Matrix-F5 algorithms and tropical Gröbner bases computation

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ABSTRACT

Let K be a field equipped with a valuation. Tropical varieties over K can be defined with a theory of Gröbner bases taking into account the valuation of K.

We design a strategy to compute such tropical Gröbner bases by adapting the Matrix-F5 algorithm. We show that both Matrix-F5 and the signature-preserving Matrix-F5 are available to tropical computation with respective modifications.

Our study is performed both over any exact field with valuation and some inexact fields like $\mathbb{Q}_p$ or $\mathbb{F}_q[[t]]$. In the latter case, we track the loss in precision, and show that the numerical stability compare favorably to the case of classical Gröbner bases. Numerical examples are provided.

Categories and Subject Descriptors

I.1.2 [**Computing Methodologies**]: Symbolic and Algebraic Manipulations—*Algebraic Algorithms*

General Terms

Algorithms, Theory

Keywords

Gröbner bases, tropical geometry, F5 algorithm, p-adic precision, p-adic algorithm

1. INTRODUCTION

Despite its young age, tropical geometry has revealed to be of significant value, with applications in algebraic geometry, combinatorics, computer science, and more recently non-archimedean geometry (see [4]).

Effective computation over tropical varieties make decisive usage of Gröbner bases, but before Chan and Maclagan's definition of tropical Gröbner bases in [3], computations were only available over exact fields where standard Gröbner bases techniques applied. In this document, we show that following Chan and Maclagan's definition, both Matrix-F5 and signature-based Matrix-F5 algorithms can be performed to compute tropical Gröbner bases. Moreover, if the input polynomials are in, for example, $\mathbb{Q}_p$ or $\mathbb{Q}[[t]]$, and known with enough precision, then Matrix-F5 can still be performed. Tropical Gröbner bases provide therefore a numericaly more stable substitute for classical Gröbner bases.

Related works on tropical Gröbner bases

We refer to the book of Maclagan and Sturmfels [9] for an introduction to computational tropical algrebraic geometry.

The computation of tropical varieties over $\mathbb{Q}$ with trivial valuation is available in the Gfan package by Anders Jensen (see [7]), by using standard Gröbner basis computation. Yet, if we want to compute tropical varieties over general fields, with non-trivial valuation, such techniques are not readily available. This is why Chan and Maclagan have developed in [3] a way to extend the theory of Gröbner bases to take into account the valuation and allow tropical computation. Their theory of tropical Gröbner bases is effective and allows, with a suitable division algorithm, a Buchberger algorithm.

Main results

Let K be a field equipped with a valuation. Let $\geq$ be an order on the terms of $K[X_1, \ldots, X_n]$ as in definition 2, defined with $\omega \in Im(val)^n$ and a monomial ordering $\geq_1$. Following [3], we define tropical D-Gröbner bases as for classical Gröbner bases.

Then, we provide as algorithm 1 a tropical row-echelon form computation algorithm for Macaulay matrices. We show that the F5 criterion still holds in a tropical setting. We therefore define the tropical Matrix-F5 algorithm (algorithm 2) as the adaptation of the classical Matrix-F5 algorithm with the tropical row-echelon form computation. We then have the following result :

PROPOSITION 1.1. *Let $(f_1, \ldots, f_s) \in K[X_1, \ldots, X_n]^s$ be a sequence of homogeneous polynomials. Then, the tropical Matrix-F5 algorithm computes a tropical D-Gröbner basis of $\langle f_1, \ldots, f_s \rangle$. Time-complexity is asymptotically the same as in the classical case : $O\left(s^2 D \binom{n+D-1}{D}^3\right)$ operations in K, as $D \to +\infty$.*

Not only does the tropical Matrix-F5 algorithm computes tropical D-Gröbner bases, but it is compatible with finite-precision coefficients, under the assumption that the sequence is regular. Let us assume that $K = \mathbb{Q}_p$ or $\mathbb{F}_q[[t]]$. Let $(f_1, \ldots, f_s) \in K[X_1, \ldots, X_n]^s$.

We define a bound on the precision, $prec_{F5trop}(\{f_1,\dots,f_s\},D,\geq)$, and one on the loss in precision, $loss_{F5trop}(\{f_1,\dots,f_s\},D,\geq)$, which depend explicitly on the coefficients of the f_i's.

Then we have the following proposition regarding to numerical stability of tropical Gröbner bases :

PROPOSITION 1.2. *Let $(f_1,\dots,f_s)\in K[X_1,\dots,X_n]^s$ be a regular sequence of homogeneous polynomials.*

Let $(f'_1,\dots,f'_s)$ be some approximations of $(f_1,\dots,f_s)$, with precision l better than $prec_{F5trop}(\{f_1,\dots,f_s\},D,\geq)$.

Then, with the tropical Matrix-F5 algorithm, one can compute an approximation $g'_1,\dots,g'_t$ of a Gröbner basis of $\langle f_1,\dots,f_s\rangle$, up to precision $l-loss_{F5trop}(\{f_1,\dots,f_s\},D,\geq)$.

This contrasts with the case of classical Gröbner bases over p-adics (or complete discrete valuation fields) considered in [12]. Indeed, the structure hypothesis **H2** which requires that the ideals$\langle f_1,\dots,f_i\rangle$ are weakly-w is no longer necessary. It is only replaced by the stronger assumption on the initial precision $prec_{F5trop}$ and $loss_{F5trop}$.

Finally, we show that the signature-based F5M algorithm can be adapted to compute tropical Gröbner bases. We first provide a tropical LUP form computation for Macaulay matrices that is compatible with signatures, and then the tropical signature-based Matrix-F5 algorithm (algorithms 3 and 4). We prove the following result :

PROPOSITION 1.3. *Let $(f_1,\dots,f_s)\in K[X_1,\dots,X_n]^s$ be a sequence of homogeneous polynomials. Then, the tropical signature-based Matrix-F5 algorithm computes a tropical D-Gröbner basis of $\langle f_1,\dots,f_s\rangle$.*

Concerning this algorithm, time-complexity is asymptotically the same as in the classical case : $O\left(s^2D\binom{n+D-1}{D}^3\right)$ operations in K, as $D\to+\infty$. Yet, a more refined analysis show that compared with the first tropical Matrix-F5 algorithm, complexity is better.

Structure of the paper

In section 2, we provide motivations from tropical geometry, and expose the ideas of Chan and Maclagan's algorithm.

In section 3, we show that matrix algorithms can be performed to compute tropical Gröbner bases. To that intent, after an introduction to matrix algorithms for Gröbner bases, we provide a row-reduction algorithm that will make the Matrix-F5 algorithm available. We then prove and analyze this tropical Matrix-F5 algorithm. Section 4 is devoted to the analysis of this algorithm over inexact fields with valuations, such as $\mathbb{Q}_p$.

In section 5, we provide a tropical LUP algorithm that is suited to a signature-based Matrix-F5 algorithm. We then present and prove a tropical signature-based Matrix-F5 algorithm. Section 6 is devoted to some numerical examples. Finally, section 7 will be a glance at some future development for tropical Gröbner bases.

2. TROPICAL MOTIVATIONS

Setting

From now on, let K be a field equipped with a valuation $val : K^*\to\mathbb{R}_+$. Let $\Gamma=Im(val)$. An example of such a field is $\mathbb{Q}$ with p-adic valuation. In that case, $\Gamma=\mathbb{Z}$.

Let also $n\in\mathbb{Z}_{>0}$, and $A=K[X_1,\dots,X_n]$. We write $|f|$ for the degree of a homogeneous polynomial $f\in A$, and $A_d=K[X_1,\dots,X_n]_d$ for the K-vector space of homogeneous polynomials of degree d.

2.1 Tropical varieties and tropical Gröbner bases

If I is an homogeneous ideal in $A=K[X_1,\dots,X_n]$, and $V(I)\subset\mathbb{P}_K^{n-1}$ is the projective variety defined by I. Then the tropical variety defined by I, or the tropicalization of $V(I)$, is $Trop(I)=val\left(V(I)\cap(K^*)^n\right)$. $Trop(I)$ is a polyhedral complex and acts as a combinatorial shadow of $V(I)$: many properties of $V(I)$ can be recovered combinatorially from $Trop(I)$.

If $w\in\Gamma^n$, we can define an order on the terms of $K[X_1,\dots,X_n]$.

Definition 1. If $a,b\in K$ and x^α and x^β two monomials in $K[X_1,\dots,X_n]$, $ax^\alpha\geq_w bx^\beta$ if $val(a)+w\cdot\alpha\geq val(b)+w\cdot\beta$. Naturally, it is possible that $ax^\alpha\neq bx^\beta$ and $val(a)+w\cdot\alpha=val(b)+w\cdot\beta$.

For any $f\in K[X_1,\dots,X_n]$, we can define $LT_{\geq_w}(f)$, and then $LT_{\geq_w}(I)$, for $I\subset K[X_1,\dots,X_n]$ an ideal, accordingly.

We shall note that $LT_{\geq_w}(f)$ might be a polynomial (with more than one term). For example, if we take $w=[1,2,3]$ in $\mathbb{Q}_2[x,y,z]$ (with 2-adic valuation), then

$$LT_{\geq_w}\left(x^4+x^2*y+2y^4+2^{-8}z^4\right)=x^4+x^2*y+2^{-8}z^4.$$

Therefore, in order to compute $Trop(I)$, one might want to use the following property :

THEOREM 2.1 (FUNDAMENTAL TH. OF TROPICAL GEOMETRY). *$Trop(I)$ is the closure in $\mathbb{R}^n$ of those $w\in\Gamma^n$ such that $LT_{\geq_w}(I)$ does not contain a monomial.*

PROOF. See Theorem 3.2.5 of the book of Maclagan and Sturmfels [9]. □

Andrew Chan and Diane Maclagan have developed in [3] a way to compute $LT_w(I)$ by adding another (classical) monomial order in order to break ties when $LT_w(f)$ has more than one monomial.

Definition 2. Let us take $\geq_1$ be a monomial order on $K[X_1,\dots,X_n]$.

Given $a,b\in K$ and x^α and x^β two monomials in $K[X_1,\dots,X_n]$, we shall write $ax^\alpha\geq bx^\beta$ if $val(a)+w\cdot\alpha<val(b)+w\cdot\beta$, or $val(a)+w\cdot\alpha\geq val(b)+w\cdot\beta$ and $x_1^\alpha\geq_1 x_1^\beta$.

We define $LT(f)$ and $LT(I)$ accordingly. We shall remark that $LT(I)=LT_{\geq_1}(LT_w(I))$. We define $LM(f)$ to be the monomial of $LT(f)$, and $LM(I)$ accordingly.

Remark 1. We use the same notations as in the article of Chan and Maclagan [3], except that we write $ax^\alpha\geq bx^\beta$, instead of $ax^\alpha\leq bx^\beta$. According to our order, the $LT_{\geq_1}(LT_w(f))=LT(f)$ of f (in contrary to Chan and Maclagan) is indeed its biggest monomial.

We will first briefly describe their *à la* Buchberger algorithm to compute $LT(I)$ for I a homogeneous ideal of $K[X_1,\dots,X_n]$, and the rest of the article will be devoted to the study of two matrix-F5 algorithms

2.2 The algorithm of Chan and Mcalagan

In their article [3], Chan & Maclagan have proved that if you modify the classical division algorithm of a polynomial by a finite family of polynomials with a variant of Mora's tangent cone algorithm, then you can get a division algorithm suited to the computation of tropical Gröbner bases. Indeed, they proved that Buchberger's algorithm using this division algorithm computes tropical Gröbner bases.

The main ideas of their division algorithm is to allow division by previous partial quotients, and chose the divisor polynomial with a suited *écart* function.

Following these ideas of tropical Buchberger algorithms, we prove in the following section that, when the ideal considered is generated by homogeneous polynomials, matrix algorithms are available.

3. A TROPICAL MATRIX-F5 ALGORITHM

From now on, and through the end of the article, we fix $w \in \Gamma^n$ and $\geq_1$ a monomial ordering on A. We denote by $\geq$ the term ordering defined by w and $\geq_1$, as in definition 2.

3.1 Matrix algorithm

Here we show that to compute a tropical Gröbner basis of an ideal given by a finite sequence of homogeneous polynomials, a matrix algorithm can be written. The first main idea is due to Daniel Lazard in [8], and comes from the following property, valid over any field :

PROPOSITION 3.1. *For an homogeneous ideal $I \subset A$, generated by homogeneous polynomials $(f_1, \ldots, f_s)$, for $d \in \mathbb{N}$, then as K-vector space :*

$$I \cap A_d = \langle x^\alpha f_i,\ |\alpha| + |f_i| = d \rangle .$$

One of the main features of this property is that it can be given in term of matrices. First, we define the matrices of Macaulay :

Definition 3. Let $B_{n,d}$ be the basis of the monomials of degree d, ordered decreasingly according to $\geq$. Then for $f_1, \ldots, f_s \in A[X_1, \ldots, X_n]$ homogeneous polynomials, $d \in \mathbb{N}$, we define $Mac_d(f_1, \ldots, f_s)$ to be the following matrix :

$$\begin{array}{c} \\ x^{\alpha_{1,1}} f_1 \\ \vdots \\ x^{\alpha_{1,\binom{n-1}{n+d-d_1-1}}} \\ x^{\alpha_{2,1}} f_2 \\ \vdots \\ x^{\alpha_{s,\binom{n-1}{n+d-d_s-1}}} f_s \end{array} \begin{array}{c} x^{d_1} > \quad \ldots \quad > \quad \ldots \quad > x^{d\binom{n-1}{n+d-1}} \\ \left[\begin{array}{ccc} & & \\ & * & \\ & & \end{array} \right] \end{array}$$

with $|\alpha_{i,*}| + |f_i| = d$. The rows of $Mac_d(f_1, \ldots, f_s)$ are the polynomials $x^\alpha f_i$ written in the basis $B_{n,d}$.

Then, naturally, if we identify the rows vectors of $k^{\binom{n+d-1}{n}}$ with homogeneous polynomials of degree d,

PROPOSITION 3.2. *$Im(Mac_d(f_1, \ldots, f_s)) = I \cap A_d$, with Im being the left image of the matrix.*

When performing classical matrix algorithm to compute Gröbner bases (see [1]), the idea is to compute row-echelon forms of the $Mac_d(f_1, \ldots, f_s)$ up to some D : if D is large enough, the reduced rows forms a Gröbner basis of I.

Although, it is not easy to guess in advance up to which D we have to perform row-reductions of Macaulay matrices. This why the idea of tropical D-Gröbner bases can be introduced.

Definition 4. Let I be an ideal of A, $\geq$ a tropical term ordering on A as in definition 2 and D an integer.

Then $(g_1, \ldots, g_l)$ is a D-Gröbner basis of I if for any $f \in I$, homogeneous of degree less than D, there exists $1 \leq i \leq l$ such that, regarding to $\geq$, $LT(g_i)$ divides $LT(f)$.

3.2 Tropical row-echelon form computation

This subsection is devoted to define an algorithm that will compute echelonized bases of the $Mac_d(f_1, \ldots, f_i)$, as long as $LT(\langle f_1, \ldots, f_i \rangle \cap A_d)$.

An algorithm

To perform this computation, we define the class of Macaulay matrices :

Definition 5. A Macaulay matrix of degree d in A is a couple (M, mon) where M is a matrix with $\binom{n+d-1}{n-1}$ columns and coefficients in K and mon is the list of the $\binom{n+d-1}{n-1}$ monomials of degree d of A, in decreasing order according to $\geq$. If mon is not ordered, (M, mon) is only called a labeled matrix.

The algorithm 1 computes tropical row-echelon form of Macaulay matrices.

Algorithm 1: The tropical row-echelon algorithm

input : M, a Macaulay matrix of degree d in $A = K[X_1, \ldots, X_n]$, with n_{row} rows and n_{col} columns.
output: $\widetilde{M}$, the tropical row-echelon form of M

begin
- $\widetilde{M} \leftarrow M$;
- **if** $n_{col} = 1$ *or* $n_{row} = 0$ *or* M *has no non-zero entry* **then**
  - Return $\widetilde{M}$;
- **else**
  - Find i, j such that $\widetilde{M}_{i,j}$ has the greatest term $\widetilde{M}_{i,j} x^{mon_j}$;
  - Swap the columns 1 and j of $\widetilde{M}$, and the 1 and j entries of mon;
  - Swap the rows 1 and i of $\widetilde{M}$;
  - By pivoting with the first row, eliminates the coefficients of the other rows on the first column;
  - Proceed recursively on the submatrix $\widetilde{M}_{i \geq 2, j \geq 2}$;
  - Return $\widetilde{M}$;

Definition 6. We define the tropical row-echelon form of a Macaulay matrix M as the result of the previous algorithm, and denote it by $\widetilde{M}$. $\widetilde{M}$ is indeed under row-echelon form.

Correctness

We prove here that $\widetilde{Mac_d(f_1,\dots,f_i)}$ provides what we expect for matrix algorithms :

PROPOSITION 3.3. *Let $M = Mac_d(f_1,\dots,f_s)$ be the Macaulay matrix of degree d in A defined by the homogeneous polynomials $f_1,\dots,f_s$. Let $I =< f_1,\dots,f_s >$ be the ideal generated by the f_i.*

Let $\widetilde{M}$ be the tropical row-echelon form of M. Then the rows of $\widetilde{M}$ form a basis of $I \cap A_d$, whose initial terms corresponds to the initial terms of the polynomials of $I \cap A_d$.

PROOF. The fact that the rows of $\widetilde{M}$ form a basis of $I \cap A_d$ is clear : it forms an echelonized basis (considering the basis *mon* of A_d).

Considering the initial terms of $I \cap A_d$, we begin by the following lemma :

LEMMA 3.4. *if $ax^\alpha > b_1x^\beta$ and $ax^\alpha > b_2x^\beta$, then $ax^\alpha > (b_1+b_2)x^\beta$.*

PROOF. Since $val(b_1+b_2) \geq \min(val(b_1), val(b_2))$, this result is clear. □

As a consequence, if piv is the pivot coefficient on the first column, then for any $1 \leq i \leq n_{row}$, for any $2 \leq j \leq n_{col}$, $piv \cdot x^{mon(1)} \geq \widetilde{M}_{i,j} \cdot x^{mon(j)}$.

The only fact to prove now is that for any $f \in I \cap A_d$, the initial monomial of f is the initial monomial of one of the rows of $\widetilde{M}$.

Since the rows of $\widetilde{M}$ generates $I \cap A_d$, then we can write $f = a_1L_1 + \cdots + a_lL_l$ with the $a_i \in K$. We can assume that the L_i are non-zero rows.

With the previous remark about $\widetilde{M}$, the lemma, and transitivity, it is clear that the initial term of $a_1L_1+a_2L_2$ is either $a_1\widetilde{M}_{1,1}x^{d_1}$ or $(a_2\widetilde{M}_{2,2} + a_1\widetilde{M}_{1,2})x^{d_2}$. The general result for $a_1L_1 + \cdots + a_lL_l$ can then be easily inductively derived.

Therefore, we can find all the polynomials of a tropical D-Gröbner basis of $< f_1,\dots,f_s >$ by computing the tropical row-echelon forms of the $Mac_i(f_1,\dots,f_s)$ for i from 1 to d.

3.3 The F5 criterion

Faugère proved in [6] that with classical monomial orderings, if we know which monomials x^α are in $LM(I_{i-1})$, we are able to discard corresponding rows $x^\alpha f_i$ of the Macaulay matrices. We prove here that his criterion is compatible with tropical initial ideals :

THEOREM 3.5 (F5-CRITERION). *Let $(f_1,\dots,f_i)$ be homogeneous polynomials of $K[X_1,\dots,X_n]$ of degree $d_1,\dots,d_i$. Let $a_{\alpha_1}x^{\alpha_1},\dots,a_{\alpha_u}x^{\alpha_u}$ be the initial terms of the rows of $\widetilde{Mac_{d-d_i}(f_1,\dots,f_{i-1})}$, ordered by decreasing order (regarding the initial term). Let x^{β_j} denote the remaining monomials of degree $d-d_i$ (i.e. the monomials that are not an initial monomial of $(f_1,\dots f_{i-1}) \cap A_{d-d_i}$). Then, the row $x^{\alpha_k}f_i$ of $Mac_d(f_1,\dots,f_i)$ is a linear combination of the rows of the form $x^{\alpha_{k+k'}}f_i$ ($k' > 0$), $x^{\beta_j}f_i$ or $x^\gamma f_j$ ($j < i$) of $Mac_d(f_1,\dots,f_i)$.*

PROOF. Since $a_{\alpha_u}x^{\alpha_u}$ is the initial term of the last non-zero row operated during the computation of $\widetilde{Mac_{d-d_i}(f_1,\dots,f_{i-1})}$, we can write $a_{\alpha_u}x^{\alpha_u} + \sum_{j=1}^{v} b_jx^{\beta_j} = \sum_{j=1}^{i-1} c_jf_j$ for some $v \in \mathbb{N}$, β_j monomials of degree $d-d_i$, $b_j \in K$ and $c_j \in K[X_1,\dots,X_n]$, with $b_jx^{\beta_j} < a_{\alpha_u}x^{\alpha_u}$.

Thus, $a_{\alpha_u}x^{\alpha_u}f_i = -\sum_{j=1}^{v} b_jx^{\beta_j}f_i + \sum_{j=1}^{i-1}(c_jf_i)f_j$, and the row $x^{\alpha_u}f_i$ is indeed a linear combination of the rows of the form $x^{\beta_k}f_i$ or $x^\gamma f_j$ ($j < i$) of $Mac_d(f_1,\dots,f_i)$.

In the same way, we can write $a_{\alpha_{u-1}}x^{\alpha_{u-1}} + a'_{\alpha_u}x^{\alpha_u} + \sum_{j=1}^{v} b_jx^{\beta_j} = \sum_{j=1}^{i-1} c_jf_j$ as $a_{\alpha_{u-1}}x^{\alpha_{u-1}}$ is the initial term of the second to last non-zero row computed when dealing with $\widetilde{Mac_{d-d_i}(f_1,\dots,f_{i-1})}$, thus $a_{\alpha_{u-1}}x^{\alpha_{u-1}}f_i = -a'_{\alpha_u}x^{\alpha_u}f_i - \sum_{j=1}^{v} b_jf_ix^{\beta_j} + \sum_{j=1}^{i-1}(f_ic_j)f_j$. The result shall therefore be proven inductively. □

COROLLARY 3.6. *As a consequence,*

$$I_i \cap A_d = Vect\left(\{x^\alpha f_k,\ s.t.\ 1 \leq k \leq i,\ |x^\alpha f_k| = d \text{ and } x^\alpha \notin LM(I_{k-1})\right)$$

Thus, it is now clear which rows we can remove with the F5 criterion. The following subsection will provide an effective way of taking advantage of this criterion.

3.4 A matrix-F5 algorithm

The tropical F5M algorithm

We apply Faugère's idea to the tropical setting and therefore provide a tropical F5 algorithm :

Algorithm 2: A tropical F5 algorithm

input : $F = (f_1,\dots,f_s) \in A^s$, with respective degrees $d_1,\dots,d_s$, and $D \in \mathbb{N}$
output: $(g_1,\dots,g_k) \in A^k$, a Gröbner basis of $Id(F)$, if D is large enough.

```
begin
    G ← F
    for d ∈ [[0, D]] do
        M̃_{d,0} := ∅
        for i ∈ [[1, s]] do
            M_{d,i} := M̃_{d,i-1}
            for α such that |α| + d_i = d do
                if x^α is not the leading term of a row
                of M̃_{d-d_i,i-1} then
                    Add x^α f_i to M_{d,i}
            Compute M̃_{d,i}, the tropical row-echelon
            form of M_{d,i}
            Add to G all the rows with a new leading
            monomial.
    Return G
```

Correctness

What we have to show is that for any $d \in [\![0, D]\!]$ and $i \in [\![1, s]\!]$, $Im(\mathscr{M}_{d,i}) = I_i \cap A_d$.

This can be proved by induction on d and i.

We shall remark that there is nothing to prove for $i = 1$ and any d.

Now let us assume that there exists some $i \in\in [\![1, s]\!]$ such that for any j with $1 \leq j < i$ and for any d, $0 \geq d \geq D$, $Im(\mathscr{M}_{d,i}) = I_i \cap A_d$.

Then, for i, the first d such that $\mathscr{M}_{d,i} \neq \mathscr{M}_{d,i-1}$ is d_i.

Let d be such that $d_i \leq d \leq D$.

Then, with the induction hypothesis and corollary 3.6 :

$$I_i \cap A_d = Im(\mathscr{M}_{d,i-1}) + Vect\left(\{x^\alpha f_i, \text{ s.t. } x^\alpha \notin LM(I_{i-1})\}\right). \quad (1)$$

Besides, by the induction hypothesis and the correctness of the row-echelon algorithm (see proposition 3.3), the leading terms of $I_{i-1} \cap A_{d-d_i}$ are exactly the leading terms of rows of $\widetilde{\mathscr{M}_{d-d_i,i-1}}$. Thus, the rows that we add to $\widetilde{\mathscr{M}_{d,i-1}}$ in order to build $\mathscr{M}_{d,i}$ are exactly the $x^\alpha f_i$, such that $x^\alpha \notin LM(I_{i-1})$.

Finally, we remark that $Im(\mathscr{M}_{d,i-1}) = Im(\widetilde{\mathscr{M}_{d,i}})$. Therefore, $Im(\mathscr{M}_{d,i})$ contains both the summands of (1), and since it is clearly included in $I_i \cap A_d$, we have proved that $I_i \cap A_d = Im(\mathscr{M}_{d,i})$.

To conclude the correctness of the tropical F5M algorithm, we point out that the correctness of the tropical row-echelon computation (see prop 3.3) show that the leading terms of rows of $\widetilde{\mathscr{M}_{d,i}}$ do indeed corresponds to the leading terms of $I_i \cap A_d$.

3.5 Regular sequences and complexity

Principal syzygies and regularity

The behavior of this algorithm with respect to principal syzygies is the same as the classical F5 algorithm to compute Gröbner bases. See [1] for a precise description of the link between syzygies and row-reduction. We shall instead only prove the main result linking principal syzygies and tropical row-reduction of Macaulay matrices.

PROPOSITION 3.7. *If a row reduces to zero during the tropical row-echelon form computation of the tropical F5 algorithm, then the sizygy it yields is not in the module of principal syzygies.*

PROOF. We use the concept of label and signature of definitions 9 and 10 in section 5.

If a row L reduces to zero during the tropical row-echelon form computation of $\mathscr{M}_{d,i}$ such that the sizygy it yields is in the module of principal syzygies, then, the label of this row is of the form $(a_1, \dots, a_i)$ and without loss of generality, we might assume $a_i \neq 0$.

Then, by definition of the module of principal syzygies, $a_i \in (f_1, \dots, f_{i-1})$, therefore, the signature of L is x^α, an initial monomial of $(f_1, \dots, f_{i-1}) \cap A_{d-d_i}$, and so, of the rows of $\widetilde{\mathscr{M}_{d-d_i,i-1}}$. Yet, there is no row in $\mathscr{M}_{d,i}$ with a monomial x^α on the $i-th$ component of its label that is the initial term of a row of $\mathscr{M}_{d-d_i,i-1}$, because of the F5 criterion in the building of the matrix. Therefore, no linear combination of rows of $\mathscr{M}_d(f_1, \dots, f_i)$ can yield a row of with such a label. Thus, no row of $\mathscr{M}_d(f_1, \dots, f_i)$ can reduce into a principal syzygy. □

COROLLARY 3.8. *If the sequence $(f_1, \dots, f_s)$ is regular, then no row of a Macaulay matrix in the tropical F5M algorithm reduces to zero. In other words, the $\mathscr{M}_{d,i}$ are all injective, and have less rows than columns.*

PROOF. For a regular sequence of homogeneous polynomials, all syzygies are principal. See [5] page 69 for more about this. □

Complexity

Complexity has exactly the same behavior as in the classical case, for which we refer to [2] :

- $O\left(s^2 D\binom{n+D-1}{D}^3\right)$ operations in K, as $D \to +\infty$.
- $O\left(sD\binom{n+D-1}{D}^3\right)$ operations in K, as $D \to +\infty$, in the special case where $(f_1, \dots f_s)$ is regular, because of corollary 3.8.

4. THE CASE OF FINITE-PRECISION CDVF

4.1 Setting

Throughout this section, K is a complete discrete valuation field, whose valuation is also denoted by $val : K^* \to \mathbb{R}_+$. We refer to Serre [10] for an introduction to such fields. We denote by $R = O_K$ its ring of integers, m_K its maximal ideal and $k = O_K/m_K$ its fraction field. Let $\pi \in R$ be a uniformizer for K and let $S_K \subset R$ be a system of representatives of $k = O_K/m_K$.

All numbers of K can be written uniquely under its π-adic power series development form : $\sum_{k \geq l} a_k \pi^l$ for some $l \in \mathbb{Z}$, $a_k \in S_K$.

We assume that K is not an exact field, but k is, and symbolic computation can only be performed on truncation of π-adic power series development. We shall denote by finite-precision CDVF such a field. An example of such a CDVF is $K = \mathbb{Q}_p$, with p-adic valuation.

We are interested in the computation of tropical Gröbner bases over finite-precision CDVF. We first study when does the leading monomial of a polynomial known only up to finite-precision is well-defined, and which row-echelon form computation can be performed when the precision is finite.

4.2 Precision issues

Indeed, if the precision on $f \in A$ is not enough, then one can not determine what the leading term of f is. For example, on $\mathbb{Q}_p[X_1, X_2]$, with $w = (0, 4)$ and lexicographical order, then one can not compare $O(p^2) * X_1$ and X_2.

Yet, if the precision is enough, such an issue should not occur when computing tropical row-echelon form. The following proposition provides a bound on the precision needed on f to determine its leading term.

PROPOSITION 4.1. *Let $f \in A$ be an homogeneous polynomial, and let aX^α be its leading term.*

Then the precision needed on f for its leading term to be well-defined is $val(a) + \max_{|\beta|=d}((\alpha - \beta) \cdot w)$. The leading monomials according to $\geq_w$ are then also well-defined.

PROOF. We only have to remark that $O(p^n)X^\beta < aX^\alpha$ if and only if $n > val(a) + (\alpha - \beta) \cdot w$. □

4.3 Row-echelon form computation

Regular sequences

As usual when dealing with finite-precision coefficients, one can not decide whether a coefficient $O(\pi^k)$ is zero or not. Fortunately, thanks to corollary 3.8, when the input polynomials form a regular sequence, all matrices in the tropical F5M algorithm are injective. It means that if the precision is enough, the tropical row-echelon form computation performed over these matrices will have no issue with finding pivots and deciding what the leading terms of the rows are.

We estimate which precision is "enough" in order to be able to compute D-Gröbner bases of such a sequence.

A sufficient precision

PROPOSITION 4.2. *Let M be an injective tropical Macaulay matrix, of degree d. Let $a_1, \dots, a_u$ be the pivots chosen during the computation of its tropical row-echelon form. Let x^{α_k} be the corresponding monomials. Let prec be :*

$$prec = \sum_k val(a_k) + \max_k val(a_k) + \max_{k,|\beta|=d} (\alpha_k - \beta) \cdot w.$$

Then, if the rows are known up to flat precision $O(\pi^{prec})$, the tropical row-echelon form computation of M can be computed.

PROOF. We begin with a matrix M with coefficients known with finite flat precision $O(\pi^l)$, and we first assume that there is no issue with finding the pivots. Thus, we shall first understand what the loss in precision is when we pivot.

That is, we wish to put a "real zero" on the coefficient $M_{i,j} = \varepsilon\pi^{n_1} + O(\pi^n)$, by pivoting with a pivot $piv = \mu\pi^{n_0} + O(\pi^n)$ on row L, with $n_0, n_1 < n$ be integers, and $\varepsilon = \sum_{j=0}^{n-n_1-1} a_j\pi^j$, $\mu = \sum_{j=0}^{n-n_0-1} b_j\pi^j$, with $a_j, b_j \in S_K$, and $a_0, b_0 \neq 0$. We remark that by definition of the pivot, necessarily, $n_0 \leq n_1$.

Now, this can be performed by the following operation on the i-th row L_i :

$$L_i \leftarrow L_i - \frac{M_{i,j}}{piv} L = L_i + (\varepsilon\mu^{-1}\pi^{n_1-n_0} + O(\pi^{n-n_0}))L,$$

along with the symbolic operation $M_{i,j} \leftarrow 0$.

Indeed, $\frac{M_{i,j}}{piv} = \frac{\varepsilon\pi^{n_1}+O(\pi^{m_1})}{\mu\pi^{n_0}+O(\pi^{m_0})}$, therefore $\frac{M_{i,j}}{piv} = \varepsilon\mu^{-1}\pi^{n_1-n_0} + O(\pi^{n-n_0})$.

As a consequence, after the first pivot is chosen and other coefficient of the first column have been reduced to zero, the coefficients of the submatrix $\widetilde{M}_{i\geq 2, j\geq 2}$ are known up to $O(\pi^{l-val(a_1)})$.

We can then proceed inductively to prove that after the termination of the tropical row-echelon form computation, coefficients of $\widetilde{M}$ are known up to $O(\pi^{l-val(a_1,\dots,a_u)})$.

Now, since we have to be able to determine what the leading terms of the rows are in order to determine what the pivots are, then, with proposition 4.1, it is enough that $l - val(a_1, \dots, a_u)$ is bigger than $\max_{k,|\beta|=d} (\alpha - \beta) \cdot w$.

Hence, the result is proved. □

4.4 Tropical F5M algorithm

We apply this study of the row-echelon computation to prove proposition 1.2 concerning the tropical Matrix-F5 algorithm over CDVF. To facilitate this investigation, and only for section 4, the step $\widetilde{\mathscr{M}_{d,i}} := \widetilde{\mathscr{M}_{d,i-1}}$ in algorithm 2 is replaced with $\mathscr{M}_{d,i} := \mathscr{M}_{d,i-1}$. We first define bounds on the initial precision and loss in precision. Let $(f_1, \dots, f_s) \in K[X_1, \dots, X_n]^s$ be a regular sequence of homogeneous polynomials.

Definition 7. Let $d \geq 1$ and $1 \leq i \leq s$. Let $x^{\alpha_1}, \dots, x^{\alpha_u}$ be the monomials of the leading terms of $\langle f_1, \dots, f_i\rangle \cap A_d$.

Let $\Delta_{d,i}$ be the minor over the columns corresponding to the x^{α_l} with smallest valuation. Let

$$\square_{d,i} = 2\Delta_i + \max_{k,|\beta|=d} (\alpha_k - \beta) \cdot w.$$

We define $prec_{F5trop}(\{f_1, \dots, f_s\}, D, \geq) = \max_{d\leq D,i} \square_{d,i}$, and $loss_{F5trop}(\{f_1, \dots, f_s\}, D, \geq) = \max_{d\leq D,i} \Delta_{d,i}$.

As a consequence of proposition 4.2, these bounds are enough for proposition 1.2.

Furthermore, we can precise the special case of $w = 0$:

PROPOSITION 4.3. *If $w = 0$, then the loss in precision corresponds to the maximal minors of the $\mathscr{M}_{d,i}$ with the smallest valuation. In particular, $w = 0$ corresponds to the smallest $loss_{F5trop}$.*

4.5 Precision versus time-complexity

We might remark that if one want to achieve a smaller loss in precision, one might want to drop the F5 criterion and use the tropical row-reduction algorithm on the whole Macaulay matrices until enough linearly-free rows are found. The required number of rows can be computed thanks to the F5-criterion and corollary 3.6 if Macaulay matrices are operated iteratively in d and i.

This way, one would be assured that its pivots will yield the smallest loss of precision possible over $Mac_d(f_1, \dots, f_s)$. Yet, such an algorithm would be more time-consuming, being in $O\left(s^2 D\binom{n+D-1}{D}^3\right)$ instead of $O\left(sD\binom{n+D-1}{D}^3\right)$.

4.6 Comparison with classical Gröbner bases

We compare here the results over finite-precision CDVF for computation of tropical Gröbner bases and for computation of classical Gröbner bases, as it was performed in [12].

We recall the main result of [12] :

Definition 8. Let $(f_1, \dots, f_s) \in K[X_1, \dots, X_n]^s$ be homogeneous polynomials. Let $\mathscr{M}_{d,i}$ be the Macaulay matrix in degree d for $(f_1, \dots, f_i)$, without the rows discarded by the F5-criterion. Let $l_{d,i}$ be the maximum of the $l \in \mathbb{Z}_{\geq 0}$ such that the l-first columns of $\mathscr{M}_{d,i}$ are linearly free. We define

$$\Delta_{d,i} = \min\left(val\left(\{\text{minor over the } l_{d,i}\text{-first columns of } \mathscr{M}_{d,i}\}\right)\right).$$

We define the Matrix-F5 precision of $(f_1, \dots, f_s)$ regarding to w and D as :

$$prec_{MF5}(\{f_1, \dots, f_s\}, D, w) = \max_{d\leq D,\, 1\leq i\leq s} val\left(\Delta_{d,i}\right).$$

Then, $prec_{MF5}(\{f_1, \dots, f_s\}, D, w)$ is enough to compute approximate D-Gröbner bases :

THEOREM 4.4. *Let $(f'_1, \dots, f'_s)$ be approximations of the homogeneous polynomials $(f_1, \dots, f_s) \in K[X_1, \dots, X_n]^s$, with precision better than $prec_{MF5} = prec_{MF5}(\{f_1, \dots, f_s\}, D, w)$. We assume that $(f_1, \dots, f_s)$ is a regular sequence (**H1**) and all the $\langle f_1, \dots, f_i\rangle$ are weakly-w-ideals (**H2**). Then, the weak Matrix-F5 algorithm computes an approximate D-Gröbner basis of $(f'_1, \dots, f'_s)$, with loss in precision upper-bounded by $prec_{MF5}$. The complexity is in $O\left(sD\binom{n+D-1}{D}^3\right)$ operations in K, as $D \to +\infty$.*

We remark that for tropical Gröbner bases, the structure hypothesis **H2** is compensated by the precision requirement for the tropical row-echelon computation : $\max_k val(a_k) + \max_{k,|\beta|=d} (\alpha_k - \beta) \cdot w$. There is no position problem for the leading terms when a tropical Gröbner basis is computed, as long as the precision is enough. This leads to a sufficient precision $prec_{F5trop}(\{f_1, \dots, f_s\}, D, \geq)$ that might be bigger than $prec_{MF5}$.

Yet, for tropical Gröbner bases, the only structure hypothesis is **H1**, and is clearly generic, whereas for classical

Gröbner bases, **H1** and **H2** might be generic only for the grevlex ordering, if Moreno-Socias' conjecture holds.

Therefore, tropical Gröbner bases computation may require a bigger precision on the input than classical Gröbner bases, but it can be performed generically, while it is not clear for classical Gröbner bases.

5. A FASTER TROPICAL F5M ALGORITHM

In this section, we see that one can perform a signature-based tropical Matrix-F5 algorithm where the fact that $\widetilde{\mathscr{M}_{d,i}}$ is under echelon form is used to build a $\mathscr{M}_{d,i}$ closer to its echelon-form.

To that intent, we introduce labels and signatures for polynomials, and a tropical LUP form computation. They allow us to adapt the classical signature-based Matrix-F5 algorithm (see [1]).

5.1 Label and signature

Definition 9. Given $(f_1, \dots, f_s) \in A^s$, a *labeled polynomial* is a couple (u, p) with $u = (l_1, \dots, l_s) \in A^s$, $p \in A$ and $\sum_{i=1}^{s} l_i f_i = p$.

u is called the *label* of the labeled polynomial. We write $(e_1, \dots, e_s)$ to be the canonical basis of A^s.

If $u = (l_1, \dots, l_i, 0, \dots, 0)$ with $l_i \neq 0$, then the *signature* of the labeled polynomial (u, p) is $(HM(l_i), i)$, with the following definition : $HM(l_i)$ is the highest monomial, regarding to $\leq$, that appears in l_i with a non-zero coefficient.

Remark 2. We must point out that in the definition of the signature, we *do not* take into account the valuations of the coefficients in the label, hence the $HM(l_i)$ instead of $LT(l_i)$ or $LM(l_i)$. $HM(l_i)$ is not, in general, the monomial of the leading term of l_i.

Signatures can be compared :

Definition 10. We define a total order on the set {monomials in R}$\times$ $\{1, \dots, s\}$ of the signatures with the following definition : $(x^\alpha, i) \leq (x^\beta, k)$ if $i < k$, or $x^\alpha \leq x^\beta$ and $i = k$.

We shall also remark some compatibility of signatures with operations over labeled polynomials :

PROPOSITION 5.1. *Let (u, p) be a labeled polynomial, and let x^α be a monomial in A. Then*

$$sign((x^\alpha u, x^\alpha p)) = x^\alpha sign((u, p)).$$

If (v, q) is another labeled polynomial such that $sign((v, q)) < sign((u, p))$, and if $\mu \in K$, then $sign((u + \mu v, p + \mu q)) = sign((u, p))$.

5.2 Signature-preserving LUP form computation

From now on throughout this subsection, an additional datum will be attached to the rows of the Macaulay matrices : its label and signature. We make the further assumption that the rows are ordered with increasing signature. Such a matrix will be called a labeled Macaulay matrix. When adding a row, both its label and its signature will be noted, and all the operations on the rows shall be carried on to the labels of these rows.

Algorithm 3: The tropical LUP algorithm

input : M, a labeled Macaulay matrix of degree d in $A = K[X_1, \dots, X_n]$, with n_{row} rows and n_{col} columns.
output: $\widetilde{M}$, the tropical LUP form of M

```
begin
    M̃ ← M ;
    if n_col = 1 or n_row = 0 or M has no non-zero entry then
        Return M̃ ;
    else
        for i = 1 to n_row do
            Find j such that M̃_{i,j} has the greatest term M̃_{i,j} x^{mon_j} over the row;
            Swap the columns 1 and j of M̃, and the 1 and j entries of mon;
            By pivoting with the first row, eliminates the coefficients of the other rows on the first column;
            Proceed recursively on the submatrix M̃_{i≥2,j≥2};
        Return M̃;
```

The algorithm

We provide a tropical LUP algorithm for labeled Macaulay matrices.

We remark :

- At the end of the algorithm, there exist a unipotent lower-triangular matrix L, a permutation matrix P, such that $\widetilde{M} = LMP$.
- Moreover, $\widetilde{M}$ is under row-echelon form up to permutation.
- Since we only add to a row L a linear combination of rows that are above L, those rows have a strictly lower signature than L, and therefore the operations performed on the rows (and on the columns) preserve the signature.

Furthermore,

PROPOSITION 5.2. *For any $1 \leq i \leq n_{row}(M)$, if j is the index of the i-th row of $\widetilde{M}$, then $\widetilde{M}_{i,j} x^{mon_j}$ is the leading term of the polynomial corresponding to this row.*

Those remarks justify the name of tropical LUP algorithm, and the facts that this algorithm computes the leading terms of $Vect(rows(M))$.

Finally, since signature remains unchanged throughout the tropical LUP reduction, we can omit the labels and only handle Macaulay matrices on which the signatures of the rows are marked.

5.3 A signature-based tropical F5M algorithm

The signature-based F5 criterion

PROPOSITION 5.3. *Let (u, f) be a labeled homogeneous polynomial of degree d, such that $sign(u) = x^\alpha e_i$, with $1 < i \leq s$*

and $x^\alpha \in I_{i-1}$. Then,

$$x^\alpha \in Vect\left(\left\{x^\beta f_k, |x^\beta f_k| = d,\ k < i,\ \text{or } i = k \text{ and } x^\beta < x^\alpha\right\}\right).$$

As a consequence, if (u, f) is a labeled homogeneous polynomial of degree d with $sign(u) = x^\alpha e_i$ and $x^\alpha \notin LM(I_{i-1})$. Then f can be written $f = x^\alpha f_i + g$, with

$$g \in Vect\left(\left\{x^\beta f_k, |x^\beta f_k| = d,\ k < i,\ \text{or } i = k \text{ and } x^\beta < x^\alpha\right\}\right),$$

i.e., g is a linear combination of $x^\beta f_k$ such that $x^\beta e_k < x^\alpha e_i$.

A faster tropical Matrix-F5 algorithm

Algorithm 4: The tropical signature-based Matrix-F5 algorithm

input : $F = (f_1, \dots, f_s) \in A^s$, with respective degrees $d_1, \dots, d_s$, and $D \in \mathbb{N}$
output: $(g_1, \dots, g_k) \in A^k$, a D-tropical Gröbner basis of $Id(F)$, if D is large enough.

begin
  $G \leftarrow F$
  for $d \in [\![0, D]\!]$ **do**
    $\widetilde{\mathscr{M}_{d,0}} := \emptyset$
    for $i \in [\![1, s]\!]$ **do**
      $\mathscr{M}_{d,i} := \widetilde{\mathscr{M}_{d,i-1}}$
      for *L a row of* $\widetilde{\mathscr{M}_{d-1,i}}$ **do**
        for $x \in \{X_1, \dots, X_n\}$ **do**
          $x^\alpha e_k := sign(xL)$
          if *$k = i$, x^α is not the leading term of a row of* $\widetilde{\mathscr{M}_{d-d_i,i-1}}$, *and $\mathscr{M}_{d,i}$ has not already a row with signature $x^\alpha e_i$* **then**
            Add xL to $\mathscr{M}_{d,i}$.
      Compute $\widetilde{\mathscr{M}_{d,i}}$, the tropical LUP form of $\mathscr{M}_{d,i}$.
      Add to G all the rows with a new leading monomial.
  Return G

Correctness

PROPOSITION 5.4. *This algorithm indeed compute a tropical D-Gröbner basis.*

PROOF. The first thing to prove is that with the building of the Macaulay matrices suggested in the algorithm, the two following properties are satisfied : $Im(\mathscr{M}_{d,i}) = I_i \cap A_d$ and for any monomial x^α of degree $d - d_i$ such that $x^\alpha \notin LM(I_{i-1})$, $\mathscr{M}_{d,i}$ has a row with signature $x^\alpha e_i$.

This can be proved by induction on d and i.

If $i = 1$, the result is clear, for any d. Now, we may assume that this result is true for $i - 1 \in \mathbb{Z}_{\geq 0}$.

The first d for which $\mathscr{M}_{d,i} \neq \mathscr{M}_{d,i-1}$ is d_i, where the result is also clear.

Let us assume that our inclusions are true for some $d \geq d_i$.

Let x^α be a monomial of degree $d - d_i$ such that $x^\alpha \notin LM(I_{i-1})$. If $d - d_i = 0$, there is nothing to prove. Otherwise, let x^β be a monomial of degree $d - d_i - 1$ such that x^β divides x^α. Then, necessarily, $x^\beta \notin LM(I_{i-1})$, and by the induction hypothesis, $\mathscr{M}_{d-1,i}$ has a row of signature x^β.

The tropical LUP algorithm preserves the signature, therefore, $\widetilde{\mathscr{M}_{d-1,i}}$ also has a row with signature x^β. The building of $\mathscr{M}_{d,i}$ from $\widetilde{\mathscr{M}_{d-1,i}}$ will thus provide a row with signature x^α for $\mathscr{M}_{d,i}$.

Now, let $\overline{M_{d,i}}$ be the Macaulay matrix whose rows are $x^\alpha f_k$, with $k \leq i$, $x^\alpha \notin LM(I_{i-1})$, ordered as in definition 3. By corollary 3.6, $Im(\overline{M_{d,i}}) = I_i \cap A_d$.

Then, with the second F5 criterion, proposition 5.3, and the definition of tropical LUP reduction, there exist some permutation matrix P and some lower-triangular, unipotent, matrix L such that $\mathscr{M}_{d,i} = L\overline{M_{d,i}}P$.

It is therefore clear that $Im(\mathscr{M}_{d,i}) = I_i \cap A_d$, which terminates the proof by induction.

Now, since the tropical LUP reduction indeed computes an echelon-basis of the $\mathscr{M}_{d,i}$, as in the previous tropical F5M algorithm, one can directly prove that the signature-based tropical F5M algorithm computes a tropical D-Gröbner basis. □

Complexity

Asymptotically, the complexity to compute a tropical D-Gröbner basis of $(f_1, \dots, f_s)$ is the same as the previous tropical F5M algorithm, that is to say, $O\left(s^2 D \binom{n+D-1}{D}^3\right)$ operations in K, as $D \to +\infty$. Yet, the, quadratic, step where one look for a pivot is replaced by finding the leading term of a row, which is linear (in the size of the matrix). Moreover, We use $\widetilde{\mathscr{M}_{d-1,i}}$, which is already under row-echelon form, to build $\mathscr{M}_{d,i}$. Thus, $\mathscr{M}_{d,i}$, is "closer" to be under row-echelon form, and the row-reduction is easier.

6. IMPLEMENTATION

A toy implementation in Sage [11] of the previous algorithm is available at `http://perso.univ-rennes1.fr/tristan.vacco`

The purpose of this implementation was the study of the precision. It is therefore not optimized regarding to time-complexity.

We have experimented the tropical Matrix-F5 algorithm with homogeneous polynomials with varying degrees and random coefficients in $\mathbb{Z}_p$: $f_1, \dots, f_s$, of degree $d_1, \dots, d_s$ in $\mathbb{Q}_p[X_1, \dots, X_s]$, known up to initial precision 30, with a given weight w and the grevlex ordering to break the ties. We present the results in the following array :

$d =$	w	D	p	iterations	maximal loss	mean loss
[3,4,7]	[1,-3,2]	12	2	20	17	.81
[3,4,7]	[0,0,0]	12	2	20	0	0
[3,4,7]	[1,-3,2]	12	7	20	1	.01
[2,3,4,5]	[1,4,1,1]	11	2	2	2	.3
[2,3,4,5]	[0,0,0]	11	2	1	0	0
[2,3,4,5]	[1,4,1,1]	11	7	2	0	0
[2,4,5,6]	[1,4,1,1]	14	2	2	19	6.13

We emphasize that the precision was enough on every attempt.

We remark that these results suggest that the loss in precision is less whean working with bigger primes. It seems reasonnable since the loss in precision comes from pivots with positive valuation, wheras the probability that $val(x) = 0$ for $x \in \mathbb{Z}_p$ is $\frac{p-1}{p}$. Those results also corroborate the facts that $w = [0, \dots, 0]$ lead to significantly smaller loss in precision.

7. FUTURE WORKS

Since both the Buchberger algorithm and the signature-based Matrix-F5 algorithm are available, it is reasonnable to consider a tropical F5-algorithm. We conjecture that Faugère's algorithm applied in a tropical setting will computes tropical Gröbner bases. We might have to use signatures as in section 5 and adapt the TopReduction of [6].

With the numerical stability of proposition 1.2, one might want to study which computations with tropical Gröbner bases can be performed under a finite-precision setting. One could be interested in investigating the computation of the degree of an ideal or the elimination by using block-order.

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