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MEROMORPHIC ALMOST RIGID GEOMETRIC STRUCTURES

SORIN DUMITRESCU

Dedicated to Bob Zimmer

ABSTRACT. We study the local Killing Lie algebra of meromorphic almost rigid geometric structures on complex manifolds. This leads to classification results for compact complex manifolds admitting holomorphic rigid geometric structures.

1. INTRODUCTION

Zimmer and Gromov have conjectured that “big” actions of Lie groups that preserve unimodular rigid geometric structures (for example, a pseudo-Riemannian metric or an affine connection together with a volume form) are “essentially classifiable” [2, 28, 64, 65, 66, 67].

We try to address here this general question in the framework of the complex geometry. We study holomorphic (and meromorphic) unimodular geometric structures on compact complex manifolds. We show that in many cases the holomorphic rigidity implies that the situation is classifiable (even

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without assuming any isometric group action). This is a survey paper, but some results are new (see theorem 2.1, corollary 2.2 and theorem 5.10).

In the sequel all complex manifolds are supposed to be smooth and connected.

Consider a complex n -manifold M and, for all integers $r \geq 1$, consider the associated bundle $R^r(M)$ of r -frames, which is a $D^r(\mathbb{C}^n)$ -principal bundle over M , with $D^r(\mathbb{C}^n)$ -the (algebraic) group of r -jets at the origin of local biholomorphisms of \mathbb{C}^n fixing 0 [2, 28, 7, 12, 22, 63].

Let us consider, as in [2, 28], the following

Definition 1.1. A *meromorphic geometric structure* (of order r) ϕ on a complex manifold M is a $D^r(\mathbb{C}^n)$ -equivariant meromorphic map from $R^r(M)$ to a quasiprojective variety Z endowed with an algebraic action of $D^r(\mathbb{C}^n)$. If Z is an affine variety, we say that ϕ is of *affine type*.

In holomorphic local coordinates U on M , the geometric structure ϕ is given by a meromorphic map $\phi_U : U \rightarrow Z$.

Denote by $P \subset M$ the analytic subset of poles of ϕ and by $M^* = M \setminus P$ the open dense subset of M where ϕ is holomorphic. If P is the empty set, then ϕ is a *holomorphic geometric structure* on M .

If ϕ uniquely determines a holomorphic volume form on M^* , then ϕ is called *unimodular*.

Definition 1.2. The meromorphic geometric structure ϕ is called *almost rigid* if, away from an analytic subset of M of positive codimension, ϕ is (holomorphic and) rigid in Gromov's sense [2, 28].

A holomorphic geometric structure on M is called *rigid* if it is rigid in Gromov's sense on all of M .

Notice that our definition of almost rigidity is not exactly the same as in [9].

Examples. On a complex connected n -manifold M the following meromorphic geometric structures are almost rigid.

- A meromorphic map ϕ from M to a projective space $P^N(\mathbb{C})$, which is an embedding on some nontrivial open subset of M . In this case M is of algebraic dimension n (see section 2). If ϕ is a holomorphic embedding, then ϕ is rigid.
- A family X_1, X_2, \dots, X_n of meromorphic vector fields such that there exists an open (dense) subset of M where the X_i span the holomorphic tangent bundle TM . Any homogeneous manifold admits a holomorphic family of such vector fields. They pull-back after the blow up of a point (or a submanifold) in the homogeneous manifold in a meromorphic family of such vector fields.

In the special case where X_1, X_2, \dots, X_n are holomorphic and span TM on all of M , the corresponding geometric structure is called a *holomorphic parallelisation of the tangent bundle* and it is a (unimodular) holomorphic rigid geometric structure of affine type.

- A meromorphic section g of the bundle $S^2(T^*M)$ of complex quadratic forms such that g is nondegenerate on an open (dense) subset. If g is holomorphic and nondegenerate on M then g is a holomorphic

rigid geometric structure of affine type called *holomorphic Riemannian metric*. Up to a double cover of M , a holomorphic Riemannian metric is unimodular.

A holomorphic Riemannian metric has nothing to do with the more usual Hermitian metric. It is in fact nothing but the complex version of a (pseudo-)Riemannian metric. Observe that since complex quadratic forms have no signature, there is no distinction here between the Riemannian and pseudo-Riemannian cases. This observation was at the origin of the nice use by F. Gauß of the complexification technic of (analytic) Riemannian metrics on surfaces, in order to find conformal coordinates for them. Actually, the complexification of analytic Riemannian metrics leading to holomorphic ones, is becoming a standard trick (see for instance [23]).

In the blow up process, the pull-back of a holomorphic Riemannian metric will stay nondegenerate away from the exceptional set and will vanish on the exceptional set.

More general, if g is a meromorphic section of $S^2(T^*M)$ nondegenerate on an open dense set, then any complex manifold \tilde{M} bimeromorphic to M inherits a similar section \tilde{g} . Moreover, if g is holomorphic, then \tilde{g} is also holomorphic. This comes from the fact that indeterminacy points of a meromorphic map are of codimension at least 2 (see theorem 2.5 in [57]) and from Levi's extension principle (see [57] for the details).

- Meromorphic affine or projective connections or meromorphic conformal structures in dimension ≥ 3 . For example, in local holomorphic coordinates (z_1, \dots, z_n) on the manifold M , a meromorphic affine connection ∇ is determined by the meromorphic functions Γ_{ij}^k such that $\nabla_{\frac{\partial}{\partial z_i}} \frac{\partial}{\partial z_j} = \Gamma_{ij}^k \frac{\partial}{\partial z_k}$, for all $i, j, k \in \{1, \dots, n\}$.

Affine connections are geometric structures of affine type, but projective connections or conformal structures are not of affine type.

- If ϕ is a meromorphic almost rigid geometric structure of order r_1 and g is a meromorphic geometric structure of order r_2 , then we can put together ϕ and g in some meromorphic almost rigid geometric structure of order $\max(r_1, r_2)$, denoted by (ϕ, g) [28, 2, 7, 12, 22, 63].
- The s -jet (prolongation) $\phi^{(s)}$ of a meromorphic almost rigid geometric structure ϕ is still a meromorphic almost rigid geometric structure [28, 2, 7, 12, 22, 63].

Recall that local biholomorphisms of M which preserve a meromorphic geometric structure ϕ are called *local isometries*. Note that local isometries of (ϕ, g) are the local isometries of ϕ which preserve also g .

The set of local isometries of a holomorphic rigid geometric structure ϕ is a Lie pseudogroup $Is^{loc}(\phi)$ generated by a Lie algebra of local vector fields called *(local) Killing Lie algebra*. If the local Killing Lie algebra is transitive on M , then ϕ is called *locally homogeneous*.

2. LOCAL ISOMETRIES AND MEROMORPHIC FUNCTIONS

The maximal number of algebraically independent meromorphic functions on a complex manifold M is called *the algebraic dimension* $a(M)$ of M .

Recall that a theorem of Siegel proves that a complex n -manifold M admits at most n algebraically independent meromorphic functions [57]. Then $a(M) \in \{0, 1, \dots, n\}$ and for algebraic manifolds $a(M) = n$.

We will say that two points in M are in the same *fiber of the algebraic reduction* of M if any meromorphic function on M takes the same value at the two points. There exists some open dense set in M where the fibers of the algebraic reduction are the fibers of a holomorphic fibration on an algebraic manifold of dimension $a(M)$ and any meromorphic function on M is the pull-back of a meromorphic function on the basis [57].

The following theorem shows that the fibers of the algebraic reduction are in the same orbit of the pseudogroup of local isometries for any meromorphic almost rigid geometric structure on M . This is a meromorphic version of the celebrated Gromov's open-dense orbit theorem [2, 28] (see also [7, 12, 22, 63]).

Theorem 2.1. *Let M be a connected complex manifold of dimension n which admits a meromorphic almost rigid geometric structure ϕ . Then, there exists a nowhere dense analytic subset S in M , such that $M \setminus S$ is $Is^{loc}(\phi)$ -invariant and the orbits of $Is^{loc}(\phi)$ in $M \setminus S$ are the fibers of a holomorphic fibration of constant rank. The dimension of the fibers is $\geq n - a(M)$, where $a(M)$ is the algebraic dimension of M .*

Recall that $g' = (\phi, g)$ is still almost rigid for any meromorphic (not necessarily almost rigid) geometric structure g on M . This yields to the following:

Corollary 2.2. *If g is a meromorphic geometric structure on M , then, away from a nowhere dense analytic subset, the $Is^{loc}(g)$ -orbits are of dimension $\geq n - a(M)$.*

In particular, if $a(M) = 0$, then g is locally homogeneous outside a nowhere dense analytic subset of M (the Killing Lie algebra \mathcal{G} of $g' = (\phi, g)$ is transitive on an open dense set). Moreover, if ϕ is unimodular and \mathcal{G} is unimodular and simply transitive, then g is locally homogeneous on all of the open dense set where ϕ and g are holomorphic and ϕ is rigid. In this case, if ϕ is unimodular holomorphic and rigid and g is holomorphic, then g is locally homogeneous on M .

Proof. Let ϕ be of order r , given by a map $\phi : R^r(M) \rightarrow Z$. For each positive integer s we consider the s -jet $\phi^{(s)}$ of ϕ . This is a $D^{(r+s)}(\mathbb{C}^n)$ -equivariant meromorphic map $R^{(r+s)}(M) \rightarrow Z^{(s)}$, where $Z^{(s)}$ is the algebraic variety of the s -jets at the origin of holomorphic maps from \mathbb{C}^n to Z . One can find the expression of the (algebraic) $D^{(r+s)}(\mathbb{C}^n)$ -action on $Z^{(s)}$ in [12, 22].

Since ϕ is almost rigid, there exists a nowhere dense analytic subset S' in M , containing the poles of ϕ , and a positive integer s such that two points m, m' in $M \setminus S'$ are in the same orbit of $Is^{loc}(\phi)$ if and only if $\phi^{(s)}$ sends the fibers of $R^{(r+s)}(M)$ above m and m' on the same $D^{(r+s)}(\mathbb{C}^n)$ -orbit in $Z^{(s)}$ [28, 2].

Rosenlicht's theorem (see [49]) shows that there exists a $D^{(r+s)}(\mathbb{C}^n)$ -invariant stratification

$$Z^{(s)} = Z_0 \supset \dots \supset Z_l,$$

such that Z_{i+1} is Zariski closed in Z_i , the quotient of $Z_i \setminus Z_{i+1}$ by $D^{(r+s)}(\mathbb{C}^n)$ is a complex manifold and rational $D^{(r+s)}(\mathbb{C}^n)$ -invariant functions on Z_i separate orbits in $Z_i \setminus Z_{i+1}$.

Consider the open dense $Is^{loc}(\phi)$ -invariant subset U in $M \setminus S'$, such that $\phi^{(s)}$ is of constant rank above U and the image of $R^{(r+s)}(M)|_U$ through $\phi^{(s)}$ lies in $Z_i \setminus Z_{i+1}$, but not in Z_{i+1} . Then the orbits of $Is^{loc}(\phi)$ in U are the fibers of a fibration of constant rank (on the quotient of $Z_i \setminus Z_{i+1}$ by $D^{(r+s)}(\mathbb{C}^n)$). Obviously, $U = M \setminus S$, where S is a nowhere dense analytic subset in M .

If m and n are two points in U which are not in the same $Is^{loc}(\phi)$ -orbit, then the corresponding fibers of $R^{(r+s)}(M)|_U$ are sent by $\phi^{(s)}$ on two distinct $D^{(r+s)}(\mathbb{C}^n)$ -orbits in $Z_i \setminus Z_{i+1}$. By Rosenlicht's theorem there exists a $D^{(r+s)}(\mathbb{C}^n)$ -invariant rational function $F : Z_i \setminus Z_{i+1} \rightarrow \mathbb{C}$, which takes distinct values at these two orbits.

The meromorphic function $F \circ \phi^{(s)} : R^{(r+s)}(M) \rightarrow \mathbb{C}$ is $D^{(r+s)}(\mathbb{C}^n)$ -invariant and descends in a $Is^{loc}(\phi)$ -invariant meromorphic function on M which takes distinct values at m and at n .

Consequently, the complex codimension in U of the $Is^{loc}(\phi)$ -orbits is $\leq a(M)$, which finishes the proof. \square

We deduce now the corollary:

Proof. It is convenient to put together ϕ and g in some extra geometric structure $g' = (\phi, g)$. Now g' is a meromorphic almost rigid geometric structure and theorem 2.1 shows that the complex dimension of a generic

$Is^{loc}(g')$ -orbit is $\geq n - a(M)$. As each $Is^{loc}(g')$ -orbit is contained in a $Is^{loc}(g)$ -orbit, the result follows also for g .

If $a(M) = 0$, the Killing Lie algebra \mathcal{G} of g' is transitive on a maximal open dense subset U in M . Suppose now that ϕ is unimodular and also that \mathcal{G} is unimodular. The other inclusion being trivial, it is enough to show that U contains the maximal subset of M where ϕ and g are holomorphic and ϕ is rigid.

Pick up a point m in the previous subset. We want to show that m is in U . Since $g' = (\phi, g)$ is holomorphic and rigid in the neighborhood of m , it follows that m admits an open neighborhood U_m in M such that any local holomorphic Killing field of g' defined on a connected open subset in U_m extends on all of U_m [48, 1].

Since \mathcal{G} acts transitively on U , choose local linearly independent Killing fields X_1, \dots, X_n on a connected open set included in $U \cap U_m$. As ϕ is unimodular, it determines a holomorphic volume form vol on U_m (if necessary we restrict to a smaller U_m) which is preserved by $Is^{loc}(g')$. But $Is^{loc}(g')$ acts transitively on $U \cap U_m$ and \mathcal{G} is supposed to be unimodular. This implies that the function $vol(X_1, \dots, X_n)$ is \mathcal{G} -invariant and, consequently, a (non-zero) constant on $U \cap U_m$.

On the other hand X_1, \dots, X_n extend in some holomorphic Killing fields $\tilde{X}_1, \dots, \tilde{X}_n$ defined on all of U_m . The holomorphic function $vol(\tilde{X}_1, \dots, \tilde{X}_n)$ is a non-zero constant on U_m : in particular, $\tilde{X}_1(m), \dots, \tilde{X}_n(m)$ are linearly independent. We proved that \mathcal{G} acts transitively in the neighborhood of m and thus $m \in U$. □

3. CLASSIFICATION RESULTS

Parallelisations of the tangent bundle. We begin with the classification of compact complex manifolds which admit a holomorphic parallelisation of the tangent bundle [60].

Theorem 3.1. *(Wang) Let M be a compact connected complex manifold with a holomorphic parallelisation of the tangent bundle. Then M is a quotient $\Gamma \backslash G$, where G is a connected simply connected complex Lie group and Γ is a uniform lattice in G .*

Moreover, M is Kaehler if and only if G is abelian (and M is a complex torus).

Proof. Let n be the complex dimension of M and consider X_1, X_2, \dots, X_n global holomorphic vector fields on M which span TM . Then, for all $1 \leq i, j \leq n$, we have

$$[X_i, X_j] = f_1^{ij} X_1 + f_2^{ij} X_2 + \dots + f_n^{ij} X_n,$$

with f_k^{ij} holomorphic functions on M . Since M is compact, these functions have to be constant and, consequently, X_1, X_2, \dots, X_n generate a n -dimensional Lie algebra \mathcal{G} which acts simply transitively on M . By Lie's theorem there exists a unique connected simply connected complex Lie group G corresponding to \mathcal{G} .

In particular, the holomorphic parallelisation is locally homogenous, locally modelled on the parallelisation given by translation-invariant vector fields on the Lie group G .

Since M is compact, the X_i are complete and they define a holomorphic simply transitive action of G on M . Hence M is a quotient of G by a cocompact discrete subgroup Γ in G .

Assume now M is Kaehler. Then, any holomorphic form on M has to be closed [27]. Consider $\omega_1, \omega_2, \dots, \omega_n$ the dual basis with respect to X_1, X_2, \dots, X_n . The one forms ω_i are holomorphic and translation-invariant on G (they descend on $\Gamma \backslash G$). The Lie-Cartan formula

$$d\omega_i(X_j, X_k) = -\omega_i([X_j, X_k]),$$

shows that the one forms ω_i are all closed if and only if G is abelian and thus M is a complex torus. \square

Holomorphic Riemannian metrics. As in the real case, a holomorphic Riemannian metric on M gives rise to a covariant differential calculus, i.e. a Levi-Civita (holomorphic) affine connection, and to geometric features: curvature tensors, geodesic (complex) curves [40, 41].

Locally, a holomorphic Riemannian metric has the expression $\Sigma g_{ij}(z) dz_i dz_j$, where $(g_{ij}(z))$ is a complex invertible symmetric matrix depending holomorphically on z . The standard example is that of the global flat holomorphic Riemannian metric $dz_1^2 + dz_2^2 + \dots + dz_n^2$ on \mathbb{C}^n . This metric is translation-invariant and thus descends to any quotient of \mathbb{C}^n by a lattice. Hence complex tori possess (flat) holomorphic Riemannian metrics. This is however a very special situation since, contrary to real case, *only few compact complex manifolds admit holomorphic Riemannian metrics*. In fact, Yau's proof

of the Calabi conjecture shows that, up to finite unramified covers, complex tori are the only compact *Kaehler* manifolds admitting holomorphic Riemannian metrics [34].

However, very interesting examples, constructed by Ghys in [25], do exist on 3-dimensional complex non-Kaehler manifolds and deserve classification. Notice that parallelisable manifolds admit holomorphic Riemannian metrics coming from left invariant holomorphic Riemannian metrics on G (which can be constructed by left translating any complex nondegenerate quadratic form defined on the Lie algebra \mathcal{G}).

Ghys' examples of 3-dimensional compact complex manifolds endowed with holomorphic Riemannian metrics are obtained by deformation of the complex structure on parallelisable manifolds $\Gamma \backslash SL(2, \mathbb{C})$ [25]. They are *nonstandard*, meaning they do not admit parallelisable manifolds as finite unramified covers. Those nonstandard examples will be described latter on.

A first obstruction to the existence of a holomorphic Riemannian metric on a compact complex manifold is the vanishing of its first Chern class. Indeed, a holomorphic Riemannian metric on M provides an isomorphism between TM and T^*M . In particular, the canonical bundle K is isomorphic to the anti-canonical bundle K^{-1} and thus K^2 is trivial. This means that the first Chern class of M vanishes and, up to a double unramified cover, M possesses a holomorphic volume form.

The following proposition describes holomorphic Riemannian metrics on parallelisable manifolds:

Proposition 3.2. *Let $M = \Gamma \backslash G$ a compact parallelisable manifold, with G a simply connected complex Lie group and Γ a uniform lattice in G . Then, any holomorphic Riemannian metric g on M comes from a nondegenerate complex quadratic form on the Lie algebra \mathcal{G} of G . In particular, the pull-back of g is left invariant on the universal cover G (and g is locally homogeneous on M).*

Moreover, any compact parallelisable 3-manifold admits a holomorphic Riemannian metric of constant sectional curvature. The metric is flat exactly when G is solvable.

Proof. Consider X_1, X_2, \dots, X_n the fundamental vector fields corresponding to the locally free G -action on M . Let g be a holomorphic Riemannian metric on M and denote also by g the associated complex symmetric bilinear form. Then $g(X_i, X_j)$ is a holomorphic function on M and thus constant, for all $1 \leq i, j \leq n$. This implies that g comes from a left invariant holomorphic Riemannian metric on G .

Assume now G is a connected simply connected complex unimodular Lie group of dimension 3. We have only four such Lie groups: \mathbb{C}^3 , the complex Heisenberg group, the complex SOL group and $SL(2, \mathbb{C})$ [35]. Note that the group SOL is the complexification of the affine isometry group of the Minkowski plane $\mathbb{R}^{1,1}$ or equivalently the isometry group of \mathbb{C}^2 endowed with its flat holomorphic Riemannian metric.

We begin with the case $G = \mathbb{C}^3$. We have seen that \mathbb{C}^3 admits a flat translation-invariant holomorphic Riemannian metric. The isometry group

of this metric is $O(3, \mathbb{C}) \ltimes \mathbb{C}^3$. We recall here that $O(3, \mathbb{C})$ and $SL(2, \mathbb{C})$ are locally isomorphic.

Consider now the case $G = SL(2, \mathbb{C})$. The Killing form of the Lie algebra $sl(2, \mathbb{C})$ is a nondegenerate complex quadratic form which endows $SL(2, \mathbb{C})$ with a left invariant holomorphic Riemannian metric of constant sectional curvature. Since the Killing quadratic form is invariant by the adjoint representation, the isometry group contains also all right translations. In fact, the connected component of the isometry group is $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ acting by left and right translation. On the other hand, $\Gamma \backslash SL(2, \mathbb{C})$ doesn't admit flat torsion-free affine connections (neither flat holomorphic Riemannian metrics) [19].

It is an easy exercise to exhibit in the isometry group $O(3, \mathbb{C}) \ltimes \mathbb{C}^3$ of the flat holomorphic Riemannian space, copies of the complex Heisenberg group and of the complex SOL group which act simply transitively [19]. Thus the flat holomorphic Riemannian space also admits models which are given by left invariant metrics on the Heisenberg group and on the SOL group. One can get explicit expression of these holomorphic Riemannian metrics by complexification of flat left invariant Lorentz metrics on the real Heisenberg and SOL groups [51, 52].

On the other hand, there exists no 3-dimensional solvable subgroups in $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ acting with an open orbit on $SL(2, \mathbb{C})$. It follows that the compact quotients of 3-dimensional solvable groups don't admit holomorphic Riemannian metrics of non-zero constant sectional curvature. \square

Ghys nonstandard examples. As above, for any cocompact lattice Γ in $SL(2, \mathbb{C})$, the quotient $M = \Gamma \backslash SL(2, \mathbb{C})$ admits a holomorphic Riemannian metric of non-zero constant sectional curvature. It is convenient to consider M as a quotient of $S_3 = O(4, \mathbb{C})/O(3, \mathbb{C}) = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})/SL(2, \mathbb{C})$ by Γ , seen as a subgroup of $SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ by the trivial embedding $\gamma \in \Gamma \mapsto (\gamma, 1) \in SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$.

New interesting examples of manifolds admitting holomorphic Riemannian metrics of non-zero constant sectional curvature have been constructed in [25] by deformation of this embedding of Γ .

Those deformations are constructed choosing a morphism $u : \Gamma \rightarrow SL(2, \mathbb{C})$ and considering the embedding $\gamma \mapsto (\gamma, u(\gamma))$. Algebraically, the action is given by:

$$(\gamma, m) \in \Gamma \times SL(2, \mathbb{C}) \rightarrow \gamma m u(\gamma^{-1}) \in SL(2, \mathbb{C}).$$

It is proved in [25] that, for u close enough to the trivial morphism, Γ acts properly (and freely) on $S_3 (\cong SL(2, \mathbb{C}))$ such that the quotient M_u is a complex compact manifold (covered by $SL(2, \mathbb{C})$) admitting a holomorphic Riemannian metric of non-zero constant sectional curvature. *In general, these examples do not admit parallelisable manifolds as finite covers.*

Note that left invariant holomorphic Riemannian metrics on $SL(2, \mathbb{C})$ which are not right invariant, in general, will not descend on M_u .

Let us notice that despite this systematic study in [25], there are still many open questions regarding these examples (including the question of completeness). A real version of this study is in [39, 26, 53].

Dimension 3. The classification of complex compact manifolds admitting holomorphic Riemannian metrics is simple in complex dimension 2 (see section 5). An important step toward the classification in dimension 3 was made in [17] with the following result:

Theorem 3.3. *Any holomorphic Riemannian metric on a compact connected complex 3-manifold is locally homogeneous. More generally, if a compact connected complex 3-manifold M admits a holomorphic Riemannian metric, then any holomorphic geometric structure of affine type on M is locally homogeneous.*

Observe that the previous result is trivial in dimension 2, since the sectional curvature is a holomorphic function and thus constant on compact complex surfaces: this implies the local homogeneity [61]. In dimension 3, the sectional curvature will be, in general, a non constant meromorphic function on the 2-grassmanian of the holomorphic tangent space with poles at the degenerate planes.

Thanks to theorem 3.3, our manifold M is locally modelled on a $(G, G/I)$ -geometry in Thurston's sense [55], where I is a closed subgroup of the Lie group G such that the G -action on G/I preserves some holomorphic Riemannian metric (see [18] for details and notice that the local Killing Lie algebra of the holomorphic Riemannian metric is the Lie algebra of G). In this context we have a *developing map* from the universal cover of M into G/I which is a local diffeomorphism and which is equivariant with respect to the action of the fundamental group on \tilde{M} by deck transformations and on G/I via the *holonomy morphism* $\rho : \pi_1(M) \rightarrow G$ [55].

Recall that the $(G, G/I)$ -geometry is called *complete* if the developing map is a diffeomorphism and, consequently, $\Gamma = \rho(\pi_1(M))$ acts properly on G/I such that M is a compact quotient $\Gamma \backslash G/I$.

A second step was achieved in [18] were, in a commun work with Zeghib, we proved the following result which can be seen, in particular, as a completeness result in the case where G is solvable.

Theorem 3.4. *Let M be a compact connected complex 3-manifold which admits a (locally homogeneous) holomorphic Riemannian metric g . Then:*

(i) *If the Killing Lie algebra of g has a nontrivial semi-simple part, then it preserves some holomorphic Riemannian metric on M with constant sectional curvature.*

(ii) *If the Killing Lie algebra of g is solvable, then, up to a finite unramified cover, M is a quotient either of the complex Heisenberg group, or of the complex SOL group by a lattice.*

Remark 3.5. *If g is flat, its Killing Lie algebra corresponds to $O(3, \mathbb{C}) \times \mathbb{C}^3$, which has nontrivial semi-simple part. Thus, flat holomorphic Riemannian metrics on complex tori are part of point (i) in theorem 3.4.*

The point (ii) of the previous theorem is not only about completeness, but also gives a rigidity result in Bieberbach's sense [61]: G contains a 3-dimensional closed subgroup H (either isomorphic to the complex Heisenberg group, or to the complex SOL group) which acts simply transitively (and so identifies) with G/I and (up to a finite index) the image Γ of the holonomy morphism lies in H . It follows that, up to a finite cover, M identifies with $\Gamma \backslash H$.

Since the Heisenberg and SOL groups admit left invariant flat holomorphic Riemannian metrics this leads to:

Corollary 3.6. *If a compact connected complex 3-manifold M admits a holomorphic Riemannian metric, then, up to a finite unramified cover, M admits a holomorphic Riemannian metric of constant sectional curvature.*

Theorem 3.4 does not end the story, even in dimension 3, essentially because of remaining *completeness* questions, and those on the algebraic structure of the fundamental group.

It remains to classify the compact complex 3-manifolds endowed with a holomorphic Riemannian metric of constant sectional curvature.

Flat case. In this case M admits a $(O(3, \mathbb{C}) \times \mathbb{C}^3, \mathbb{C}^3)$ -geometry. The challenge remains:

- 1) *Markus conjecture:* Is M complete?
- 2) *Auslander conjecture:* Assuming M complete, is Γ solvable?

Note that these questions are settled in the setting of (real) flat Lorentz manifolds [13, 24], but unsolved for general (real) pseudo-Riemannian metrics. The real part of the holomorphic Riemannian metric is a (real) pseudo-Riemannian metric of signature $(3, 3)$ for which both previous conjectures are still open.

Nonflat case. In this case $G = SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ and $I = SL(2, \mathbb{C})$ is diagonally embedded in the product. The completeness of this geometry on compact complex manifolds is still an open problem, despite a local result of Ghys [25]. Recall that the real analogous of this problem, i.e. the

completeness of compact manifolds endowed with Lorentz metrics of negative constant sectional curvature, was solved in [37], but the proof cannot generalize to other signatures.

Higher dimension. One interesting problem in differential geometry is to decide whether a given homogeneous space G/I possesses or not a compact quotient. A more general related question is to decide whether there exists compact manifolds locally modelled on $(G, G/I)$ (see, for instance, [6, 8, 38]).

The case $I = 1$, or more generally I compact, reduces to the classical question of existence of cocompact lattices in Lie groups. For homogeneous spaces of non-Riemannian type (i.e. I non compact) the problem is much harder.

The case $S_n = O(n+1, \mathbb{C})/O(n, \mathbb{C})$ is a geometric situation where these questions can be tested. It turns out that compact quotients of S_n are known to exist only for $n = 1, 3$ or 7 . We discussed the case $n = 3$ above, and the existence of a compact quotient of S_7 was proved in [38]. Here, we dare ask with [38]:

Conjecture 3.7. [38] S_n has no compact quotients, for $n \neq 1, 3, 7$.

A stronger version of this question was proved in [6] for S_n , if n has the form $4m+1$, with $m \in \mathbb{N}$.

Keeping in mind our geometric approach, we generalize the question to manifolds locally modelled on S_n . More exactly:

Conjecture 3.8. [18] *A compact complex manifold endowed with a holomorphic Riemannian metric of constant non-vanishing curvature is complete. In particular, such a manifold has dimension 3 or 7.*

4. APPLICATIONS TO SIMPLY CONNECTED MANIFOLDS

Remark first that theorem 3.3 has the following direct consequence:

Corollary 4.1. *A compact connected simply connected 3-manifold admits no holomorphic Riemannian metrics.*

Proof. Assume, by contradiction, (M, g) being as in the statement of the corollary. Then theorem 3.3 implies that g is locally homogeneous. Since M is simply connected, the local Killing fields of the Killing algebra \mathcal{G} extend on all of M : the unique connected simply connected complex Lie group G associated to \mathcal{G} acts isometrically and transitively on M . Then M is a homogeneous space G/H . Moreover, up to a double cover, G/H admits a holomorphic volume form vol coming from the holomorphic Riemannian metric.

Take X_1, X_2, X_3 three global Killing fields on M which are linearly independent at some point. Since $vol(X_1, X_2, X_3)$ is a holomorphic function on M , it is a non-zero constant and, consequently, X_1, X_2, X_3 are linearly independent on M . Hence Wang's theorem implies that M is a quotient of a three dimensional connected simply connected complex Lie group G_1 by a discrete subgroup. Since M is simply connected, this discrete subgroup has to be trivial and M identifies with G_1 . But there is no compact simply connected complex Lie group: a contradiction. \square

Theorem 4.2. *A compact connected simply connected complex n -manifold without nonconstant meromorphic functions admits no holomorphic unimodular rigid geometric structures.*

Proof. Assume, by contradiction, that (M, ϕ) verifies the hypothesis. Since $a(M) = 0$, theorem 2.1 implies ϕ is locally homogeneous on an open dense set U . As M is simply connected, elements in the Killing algebra \mathcal{G} extend on all of M : the connected simply connected complex Lie group G associated to \mathcal{G} acts isometrically on M with an open dense orbit. The open dense orbit U identifies with a homogeneous space G/H , where H is a closed subgroup of G .

Consider X_1, X_2, \dots, X_n global Killing fields on M which are linearly independent at some point of the open orbit U . As before, $vol(X_1, X_2, \dots, X_n)$ is a non-zero constant, where vol is the holomorphic volume form associated to ϕ . Thus the X_i give a holomorphic parallelisation of TM and Wang's theorem enables us to conclude as in the previous proof. \square

For non-unimodular rigid geometric structures we have the following less precise:

Theorem 4.3. [15] *Let M be a compact connected simply connected complex n -manifold without nonconstant meromorphic functions and admitting a holomorphic rigid geometric structure ϕ . Then M is a equivariant compactification of $\Gamma \backslash G$, where Γ is a discrete non cocompact subgroup in a complex Lie group G .*

Proof. Since $a(M) = 0$, theorem 2.1 implies ϕ is locally homogeneous on an open dense set U . As before, the extension property of local Killing fields implies U is a complex homogeneous space G/H , where G is a connected simply connected complex Lie group and H is a closed subgroup in G .

We show now that H is a discrete subgroup of G . Assume by contradiction the Lie algebra of H is nontrivial. Take at any point $u \in U$, the isotropy subalgebra \mathcal{H}_u (i.e. the Lie subalgebra of Killing fields vanishing at u). Remark that $\mathcal{H}_{gu} = Ad(g)\mathcal{H}_u$, for any $g \in G$ and $u \in U$, where Ad is the adjoint representation.

The map $u \rightarrow \mathcal{H}_u$ is a meromorphic map from M to the grassmanian of d -dimensional vector spaces in \mathcal{G} . But M doesn't admit any nonconstant meromorphic function and this map has to be constant. It follows that \mathcal{H}_u is $Ad(G)$ -invariant and H is a normal subgroup of G : a contradiction, since the G -action on M is faithful. Thus G is of dimension n and H identifies to a lattice Γ in G .

As M is simply connected, U has to be strictly contained in M and M is a equivariant compactification of $\Gamma \backslash G$. □

We don't know if such equivariant compactifications of $\Gamma \backslash G$ admit equivariant holomorphic rigid geometric structures, but the previous result has the following application.

Recall that a well-known open question asks whether the 6-dimensional real sphere S^6 admits complex structures or not. In this context, we have the following:

Corollary 4.4. *If S^6 admits a complex structure M , then M admits no holomorphic rigid geometric structures.*

Proof. The starting point of the proof is a result of [11] where it is proved that M doesn't admit nonconstant meromorphic functions. If M supports holomorphic rigid geometric structures, then theorem 4.3 implies that M is a equivariant compactification of a homogeneous space. This is in contradiction with the main theorem of [30]. \square

As for *Kaehler manifolds* we have proved in [15] the following more precise:

Theorem 4.5. *Let M be a compact connected Kaehler manifold endowed with a holomorphic unimodular rigid geometric structure ϕ of affine type. Then, up to a finite unramified cover, M is a complex torus (quotient of \mathbb{C}^n by a lattice) and ϕ is translation-invariant.*

The proof is done in two steps. First we prove that ϕ is locally homogeneous. Then we use a splitting theorem [5] which asserts that such compact Kaehler manifolds with a holomorphic volume form (Calabi-Yau manifolds) are biholomorphic, up to a finite cover, to a direct product of a complex torus and a compact *simply connected* Kaehler manifold with a holomorphic volume form. Starting with ϕ and using the product structure, we construct a holomorphic unimodular rigid geometric structure on the simply connected factor which is locally homogeneous. We conclude as in the proof of theorem 4.2 that the simply connected factor is trivial.

Recently, we adapted this proof in [20] to all holomorphic Cartan geometries of algebraic type on Calabi-Yau manifolds. A similar result was independently proved in [10].

One can also find a classification of certain holomorphic G -structures of order one on *uniruled* projective manifolds in [31].

5. APPLICATIONS TO COMPLEX SURFACES

Theorem 5.1. *Let S be a compact complex surface endowed with a holomorphic unimodular rigid geometric structure ϕ of affine type. Then the Killing Lie algebra of ϕ is nontrivial.*

The assumption on ϕ to be affine is essential:

Remark 5.2. *If S is a compact complex algebraic surface with trivial canonical bundle (for example, a complex torus or an algebraic K3 surface), the geometric structure given by a holomorphic volume forme on S together with a holomorphic embedding of S in a complex projective space doesn't admit any nontrivial local isometry. However this geometric structure is not of affine type.*

Proof. The proof is a simply corollary of theorems 4.5 and 2.1. Indeed, theorem 2.1 implies the Killing Lie algebra of ϕ is trivial only if the algebraic dimension $a(S)$ equals 2. But in this case S is algebraic [4] and thus Kaehler. Then theorem 4.5 applies and the Killing Lie algebra is transitive on S and hence of dimension at least 2. \square

Recall that a complex surface is called *minimal* if it does not contain any copy of $P^1(\mathbb{C})$ with self-intersection -1 (see [4], page 91). Then we have:

Theorem 5.3. *Let S be a compact minimal complex surface which is not biholomorphic to a non-algebraic K3 surface or to a non-affine Hopf surface. Then S admits holomorphic rigid geometric structures.*

Remark 5.4. *By definition, a complex algebraic manifold admits an embedding in a complex projective space, which was seen to be a holomorphic rigid geometric structure (of order zero).*

Proof. By the previous remark, it remains to consider the case of non-algebraic complex surfaces. Then the Enriques-Kodaira classification shows that, up to a finite unramified cover, any minimal non-algebraic complex compact surface is biholomorphic to one of the following complex surfaces: a complex tori, a Hopf surface, an Inoue surface, a K3 surface, a principal elliptic principal bundle over an elliptic curve or a principal elliptic bundle over a Riemann surface of genus $g \geq 2$, with odd first Betti number (see [4], p. 244).

However, it is known that complex tori, Inoue surfaces, affine Hopf surfaces, principal elliptic bundles over elliptic curves and principal elliptic bundles over a Riemann surface of genus $g \geq 2$, with odd first Betti number, admit flat holomorphic affine connections [34, 36, 43, 54, 58, 59]. \square

The classification of compact complex surfaces admitting holomorphic affine connections (see [34, 36, 43, 58, 59]) implies the following result:

Theorem 5.5. *Let S be a compact complex surface admitting a holomorphic unimodular affine connection. Then, up to a finite unramified cover, either S is a complex torus and the connection is translation-invariant, or S is an*

elliptic principal bundle over an elliptic curve and the connection is locally modelled on a translation-invariant connection on a complex torus.

Since any complex manifold endowed with a holomorphic Riemannian metric inherits a holomorphic (unimodular) affine (Levi-Civita) connection [40, 41], this easily implies the following [17]:

Corollary 5.6. *Let S be a complex compact surface admitting a holomorphic Riemannian metric g . Then, up to a finite unramified cover, S is a complex torus and g is (flat) translation-invariant.*

In [21] we classified the local geometry of all torsion-free holomorphic affine connections ∇ on compact complex surfaces. In particular, we proved that either ∇ is locally homogeneous, locally isomorphic to a translation invariant connection on \mathbb{C}^2 , or ∇ is a nonflat connection on a principal elliptic bundle over a Riemann surface of genus $g \geq 2$ with odd first Betti number and the local Killing Lie algebra of ∇ is one-dimensional generated by the fundamental vector field of the principal fibration. In all cases, ∇ is projectively flat.

As a consequence we proved in [21] the following:

Theorem 5.7. *Normal holomorphic projective connections on compact complex surfaces are flat.*

Inoue surfaces. Recall that Inoue surface are compact complex surfaces in the class VII_0 , which are not Hopf, and have a vanishing second Betti number [4, 33].

In [34, 36] it is proved that any Inoue surface admits a (unique) flat torsion-free holomorphic affine connection. Here we prove the following:

Theorem 5.8. *Holomorphic geometric structures on Inoue surfaces are locally homogeneous.*

Remark 5.9. *Inoue surfaces don't admit nonconstant meromorphic functions [4].*

Proof. Let τ be a holomorphic geometric structure on a Inoue surface S .

Let ∇_0 be a flat torsion-free holomorphic affine connection on S . We prove that the holomorphic rigid geometric structure $\tau' = (\tau, \nabla_0)$ is locally homogeneous. Since $a(S) = 0$, theorem 2.1 implies τ' is locally homogeneous on some maximal open dense set $S \setminus E$, where E is a compact analytic subset of S of positive codimension.

We want to show that E is empty.

We prove first that, up to a double cover of S , the subset E is a smooth submanifold in S (this is always true if S is a finite set; but here S might have components of complex dimension one).

Assume, by contradiction, that E is not a smooth submanifold in S .

Choose $p \in E$ a singular point in E . In particular, p is not isolated in E , but p is isolated among the singular points of E . Since $I_S^{loc}(\tau')$ preserves E , it has to preserve the set of its singular points and thus it fixes p . Consequently any local Killing field defined in the neighborhood of p , has to vanish at p .

Denote by \mathcal{G} the Lie algebra of local Killing fields in the neighborhood of p . Since \mathcal{G} acts transitively on an open set, its dimension is at least 2.

Each element of \mathcal{G} preserves ∇ and fixes p . In exponential coordinates the \mathcal{G} -action in the neighborhood of p is linear. This gives an embedding of \mathcal{G} in the Lie algebra of $GL(2, \mathbb{C})$ (the image of the isotropy representation at p). In particular, \mathcal{G} is of dimension ≤ 4 .

Suppose first that \mathcal{G} is of dimension 2. The corresponding subgroups of $GL(2, \mathbb{C})$ are conjugated either to the group of diagonal matrices, or to one of the following subgroups $\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$, with $a \in \mathbb{C}^*$ and $b \in \mathbb{C}$, $\begin{pmatrix} 1 & m \\ 0 & n \end{pmatrix}$, with $m \in \mathbb{C}$ and $n \in \mathbb{C}^*$, or $\begin{pmatrix} m' & n' \\ 0 & 1 \end{pmatrix}$, with $m' \in \mathbb{C}^*$ and $n' \in \mathbb{C}$.

In the first case, the invariant closed subset E lies, in exponential coordinates, in the union of the two eigendirections. In the last two cases, E locally lies in the invariant line $y = 0$. In all situations, up to a double cover of S , the analytic set E is smooth: a contradiction.

We settle now the case where \mathcal{G} is of dimension 3 or 4. Then the image of \mathcal{G} by the isotropy representation in p is conjugated in $GL(2, \mathbb{C})$ to one of the following subgroups: $SL(2, \mathbb{C})$, $GL(2, \mathbb{C})$ or the group of invertible upper triangular matrices. But $GL(2, \mathbb{C})$ and $SL(2, \mathbb{C})$ don't admit invariant subsets other than p , which will be an isolated point in E : impossible.

In the last situation, E locally coincides, as before, with the unique invariant line and it is smooth.

Up to a double cover, E is a holomorphic submanifold in S . If E admits a component of dimension one, then this component will be a union of closed curves. But Inoue surfaces contain no curves [33].

This proves that E is a finite set. Assume, by contradiction, that E is not empty and consider $p \in E$.

The previous arguments show that the Lie algebra \mathcal{G} is isomorphic either to $sl(2, \mathbb{C})$, or to $gl(2, \mathbb{C})$.

Assume first $\mathcal{G} = sl(2, \mathbb{C})$. As before, the local action of \mathcal{G} in the neighborhood of p is conjugated to the action of \mathcal{G} on $T_p S$ which coincides with the standard linear action of $sl(2, \mathbb{C})$ on \mathbb{C}^2 . This action has two orbits: the point p and $\mathbb{C}^2 \setminus \{p\}$.

The stabilizer H in $SL(2, \mathbb{C})$ of a non-zero vector $x \in T_p S$ is conjugated to the following one-parameter subgroup of $SL(2, \mathbb{C})$: $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$, with $b \in \mathbb{C}$. Observe that the action of G on G/H preserves a nontrivial holomorphic vector field. The expression of this vector field in linear coordinates (z_1, z_2) in the neighborhood of p is $z_1 \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_2}$.

Since $S \setminus E$ is locally modelled on $(G, G/H)$, this vector field is well defined on $S \setminus E$. But, E is of complex codimension 2 in S and the extension theorem of Hartogs [27] shows that the vector field extends to a global nontrivial holomorphic vector field X on S . The vector field X is \mathcal{G} -invariant on $S \setminus E$ and hence on all of S . Since the isotropy action of $SL(2, \mathbb{C})$ at p doesn't preserve nontrivial vectors in $T_p S \simeq \mathbb{C}^2$, it follows that $X(p) = 0$. This is impossible, since Inoue surfaces don't admit *singular* nontrivial holomorphic vector fields [33].

The proof is the same in the case $\mathcal{G} = gl(2, \mathbb{C})$. □

Meromorphic affine connections. In the following theorem we describe some meromorphic affine connections on non-Kaehler principal elliptic bundles over a compact Riemann surface of genus $g \geq 2$.

Theorem 5.10. *Let S be a principal elliptic bundle over a compact Riemann surface Σ of genus $g \geq 2$, with odd first Betti number.*

i) A meromorphic torsion-free affine connection ∇ on S is invariant by the principal fibration (i.e. the fundamental vector field of the principal fibration is a Killing field) if and only if the set of its poles intersects only a finite set of fibers.

ii) The space of those previous connections admitting simple poles on a single fiber, above a point $\xi_0 \in \Sigma$, is a complex affine space of dimension $5g+1$ and its underlying vector space identifies with $H_{\xi_0}^0(K_{\Sigma}^2) \times M_{2,\xi_0}^2$, where $H_{\xi_0}^0(K_{\Sigma}^2)$ is the vector space of meromorphic quadratic differentials on Σ with a single pole of order at most two at ξ_0 and M_{2,ξ_0}^2 is the vector space of quasimodular forms of weight 2 on Σ with a single simple pole at ξ_0 .

iii) If ∇ is generic among the meromorphic affine connections which satisfy ii), then the local Killing Lie algebra of ∇ is generated by the fundamental vector field of the principal fibration.

iv) The meromorphic affine connections which satisfy ii) are projectively flat away from the poles.

Recall that a quasimodular form of weight 2 on Σ is a holomorphic function f defined on the upper half-plane H such that, for some $K \in \mathbb{C}$, we have $f(\xi) = f(\gamma\xi)(c_\gamma\xi + d_\gamma)^{-2} - K(c_\gamma\xi + d_\gamma)^{-1}$, for all $\gamma = \begin{pmatrix} a_\gamma & b_\gamma \\ c_\gamma & d_\gamma \end{pmatrix} \in$

$SL(2, \mathbb{R})$ in the fundamental group of Σ [3, 62]. If $K = 0$, then f is a classical modular form. Theorem 9 in [3] shows that the space of quasimodular forms of weight 2 on Σ which admit a simple pole in a single orbit is a complex vector space of dimension $g + 1$ (i.e. the quotient of the space of such quasimodular forms over those which are modular is one-dimensional).

Remark 5.11. *The proof below shows that S admits a (flat) holomorphic affine connection. Theorem 2.1 applies and, since any meromorphic function on S is a pull-back of a meromorphic function on Σ [4], the orbits of the pseudogroup of local isometries of any meromorphic geometric structure on S contain the fibers of S .*

Proof. i) The projection of the set of poles of ∇ on Σ is a closed analytic set which is either a finite set of points, or all of Σ . If this projection is Σ and ∇ is invariant by the principal fibration, then each point of S is a pole, which is impossible. This proves the easy sense of the implication.

We will describe now the space of connections ∇ , such that only a finite set of fibers contain poles of ∇ . Up to a finite unramified cover and a finite quotient, S admits holomorphic affine structures (i.e. flat torsion-free holomorphic affine connections) which can be built in the following way [36]:

Consider Γ a discrete torsion-free subgroup in $PSL(2, \mathbb{R})$ such that $\Sigma = \Gamma \backslash H$, with H the upper-half plane. Take any holomorphic projective structure on Σ , its developping map $\tau : H \rightarrow P^1(\mathbb{C})$ and its holonomy morphism $\rho : \Gamma \rightarrow PSL(2, \mathbb{C})$. This embedding of Γ into $PSL(2, \mathbb{C})$ lifts to $SL(2, \mathbb{C})$ (this follows from the fact that orientable (real) closed 3-manifolds have a trivial second Stiefel-Whitney class [47]). Choose such a lift and consider Γ

as a subgroup of $SL(2, \mathbb{C})$. Denote by $W = \mathbb{C}^2 \setminus \{0\}$ the \mathbb{C}^* -tautological bundle over $P^1(\mathbb{C})$. The canonical affine structure of \mathbb{C}^2 induces a Γ -invariant affine structure on W and hence a Γ -invariant holomorphic affine structure on the pull-back $\tau^*(W) \simeq \mathbb{C}^* \times H$. The Γ -action on $\tau^*(W)$ comes from the action by deck transformation on H and from the ρ -action on W .

The previous holomorphic affine structure on $\tau^*(W)$ is also invariant by the homotheties in the fibers (which commute with the Γ -action). Consider now $\Delta \simeq \mathbb{Z}$ a lattice in \mathbb{C}^* which acts by multiplication on the fibers of $\tau^*(W)$ and take the quotient of $\tau^*(W)$ by $\Delta \times \Gamma$. The quotient is a principal elliptic bundle over Σ , with fiber $\Delta \setminus \mathbb{C}^*$, biholomorphic to S .

The affine structure inherited by the universal cover $\mathbb{C} \times H$ of S is the pull-back of the previous affine structure on $\mathbb{C}^* \times H$ by the map

$$\mathbb{C} \times H \rightarrow \mathbb{C}^* \times H$$

$$(z, \xi) \rightarrow (e^z, \xi).$$

In the following we will consider the flat torsion-free holomorphic affine connection ∇_0 on S given by the case where τ is the standard embedding of H into $P^1(\mathbb{C})$. In this case, the action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \subset SL(2, \mathbb{R})$ on the universal cover $\mathbb{C} \times H$ of S is easily seen to be given by [36]:

$$\gamma(z, \xi) = (z + \log(c\xi + d), \gamma\xi), \forall (z, \xi) \in \mathbb{C} \times H,$$

where \log is a determination of the logarithm and the γ -action on H comes from the standard action of $SL(2, \mathbb{R})$ on H .

The difference $\nabla - \nabla_0$ is a meromorphic $(2, 1)$ -tensor ω on S , or, equivalently, a $\Delta \times \Gamma$ -invariant meromorphic $(2, 1)$ -tensor $\tilde{\omega}$ on the universal cover $\mathbb{C} \times H$. Moreover, $\tilde{\omega}$ is invariant by $t(z, \xi) = (z + 2i\pi, \xi)$.

Then we have,

$$\begin{aligned} \tilde{\omega} = & f_{11}(z, \xi)dz \otimes dz \otimes \frac{\partial}{\partial z} + f_{12}(z, \xi)dz \otimes d\xi \otimes \frac{\partial}{\partial z} + f_{21}(z, \xi)dz \otimes dz \otimes \frac{\partial}{\partial \xi} + f_{22}(z, \xi)dz \otimes d\xi \otimes \frac{\partial}{\partial \xi} + \\ & g_{11}(z, \xi)d\xi \otimes dz \otimes \frac{\partial}{\partial z} + g_{12}(z, \xi)d\xi \otimes d\xi \otimes \frac{\partial}{\partial z} + g_{21}(z, \xi)d\xi \otimes dz \otimes \frac{\partial}{\partial \xi} + g_{22}(z, \xi)d\xi \otimes d\xi \otimes \frac{\partial}{\partial \xi}, \end{aligned}$$

with f_{ij}, g_{ij} meromorphic functions on $\mathbb{C} \times H$, and $f_{ij}(\cdot, \xi), g_{ij}(\cdot, \xi)$ holomorphic except for ξ lying in the union of a finite number of Γ -orbits in H .

Notice that the difference between ∇_0 and the standard affine structure of $\mathbb{C} \times H$ is given by $f_{11} = f_{22} = g_{21} = 1$, the others f_{ij}, g_{ij} being trivial (see the straightforward computation in [36]).

Since $\tilde{\omega}$ is Δ -invariant and t -invariant, the functions $f_{ij}(\cdot, \xi)$ and $g_{ij}(\cdot, \xi)$ descend on a elliptic curve, for all $\xi \in H$. They are constant for all $\xi \in H$ for which they are holomorphic. It follows that $f_{ij}(\cdot, \xi)$ and $g_{ij}(\cdot, \xi)$ are constant for ξ lying in a open dense subset of H and, consequently, for all $\xi \in H$. It follows that the functions f_{ij} and g_{ij} depend only on ξ . Consequently, the flow of $\frac{\partial}{\partial z}$ preserves $\tilde{\omega}$. Since the fundamental generator of the principal fibration $\frac{\partial}{\partial z}$ preserves also ∇_0 , it is a Killing field for ∇ .

ii) In the following we consider only torsion-free connections: $f_{12} = g_{11}$ and $f_{22} = g_{21}$.

The Γ -invariance of $\tilde{\omega}$ yields the following equations:

$$(1) f_{11}(\xi) = f_{11}(\gamma\xi) - cf_{21}(\gamma\xi)(c\xi + d)$$

$$(2) f_{12}(\xi) = f_{12}(\gamma\xi)(c\xi + d)^{-2} - 2c^2 f_{21}(\gamma\xi) - cf_{22}(\gamma\xi)(c\xi + d)^{-1} + 2cf_{11}(\gamma\xi)(c\xi + d)^{-1}$$

$$(3) f_{21}(\xi) = f_{21}(\gamma\xi)(c\xi + d)^2$$

$$(4) f_{22}(\xi) = 2f_{21}(\gamma\xi)c(c\xi + d) + f_{22}(\gamma\xi)$$

$$(5) g_{12}(\xi) = g_{12}(\gamma\xi)(c\xi + d)^{-4} + c^2 f_{11}(\gamma\xi)(c\xi + d)^{-2} + cf_{12}(\gamma\xi)(c\xi + d)^{-3} - c^3 f_{21}(\gamma\xi)(c\xi + d)^{-1} - c^2 f_{22}(\gamma\xi)(c\xi + d)^{-2} - cg_{22}(\gamma\xi)(c\xi + d)^{-3}$$

$$(6) g_{22}(\xi) = g_{22}(\gamma\xi)(c\xi + d)^{-2} + cf_{22}(\gamma\xi)(c\xi + d)^{-1} + c^2 f_{21}(\gamma\xi),$$

$$\text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

The equation (3) implies that f_{21} is a meromorphic vector field on the compact Riemann surface Σ . Since a single pole of order at most one is allowed, we get $f_{21} = 0$ as a direct consequence of Riemann-Roch theorem and Serre duality [27] (page 245).

The equations (1) and (4) imply then that f_{11} and f_{22} are meromorphic functions on Σ with a single pole of order at most one. If f_{11}, f_{22} are not constant, they give a biholomorphism between Σ and the projective line $P^1(\mathbb{C})$ [27]: a contradiction, since the genus of Σ is ≥ 2 .

We conclude that f_{11} and f_{22} are constants.

Then we have :

$$(2')f_{12}(\xi) = f_{12}(\gamma\xi)(c\xi + d)^{-2} - cf_{22}(c\xi + d)^{-1} + 2cf_{11}(c\xi + d)^{-1}$$

$$(5')g_{12}(\xi) = g_{12}(\gamma\xi)(c\xi + d)^{-4} + c^2(f_{11} - f_{22})(c\xi + d)^{-2} + c(f_{12} - g_{22})(\gamma\xi)(c\xi + d)^{-3}$$

$$(6')g_{22}(\xi) = g_{22}(\gamma\xi)(c\xi + d)^{-2} + cf_{22}(c\xi + d)^{-1}.$$

It follows from (2') and (6') that f_{12} and g_{22} are quasimodular forms of weight two on Σ with a single simple pole. The space of those quasimodular forms is a complex vector space of dimension $g + 1$ (see [3], theorem 9).

Equation (5') is equivalent to the Γ -invariance of the quadratic differential $w(\xi)d\xi^2$, where $w = 2g_{12} + f'_{12} - g'_{22}$. It follows that $2g_{12} + f'_{12} - g'_{22}$ is a meromorphic quadratic differential on Σ with a single pole of order at most two.

It is classically known (as an application of Riemann-Roch theorem and Serre duality) that the space of quadratic differentials with a single pole of order at most 2 is of complex dimension $3g - 1$ (see, for example, [27]).

iii) Let $X = a(z, \xi)\frac{\partial}{\partial z} + b(z, \xi)\frac{\partial}{\partial \xi}$ be a local holomorphic Killing field of a generic connection ∇ , in the neighborhood of a point where ∇ is holomorphic (a and b are holomorphic local functions on $\mathbb{C} \times H$).

The equation of the Killing field is

$$[X, \nabla_Y Z] = \nabla_{[X, Y]} Z + \nabla_Y [X, Z],$$

for all Y, Z tangents to S . It is enough to verify the equation for (Y, Z)

corresponding to $(\frac{\partial}{\partial z}, \frac{\partial}{\partial \xi}), (\frac{\partial}{\partial z}, \frac{\partial}{\partial z})$ et $(\frac{\partial}{\partial \xi}, \frac{\partial}{\partial \xi})$.

This leads to the following PDE system:

$$(1) \quad a_{zz} + (1 + f_{11})a_z + 2f_{12}b_z = 0$$

$$(2) \quad b_{zz} + (1 + 2f_{22} - f_{11})b_z = 0$$

$$(3) \quad a_{z\xi} + (f_{11} - f_{22})a_\xi + g_{12}b_z + g_{11}b_\xi + \frac{\partial f_{12}}{\partial \xi}b = 0$$

$$(4) \quad b_{z\xi} + (1 + f_{22})a_z + (g_{22} - f_{12})b_z = 0$$

$$(5) \quad a_{\xi\xi} - g_{12}a_z + (2f_{12} - g_{12})a_\xi + 2g_{12}b_\xi + \frac{\partial g_{12}}{\partial \xi}b = 0$$

$$(6) \quad b_{\xi\xi} + 2(1 + f_{22})a_\xi - g_{12}b_z + g_{22}b_\xi + \frac{\partial g_{22}}{\partial \xi}b = 0.$$

The general solution of the first equation is $b = \nu(\xi)e^{-\mu z} + C(\xi)$, with ν, C holomorphic functions of ξ and $\mu = 1 + 2f_{22} - f_{11}$.

We replace b_z in the first equation and we get

$$a_z = \frac{\mu}{f_{11} - f_{22}} f_{12}(\xi)\nu(\xi)e^{-\mu z} + A(\xi)e^{-(1+f_{11})z},$$

with A a holomorphic function of ξ .

Then equation (4) leads to

$$\mu[-\nu'(\xi) + (\frac{1 + f_{11}}{f_{11} - f_{22}} f_{12} - g_{22})\nu(\xi)]e^{-\mu z} + (1 + f_{22})A(\xi)e^{-(1+f_{11})z} = 0.$$

For a generic ∇ , we have $f_{11} \neq f_{22}$, thus $\mu \neq 1 + f_{11}$ and the functions $e^{-\mu z}, e^{-(1+f_{11})z}$ are \mathbb{C} -linearly independent. This implies $(1 + f_{22})A(\xi) = 0$ and, since for a generic ∇ , $f_{22} \neq -1$, we have $A(\xi) = 0$.

We also get

$$(I) \nu'(\xi) = \nu(\xi) \left(\frac{1 + f_{11}}{f_{11} - f_{22}} f_{12} - g_{22} \right).$$

Now we check equation (3). We replace the partial derivatives of a and b in (3) and we get the following

$$\mu \left[\frac{f_{12}}{f_{11} - f_{22}} \nu'(\xi) - \left(g_{12} - \frac{f'_{12}}{f_{11} - f_{22}} \right) \nu(\xi) \right] e^{-\mu z} + (f_{12}C)' = 0.$$

Since generically $\mu \neq 0$, the functions $e^{-\mu z}$ and 1 are linearly independent, which yields to $(f_{12}C)' = 0$ and to

$$(II) \nu'(\xi) = \nu(\xi) \frac{1}{f_{12}} [(f_{11} - f_{22})g_{12} - f'_{12}].$$

Relations (I) et (II) are compatible, for a generic connection, only if $\nu = 0$. This implies $b = C(\xi)$, $a = B(\xi)$.

Our PDE system becomes:

$$(3') (f_{11} - g_{21})a' + f'_{12}b + f_{12}b' = 0$$

$$(5') a'' + (2f_{12} - g_{12})a' + 2g_{12}b' - g'_{12}b = 0$$

$$(6') b'' + 2(1 + 2g_{21})a' + g_{22}b' + g'_{22}b = 0.$$

Since $(f_{12}b)' = 0$, we have $a' = 0$ and thus a is a constant function. Equation (5') implies then $2g_{12}b' - g'_{12}b = 0$, which, for a generic ∇ , is compatible with $f_{12}b' + f'_{12}b = 0$ only if $b = 0$.

It follows that X is a constant multiple of $\frac{\partial}{\partial z}$.

iv) The projective connection associated to the affine connection ∇ is given by the following second order ODE [14]:

$$\xi'' = K^0(z, \xi) + K^1(z, \xi)\xi' + K^2(z, \xi)(\xi')^2 + K^3(z, \xi)(\xi')^3,$$

where $K^0 = -f_{21} = 0$, $K^1 = (1 + f_{11}) - 2(1 + f_{22})$, $K^2 = -(g_{22} - 2f_{12})$ et $K^3 = g_{12}$ (see [14]).

Liouville [42], followed by Tresse [56] and Cartan [14], proved that this projective connection is projectively flat if and only if both of the following invariants vanish:

$$L_1 = 2K_{z\xi}^1 - K_{zz}^2 - 3K_{\xi\xi}^0 - 6K^0 K_z^3 - 3K^3 K_z^0 + 3K^0 K_\xi^2 + 3K^2 K_\xi^0 + K^1 K_z^2 - 2K^1 K_\xi^1,$$

$$L_2 = 2K_{z\xi}^2 - K_{\xi\xi}^1 - 3K_{zz}^3 + 6K^3 K_\xi^0 + 3K^0 K_\xi^3 - 3K^3 K_z^1 - 3K^1 K_z^3 - K^2 K_\xi^1 + 2K^2 K_z^2.$$

Here $K^0 = 0$, K^1 is a constant function and K^2, K^3 depend only of ξ .

This implies the vanishing of both invariants L_1 and L_2 . \square

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