When are increment-stationary random point sets stationary?

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WHEN ARE INCREMENT-STATIONARY RANDOM POINT SETS STATIONARY?

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Abstract. In a recent work, Blanc, Le Bris, and Lions defined a notion of increment-stationarity for random point sets, which allowed them to prove the existence of a thermodynamic limit for two-body potential energies on such point sets (under the additional assumption of ergodicity), and to introduce a variant of stochastic homogenization for increment-stationary coefficients. Whereas stationary random point sets are increment-stationary, it is not clear a priori under which conditions increment-stationary random point sets are stationary. In the present contribution, we give a characterization of the equivalence of both notions of stationarity based on elementary PDE theory in the probability space. This allows us to give conditions on the decay of a covariance function associated with the random point set, which ensure that increment-stationary random point sets are stationary random point sets up to a random translation with bounded second moment in dimensions $d > 2$. In dimensions $d = 1$ and $d = 2$, we show that such sufficient conditions cannot exist.

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1. Introduction and main results

1.1. **Increment-stationarity.** Let d be the dimension, and denote by $\mathcal{L}(\mathbb{R}^d)$ the set of locally finite simple point sets of $\mathbb{R}^d$. In what follows we consider random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $\mathcal{L}(\mathbb{R}^d)$. We call such random variables random point sets. In [2], Blanc, Le Bris and Lions addressed the issue of defining the thermodynamic limit of the energy of random sets ℓ of particles (seen as simple random point sets). Typical energies to be considered are given by two-body potentials $V : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$, so that the energy in some bounded region $D \subset \mathbb{R}^d$ writes

$$\mathcal{E}_{L,T}(\ell, D) := \frac{1}{2} \sum_{x,y \in \ell, x \neq y} V(x-y).$$

As noticed in [2, 3], the existence of a deterministic thermodynamic limit

$$\lim_{D \uparrow \mathbb{R}^d} \frac{1}{|D|} \mathcal{E}_{L,T}(\ell, D) \tag{1.1}$$

(when D properly invades $\mathbb{R}^d$) does not require the point set ℓ to be stationary, which lead the authors to define a notion of increment-stationarity.

To motivate the definition of increment-stationarity, let us momentarilly focus on the case $d = 1$, and consider a random point set ℓ in the real line. Recall (see for instance [6,

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Chapter 3]) that the random point set ℓ is characterized by the sequence of (measurable) random variables $\{X_i\}_{i\in\mathbb{Z}}$, where $X_i < X_{i+1}$ and X_0 is the closest point to the origin on the negative axis. The associated sequence of intervals is denoted by $\{\tau_i\}_{i\in\mathbb{Z}^d}$ and defined by $\tau_i = X_{i+1} - X_i$. The random point set ℓ is said to be *interval-stationary* if the sequence $\{\tau_i\}_{i\in\mathbb{Z}^d}$ is stationary in the following sense: For all $m \in \mathbb{N}$, $i_1, \dots, i_m \in \mathbb{Z}$, the distribution of $(\tau_{i_1+k}, \dots, \tau_{i_m+k})$ does not depend on $k \in \mathbb{Z}$. In turn, this implies (and is indeed equivalent to) the existence of a discrete group action $\{\theta_z\}_{z\in\mathbb{Z}}$ which preserves the probability measure and is such that for all $i, k \in \mathbb{Z}$ and almost every $\omega \in \Omega$,

$$\tau_i(\theta_k\omega) \;=\; \tau_{i+k}(\omega).$$

In terms of the random point set ℓ, this turns into: For all $i, j, k \in \mathbb{Z}$ and almost every $\omega \in \Omega$,

$$X_{i+k}(\omega) - X_{j+k}(\omega) \;=\; X_i(\theta_k\omega) - X_j(\theta_k\omega).$$

The latter implies that the random point set ℓ satisfies for all $k \in \mathbb{Z}$ and almost every $\omega \in \Omega$

$$\ell(\theta_k\omega) \;=\; \ell(\omega) + Y_k(\omega) \tag{1.2}$$

for some sequence of random variables $\{Y_k\}_{k\in\mathbb{Z}}$ (indeed, $Y_k(\omega) = X_j(\theta_k\omega) - X_j(\omega)$ for any $j \in \mathbb{Z}$), that is, $\ell(\theta_k\omega)$ is the translation of $\ell(\omega)$ by the vector $Y_k(\omega)$. In addition this sequence satisfies

$$Y_0 \;\equiv\; 0, \tag{1.3}$$

$$Y_j(\theta_k\omega) - Y_i(\theta_k\omega) \;=\; Y_{j+k}(\omega) - Y_{i+k}(\omega) \tag{1.4}$$

for all $i, j, k \in \mathbb{Z}$ and almost every $\omega \in \Omega$. We then say that a random point set ℓ is *increment-stationary* if it satisfies (1.2)—(1.4) for some measure-preserving group action $\{\theta_k\}_{k\in\mathbb{Z}}$. A random point set ℓ is *lattice-stationary* (or simply *stationary* in what follows) if there exists a vector $T \in \mathbb{R}$ such that ℓ and $\ell + kT$ have the same distribution for all $k \in \mathbb{Z}$, or equivalently if there exists a measure-preserving group action $\{\theta_k\}_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$ and almost every $\omega \in \Omega$, $\ell(\theta_k\omega) = \ell(\omega) + kT$. Increment-stationarity is a generalization of stationarity and of interval-stationarity in the sense that random point sets which are stationary or interval-stationary are increment-stationary. Unlike interval-stationarity (which relies on the order property of $\mathbb{R}$), increment-stationarity extends in a straightforward way to higher dimension.

We are now in position to give the definitions of *increment-stationarity* and of *stationarity* for random point sets ℓ in $\mathbb{R}^d$. We say that ℓ is *increment-stationary* if there exist a sequence of random vectors $\{Y_k\}_{k\in\mathbb{Z}^d}$ of $L^2(\Omega, \mathbb{R}^d)$ and a measure-preserving group action $\{\theta_k\}_{k\in\mathbb{Z}^d}$ such that ℓ satisfies (1.2)—(1.4) for all $i, j, k \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$. We say that ℓ is *stationary* if in addition Y_k is given for all $k \in \mathbb{Z}^d$ by

$$Y_k := Tk \tag{1.5}$$

for some deterministic $d \times d$ matrix T.

The interest of the notion of increment-stationarity is the following: If the point set ℓ is increment-stationary for some *ergodic* measure-preserving group action $\{\theta_z\}_{z\in\mathbb{Z}^d}$, then (1.1) exists as a consequence of the ergodic theorem. In [2], the authors prove that if the positions of an infinite set of nuclei are given by a stationary random point set (satisfying in addition uniform hard-core and non-empty space properties), then the thermodynamic limit of the associated electronic cloud exists in the sense that the notions of averaged energy and cloud density are well-defined, in the case of Thomas-Fermi models. This was later extended by Blanc and Lewin [4], and Cancès, Lahbabi and Lewin [5], to quantum models with Coulomb forces and to Hartree-Fock and Kohn-Sham type models, respectively. In terms of point sets, their proofs essentially rely on the stationarity of two-body interactions and an ergodic theorem, so that these proofs should extend to the more general case of increment-stationary random point sets.

The aim of this contribution is to investigate in which respect *increment-stationarity* is more general than *stationarity*, and identify under which conditions on the sequence Y_k one can conclude that an *increment-stationary* random point set is *stationary*. In particular, both the probability space and the group action are fixed, and we are indeed investigating the *rigidity* of increment-stationary random point sets. This question is of different nature than the Palm-Khinchin theory (see for instance [6, Theorem 13.3.I] which establishes a one-to-one relation between interval-stationary and stationary random point sets). As we shall see, the validity of such a rigidity result depends on the dimension.

1.2. **Main results.** In what follows we endow $(\Omega, \mathcal{F}, \mathbb{P})$ with an ergodic measure-preserving discrete group action $\{\theta_k\}_{k\in\mathbb{Z}^d} : \Omega \to \Omega$, and we denote by $\langle\cdot\rangle$, $\mathrm{var}\,[\cdot]$, and $\mathrm{cov}\,[\cdot;\cdot]$ the associated expectation, variance, and covariance, respectively. We denote by $L^2_0(\Omega, \mathbb{R}^d)$ the space of random vectors with bounded second moments and vanishing expectations. We let $\{e_l\}_{l\in\{1,\dots,d\}}$ denote the canonical basis of $\mathbb{R}^d$.

Let ℓ be a stationary point set in $\mathbb{R}^d$ and $Y \in L^2_0(\Omega, \mathbb{R}^d)$ be some non-identically zero random vector. Then $\ell + Y$ is not stationary but is clearly increment-stationary. We shall say that a random point set ℓ is *stationary up to translation* if there exists some $\tilde{Y} \in L^2_0(\Omega, \mathbb{R}^d)$ such that $\ell + \tilde{Y}$ is a stationary random point set. The following theorem gives a characterization of the equivalence between *increment-stationarity* and *stationarity up to translation.* This result is directly inspired by the treatment of the corrector equation in stochastic homogenization by Papanicolaou and Varadhan in [12] (see [10] for the case of discrete elliptic equations). It relies on the differential calculus in the probability space generated by the group action $\{\theta_k\}_{\theta\in\mathbb{Z}^d}$.

Proposition 1. *Let $\ell : \Omega \to \mathcal{L}(\mathbb{R}^d)$ be an increment-stationary random point set for the group action $\{\theta_k\}_{k\in\mathbb{Z}^d}$. For all $\mu > 0$ and $i \in \{1, \dots, d\}$, consider the unique weak solution $\phi_{\mu,i} \in L^2(\Omega)$ of the equation: For all $\psi \in L^2(\Omega)$,*

$$\langle \mu\phi_{\mu,i}\psi + D\psi \cdot D\phi_{\mu,i}\rangle \;=\; \langle D\psi \cdot \zeta_i\rangle\,, \tag{1.6}$$

where $\zeta_i := (Z_{e_1} \cdot e_i, \dots, Z_{e_d} \cdot e_i) \in L^2(\Omega, \mathbb{R}^d)$, $Z_{e_l} := Y_{e_l} - \langle Y_{e_l}\rangle$ for all $l \in \{1, \dots, d\}$, and $D := (D_1, \dots, D_d)$ is the differential operator from $L^2(\Omega) \to L^2(\Omega, \mathbb{R}^d)$ defined by $D_l\psi(\omega) = \psi(\theta_{e_l}\omega) - \psi(\omega)$. Then, ℓ is stationary up to translation if and only if *the family $\{\phi_{\mu,i}\}_{i\in\{1,\dots,d\}}$ is bounded in $L^2(\Omega)$ uniformly wrt $\mu > 0$. In addition, if ℓ is stationary up to translation, then the associated random translation $\tilde{Y} \in L^2_0(\Omega, \mathbb{R}^d)$ is uniquely defined.*

It remains to identify sufficient conditions on $\{Y_{e_l}\}_{l\in\{1,\dots,d\}}$ for the boundedness of the functions $\phi_{\mu,i}$. These conditions are written in terms of the decay of a covariance function as follows.

Hypothesis 1 (Decay of order $\alpha > 0$)**.** The random point set $\ell : \Omega \to \mathcal{L}(\mathbb{R}^d)$ is increment-stationary for the group action $\{\theta_k\}_{k\in\mathbb{Z}^d}$, and the associated random vectors $Y_{e_1}, \dots, Y_{e_d}$

display the following covariance decays: There exists $\alpha > 0$ such that for all $k \in \mathbb{Z}^d$ and $l, l', n \in \{1, \dots, d\}$

$$\operatorname{cov}\left[Y_{e_l} \circ \theta_k \cdot e_n ; Y_{e_{l'}} \cdot e_n\right] \lesssim \frac{1}{1+|k|^\alpha},$$

where $\lesssim$ means $\leq$ up to a universal multiplicative constant.

As the following result shows, there are two types of behavior, depending on the dimension d ($d = 2$ is critical).

Theorem 1. *We have:*

- *For $d = 1$ and $d = 2$: For all $\alpha > 0$ there exists an increment-stationary random point set $\ell : \Omega \to \mathcal{L}(\mathbb{R}^d)$ satisfying Hypothesis 1 and which is not stationary up to translation.*
- *For all $d > 2$: If $\ell : \Omega \to \mathcal{L}(\mathbb{R}^d)$ is increment-stationary and satisfies Hypothesis 1 for some $\alpha > 2$, then ℓ is stationary up to translation.*

Note that this type of behavior is similar to that of the corrector in stochastic homogenization: In dimension $d = 1$, there cannot exist stationary correctors in $L^2(\Omega)$, dimension $d = 2$ is critical and stationary correctors do not exist either, whereas in dimensions $d > 2$, stationary correctors exist under some assumptions on the statistics (a spectral gap estimate, see [7]). Equation (1.7) can indeed be seen as the corrector equation in the regime of vanishing ellipticity contrast (the variable-coefficients elliptic operator is replaced by a constant-coefficients elliptic operator). In particular, in dimensions $d = 1$ and $d = 2$, the corrector in the regime vanishing ellipticity contrast for independent and identically distributed random conductivities provides with an example of increment-stationary point set which is not stationary up to translation, see Step 2 in the proof of Theorem 1 for details.

Before we turn to the proofs of these results, let us focus on a specific example of increment-stationary random point sets given by the image of $\mathbb{Z}^d$ by an "increment-stationary stochastic diffeomorphism". This case is of interest for disordered crystals and their thermodynamic limits, and was a motivation for [3]. It will also allow us to emphasize that, although Hypothesis 1 can be interpreted as a condition on the decay of the correlation between "stationary increments" (whenever this notion is well-defined), the distance is the one given by the group action (that is, $k \in \mathbb{Z}^d$ associated with θ_k). In particular Hypothesis 1 is *not* a condition on the decay of the correlation between the "stationary increments" wrt to the Euclidean distance.

1.3. **Increment-stationary stochastic diffeomorphisms.** In [3], Blanc, Le Bris and Lions introduced a variant of stochastic homogenization of linear elliptic equations where the diffusion coefficients are random but not necessarily stationary. These diffusion coefficients are obtained using a stochastic diffeomorphism $\Phi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ which is increment-stationary in the following sense: For all $k \in \mathbb{Z}^d$, all $x \in \mathbb{R}^d$, and almost every $\omega \in \Omega$,

$$\nabla\Phi(x + k, \omega) \;=\; \nabla\Phi(x, \theta_k\omega).$$

Such random fields Φ allow one to define a specific class of increment-stationary random point sets. Set $\ell := \Phi(\mathbb{Z}^d)$. Since Φ is a diffeomorphim, ℓ is a simple point process almost surely. The interest of such a definition is the natural labelling of the points by $\mathbb{Z}^d$. Define

$X_i = \Phi(i)$ for all $i \in \mathbb{Z}^d$ so that $\ell = \cup_{i\in\mathbb{Z}^d}\{X_i\}$. The increment-stationarity of Φ then implies that for all $i, j, k \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$ we have

$$X_i(\theta_k\omega) - X_j(\theta_k\omega) = X_{i+k}(\omega) - X_{j+k}(\omega),$$

from which we deduce that ℓ is increment-stationary (cf. the case $d = 1$ in Paragraph 1.1).

Such increment-stationary point sets are very specific. Under the assumption that Φ satisfies uniform bounds (that is, $|\nabla\Phi|$ is uniformly bounded from above and $\det\nabla\Phi$ uniformly bounded from below), then $\ell = \Phi(\mathbb{Z}^d)$ uniformly satisfies the hard-core and non-empty space conditions (that is, positive minimal distance between any point $x \in \ell$ and $\ell\setminus\{x\}$, and existence of $R < \infty$ such that any ball of radius R contains at least a point of ℓ almost surely). Even within the class of random point sets satisfying the uniform hard-core and non-empty space conditions, such random point sets are not generic. Indeed, on the one hand, as proved in [1, Theorem 10], such random point sets cannot be statistically isotropic, whereas, on the other hand, there exist stationary random point sets which are statistically isotropic and satisfy the uniform hard-core and non-empty space conditions (for instance the random parking point set, defined in [13], the properties of which are listed in [8, Proposition 2.1]).

As opposed to increment-stationary random point sets in general, the labelling of $\ell = \Phi(\mathbb{Z}^d)$ allows one to define the notion of increments in the form of the quantities $X_i - X_j$. As mentioned above, these increments are stationary, and one may consider the associated covariances, that is, $\mathrm{cov}\,[(\Phi(k+e_l) - \Phi(k))\cdot e_n; (\Phi(e_l) - \Phi(0))\cdot e_n]$. As we shall show, conditions on the decay of such quantities may ensure the stationarity of the random field Φ, but *not* the stationarity of the point process $\ell = \Phi(\mathbb{Z}^d)$. More precisely, we have the following counterpart of Proposition 1 and Theorem 1 for increment-stationary random fields.

Proposition 2. *Let $\Phi : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ be an increment-stationary (discrete) random field for the group action $\{\theta_k\}_{k\in\mathbb{Z}^d}$, that is, such that its discrete gradient $\partial\Phi = (\partial_1\Phi, \dots, \partial_d\Phi)$, with $\partial_i\Phi := \Phi(\cdot + e_i) - \Phi(\cdot)$, is stationary: For all $k, z \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$,*

$$\partial\Phi(z + k, \omega) \;=\; \partial\Phi(z, \theta_k\omega).$$

Assume that $\partial\Phi(0) \in L^2(\Omega, \mathbb{R}^{d\times d})$, and for all $\mu > 0$ and $i \in \{1, \dots, d\}$ consider the unique weak solution $\phi_{\mu,i} \in L^2(\Omega)$ of the equation: For all $\psi \in L^2(\Omega)$,

$$\langle \mu\phi_{\mu,i}\psi + D\psi \cdot D\phi_{\mu,i}\rangle \;=\; \langle D\psi \cdot \partial(\Phi\cdot e_i)(0)\rangle\,. \tag{1.7}$$

Then, Φ is stationary up to translation, that is, there exists a unique random vector $\tilde{X} \in L^2_0(\Omega, \mathbb{R}^d)$ such that $(z, \omega) \mapsto \Phi(z, \omega) + \tilde{X}(\omega)$ is stationary in the sense that for all $k, z \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$,

$$\Phi(z + k, \omega) + \tilde{X}(\omega) \;=\; \Phi(z, \theta_k\omega) + \tilde{X}(\theta_k\omega),$$

if and only if *the family $\{\phi_{\mu,i}\}_{i\in\{1,\dots,d\}}$ is bounded in $L^2(\Omega)$ uniformly wrt $\mu > 0$.*

The conditions corresponding to Hypothesis 1 are now

Hypothesis 2 (Decay of order $\alpha > 0$)**.** The random field $\Phi : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ is increment-stationary for the ergodic group action $\{\theta_k\}_{k\in\mathbb{Z}^d}$, and there exists $\alpha > 0$ such that for all $k \in \mathbb{Z}^d$ and $l, l', n \in \{1, \dots, d\}$

$$\mathrm{cov}\,[\partial_l\Phi(k)\cdot e_n; \partial_{l'}\Phi(0)\cdot e_n] \;\lesssim\; \frac{1}{1 + |k|^\alpha},$$

where $\lesssim$ means $\leq$ up to a universal multiplicative constant.

Theorem 2. *We have:*

- *For $d = 1$ and $d = 2$: For all $\alpha > 0$ there exists an increment-stationary random field $\Phi : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ satisfying Hypothesis 2 and which is not stationary up to translation.*
- *For all $d > 2$: If $\Phi : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ is increment-stationary and satisfies Hypothesis 2 for some $\alpha > 2$, then it is stationary up to translation.*

As a direct corollary of Theorem 2, we have the following result on random Lipschitz fields with stationary gradients:

Corollary 1. *Let $d > 2$ and $\Phi : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ be a random Lipschitz field such that its (continuum) gradient $\nabla\Phi$ is stationary and uniformly bounded. If there exists $\alpha > 2$ such that for all $k \in \mathbb{Z}^d, x \in [0,1)^d$ and $l, l', n \in \{1, \dots, d\}$*

$$\mathrm{cov}\left[\partial_l \Phi(x+k) \cdot e_n; \partial_{l'} \Phi(x) \cdot e_n\right] \lesssim \frac{1}{1+|k|^\alpha},$$

then there exists a $[0,1)^d$-periodic random field $\tilde{X} \in W^{1,\infty}_{\mathrm{per}}([0,1)^d, L^2_0(\Omega, \mathbb{R}^d))$ such that $(x,\omega) \mapsto \Phi(x,\omega) + \tilde{X}(x,\omega)$ is a stationary field: For all $x \in \mathbb{R}^d$, $k \in \mathbb{Z}^d$, and almost every $\omega \in \Omega$,

$$\Phi(x+k,\omega) + \tilde{X}(x+k,\omega) \;=\; \Phi(x,\theta_k\omega) + \tilde{X}(x,\theta_k\omega).$$

To conclude, let us compare the two notions of stationarity for a random field Φ and for the associated random point set $\ell = \Phi(\mathbb{Z}^d)$. The following result shows that these notions are essentially incompatible.

Proposition 3. *Let $\Phi : \mathbb{Z}^d \times \Omega \to \mathbb{R}^d$ be an increment-stationary (discrete) injective random field for the ergodic group action $\{\theta_k\}_{k\in\mathbb{Z}^d}$, that is, such that its discrete gradient $\partial\Phi = (\partial_1\Phi, \dots, \partial_d\Phi)$, with $\partial_i\Phi := \Phi(\cdot + e_i) - \Phi(\cdot)$, is stationary. If Φ and $\ell = \Phi(\mathbb{Z}^d)$ are stationary up to translation, then Φ is linear and ℓ is periodic up to a random translation. Indeed, we then have $\Phi(\cdot,\omega) : x \mapsto \langle\partial\Phi\rangle\, x + \Phi(0,\omega)$, and $\ell(\omega) = \langle\partial\Phi\rangle\,\mathbb{Z}^d + \Phi(0,\omega)$ for almost every $\omega \in \Omega$.*

This proposition illustrates that even in the case when one may properly define the notion of "stationary increments", Hypothesis 1 cannot be turned into a condition on the decay of the correlation between the "stationary increments" wrt to the Euclidean distance (which corresponds to Hypotheses 2).

2. Proofs

The proofs of Proposition 2 and Theorem 2 are straightforward adaptations of the proofs of Proposition 1 and Theorem 1, and we only prove the latter.

2.1. **Proof of Proposition 1.** We split the proof into two steps.

Step 1. Proof that boundedness of $\{\phi_{\mu,i}\}_{i\in\{1,\dots,d\}}$ implies stationarity up to translation.
Let ℓ be an increment-stationary random point set. Since the action group is ergodic, it generates a Weyl decomposition of $L^2(\Omega, \mathbb{R}^d)$ into potential and solenoidal fields (see for instance [9, Lemma 7.3], the adaptation of which is straightforward in the case of a discrete group action). The energy estimate $\mu\left\langle \phi_{\mu,i}^2 \right\rangle + \left\langle |D\phi_{\mu,i}|^2 \right\rangle \lesssim 1$ associated with the

equation ensures by weak compactness that $D\phi_{\mu,i}$ converges weakly in $L^2(\Omega,\mathbb{R}^d)$ to some potential field $\chi_i \in L^2(\Omega,\mathbb{R}^d)$. Passing to the limit along the subsequence in the defining equation for $\phi_{\mu,i}$ and using the a priori estimate $\mu\left\langle \phi_{\mu,i}^2\right\rangle \lesssim 1$ yield for all $\psi \in L^2(\Omega)$,

$$\langle D\psi \cdot \chi_i\rangle \;=\; \langle D\psi \cdot \zeta_i\rangle\,.$$

The above identity for arbitrary $\psi \in L^2(\Omega)$ shows that the $L^2(\Omega)$-projections of χ_i and ζ_i onto potential fields coincide. Since $\zeta_i = ((Z_{e_1}-Z_0)\cdot e_i,\ldots,(Z_{e_d}-Z_0)\cdot e_i) = D(Z\cdot e_i)$ and χ_i (as limit of potential fields $D\phi_{\mu,i}$) are potential fields themselves, this implies $\chi_i=\zeta_i$ and yields the uniqueness of the limit. Assume in addition that the family $\phi_{\mu,i}$ is bounded in $L^2(\Omega)$ uniformly wrt μ. Then, by weak compactness, there exists some $\phi_i \in L^2(\Omega)$ such that, up to extraction, $\phi_{\mu,i}$ converges to ϕ_i weakly in $L^2(\Omega)$. Note that $\langle\phi_i\rangle = 0$ since $\langle\phi_{\mu,i}\rangle = 0$ for all $\mu>0$. Combined with the argument above, this shows that $D\phi_i=\zeta_i$, and this implies in turn the uniqueness of ϕ_i and the convergence of the entire sequence. Indeed, let $\varphi \in L^2(\Omega)$ be such that $\langle\varphi\rangle = \langle\phi_i\rangle = 0$ and $D\varphi = \zeta_i$. Then $\varphi-\phi_i \in L^2(\Omega)$ is such that $\left\langle |D(\varphi-\phi_i)|^2\right\rangle = 0$. This implies by ergodicity that $\varphi-\phi_i$ is constant, and therefore $\varphi=\phi_i$ by the mean-free condition.

We then define $\tilde{Y}:\Omega\to\mathbb{R}^d, \omega\mapsto \sum_{i=1}^d \phi_i(\omega)e_i$, and for all $i\in\{1,\ldots,d\}$, we set $T_i := \langle Y_{e_i}\rangle$. It remains to check that $\ell+\tilde{Y}$ is stationary.

Using that $Y_0\equiv 0$ one can decompose Y_k as a sum of differences along the d canonical directions, i. e.

$$\begin{aligned}Y_k \;=\; &\sum_{i_1=1}^{k_1}\big(Y_{k_1+1-i_1,k_2,\ldots,k_d}-Y_{k_1-i_1,k_2,\ldots,k_d}\big)+\sum_{i_2=1}^{k_2}\big(Y_{0,k_2+1-i_2,k_3,\ldots,k_d}-Y_{0,k_2-i_2,k_3,\ldots,k_d}\big)\\ &+\cdots+\sum_{i_d=1}^{k_d}\big(Y_{0,\ldots,0,k_d+1-i_d}-Y_{0,\ldots,0,k_d-i_d}\big).\end{aligned}$$

By stationarity of the increments and definition of $\{Z_{e_i}\}_{i\in\{1,\ldots,d\}}$, this yields

$$\begin{aligned}\langle Y_k\rangle \;&=\; \sum_{i=1}^d k_i\,\langle Y_{e_i}\rangle\,,\\ Y_k(\omega)-\langle Y_k\rangle \;&=\; \sum_{i_1=1}^{k_1} Z_{e_1}(\theta_{k_1-i_1,k_2,\ldots,k_d}\omega)+\sum_{i_2=1}^{k_2} Z_{e_2}(\theta_{0,k_2-i_2,k_3,\ldots,k_d}\omega)\\ &\quad+\cdots+\sum_{i_d=1}^{k_d} Z_{e_d}(\theta_{0,\ldots,0,k_d-i_d}\omega).\end{aligned}$$

Using $Z_0\equiv 0$ and $D\phi_i=\zeta_i$, this turns into

$$\begin{aligned}Y_k(\omega)-\langle Y_k\rangle \;=\; &\sum_{i_1=1}^{k_1}\sum_{l=1}^{d} D_1\phi_l(\theta_{k_1-i_1,k_2,\ldots,k_d}\omega)e_l+\sum_{i_2=1}^{k_2}\sum_{l=1}^{d} D_2\phi_l(\theta_{0,k_2-i_2,k_3,\ldots,k_d}\omega)e_l\\ &+\cdots+\sum_{i_d=1}^{k_d}\sum_{l=1}^{d} D_d\phi_l(\theta_{0,\ldots,0,k_d-i_d}\omega)e_d.\end{aligned}$$

By definition of the difference operators D_i, terms cancel two by two, and the sum simplifies to

$$Y_k(\omega) - \langle Y_k \rangle \;=\; \sum_{l=1}^{d} \phi_l(\theta_k \omega) e_l - \sum_{l=1}^{d} \phi_l(\omega) e_l.$$

This implies the desired property by the choice of $\tilde{Y}$ and T_i: For all $k \in \mathbb{Z}^d$,

$$\ell(\theta_k \omega) + \tilde{Y}(\theta_k \omega) \;=\; \ell(\omega) + \tilde{Y}(\omega) + \sum_{i=1}^{d} k_i T_i.$$

Step 2. Proof that stationarity up to translation implies boundedness of $\{\phi_{\mu,i}\}_{i\in\{1,\dots,d\}}$.
Assume that ℓ is stationary up to translation. Then Z_{e_l} takes the form

$$Z_{e_l}(\omega) \;=\; \tilde{Y}(\theta_{e_l}\omega) - \tilde{Y}(\omega).$$

Recall that for all $\mu > 0$ and $i \in \{1,\dots,d\}$, $\phi_{\mu,i} \in L^2(\Omega)$ is solution of: For all $\psi \in L^2(\Omega)$,

$$\mu \langle \phi_{\mu,i} \psi \rangle + \langle D\psi \cdot D\phi_{\mu,i} \rangle \;=\; \langle D\psi \cdot \zeta_i \rangle$$

with $\zeta_i = (Z_{e_1} \cdot e_i, \dots, Z_{e_d} \cdot e_i)$. For all $i \in \{1,\dots,d\}$, set $\tilde{\phi}_i := \tilde{Y} \cdot e_i \in L^2(\Omega)$, so that $D\tilde{\phi}_i = \zeta_i$. We then have for all $\psi \in L^2(\Omega)$,

$$\mu \left\langle (\phi_{\mu,i} - \tilde{\phi}_i)\psi \right\rangle + \left\langle D\psi \cdot D(\phi_{\mu,i} - \tilde{\phi}_i) \right\rangle \;=\; -\mu \left\langle \tilde{\phi}_i \psi \right\rangle,$$

whence the a priori estimate

$$\left\langle (\phi_{\mu,i} - \tilde{\phi}_i)^2 \right\rangle \;\le\; \left\langle \tilde{\phi}_i^2 \right\rangle = \left\langle (\tilde{Y} \cdot e_i)^2 \right\rangle,$$

which yields the desired uniform-in-μ boundedness estimate by the triangle inequality:

$$\langle \phi_{\mu,i}^2 \rangle \;\lesssim\; \left\langle (\tilde{Y} \cdot e_i)^2 \right\rangle.$$

2.2. **Proof of Theorem 1.** Let ℓ be an increment-stationary random point set, and for all $i \in \{1,\dots,d\}$ and $\mu > 0$, let ζ_i and $\phi_{\mu,i}$ be as in Proposition 1. We first derive an integral representation for $\phi_{\mu,i}$ in physical space, then treat the case $d \le 2$ in Step 2, and the case $d > 2$ in Step 3.

Step 1. Green representation formula for $\phi_{\mu,i}$.
In this step we derive a Green representation formula for $\phi_{\mu,i}$, see [7, Lemma 2.6]. Equation (1.7) indeed admits an equivalent form in the physical space. Let $\bar{\phi}_{\mu,i}, \bar{\zeta}_i : \mathbb{Z}^d \times \Omega \to \mathbb{R}$ be the stationary extensions of $\phi_{\mu,i}$ and ζ_i, that is, the random fields defined by $\bar{\phi}_{\mu,i}(k,\omega) := \phi_{\mu,i}(\theta_k\omega)$ and $\bar{\zeta}_i(k,\omega) := \zeta_i(\theta_k\omega)$, respectively. Then, $\bar{\phi}_{\mu,i}$ solves almost surely the elliptic PDE

$$\mu\bar{\phi}_{\mu,i} - \triangle\bar{\phi}_{\mu,i} \;=\; \partial^* \cdot \bar{\zeta}_i \qquad \text{in } \mathbb{Z}^d, \tag{2.1}$$

where ∂ is the forward discrete gradient, $\partial^* \cdot$ the backward discrete divergence, and $-\triangle := -\partial^* \cdot \partial$ the discrete Laplace operator on $\mathbb{Z}^d$.

Let $G_\mu : \mathbb{Z}^d \to \mathbb{R}$ denote the Green's function associated with the elliptic operator $\mu - \triangle$ on $\mathbb{Z}^d$, that is, the only solution in $L^2(\mathbb{Z}^d)$ of

$$\mu G_\mu(x) - \triangle G_\mu(x) \;=\; \delta(x),$$

where δ is such that $\delta(0) = 1$ and $\delta(x) = 0$ for all $x \neq 0$ (the existence and uniqueness of G_μ follows from the Riesz representation theorem).

Testing equation (2.1) with $y \mapsto G_\mu(y-x)$ yields the desired Green representation formula

$$\bar\phi_{\mu,i}(x) \;=\; \int_{\mathbb{Z}^d} \partial G_\mu(y-x)\cdot\bar\zeta_i(y)dy,$$

where $\int_{\mathbb{Z}^d} dy$ stands for the sum over $y \in \mathbb{Z}^d$.

Step 2. Case $d \le 2$.
In this step we prove that even in the case when $\{\bar\zeta_i(z)\}_{z\in\mathbb{Z}^d}$ is a field of independent and identically distributed (iid) variables, the family $\left\langle \phi_{\mu,i}^2 \right\rangle$ may be unbounded in μ. Consider in particular the field Y_k characterized by: $Y_0 \equiv 0$ and $Y_{e_l}\circ\theta_k = Y_{k+e_l} - Y_k = a_l(k)e_l$, where $\{a_l(k)\}_{l\in\{1,\dots,d\},k\in\mathbb{Z}^d}$ are iid variables following the law of some $a \in L^2(\Omega)$. Then, the random point set ℓ satisfies Hypothesis 1 for any $\alpha > 0$ (it has finite correlation-length). In view of Step 1, we have

$$\left\langle \phi_{\mu,i}^2 \right\rangle \;=\; \left\langle (\bar\phi_{\mu,i}(0))^2 \right\rangle \;=\; \int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} \partial G_\mu(y)\otimes\partial G_\mu(y') : \left\langle \bar\zeta_i(y)\otimes\bar\zeta_i(y')\right\rangle dydy'. \tag{2.2}$$

Since $\bar\zeta_i(y) = (a_i(y) - \langle a\rangle)e_i$, the sum reduces by independence to

$$\left\langle \phi_{\mu,i}^2 \right\rangle \;=\; \mathrm{var}\,[a]\int_{\mathbb{Z}^d} (\partial_i G_\mu(y))^2 dy. \tag{2.3}$$

Since ∂G_μ converges locally to the gradient ∂G of the whole space Green's function of the discrete Laplace operator $-\triangle$ on $\mathbb{Z}^d$ as $\mu \downarrow 0$, the RHS of (2.3) cannot be bounded in dimensions $d \le 2$. If it were, this would imply that $\partial G \in L^2(\mathbb{Z}^d)$, which is not true for $d \le 2$ (as can be seen in Fourier space [11], or by comparison to the large scale behavior of the continuum Green's function). This qualitative behavior is enough for the proof of Theorem 1. To be more quantitative, one indeed expects $\left\langle \phi_{\mu,i}^2 \right\rangle \sim \mu^{-\frac{1}{2}}$ for $d = 1$, and $\left\langle \phi_{\mu,i}^2 \right\rangle \sim |\ln\mu|$ for $d = 2$. The proof of these estimates would require a more careful analysis of the Green's functions.

Step 3. Case $d > 2$.
The starting point in dimensions $d > 2$ is again (2.2) which in view of Hypothesis 1 implies

$$\left\langle \phi_{\mu,i}^2 \right\rangle \;\lesssim\; \int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} |\partial G_\mu(y)||\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy'.$$

Without loss of generality, we assume in addition that $\alpha < d$. We shall use the following uniform-in-μ bounds on ∂G_μ: $\|\partial G_\mu\|_{L^\infty(\mathbb{Z}^d)} \lesssim 1$ (cf. [7, Corollary 2.3]), and for all exponents $1 \le p < \infty$, all $i \in \mathbb{N}$ and $d > 2$,

$$\int_{2^i<|y|\le 2^{i+1}} |\partial G_\mu(y)|^p dy \;\lesssim\; (2^i)^d(2^i)^{p(1-d)}, \tag{2.4}$$

which is optimal in terms of scaling. This estimate is standard and relies on the L^p-regularity theory for the operator $\mu - \triangle$. For a proof, we refer to [7, Lemma 2.9], which treats in addition the variable-coefficients case using the perturbation approach by Meyers. Indeed, Steps 3–6 of that proof show that if for some $p > 2$ the operator has an L^p-regularity theory, then (2.4) holds, whereas Step 1 shows that $\mu - \triangle$ has an L^p-regularity theory for all $1 < p < \infty$. The case $1 \le p < 2$ in (2.4) follows from the estimate for $p = 2$ by Hölder's inequality.

We now prove the boundedness of $\left\langle \phi_{\mu,i}^2 \right\rangle$ if $\alpha > 2$ by estimating the integrals using a doubly dyadic decomposition of $\mathbb{Z}^d \times \mathbb{Z}^d$. Note that the exponent $\alpha = 2$ is borderline in terms of integrability.

We write the integral as:

$$\begin{aligned}
&\int_{\mathbb{Z}^d}\int_{\mathbb{Z}^d} |\partial G_\mu(y)||\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy' \\
&= \int_{|y|\le 2}|\partial G_\mu(y)|\int_{|y-y'|\le 2}|\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy' && \text{(2.5a)}\\
&\quad + \int_{|y|\le 2}|\partial G_\mu(y)|\sum_{j=1}^\infty\int_{2^j<|y-y'|\le 2^{j+1}}|\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy' && \text{(2.5b)}\\
&\quad + \sum_{i=1}^\infty\int_{2^i<|y|\le 2^{i+1}}|\partial G_\mu(y)|\int_{|y-y'|\le 2}|\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy' && \text{(2.5c)}\\
&\quad + \sum_{i=1}^\infty\int_{2^i<|y|\le 2^{i+1}}|\partial G_\mu(y)|\sum_{j=1}^\infty\int_{2^j<|y-y'|\le 2^{j+1}}|\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy' && \text{(2.5d)}
\end{aligned}$$

By the uniform bound on $\|\partial G_\mu\|_{L^\infty(\mathbb{Z}^d)}$, the RHS term (2.5a) is of order 1. For the RHS term (2.5b), this uniform bound and the triangle inequality yield

$$\text{(2.5b)} \lesssim 1 + \sum_{j=1}^\infty \int_{2^j-2<|y'|\le 2^{j+1}+2}|\partial G_\mu(y')|\frac{1}{1+|y'|^\alpha}dy'.$$

Estimate (2.4) for $p = 1$ then yields that (2.5b) $\lesssim 1$ since $\alpha > 1$. The proof of the boundedness of the RHS term (2.5c) is similar. The most subtle part is the RHS term (2.5d). We split the double sum into two parts: $\sum_{i=1}^\infty\sum_{j\le i}$ and $\sum_{i=1}^\infty\sum_{j>i}$, and start with the latter. If $j > i$, and y, y' are such that $2^i < |y| \le 2^{i+1}$ and $2^j < |y-y'| \le 2^{j+1}$, then $2^{j-1} < |y'| \le 2^{j+2}$. Hence, by (2.4) for $q = 1$,

$$\begin{aligned}
&\int_{2^i<|y|\le 2^{i+1}}|\partial G_\mu(y)|\int_{2^j<|y-y'|\le 2^{j+1}}|\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy'\\
&\le\ (2^j)^{-\alpha}\int_{2^i<|y|\le 2^{i+1}}|\partial G_\mu(y)|dy\int_{2^{j-1}<|y'|\le 2^{j+2}}|\partial G_\mu(y')|dy'\\
&\lesssim\ (2^i)^{d-(d-1)}(2^j)^{d-(d-1)-\alpha}\ =\ (2^i)(2^j)^{1-\alpha}.
\end{aligned}$$

Since $\alpha > 1$, summing over $j > i$ yields

$$\int_{2^i<|y|\le 2^{i+1}}|\partial G_\mu(y)|\sum_{j<i}\int_{2^j<|y-y'|\le 2^{j+1}}|\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy' \lesssim (2^i)^{2-\alpha},$$

and therefore, using that $\alpha > 2$,

$$\sum_{i=1}^\infty\int_{2^i<|y|\le 2^{i+1}}|\partial G_\mu(y)|\sum_{j<i}\int_{2^j<|y-y'|\le 2^{j+1}}|\partial G_\mu(y')|\frac{1}{1+|y-y'|^\alpha}dydy' \lesssim 1. \qquad (2.6)$$

We now treat the sum $\sum_{i=1}^\infty\sum_{j\le i}$. If $j \le i$, and y, y' are such that $2^i < |y| \le 2^{i+1}$ and $2^j < |y-y'| \le 2^{j+1}$, then $2^{i-1} < |y'| \le 2^{i+2}$. Let $q > 1$ be such that that $\frac{d}{q} > \alpha$ (which is possible since $d > \alpha$), and $p > 1$ be the associated dual exponent, i. e. $\frac{1}{p} + \frac{1}{q} = 1$. Then, by Hölder's inequality with exponents (p, q), and (2.4) with exponents 1 and p, we have

$$\begin{aligned}
&\int_{2^i<|y|\le 2^{i+1}} |\partial G_\mu(y)| \int_{2^j<|y-y'|\le 2^{j+1}} |\partial G_\mu(y')| \frac{1}{1+|y-y'|^\alpha} dy dy' \\
&\le \int_{2^i<|y|\le 2^{i+1}} |\partial G_\mu(y)| dy \left(\int_{2^{i-1}<|y'|\le 2^{i+2}} |\partial G_\mu(y')|^p dy' \right)^{\frac{1}{p}} \\
&\qquad \times \left(\int_{2^j<|y-y'|\le 2^{j+1}} \frac{1}{1+|y-y'|^{q\alpha}} dy' \right)^{\frac{1}{q}} \\
&\lesssim (2^i)^{d-(d-1)} (2^i)^{\frac{1}{p}(d-p(d-1))} (2^j)^{\frac{1}{q}(d-q\alpha)} = (2^i)^{2-d(1-\frac{1}{p})} (2^j)^{\frac{d}{q}-\alpha}.
\end{aligned}$$

Summing over $j \le i$ and using that $\frac{d}{q} - \alpha > 0$ then yields

$$\begin{aligned}
&\int_{2^i<|y|\le 2^{i+1}} |\partial G_\mu(y)| \sum_{j\le i} \int_{2^j<|y-y'|\le 2^{j+1}} |\partial G_\mu(y')| \frac{1}{1+|y-y'|^\alpha} dy dy' \\
&\qquad \lesssim (2^i)^{2-d(1-\frac{1}{p})} \sum_{j\le i} (2^j)^{\frac{d}{q}-\alpha} \lesssim (2^i)^{2-d(1-\frac{1}{p})+\frac{d}{q}-\alpha} = (2^i)^{2-\alpha}.
\end{aligned}$$

Since $\alpha > 2$, this yields a bound for the first sum $\sum_{i=1}^{\infty} \sum_{j\le i}$:

$$\sum_{i=1}^{\infty} \int_{2^i<|y|\le 2^{i+1}} |\partial G_\mu(y)| \sum_{j\le i} \int_{2^j<|y-y'|\le 2^{j+1}} |\partial G_\mu(y')| \frac{1}{1+|y-y'|^\alpha} dy dy' \lesssim 1. \tag{2.7}$$

The combination of (2.6) and (2.7) shows that (2.5d) $\lesssim 1$, which, combined with the estimates of (2.5a), (2.5b), and (2.5c), implies the uniform-in-μ bound

$$\int_{\mathbb{Z}^d} \int_{\mathbb{Z}^d} |\partial G_\mu(y)| |\partial G_\mu(y')| \frac{1}{1+|y-y'|^\alpha} dy dy' \lesssim 1,$$

as desired.

2.3. **Proof of Corollary 1.** For all $x \in \mathbb{R}^d$, consider the random field $\Phi_x : \mathbb{Z}^d \times \Omega : (k, \omega) \mapsto \Phi(x + k, \omega)$. By Theorem 2, there exists some random vector $\tilde{X}(x, \cdot) \in L_0^2(\Omega)$ such that $(z, \omega) \mapsto \Phi_x(z, \omega) + \tilde{X}(x, \omega)$ is stationary: For all $z, k \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$,

$$\Phi_x(z + k, \omega) + \tilde{X}(x, \omega) = \Phi_x(z, \theta_k \omega) + \tilde{X}(x, \theta_k \omega),$$

which we may rewrite by definition of Φ_x as

$$\Phi(x + z + k, \omega) + \tilde{X}(x, \omega) = \Phi(x + z, \theta_k \omega) + \tilde{X}(x, \theta_k \omega). \tag{2.8}$$

Since $\Phi_{x+k}(k') = \Phi_x(k + k')$ for all $k, k' \in \mathbb{Z}^d$, the uniqueness of $\tilde{X}$ (cf. uniqueness of ϕ_i in Step 1 of the proof of Proposition 1) shows that $\tilde{X}(x + k, \cdot) = \tilde{X}(x, \cdot)$ for all $k \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$. Hence, (2.8) turns into: For all $x \in \mathbb{R}^d$, there exists a set of full measure Ω_x such that for all $z, k \in \mathbb{Z}^d$ and $\omega \in \Omega_x$, we have

$$\Phi(x + z + k, \omega) + \tilde{X}(x + z + k, \omega) = \Phi(x + z, \theta_k \omega) + \tilde{X}(x + z, \theta_k \omega).$$

To conclude, it remains to prove the measurability of $\tilde X : \mathbb{R}^d \times \Omega \to \mathbb{R}^d$ (where $\mathbb{R}^d$ is endowed with the Borel sets). We shall indeed prove that $\tilde X$ is a Carathéodory function. There exists a set $\Omega' \in \mathcal{F}$ of full measure such that for all $x \in \mathbb{Q}^d$, $z, k \in \mathbb{Z}^d$ and $\omega \in \Omega'$,

$$\Phi(x+z,\omega) + \tilde X(x+z,\omega) \;=\; \Phi(x,\theta_k\omega) + \tilde X(x,\theta_k\omega). \tag{2.9}$$

The uniform Lipschitz assumption on Φ yields: There exists $C<\infty$ such that for all $k \in \mathbb{Z}^d$ and $x,h \in \mathbb{R}^d$, $|\Phi(x+h+k,\cdot) - \Phi(x+k,\cdot)| \le C|h|$. Hence, substracting (2.9) once with $x \rightsquigarrow x+h$ and once with x implies that for all $x,h \in \mathbb{Q}^d$, $k \in \mathbb{Z}^d$, and $\omega \in \Omega'$,

$$\tilde X(x+h,\omega) - \tilde X(x,\omega) - \Big(\tilde X(x+h,\theta_k\omega) - \tilde X(x,\theta_k\omega)\Big) \;\le\; C|h|.$$

By summation over k, this yields for all $N \in \mathbb{N}$

$$\tilde X(x+h,\omega) - \tilde X(x,\omega) - \frac{1}{\#([-N,N)\cap\mathbb{Z})^d} \sum_{k\in([-N,N)\cap\mathbb{Z})^d} \tilde X(x+h,\theta_k\omega) - \tilde X(x,\theta_k\omega) \;\le\; C|h|.$$

By the ergodic theorem, and since $\Big\langle \tilde X(x,\cdot)\Big\rangle = \Big\langle \tilde X(x+h,\cdot)\Big\rangle = 0$, there exists some $\Omega'' \in \mathcal{F}$ with full measure such that for all $x \in \mathbb{Q}^d$, $h \in \mathbb{Q}^d$, $k \in \mathbb{Z}^d$ and $\omega \in \Omega''$ the limit $N \uparrow \infty$ yields

$$\tilde X(x+h,\omega) - \tilde X(x,\omega) \;\le\; C|h|.$$

By symmetry, this implies

$$|\tilde X(x+h,\omega) - \tilde X(x,\omega)| \;\le\; C|h|,$$

so that $\tilde X|_{\mathbb{Q}^d\times\Omega''}$ can be extended to a Lipschitz function on $\mathbb{R}^d$ for all $\omega \in \Omega''$. Hence, $\tilde X$ is a Carathéodory function. It is therefore equivalent to a measurable function on $\mathbb{R}^d \times \Omega$.

2.4. **Proof of Proposition 3.** We split the proof into two steps, using the stationarity of $\ell + \tilde Y = \Phi(\mathbb{Z}^d) + \tilde Y$, and of $\Phi + \tilde X$, respectively. Recall that $T = \langle \partial\Phi\rangle$.

Step 1. Stationarity of ℓ.
Since ℓ is stationary up to translation, there exists some $\tilde Y \in L^2_0(\Omega)$ such that for all $k \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$,

$$\ell(\theta_k\omega) + \tilde Y(\theta_k\omega) \;=\; \ell(\omega) + \tilde Y(\omega) + Tk.$$

Hence there is a measurable field $\gamma : \mathbb{Z}^d \times \Omega \times \mathbb{Z}^d \to \mathbb{Z}^d$ such that for all $k \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$, $\gamma(k,\omega,\cdot)$ is bijective on $\mathbb{Z}^d$ and such that

$$X_y(\theta_k\omega) + \tilde Y(\theta_k\omega) \;=\; X_{\gamma(k,\omega,y)}(\omega) + \tilde Y(\omega) + Tk,$$

where $X_y(\omega) = \Phi(y,\omega)$ as before. Combined with the stationarity of the increment $X_y - X_0$, this turns into

$$\begin{aligned} X_y(\theta_k\omega) + \tilde Y(\theta_k\omega) \;&=\; X_0(\theta_k\omega) + \tilde Y(\theta_k\omega) + X_y(\theta_k\omega) - X_0(\theta_k\omega) \\ &=\; X_y(\omega) + \tilde Y(\omega) + Tk + X_{\gamma(k,\omega,0)}(\omega) - X_0(\omega) \end{aligned} \tag{2.10}$$

Step 2. Stationarity of Φ and conclusion.
Since Φ is stationary up to translation, there exists $\tilde X \in L^2_0(\Omega)$ such that for all $y, k \in \mathbb{Z}^d$ and almost every $\omega \in \Omega$,

$$X_{y+k}(\omega) + \tilde X(\omega) \;=\; X_y(\theta_k\omega) + \tilde X(\theta_k\omega).$$

Combined with (2.10) this yields

$$X_{y+k}(\omega) - X_y(\omega) \;=\; Tk + \tilde{X}(\theta_k\omega) - \tilde{X}(\omega) + \tilde{Y}(\omega) - \tilde{Y}(\theta_k\omega) + X_{\gamma(k,\omega,0)}(\omega) - X_0(\omega).$$

Since the RHS does not depend on y, the increment $X_{y+k} - X_y$ does not depend on y either. By the ergodic theorem, and since the increment is stationary, this yields for almost every $\omega \in \Omega$ and for all $y, k \in \mathbb{Z}^d$,

$$\begin{aligned} X_{y+k} - X_y \;&=\; \lim_{R\to\infty} \frac{1}{\#([-R,R)\cap\mathbb{Z})^d} \sum_{z\in([-R,R)\cap\mathbb{Z})^d} X_{z+k} - X_z \\ &=\; \langle X_k - X_0\rangle \;=\; \langle \partial\Phi\rangle\, k \;=\; Tk. \end{aligned}$$

Hence, for almost every $\omega \in \Omega$ and all $y \in \mathbb{Z}^d$,

$$X_y(\omega) \;=\; X_0(\omega) + Ty,$$

so that we obtain

$$\begin{aligned} \Phi(y,\omega) \;&=\; \langle \partial\Phi\rangle\, y + \Phi(0,\omega), \\ \ell(\omega) \;&=\; \langle \partial\Phi\rangle\, \mathbb{Z}^d + \Phi(0,\omega), \end{aligned}$$

as desired.

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