Let \mathbf{V} be a variety such that for any alphabet A and any letter a of A , the language A^*a belongs to $\mathcal{V}(A^*)$. Then the syntactic stamp of a regular language L is in $\mathbf{V} * \mathbf{MOD}$ if and only if there exists $d > 0$ such that the languages $L_d = \pi_d^{-1}(L) \cap K_d$ and $\overline{L}_d = \pi_d^{-1}(L^c) \cap K_d$ are \mathcal{V} -separable.*

Proof. By Theorem 2, the syntactic stamp of a regular language L is in $\mathbf{V} * \mathbf{MOD}$ if and only if there exists $d > 0$ and languages L_0, \dots, L_{d-1} in $\mathcal{V}(A_d^*)$ such that

$$L = \bigcup_{i=0}^{d-1} ((A^d)^* A^i \cap \pi_d(L_i \cap K_d)).$$

For $0 \leq i < d$, we have $(A^d)^* A^i = \pi_d(A_d^*(A, i-1) \cap K_d)$. Thanks to Lemma 1, one gets

$$L = \pi_d \left(\bigcup_{i=0}^{d-1} (A_d^*(A, i-1) \cap L_i) \cap K_d \right).$$

Let denote L' the language $\bigcup_{i=0}^{d-1} (A_d^*(A, i-1) \cap L_i)$. Because $A_d^*(A, i-1)$ is in $\mathcal{V}(A_d^*)$ we have L' in $\mathcal{V}(A_d^*)$ too. Therefore, by Theorem 1, the syntactic stamp

of a regular language L is in $\mathbf{V} * \mathbf{MOD}$ if and only if there exist $d > 0$ and a language L' in $\mathcal{V}(A_d^*)$ such that $L = \pi_d(L' \cap K_d)$. The languages L_d and \overline{L}_d are \mathcal{V} -separated by L' . Conversely, if L_d and \overline{L}_d are \mathcal{V} -separable, then there exists a language L' in $\mathcal{V}(A_d^*)$ such that $L_d \subseteq L'$ and $L' \cap \overline{L}_d = \emptyset$. Because $K_d = L_d \cup \overline{L}_d$, we have $L = \pi_d(L' \cap K_d)$. \square

Next we use a general result.

Theorem 7 ([27]). *Let \mathcal{V} be a variety of languages and L and L' be two regular languages of A^* and $\eta : A^* \rightarrow M$ a stamp that recognizes both of them. Then L and L' are \mathcal{V} -separable if and only if there exists a relational morphism $\tau : M \rightarrow N$ with $N \in \mathbf{V}$ such that $\tau(x) \cap \tau(y) = \emptyset$ for all $x \in \eta(L)$ and $y \in \eta(L')$.*

Let $\eta_L : A^* \rightarrow M_L$ be the syntactic stamp of L and let $N_d = (\mathbb{Z}_d \times M_L \times \mathbb{Z}_d) \cup \{0\}$ be the monoid defined by

$$(i, m, j)(i', m', j') = \begin{cases} (i, mm', j') & \text{if } i' = j \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\mu_d : A^* \rightarrow N_d$ be the morphism defined by $\mu_d(a, i) = (i, \eta_L(a), i + 1 \bmod d)$ and let $M_d = \mu_d(A^*)$.

Lemma 4. *Let L be a regular language, η its syntactic stamp and s its stability index. Then for every positive integer k , the application $\gamma : M_{ks} \rightarrow M_s$ defined by $\gamma(i, m, j) = (i \bmod s, m, j \bmod s)$ is an onto morphism. Therefore, M_s is a quotient of M_{ks} .*

Because of the relation between the Derived Category $C_{ks}(L)$ and the monoid M_{ks} one could think that this construction gives a division of categories. In fact, this is not true since the division of categories is more rigid than the division of monoids. To prove the Delay Theorem, we first need to show some stability properties of the \mathcal{V} -separation. We first introduce some notation. Let i and j be two integers smaller than d . We denote by $L_d(i, j)$ the set of well formed words which have their first letter in (A, i) , their last letter in (A, j) and the first component in L . Similarly, we define $\overline{L}_d(i, j)$ such that the first component is a word of L^c . Since the variety $\mathcal{V}(A^*)$ contains the languages A^*a and aA^* then for $(i, j) \neq (i', j')$ we can \mathcal{V} -separate the languages $L_d(i, j)$ from $\overline{L}_d(i', j')$.

Lemma 5. *Let k be an integer. Then the languages $L_d(i, j)$ and $\overline{L}_d(i, j)$ are \mathcal{V} -separable if and only if the languages $L_d(i+k \bmod d, j+k \bmod d)$ and $\overline{L}_d(i+k \bmod d, j+k \bmod d)$ are \mathcal{V} -separable.*

Note that $L_d = \bigcup_j L_d(0, j)$ and $\overline{L}_d = \bigcup_j \overline{L}_d(0, j)$. Setting $L'_d = \bigcup_{i,j} L_d(i, j)$ and $\overline{L}'_d = \bigcup_{i,j} \overline{L}_d(i, j)$, we finally have

Corollary 5. *If L_d and \overline{L}_d are \mathcal{V} -separable, then L'_d and \overline{L}'_d are also \mathcal{V} -separable.*

Proof. We assume that L_d and \overline{L}_d are \mathcal{V} -separable. We show that for each pairs (i, j) and (i', j') the languages $L_d(i, j)$ and $\overline{L}_d(i', j')$ are \mathcal{V} -separable. This will be sufficient since \mathcal{V} is stable by Boolean operations. We distinguish two cases. If the pairs are distinct, then we can separate the languages using either $(A, i)A_d^*$ or $A_d^*(A, j)$. If the pairs are equal, then we use Lemma 5. \square

We now have all the tools to conclude the proof of the Delay Theorem. First notice that, since $\text{MOD}_k \subseteq \text{MOD}_{ks}$, if L belongs to $\mathbf{V} * \text{MOD}_k$ for some k , then it belongs to $\mathbf{V} * \text{MOD}_{ks}$. Now, this last Lemma allows us to conclude the proof.

Lemma 6. *If L belongs to $\mathbf{V} * \text{MOD}_{ks}$ then it belongs to $\mathbf{V} * \text{MOD}_s$.*

Proof. Thanks to Proposition 1, we can assume that L_{ks} and \overline{L}_{ks} are \mathcal{V} -separable. Then by Corollary 5 the languages L'_{ks} and \overline{L}'_{ks} are also \mathcal{V} -separable. By Theorem 7 there exists a relational morphism $\tau : M_{ks} \rightarrow N$ with N in \mathbf{V} such that for all $(i, m, j) \in M_{ks}$ with $m \in \eta(L)$ and $(i', m', j') \in M_{ks}$ with $m' \in \eta(L^c)$ we have $\tau(i, m, j) \cap \tau(i', m', j') = \emptyset$. Let τ' be $\tau \circ \gamma^{-1}$, where γ is as defined in Lemma 4. Since the inverse of an onto morphism is a relational morphism and

$$\begin{array}{ccc} M_{ks} & \xrightarrow{\tau} & N \\ \downarrow \gamma & & \swarrow \tau' \\ M_s & & \end{array}$$

since relational morphisms are closed under composition [18], τ' is a relational morphism. To conclude the proof, it remains to show that this relational morphism satisfies that $\tau'(x) \cap \tau'(y) = \emptyset$ for every $x \in \mu_s(L_s)$ and $y \in \mu_s(\overline{L}_s)$. Let (i, m, j) be an element of M_s . Then

$$\gamma^{-1}(i, m, j) \subseteq \{(k, m, \ell) \mid k \equiv i \pmod{s}, \ell \equiv j \pmod{s}\} \cap M_{ks}.$$

For all $(i, m, j) \in M_{ks}$ with $m \in \eta(L)$ and $(i', m', j') \in M_{ks}$ with $m' \in \eta(L^c)$ we have $\tau(i, m, j) \cap \tau(i', m', j') = \emptyset$. Then we also have that for all $(i, m, j) \in M_s$ with $m \in \eta(L)$ and $(i', m', j') \in M_s$, we have $\tau'(i, m, j) \cap \tau'(i', m', j') = \emptyset$. \square

7 Discussion

7.1 Membership problem for $\mathbf{V} * \text{MOD}$

We give here a decision process for the case where the syntactic stamps of aA^* , $(ab)^*$ and A^*a belong to a decidable (*ne*)-variety \mathbf{V} . The key argument relies on the fact that if the syntactic stamp of $(ab)^*$ is in a (*ne*)-variety then so does the syntactic stamp of K_d for any integer d .

Theorem 8. *Let L be a regular language of stability index s and let \mathbf{V} be a decidable (*ne*)-variety containing the syntactic stamps of the languages aA^* , $(ab)^*$ and A^*a . The syntactic stamp of a language L is in $\mathbf{V} * \text{MOD}$ if and only if the syntactic stamp of the language $\pi_s^{-1}(L) \cap K_s$ is in \mathbf{V} .*

Proof. Let L be a regular language of stability index s . Thanks to Proposition 1 and to Delay Theorem for **MOD** we can state the following. The regular language L has its syntactic stamp in $\mathbf{V} * \mathbf{MOD}$ if and only if the languages $\pi_s^{-1}(L) \cap K_s$ and $\pi_s^{-1}(L^c) \cap K_s$ are \mathcal{V} -separable. Assume that L' is the \mathcal{V} -separator. Then since $(ab)^*$ has its syntactic stamp in the (*ne*)-variety, any language of the form $(A_1 \dots A_s)^* A_1 \dots A_i$ is also in our variety. Thus it is also the case for the languages K_s and hence $L' \cap K_s = \pi_s^{-1}(L) \cap K_s$. Finally, L has its syntactic stamp in $\mathbf{V} * \mathbf{MOD}$ if and only if $\pi_s^{-1}(L) \cap K_s$ is in $\mathcal{V}(A_s^*)$. \square

7.2 Conclusion

We presented a study of the enrichment operation on logical fragments: $\mathcal{F}[\sigma] \rightarrow \mathcal{F}[\sigma, \mathbf{MOD}]$. For fragments defining (*ne*)-varieties, this operation exactly corresponds to the algebraic operation $\mathbf{V} \rightarrow \mathbf{V} * \mathbf{MOD}$. Our main result states that for a large class of varieties one can obtain a decision process for $\mathcal{F}[\sigma, \mathbf{MOD}]$ from a decision process for $\mathcal{F}[\sigma]$. This work subsumes several known results and leads to the decidability of new fragments. The main ingredients are the partial Delay Theorem for **MOD** and a decision process for the global of **V**. Both of them might be improved. Indeed, in the case of **MOD**, the decidability of a weaker version of the global might be sufficient for the wreath product by **MOD**. On the other hand our partial Delay Theorem only holds for varieties that contain the languages aA^* and A^*a . We conjecture that these restrictions are not necessary for the Delay Theorem for **MOD** to hold. An interesting case of study would be the variety generated by the syntactic monoid of the language $(ab)^*$, sometimes referred to as the *universal counterexample*. Indeed, this variety does not fall in the scope of any of our theorems.

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8 Appendix

8.1 Wreath Product

Here we give the definition of the wreath product by **MOD**. The wreath product of a monoid M and the cyclic group of order d , \mathbb{Z}_d , is denoted by $M \circ \mathbb{Z}_d$ and is defined on the set $M^{\mathbb{Z}_d} \times \mathbb{Z}_d$ equipped with the following product:

$$(f, i)(g, j) = (f \cdot jg, i + j)$$

with \cdot being the point wise product on $M^{\mathbb{Z}_d}$ and $jg : \mathbb{Z}_d \rightarrow M$ with $jg(t) = g(t + j)$. Having define the wreath product of two monoids, we can now define the wreath product of (*ne-*)variety by **MOD**. Let \mathbf{V} be a (*ne-*)variety. A stamp $\eta : A^* \rightarrow M$ belongs to $\mathbf{V} * \mathbf{MOD}$ if and only if there exists $d > 0$ and $\mu : (B \times \mathbb{Z}_d)^* \rightarrow N$, with $\mu \in \mathbf{V}$ such that η *lm*-divides the stamp $\mu' : B^* \rightarrow N \circ \mathbb{Z}_d$, defined by $\mu'(b) = (f, 1)$ with $f(i) = \mu(b, i)$.

Lemma 7. *Let $\mathcal{F}[\sigma, \text{MOD}]$ be a fragment of logic and φ a formula of $\mathcal{F}[\sigma, \text{MOD}_d]$. Then there exists d formulae ψ_i of $\mathcal{F}[\sigma, \text{MOD}_d]$ that do not contain any predicate D_j^d and such that $\varphi \equiv \vee_{i=0}^{d-1} (\psi_i \wedge D_i^d)$. Moreover, we have:*

$$L(\varphi) = \bigcup_{i=0}^{d-1} ((A^d)^* A^i \cap L(\psi_i)).$$

Proof. For $i < d$, we define the formula ψ_i to be the formula φ where we replaced every predicate D_i^d by *true* and every D_j^d with $j \neq i$ by *false*. One should notice that, by definition of a fragment [4], the formulae ψ_i are in $\mathcal{F}[\sigma, \text{MOD}_d]$. We can conclude the proof since the formula $(D_i^d)^*$ recognizes the language $(A^d)^* A^i$. \square

8.2 Division of category

To pursue the parallel with monoids, we recall the notion of division of categories, which extends the notion of division on monoids. First for a category C we denote by $Obj(C)$ the set of objects of C and by $C(u, v)$ the set of arrow between the objects u and v of C .

Let C, D be two categories. A *division* of categories $\tau : C \rightarrow D$ is given by a mapping $\tau : Obj(C) \rightarrow Obj(D)$, and for each pair of objects u and v , by a relation $\tau : C(u, v) \rightarrow D(\tau(u), \tau(v))$ such that

1. $\tau(x)\tau(y) \subseteq \tau(xy)$ for consecutive arrows x, y ,
2. $\tau(x) \neq \emptyset$ for any arrow x ,
3. $\tau(x) \cap \tau(y) \neq \emptyset$ implies $x = y$ if x and y are coterminal,
4. $1_{\tau(u)} \in \tau(1_u)$ for any object u of C .

One can see that this definition is exactly a generalization of a notion of division if we take, for instance, two categories C and D with only one object (ie monoids). Then C divides D in a sense of category if and only if C divide D in a sense of monoid.

8.3 Proof of the Delay Theorem

Lemma 8. *Let L be a regular language and s its stability index. Then for every positive integer k , the monoid M_s is a quotient of M_{ks} .*

Proof. Recall that η is the syntactic morphism of L . Consider the application $\gamma : M_{ks} \rightarrow M_s$ defined by $\gamma(i, m, j) = (i \bmod s, m, j \bmod s)$. First, note that this application is well defined and is a morphism. Indeed, if $(i, m, j) \in M_{ks}$ then there exists a word u such that $\eta(u) = m$ and $|u| \equiv j - i \bmod ks$. Therefore, $|u| \equiv j - i \bmod s \equiv (j \bmod s - i \bmod s) \bmod s$. Thus, $(i \bmod s, m, j \bmod s) \in M_s$. Now we have to show that γ is onto. Assume that (i, m, j) is an element of M_s . We now have several cases to treat:

1. if there exists a word u in the pre-image of m such that $|u| > s$. Then thanks to the definition of the stability index, there exists a word v which is L -equivalent to u such that $|v| \equiv j - i \bmod ks$ and finally $(i, m, j) \in M_{ks}$,
2. if $i < j$ and for all u in the pre-image of m such that $|u| \equiv j - i \bmod s$ we have $|u| < s$, then we also have $|u| \equiv j - i \bmod ks$ and finally $(i, m, j) \in M_{ks}$,
3. if $i > j$ and for all u in the pre-image of m such that $|u| \equiv j - i \bmod s$ we have $u < s$. Then $(i, m, j + s) \in M_{ks}$ and finally $\gamma(m, i, j + s) = (i, m, j)$.

In all the cases there exists an element of M_{ks} whose image is (i, m, j) . Thus, γ is onto. \square

Lemma 9. *Let k be an integer. Then the languages $L_d(i, j)$ and $\overline{L}_d(i, j)$ are \mathcal{V} -separable if and only if the languages $L_d(i+k \bmod d, j+k \bmod d)$ and $\overline{L}_d(i+k \bmod d, j+k \bmod d)$ are \mathcal{V} -separable.*

Proof. Let R be a language of $\mathcal{V}(A_d^*)$ such that $L_d(i, j) \subseteq R$ and $\overline{L}_d(i, j) \cap R = \emptyset$. Let $\theta_k : A_d^* \rightarrow A_d^*$ be the morphism defined for all $p < d$ by $\theta(a, p) = (a, p + k \bmod d)$. Since θ_k is a permutation of A_d , we have $\theta_k(R) \in \mathcal{V}(A_d^*)$. Thus $\theta_k(R)$ is a \mathcal{V} -separator of $L_d(i+k \bmod d, j+k \bmod d)$ and $\overline{L}_d(i+k \bmod d, j+k \bmod d)$. \square

8.4 Proof of Corollary 4

Corollary 6. $\mathbf{FO}^2[<, \text{LOC}, \text{MOD}] = \mathbf{LDA} * \mathbf{MOD} = \mathbf{QLDA}$

Proof. First, we use Theorem 2 and Example 2 to obtain the equality $\mathbf{FO}^2[<, \text{LOC}, \text{MOD}] = \mathbf{LDA} * \mathbf{MOD}$. The direct inclusion, $\mathbf{LDA} * \mathbf{MOD} \subseteq \mathbf{QLDA}$, is a consequence of Theorem 5.

In the opposite direction suppose that L is a regular language of stability index s with its stable stamp in **LDA**. Since $\mathbf{FO}^2[<, \text{LOC}]$ captures the languages aA^* , $(ab)^*$ and A^*a , Theorem 8 can be applied. Thus the syntactic stamp of a language L belongs to **LDA*MOD** if and only if $\pi_s^{-1}(L) \cap K_s$ is in **LDA**. Recall that a stamp $\eta : A^* \rightarrow M$ belongs to **LDA** if for any idempotent e of $\eta(A^+)$, the monoid eMe is in **DA**. Let $\mu : (A_s)^* \rightarrow M_s \subseteq (\mathbb{Z}_d \times M_L \times \mathbb{Z}_d) \cup \{0\}$ be the stamp recognizing $\pi_s^{-1}(L) \cap K_s$ defined in Section 6. Let e be an idempotent of

$\mu((A_s)^+)$. By definition of M_s , there exists $i \in \mathbb{Z}_d$ and f an idempotent of M_L such that $e = (i, f, i)$. Note that the element f is necessary in the stable monoid of L . Thus, any element in $(i, f, i)M_s(i, f, i)$ is either 0 or an element of the form (i, ftf, i) with t an element of the stable monoid of L . More precisely, if we denote by S the stable monoid of L , we have that $(i, f, i)M_s(i, f, i)$ is isomorphic to the monoid $fSf \cup \{0\}$. By hypothesis, L belongs to **QLDA**. In particular, the monoid fSf is in **DA**. Thus the monoid $fSf \cup \{0\}$ is also in **DA** and by isomorphism the monoid $(i, f, i)M_s(i, f, i)$ is also in **DA**. \square

An alternate proof can be obtained by considering the notion of locality adapted to the semigroups framework. The ne -variety **LDA** can be seen as a variety of semigroups. Therefore, one can use the notion of locality obtained in [28] for varieties of semigroups. In particular, we claim that the variety of semigroups **LDA** is local. A semigroupoid S belongs to **gLDA** if and only if the consolidated semigroup S_c belongs to **LDA**. Note that an idempotent of S_c is an idempotent loop of S . Therefore S_c belongs to **LDA** if and only if the local semigroup of S belongs to **LDA**. Since this is exactly the definition of a local variety, this shows that **LDA** is a local variety of semigroups. It is tempting to use the version of Theorem 5 for the ne -variety. Unfortunately, it is not clear that the locality of a variety of semigroups is equivalent to the locality of the corresponding ne -variety. However, the proof of Theorem 5 is easily adaptable to varieties of semigroups.