Dirichlet eigenvalues of asymptotically flat triangles

Thomas Ourmières-Bonafos

DIRICHLET EIGENVALUES OF ASYMPTOTICALLY FLAT TRIANGLES

THOMAS OURMIÈRES-BONAFOS*

Abstract

This paper is devoted to the study of the eigenpairs of the Dirichlet Laplacian on a family of triangles where two vertices are fixed and the altitude associated with the third vertex goes to zero. We investigate the dependence of the eigenvalues on this altitude. For the first eigenvalues and eigenfunctions, we obtain an asymptotic expansion at any order at the scale cube root of this altitude due to the influence of the Airy operator. Asymptotic expansions of the eigenpairs are provided, exhibiting two distinct scales when the altitude tends to zero. In addition, we generalize our analysis to the case of a shrinking symmetric polygon and we quantify the corresponding tunneling effect.

1 Introduction

1.1 Motivations and related questions

There are few planar domains for which we can have an explicit expression of the eigenpairs of the Dirichlet Laplacian. Nevertheless there has been a recent interest about it on thin domains in $\mathbb{R}^2$. In this limit, asymptotic expansions of the eigenvalues and eigenfunctions can be provided, which informs about the spectrum of the Dirichlet Laplacian.

In this spirit, Borisov and Freitas give in [3] a construction of quasimodes of the Dirichlet Laplacian expanded at any order in the height paramater on thin smooth planar domains. In [10], Friedlander and Solomyak overcome the smooth domain hypothesis: they provide a two-term asymptotics using the convergence of resolvents.

The result of Friedlander and Solomyak applies to triangles but, before investigating triangles in the thin limit, one can cite the work of McCartin [19] who gives an explicit expression of the first eigenvalue of the Dirichlet Laplacian, say μ_1, on an equilateral triangle of altitude H: $\mu_1(H) = 4\pi^2 H^{-2}$. Hillairet and Judge prove in [13] the simplicity of the eigenvalues for almost every Euclidean triangle of $\mathbb{R}^2$.

The question of an asymptotic expansion for the Dirichlet Laplacian on thin triangles has already been studied by Freitas in [9]. In this paper a finite asymptotic expansion of the first two eigenvalues is provided for a family of near isosceles triangles. We also refer to the work of Dauge and Raymond [6] in which an asymptotics at any order is given for the first eigenvalues of the Dirichlet Laplacian on a right-angled thin triangle.

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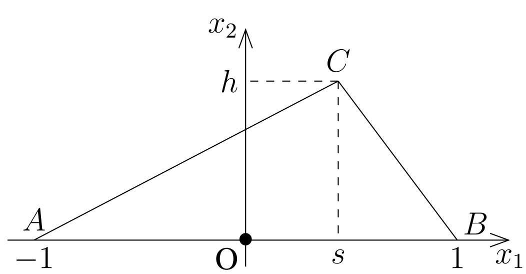
The question of finding an asymptotic expansion at any order for the Dirichlet Laplacian eigenmodes on a non-smooth thin domain is still open. We tackle the question in this paper for triangles of altitude h in the regime $h \to 0$. In a first step we construct quasimodes involving simultaneously two scales: $h^{2/3}$ and h. The scale $h^{2/3}$ is due to the singularity of type $(x \mapsto |x|)$. For smooth domains, in the case of a non-degenerate maximum, the scale which plays the same role is h (see Theorem 1 in [3]). In fact, in our case, there is also a boundary layer of scale h but this scale is not visible at first orders in the eigenpair expansions. That is why in the eigenvalue expansions of [9] and [10] this scale does not appear. Nevertheless it is present in the right-angled case exposed in [6] or in the case of a small apertured cones (see [20]). One of the motivations for this paper is to understand this boudary layer. Is it, for right-angled triangles or cones, induced by the Dirichlet boundary conditions ?

In a second step, we get the separation of eigenvalues thanks to the Feshbach method, the associated eigenfunctions being localized near the altitude providing the most space. Unlike the resolvent convergence method exposed in [10] we use Agmon localization estimates which allows to consider cases with multiple peaks. For instance in the case of a symmetric mountain we can prove an exponential decay of the eigenfunctions between the peaks which induces tunneling. This is reminiscent of the case with symmetric electric potentials studied by Helffer [11] and Helffer and Sjöstrand [12]: In the semiclassical limit $h \to 0$ there are pairs of eigenvalues exponentially close. The same method can be applied to thin planar domains containing a finite number of peaks.

1.2 The Dirichlet Laplacian

Let us denote by (x_1, x_2) the Cartesian coordinates of the space $\mathbb{R}^2$ and by $\mathbf{0} = (0,0)$ the origin. The positive Laplace operator is given by $-\partial_1^2 - \partial_2^2$. Let $s \in \mathbb{R}$ and $h > 0$ we define $\widehat{\mathsf{Tri}}(s,h)$, the convex hull of the points of coordinates $A = (-1,0)$, $B = (1,0)$ and $C = (s,h)$. We are interested in the eigenvalues of the Dirichlet Laplacian $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)} := -\partial_1^2 - \partial_2^2$ on the triangle $\widehat{\mathsf{Tri}}(s,h)$ in the regime $h \to 0$.



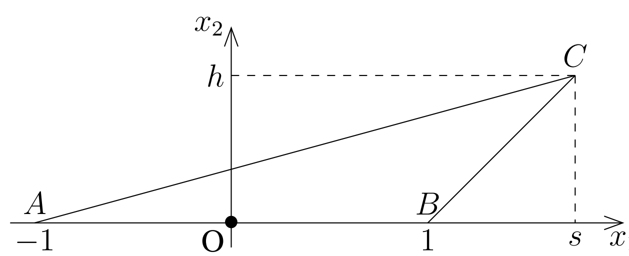


Figure 1: **The triangle $\widehat{\mathsf{Tri}}(s,h)$ in the acute and the obtuse configuration**

The values of s define different configurations for the geometry of the triangle $\widehat{\mathsf{Tri}}(s,h)$:

$s = 0$ corresponds to isoscele triangles,

$|s| < 1$ corresponds to acute triangles (see Figure 1),

$|s| = 1$ corresponds to right-angled triangles,

$|s| > 1$ corresponds to obtuse triangles (see Figure 1).

Since $\widehat{\mathsf{Tri}}(s,h)$ is convex, the domain of the operator $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ is the space of functions in $H^2(\widehat{\mathsf{Tri}}(s,h)) \cap H^1_0(\widehat{\mathsf{Tri}}(s,h))$ (see [16]). Since $\widehat{\mathsf{Tri}}(s,h)$ is a bounded domain $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ has compact resolvent and its spectrum is a non-decreasing sequence of eigenvalues denoted $(\mu_n(s,h))_{n\geq 1}$.

1.3 First properties of the eigenvalues

Before stating the main results of this paper, one can notice that in the obtuse configuration of Figure 1 if, instead of points A and B, we fix the points A and C, the regime $h \to 0$ is equivalent to the one where the altitude from B goes to zero. In fact, for $s > 1$, if we denote by

$$\tilde{s} = \frac{s}{\sqrt{(1+s)^2 + h^2}}; \quad \tilde{h} = \frac{2h}{\sqrt{(1+s)^2 + h^2}},$$

we have

$$0 < \tilde{s} < 1, \quad \tilde{h} \underset{h \to 0}{\longrightarrow} 0,$$

and we get

$$\mu_n(s,h) = \frac{(s+1)^2 + h^2}{4} \mu_n(\tilde{s}, \tilde{h}).$$

The same kind of computations can be done for $s < -1$, so that, we can take $s \in (-1, 1)$.

At fixed h, the question of the regularity of $\mu_n(\cdot, h)$ in $|s| = 1$ is considered in Section 2 but one can see that thanks to a Dirichlet bracketing (see [21, Chap. XIII]) we have, for s in a left neighborhood of 1:

$$\mu_n\Big(1, \frac{2h}{1+s}\Big) \leq \mu_n(s,h) \leq \frac{4}{(1+s)^2} \mu_n\Big(1, \frac{2h}{1+s}\Big).$$

Since $\mu_n(1, \cdot)$ is continuous for all $h > 0$ we obtain the left continuity of $\mu_n(\cdot, h)$ in $s = 1$. We can apply the same reasoning for the right continuity and we obtain the continuity of $\mu_n(\cdot, h)$ in $s = 1$.

We have the following lower bound for $\mu_n(s,h)$:

Proposition 1.1 *For all $s \in (-1, 1)$ and $h > 0$, we have:*

$$\frac{\pi^2}{h^2} + \frac{\pi^2}{4} \leq \mu_n(s,h).$$

Proof: The triangle $\widehat{\mathsf{Tri}}(s,h)$ is included in the rectangle $(-1,1) \times (0,h)$ and the conclusion follows by Dirichlet bracketing. □

1.4 Schrödinger operators in one dimension

In the analysis of $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ a one dimensional operator, constructed in the spirit of the Born-Oppenheimer approximation (see [4, 15, 18]), plays an important role: By replacing $-\partial^2_{x_2}$ in the expression of $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ by its lowest eigenvalue on each slice of $\widehat{\mathsf{Tri}}(s,h)$ at fixed x_1 we obtain an effective potential v_s and, on $L^2(-1,1)$, we arrive to the operator :

$$-\partial^2_{x_1} + h^{-2} v_s(x_1), \quad \text{with } v_s(x_1) = \begin{cases} \dfrac{(1+s)^2}{(1+x_1)^2}\pi^2, & \text{for } -1 < x_1 < s, \\ \dfrac{(1-s)^2}{(1-x_1)^2}\pi^2, & \text{for } s < x_1 < 1. \end{cases}$$

When $h \to 0$, we know that the minimum of the effective potential v_s guides the behavior of the ground eigenpairs of $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ (see [5, Chap. 11], [7] and [22]). This minimum is attained at $x_1 = s$. In a neighborhood of $x_1 = s$, v_s can be approximated by its left and right tangents, it yields the operator

$$l_s^{\mathsf{tan}}(h) := -\partial_{x_1}^2 + h^{-2} v_s^{\mathsf{tan}}(x_1), \quad \text{with } v_s^{\mathsf{tan}} = \pi^2 + 2\pi^2 \Big(\frac{1}{1+s} \mathbb{1}_{x_1<s}(x_1) + \frac{1}{1-s} \mathbb{1}_{x_1>s}(x_1) \Big) |x_1 - s| \tag{1.1}$$

To arrive to a canonical form, we perform the scaling $x_1 = \dfrac{h^{2/3}}{\sqrt[3]{2\pi^2}} u + s$ and we have:

$$l_s^{\mathsf{tan}}(h) \sim h^{-2}\pi^2 + (2\pi^2)^{2/3} h^{-4/3} l_s^{\mathsf{mod}}(u; \partial_u),$$

where the model operator l_s^{mod} is defined, on $L^2(\mathbb{R})$, as:

$$l_s^{\mathsf{mod}}(u; \partial_u) := -\partial_u^2 + v_s^{\mathsf{mod}}(u), \quad \text{with } v_s^{\mathsf{mod}}(u) := \Big(\frac{1}{1+s} \mathbb{1}_{\mathbb{R}_-}(u) + \frac{1}{1-s} \mathbb{1}_{\mathbb{R}_+}(u) \Big) |u|. \tag{1.2}$$

The parameter s introduces a skewness in the effective potential v_s^{mod}. We will see in Section 2 that this model operator is related to the Airy functions.

1.5 Asymptotic expansion of eigenvalues

We recall that $\mu_n(s,h)$ is the $n^{-\mathsf{th}}$ eigenvalue of the Dirichlet Laplacian $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ on the geometrical domain $\widehat{\mathsf{Tri}}(s,h)$. The main result of this paper is an asymptotic expansion of the eigenvalues of $\mu_n(s,h)$ as $h \to 0$. Indeed, the lowest eigenvalues of $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ admit expansions at any order in power of $h^{1/3}$. In the proof we also provide the structure of the eigenfunctions associated with these eigenvalues: at first order they are, up to some constants, almost a tensor product between the first eigenfunction of the Dirichlet Laplacian in the transverse variable x_2 and, up to normalization constants, the eigenfunctions of the model operator l_s^{mod} in the x_1 variable. Moreover they are localized near the altitude from C.

Theorem 1.2 *Let $0 < s_0 < 1$. For all $s \in [-s_0, s_0]$, the eigenvalues of $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$, denoted by $\mu_n(s,h)$, admit the expansions:*

$$\mu_n(s,h) \underset{h\to 0}{\sim} h^{-2} \sum_{j\geq 0} \beta_{j,n}(s) h^{j/3},$$

uniformly in s (see Notation 1.3). The functions $\big(s \mapsto \beta_{j,n}(s)\big)$ are analytic on $(-1,1)$ and we have: $\beta_{0,n} = \pi^2$, $\beta_{1,n} = 0$ and $\beta_{2,n}(s) = (2\pi^2)^{2/3}\kappa_n(s)$, where the $\kappa_n(s)$ are the eigenvalues of the model operator defined in (1.2). Moreover the eigenfunctions contains simultaneously the two scales $h^{2/3}$ and h as it can be seen in equation (3.8).

Notation 1.3 Let $\Lambda(s,h)$ be a function of s and h and let $\theta > 0$. We say that $\Lambda(s,h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \Gamma_j(s) h^{j\theta}$ if, for all $J \in \mathbb{N}$, there exists $C_J(s) > 0$ and $h_0 > 0$, such that for all $h \in (0, h_0)$

$$\Big|\Lambda(s,h) - \sum_{j=0}^{J} \Gamma_j(s) h^{\theta j}\Big| \leq C_J(s) h^{\theta(J+1)}.$$

We say that the asymptotic expansion is uniform in s if $C_J(s)$ does not depend on s.

1.6 Tunneling in a symmetric mountain

Thanks to the structure of the proof of Theorem 1.2, we can understand tunneling for the Dirichlet Laplacian on a thin shrinking symmetric polygon. Let us choose $s \in (0,1)$ and let $h > 0$, we consider in $\mathbb{R}^2$ the points $A = (-1,0)$, $B = (1,0)$, $C_1 = (s,h)$, $C_2 = (-s,h)$ and $D = (0,\frac{h}{2})$. Let $\Omega^{\mathsf{rig}}(h)$ be the open convex hull of $\mathbf{0}, B, C_1$ and D. We define by $\Omega^{\mathsf{lef}}(h)$ the reflection of $\Omega^{\mathsf{rig}}(h)$ with respect to the x_2-axis. Then, we define $\Omega(h)$ (see Figure 2) by:

$$\Omega(h) = \Omega^{\mathsf{rig}}(h) \cup \Omega^{\mathsf{lef}}(h) \cup [\mathbf{0}, D].$$

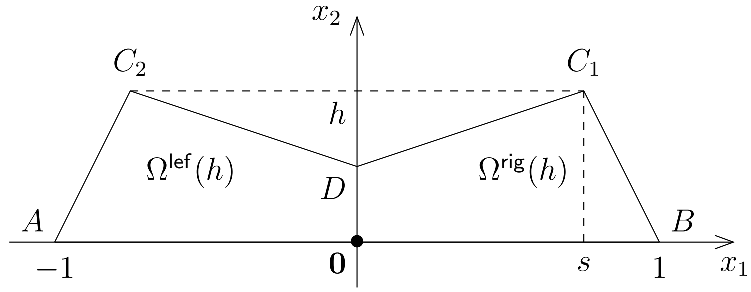


Figure 2: **The domain** $\Omega(h)$

Let $(\nu_n(h))_{n\geq 1}$ be the non-decreasing sequence of eigenvalues of $-\Delta^{\mathsf{Dir}}_{\Omega(h)}$, the Dirichlet realization of the Laplacian on $\Omega(h)$.

We consider the Dirichlet realization of the Laplacian on $\Omega^{\mathsf{lef}}(h)$ and $\Omega^{\mathsf{rig}}(h)$, respectively denoted $-\Delta^{\mathsf{Dir}}_{\Omega^{\mathsf{lef}}(h)}$ and $-\Delta^{\mathsf{Dir}}_{\Omega^{\mathsf{rig}}(h)}$. For symmetry reasons, they are isospectral and we denote by $(\mu_n(h))_{n\geq 1}$ their eigenvalues.

Let us define the operator $\mathfrak{D}(h) := -\Delta^{\mathsf{Dir}}_{\Omega^{\mathsf{lef}}(h)} \oplus -\Delta^{\mathsf{Dir}}_{\Omega^{\mathsf{rig}}(h)}$. Its eigenvalues, denoted by $(\tau_n(h))_{n\geq 1}$, verify $\tau_{2j-1}(h) = \tau_{2j}(h) = \mu_j(h)$ for all $j \geq 1$. We will prove

Theorem 1.4 *For all $N \in \mathbb{N}^*$ there exist $h_0 > 0$, $C_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and all $j \in \{1, \ldots, N\}$:*

$$|\tau_j(h) - \nu_j(h)| \leq C_0 e^{-C/h}.$$

As explained for the triangle, in the regime $h \to 0$ the eigenfunctions are localized near the altitude providing the most space. Here they are localized simultaneously near the altitude from C_1 and C_2 and they interact at a scale exponentially small.

1.7 Structure of the paper

In Section 2 we study the model operator defined in (1.2). We describe its eigenvalues and their dependence on the parameter s. In Section 3, we perform a change of variables that transforms the triangle into a rectangle. The operator is more complicated but we deal with a simpler geometrical domain. Thanks to this change of variables and some lemmas derived from the Fredholm alternative we can construct quasimodes. We finish the proof of Theorem 1.2 using a Feshbach-Grushin projection which justifies that the model operator is an actual approximation of our problem. Then we get the separation of the eigenvalues. In Section 4 we study the symmetric mountain $\Omega(h)$. We use properties derived from Theorem 1.2 to discuss the tunneling effect between the peaks.

We conclude by Appendix A and B by providing with numerical experiments which agree with the theoretical shape of the eigenfunctions.

2 Model operator

To deal with $-\Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ we first need to study some basic properties of the model operator l_s^{mod}. The difference with the operator studied by Dauge and Raymond in [6, Sec. 3] is that the model operator is not, up to a constant, the Airy reversed operator with Dirichlet boundary condition at $u = 0$. The effective potential of l_s^{mod} is a combination on $\mathbb{R}_-$ of an Airy reversed operator and on $\mathbb{R}_+$ of an Airy operator. The non-symmetry of this potential and the transmission conditions at $u = 0$ complicate the study: we cannot have an explicit expression of the eigenvalues of l_s^{mod} as zeros of an Airy function.

The behavior of the effective potential v_s^{mod} when $|u| \to +\infty$ yields the

Proposition 2.1 *For all $s \in (-1, 1)$, the model operator l_s^{mod} has compact resolvent.*

Thus, the spectrum of the model operator l_s^{mod} consists in a non-decreasing sequence of eigenvalues denoted $(\kappa_n(s))_{n\geq 1}$.

Remark 2.2 When $s = 0$, the model operator is $l_0^{\mathsf{mod}}(u; \partial_u) = -\partial_u^2 + |u|$ and its eigenvalues are, for all $n \geq 0$:

$$\begin{cases} \kappa_{2n+1}(0) = z'_{\mathsf{A}}(n+1), \\ \kappa_{2n+2}(0) = z_{\mathsf{A}}(n+1), \end{cases}$$

where, for all $k \in \mathbb{N}^*$, $z_{\mathsf{A}}(k)$ and $z'_{\mathsf{A}}(k)$ are respectively the zeros of the Airy reversed function and the zeros of the first derivative of the Airy reversed function. $\triangle$

Thanks to the theory about Sturm-Liouville operators, we get

Proposition 2.3 *For all $s \in (-1, 1)$, the eigenvalues of the model operator l_s^{mod} are simple.*

Now, we are interested in the regularity of these eigenvalues. The family $(l_s^{\mathsf{mod}})_{s\in(-1,1)}$ is an analytic family of type (A) (see [14]). Jointly with Proposition 2.3, we have the

Proposition 2.4 *For all $n \geq 1$, the functions $(s \to \kappa_n(s))$ are analytic on $(-1, 1)$. Moreover for all $n \in \mathbb{N}^*$, there exists an eigenfunction T_s^n associated with $\kappa_n(s)$, such that the functions $\left(s \to \mathsf{T}_s^n \in \mathsf{Dom}\big(l_s^{\mathsf{mod}}\big)\right)$ are also analytic on $(-1, 1)$.*

Characterization of the eigenvalues $(\kappa_n(s))_{n\geq 1}$ We have the

Proposition 2.5 *The eigenvalues $(\kappa_n(s))_{n\geq 1}$ of l_s^{mod} satisfy the following implicit equation in (s, κ):*

$$\sqrt[3]{1+s}\,\mathsf{A}(-(1+s)^{2/3}\kappa)\mathsf{A}'(-(1-s)^{2/3}\kappa) + \sqrt[3]{1-s}\,\mathsf{A}(-(1-s)^{2/3}\kappa)\mathsf{A}'(-(1+s)^{2/3}\kappa) = 0, \quad (2.1)$$

where A denotes the Airy reversed function defined by $\mathsf{A}(u) = \mathsf{Ai}(-u)$.

Proof: Let (κ, Ψ) be an eigenpair of l_s^{mod}. We define:

$$\Psi^{\pm} := \Psi \mathbb{1}_{\mathbb{R}_\pm}.$$

In order to solve

$$l_s^{\mathsf{mod}}\Psi = \kappa\Psi, \quad (2.2)$$

we consider this equation for both $u < 0$ and $u > 0$. For $u < 0$ equation (2.2) writes

$$\Big(-\partial_u^2 - \frac{1}{1+s}u - \kappa\Big)\Psi^- = 0.$$

It is an Airy reversed equation. For integrability reasons the Airy function of second kind not appear in the expression of Ψ^- and we have:

$$\Psi^-(u) = \alpha^- \mathsf{A}\big((1+s)^{-1/3}(u + \kappa(1+s))\big), \tag{2.3}$$

where $\alpha^- \in \mathbb{R}$.

For $u > 0$, the same reasoning yields:

$$\Psi^+(u) = \alpha^+ \mathsf{Ai}\big((1-s)^{-1/3}(u - \kappa(1-s))\big), \tag{2.4}$$

where $\alpha^+ \in \mathbb{R}$.

If we denote by $\mathsf{Dom}(l_s^{\mathsf{mod}})$ the domain of the model operator l_s^{mod}, the eigenfunction Ψ belongs to $\mathsf{Dom}(l_s^{\mathsf{mod}})$. In particular, $\Psi \in H^2(\mathbb{R})$ and we have the transmission conditions

$$\left\{\begin{array}{rcl} \Psi^-(0) & = & \Psi^+(0), \\ \partial_u\Psi^-(0) & = & \partial_u\Psi^+(0), \end{array}\right.$$

which becomes

$$\left\{\begin{array}{rcl} \alpha^-\mathsf{A}\big((1+s)^{2/3}\kappa\big) - \alpha^+\mathsf{A}\big((1-s)^{2/3}\kappa\big) & = & 0, \\ \alpha^-(1-s)^{1/3}\,\mathsf{A}'\big(-(1+s)^{2/3}\kappa\big) + \alpha^+(1+s)^{1/3}\,\mathsf{A}'\big(-(1-s)^{2/3}\kappa\big) & = & 0. \end{array}\right.$$

The $\kappa_n(s)$ are the values for which this system is linked and we get the implicit equation (2.1). $\square$

Thanks to the explicit equations (2.3) and (2.4) and the properties of the Airy function of first kind we have the

Proposition 2.6 *For all $n \in \mathbb{N}^*$ the eigenfunction T_s^n belongs to $H^2_{\mathsf{exp}}(\mathbb{R})$ (see Notation 2.7 below).*

Notation 2.7 For $k \in \mathbb{N}$, we define the spaces

$$H^k_{\mathsf{exp}}(\mathcal{D}) = \big\{f \in L^2(\mathcal{D}) : \exists\,\theta > 0, e^{\theta|u|^{3/2}}f \in H^k(\mathcal{D})\big\},$$

where $\mathcal{D} \subset \mathbb{R}^d$ $(d = 1, 2)$ and is unbounded in the u direction.

To understand the regularity of $(s \mapsto \kappa_n(s))$ near $s = 1$, we perform the change of variables $\sigma = (1-s)^{1/3}$ in equation (2.1). It becomes:

$$(2-\sigma^3)^{1/3}\mathsf{A}\big((2-\sigma^3)^{2/3}\kappa\big)\mathsf{A}'(\sigma^2\kappa) + \sigma\mathsf{A}(\sigma^2\kappa)\mathsf{A}'\big((2-\sigma^3)^{2/3}\kappa\big) = 0,$$

which is smooth near $\sigma = 0$.

The same study for $s > 1$ gives the implicit equation

$$4^{1/6}\mathsf{A}\Big(\frac{4}{(s+1)^{4/3}}\kappa\Big)\mathsf{A}'\Big(\Big(\frac{4(s-1)}{(s+1)^2}\Big)^{2/3}\kappa\Big) + (1+s)^{1/3}(s-1)^{1/3}\mathsf{A}\Big(\Big(\frac{4(s-1)}{(s+1)^2}\Big)^{2/3}\kappa\Big)\mathsf{A}'\Big(\frac{4}{(s+1)^{4/3}}\kappa\Big) = 0$$

which shows the same behavior of $\kappa_n(s)$ for $s > 1$. We see that κ_n is continuous at $s = 1$ with $\kappa_n(1) = 2^{-2/3}z_{\mathsf{A}}(n)$. Nevertheless κ_n has a cubic singularity on the left and on the right of $s = 1$. Thanks to the implicit expressions (both for $s < 1$ and $s > 1$) we illustrated on Figure 2 the dependence on s of $\kappa_n(s)$ for $n = 1, 2, 3$. It shows the cubic singularity at $s = 1$.

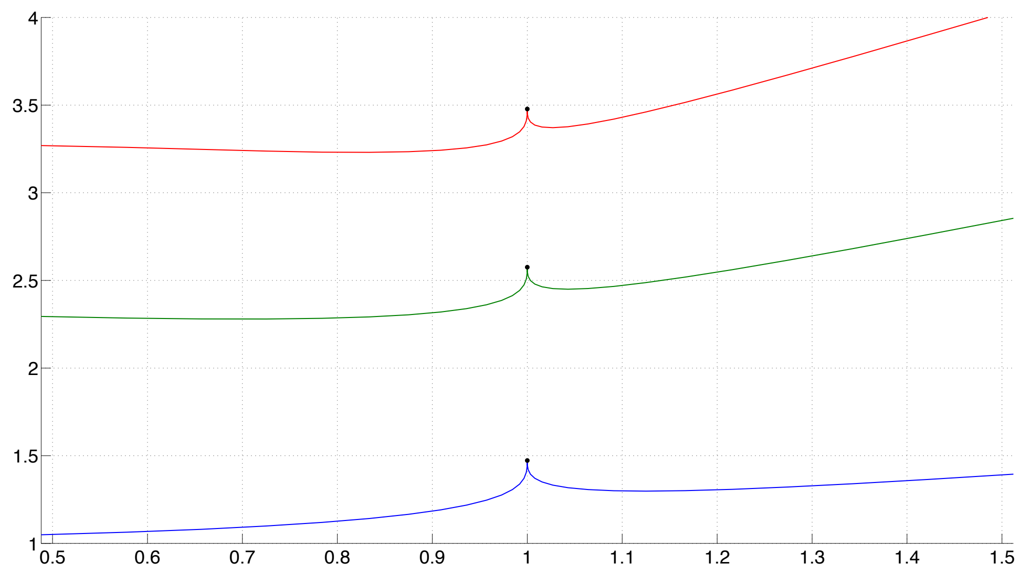


Figure 3: This figure represents the dependence of $\kappa_n(s)$ on s for $n = 1, 2, 3$. The black dots represent the values $2^{-2/3} z_{\mathsf{A}}(n)$ for $n = 1, 2, 3$.

3 Proof of the main theorem

The aim of this section is to prove Theorem 1.2. We first perform the following change of variables to transfer the dependence on h of $-\Delta_{\widehat{\mathsf{Tri}}(s,h)}$ into the coefficient of the operator:

$$x = x_1 - s; \quad y = \frac{1}{h} x_2.$$

The triangle $\widehat{\mathsf{Tri}}(s,h)$ is transformed into $\mathsf{Tri}(s) := \widehat{\mathsf{Tri}}(s,1)$. The operator $-h^2 \Delta^{\mathsf{Dir}}_{\widehat{\mathsf{Tri}}(s,h)}$ becomes

$$\mathcal{L}_s(h) := -h^2 \partial_x^2 - \partial_y^2. \tag{3.1}$$

It also has compact resolvent and we denote by $(\lambda_n(s,h))_{n\geq 1}$ its eigenvalues. They satisfy

$$\lambda_n(s,h) = \mu_n(s,h) h^2. \tag{3.2}$$

The operator $\mathcal{L}_s(h)$ defined in (3.1) is partially semiclassical in x. Its investigation follows the lines of the papers [8, 11, 12].

The proof is divided into two main steps: a construction of quasimodes and the use of the true eigenfunctions of $\mathcal{L}_s(h)$ as quasimodes for the model operator in order to obtain a lower bound for the true eigenvalues.

We first perform a change of variables to transform the triangle $\mathsf{Tri}(s)$ into the rectangle $\mathsf{Rec}(s) = (-1-s, 1-s) \times (0,1)$:

$$\begin{cases} u = x; \quad t = (1+s)\dfrac{y}{x+1+s} & \text{for } -1-s < x < 0, \\ u = x; \quad t = -(1-s)\dfrac{y}{x-(1-s)} & \text{for } 0 < x < 1-s. \end{cases} \tag{3.3}$$

For the sake of simplicity we define

$$s_- := 1 + s, \quad s_+ := s - 1.$$

In the coordinates (u, t) defined in (3.3), we denote by $\mathcal{L}_s^-(h)$ and $\mathcal{L}_s^+(h)$ the expression of $\mathcal{L}_s(h)$, respectively for $u < 0$ and $u > 0$. We have:

$$\mathcal{L}_s^{\pm}(h)(u, t; \partial_u, \partial_t) := -h^2\Big(\partial_u^2 - \frac{2t}{u + s_{\pm}}\partial_u\partial_t + \frac{2t}{(u + s_{\pm})^2}\partial_t + \frac{t^2}{(u + s_{\pm})^2}\partial_t^2\Big) - \frac{s_{\pm}}{(u + s_{\pm})^2}\partial_t^2.$$

The boundary condition are Dirichlet on $(-1, 1) \times \{0\}$ and $(-1, 1) \times \{1\}$.

3.1 Quasimodes

To prove Theorem 1.2 we first construct quasimodes at any order in power of $h^{1/3}$ and it yields the

Proposition 3.1 *Let $\mathfrak{S}(\mathcal{L}_s(h))$ denotes the spectrum of $\mathcal{L}_s(h)$ and $s_0 \in [0, 1)$. There are sequences $(\beta_{j,n}(s))_{j\geq 0}$ for any integer $n \geq 1$ so that there holds: for all $N_0 \in \mathbb{N}^*$ and $J \in \mathbb{N}$, there exist $h_0 > 0$ and $C > 0$ such that for all $s \in [-s_0, s_0)$ and all $h \in (0, h_0)$*

$$\mathsf{dist}\Big(\mathfrak{S}(\mathcal{L}_s(h)), \sum_{j=0}^{J} \beta_{j,n}(s)h^{j/3}\Big) \leq Ch^{(J+1)/3}, \quad n = 1, \dots, N_0.$$

Moreover, the functions $(s \mapsto \beta_{j,n}(s))$ are analytic on $(-1, 1)$ and we have: $\beta_{0,n}(s) = \pi^2$, $\beta_{1,n}(s) = 0$, and $\beta_{2,n}(s) = (2\pi^2)^{2/3}\kappa_n(s)$.

Proof: The proof is divided into three parts. The first one deals with the form of the Ansatz chosen to construct quasimodes. The second part deals with lemmas about operators which appear in the first part. The third part is the determination of the profiles of the Ansatz.

Ansatz We want to construct quasimodes $(\gamma_{s,h}, \psi_{s,h})$ for the operator $\mathcal{L}_s(h)(x, y; \partial_x, \partial_y)$. It will be more convenient to work in the rectangle $\mathsf{Rec}(s)$ with the operators $\mathcal{L}_s^{\pm}(h)(u, t; \partial_u, \partial_t)$. We introduce the new scales:

$$\alpha = h^{-2/3}u; \quad \beta = h^{-1}u.$$

We look for quasimodes $\hat{\psi}_{s,h}(u, t) = \psi_{s,h}(x, y)$. Such quasimodes will have the form on the left and on the right:

$$\psi_s^{\pm}(u, t) \sim \sum_{j\geq 0} \big[\Psi_{s,j}^{\pm}(\alpha, t) + \Phi_{s,j}^{\pm}(\beta, t)\big]h^{j/3}.$$

associated with quasi-eigenvalues:

$$\gamma_{s,h} \sim \sum_{j\geq 0} \beta_j(s)h^{j/3},$$

in order to solve the eigenvalue equation in the sense of formal series. An Ansatz containing the scale $h^{2/3}$ alone is not sufficient to construct quasimodes because one can see that the system is overdetermined. Expanding the operators in powers of $h^{2/3}$, we obtain the formal series:

$$\mathcal{L}_s^{\pm}(h)(h^{2/3}\alpha, t; h^{-2/3}\partial_\alpha, \partial_t) \sim \sum_{j\geq 0} \mathcal{L}_{s,2j}^{\pm}h^{j/3}, \quad \text{with leading term } \mathcal{L}_{s,0}^{\pm} := \mathcal{L}_0^{\pm} = -\partial_t^2,$$

and in power of h:

$$\mathcal{L}_s^{\pm}(h)(h\beta, t; h^{-1}\partial_\beta, \partial_t) \sim \sum_{j\geq 0} \mathcal{N}_{s,3j}^{\pm} h^{j/3}, \quad \text{with leading term } \mathcal{N}_{s,0}^{\pm} := \mathcal{N}_0^{\pm} = -\partial_\beta^2 - \partial_t^2,$$

We consider these operators on the half-strips $\mathsf{H}_- := (-\infty, 0) \times (0,1)$ and $\mathsf{H}_+ := (0,\infty) \times (0,1)$. On the left and on the right, the leading term at the scale $h^{2/3}$ acts only in the transverse variable t and is the Dirichlet Laplacian on $(0,1)$. At the scale h, the leading term is the Laplacian on a half-strip. The leading terms at both scales do not depend on s. Since $\psi_{s,h}$ has no jump across the line $x=0$, we find that ψ_s^- and ψ_s^+ should statisfy two transmission conditions on the interface $I := \{0\} \times (0,1)$:

$$\psi_s^-(0,t) = \psi_s^+(0,t) \quad \text{and} \quad \big(\partial_u - \frac{t}{s_-}\partial_t\big)\psi_s^-(0,t) = \big(\partial_u - \frac{t}{s_+}\partial_t\big)\psi_s^+(0,t),$$

for all $t \in (0,1)$. For the formal series, these conditions write for all $t \in (0,1)$ and all $j \geq 0$:

$$\Psi_{s,j}^-(0,t) + \Phi_{s,j}^-(0,t) = \Psi_{s,j}^+(0,t) + \Phi_{s,j}^+(0,t), \tag{3.4}$$

$$\begin{aligned}&\partial_\alpha \Psi_{s,j-1}^-(0,t) + \partial_\beta \Phi_{s,j}^-(0,t) - \frac{t}{s_-}\partial_t \Psi_{s,j-3}^-(0,t) - \frac{t}{s_-}\partial_t \Phi_{s,j-3}^-(0,t)\\ &= \partial_\alpha \Psi_{s,j-1}^+(0,t) + \partial_\beta \Phi_{s,j}^+(0,t) - \frac{t}{s_+}\partial_t \Psi_{s,j-3}^+(0,t) - \frac{t}{s_+}\partial_t \Phi_{s,j-3}^+(0,t),\end{aligned} \tag{3.5}$$

where we understand that the terms associated with a negative index are 0. Finally, in order to ensure the Dirichlet boundary condition on $\mathsf{Tri}(s)$ we will require for our Ansatz, for any $j \in \mathbb{N}$, the boundary conditions:

$$\Psi_{s,j}^{\pm}(\cdot, 0 \text{ and } 1) = 0; \quad \Phi_{s,j}^{\pm}(\cdot, 0 \text{ and } 1) = 0. \tag{3.6}$$

Three lemmas To start the construction of our Ansatz, we will need three lemmas, but before let us denote by $(\mathfrak{s}_j)_{j\geq 1}$ the eigenfunctions associated with the eigenvalues of the Dirichlet Laplacian on the line $(0,1)$. We have $\mathfrak{s}_j(t) = \sqrt{2}\sin(j\pi t)$ and this eigenfunction is associated with the eigenvalue $j^2\pi^2$.

The analyticity of the solutions in the following lemmas is a direct consequence of the analyticity of the datas.

Lemma 3.2 *Let $F_s^- = F_s^-(\beta,t)$ and $F_s^+ = F_s^+(\beta,t)$ be functions respectively in $L^2_{\mathsf{exp}}(\mathsf{H}_-)$ and $L^2_{\mathsf{exp}}(\mathsf{H}_+)$, depending analytically on $s \in (-1,1)$. Let $G_s \in H^{3/2}(I) \cap H_0^1(I)$ and $H_s \in H^{1/2}(I)$ be data on the interface I, depending analytically of $s \in (-1,1)$. Then, for all $s \in (-1,1)$ there exist unique coefficients ξ_s and δ_s such that the transmission problem:*

$$\begin{cases} (\mathcal{N}_0^{\pm} - \pi^2)\Phi_s^{\pm} = F_s^{\pm} \quad \text{in } \mathsf{H}_{\pm}, \quad \Phi_s^{\pm}(\cdot, 0 \text{ and } 1) = 0, \\ \Phi_s^-(0,t) - \Phi_s^+(0,t) = G_s(t) + \xi_s \mathfrak{s}_1(t) \\ \partial_\beta \Phi_s^-(0,t) - \partial_\beta \Phi_s^+(0,t) = H_s(t) + \delta_s \mathfrak{s}_1(t), \end{cases}$$

has a unique solution (Φ_s^-, Φ_s^+) in $H^2_{\mathsf{exp}}(\mathsf{H}_-) \times H^2_{\mathsf{exp}}(\mathsf{H}_+)$ and we have

$$\xi_s = -\int_{-\infty}^{0} \langle F_s^-(\beta,\cdot), \mathfrak{s}_1\rangle_t \beta \mathsf{d}\beta - \int_0^{+\infty} \langle F_s^+(\beta,\cdot), \mathfrak{s}_1\rangle_t \beta \mathsf{d}\beta - \langle G_s, \mathfrak{s}_1\rangle_t,$$

$$\delta_s = \int_{-\infty}^{0} \langle F_s^-(\beta,\cdot), \mathfrak{s}_1\rangle_t \mathsf{d}\beta - \int_0^{+\infty} \langle F_s^+(\beta,\cdot), \mathfrak{s}_1\rangle_t \mathsf{d}\beta - \langle H_s, \mathfrak{s}_1\rangle_t.$$

Moreover ξ_s, G_s and (Φ_s^-, Φ_s^+) depend analytically on $s \in (-1,1)$.

Proof of the lemma: We look for a solution (Φ_s^-, Φ_s^+) that we decompose, in the transverse coordinates, along the basis of the eigenfunction of the Dirichlet Laplacian on $(0,1)$:

$$\Phi_s^{\pm}(\beta,t) = \sum_{j\geq 1} \Phi_{s,j}^{\pm}(\beta)\mathfrak{s}_j(t).$$

For all $j \geq 1$, the following equations are satisfied:

$$\big(-\partial_\beta^2 + \pi^2(j^2-1)\big)\Phi_{s,j}^{\pm} = \langle F_s^{\pm}, \mathfrak{s}_j\rangle_t,$$

and we are looking for exponentially decaying solutions. For $j=1$, we find:

$$\Phi_{s,1}^-(\beta) = \int_{-\infty}^{\beta}\int_{-\infty}^{\beta_1} \langle F_s^-(\beta_2,\cdot), \mathfrak{s}_1\rangle_t \mathsf{d}\beta_2\mathsf{d}\beta_1, \quad \Phi_{s,1}^+(\beta) = -\int_{\beta}^{+\infty}\int_{\beta_1}^{+\infty} \langle F_s^+(\beta_2,\cdot), \mathfrak{s}_1\rangle_t \mathsf{d}\beta_2\mathsf{d}\beta_1.$$

Using the data on the interface I we find the expression of ξ_s and δ_s. For $j \geq 2$, we solve the ordinary differentials equations. Taking into account the exponential decay and data on the interface I it achieves the proof of Lemma 3.2. $\diamond$

The following lemma can be found in [6, Sec. 5]. It is a consequence of the Fredholm alternative:

Lemma 3.3 *Let $F_s^{\pm} = F_s^{\pm}(\alpha,t)$ be a function in $L^2_{\mathsf{exp}}(\mathsf{H}_{\pm})$, and depending analytically on $s \in (-1,1)$. Then, there exist solution(s) $\Psi_s^{\pm} \in H^2_{\mathsf{exp}}(\mathsf{H}_{\pm})$ such that:*

$$(\mathcal{L}_0^{\pm} - \pi^2)\Psi_s^{\pm} = F_s^{\pm} \quad \textit{in } \mathsf{H}_{\pm}, \quad \Psi_s^{\pm}(\alpha, 0 \textit{ and } 1) = 0$$

if and only if:

$$\langle F_s^{\pm}(\alpha,\cdot), \mathfrak{s}_1\rangle_t = 0 \quad \textit{for all } \alpha \in \mathbb{R}_{\pm}^*.$$

In this case, they write:

$$\Psi_s^{\pm}(\alpha,t) = \Psi_s^{\pm,\perp}(\alpha,t) + g_s^{\pm}(\alpha)\mathfrak{s}_1(t),$$

with $\Psi_s^{\pm,\perp} \in H^2_{\mathsf{exp}}(\mathsf{H}_{\pm})$. Moreover, $\Psi_s^{\pm}$, $\Psi_s^{\pm,\perp}$ and $g_s^{\pm}$ are analytic in the s-variable.

Lemma 3.4 *Let $f_s^- = f_s^-(\alpha) \in L^2_{\mathsf{exp}}(\mathbb{R}_-)$, $f_s^+ = f_s^+(\alpha) \in L^2_{\mathsf{exp}}(\mathbb{R}_+)$ and $c_s, \theta_s \in \mathbb{R}$ depending analytically on $s \in (-1,1)$. There exists a unique $\omega(s)$ such that the system*

$$\begin{cases} \Big(-\partial_\alpha^2 - \dfrac{\alpha}{1+s} - \kappa_n(s)\Big)g_s^- = f_s^- + \omega(s)\mathsf{T}_s^n & \textit{in } \mathbb{R}_-, \qquad g_s^+(0) - g_s^-(0) = c_s \\ \Big(-\partial_\alpha^2 + \dfrac{\alpha}{1-s} - \kappa_n(s)\Big)g_s^+ = f_s^+ + \omega(s)\mathsf{T}_s^n & \textit{in } \mathbb{R}_+, \qquad (g_s^{\mathsf{rig}})'(0) - (g_s^{\mathsf{lef}})'(0) = \theta_s, \end{cases}$$

has a unique solution $(g_s^-, g_s^+) \in H^2_{\mathsf{exp}}(\mathbb{R}_-) \times H^2_{\mathsf{exp}}(\mathbb{R}_+)$. Moreover g_s^-, g_s^+ and $\omega(s)$ depend analytically on s.

Proof of the lemma: Let us define $g_s := g_s^- \mathbb{1}_{\mathbb{R}_-} + g_s^+ \mathbb{1}_{\mathbb{R}_+}$. In the distribution sense we have:

$$\big(l_s^{\mathsf{mod}} - \kappa_n(s)\big)g_s = \big(l_s^{\mathsf{mod}} - \kappa_n(s)\big)(g_s^- \mathbb{1}_{\mathbb{R}_-}) + \big(l_s^{\mathsf{mod}} - \kappa_n(s)\big)(g_s^+ \mathbb{1}_{\mathbb{R}_+}).$$

After computations we find:

$$\big(l_s^{\mathsf{mod}} - \kappa_n(s)\big)g_s = f_s + \omega(s)\mathsf{T}_s^n - \theta_s\delta_0 - c_s\delta_0',$$

where $f_s := f_s^- \mathbb{1}_{\mathbb{R}_-} + f_s^+ \mathbb{1}_{\mathbb{R}_+}$ and δ_0 is the Dirac delta function at $\alpha = 0$. Then, we define the function m:

$$m(\alpha) := (\theta_s \alpha + c_s)\mathbb{1}_{\mathbb{R}_+}.$$

In the distribution sense, we have:

$$m'(\alpha) = \theta_s \mathbb{1}_{\mathbb{R}_+} + c_s \delta_0, \quad m''(\alpha) = \theta_s \delta_0 + c_s \delta_0'.$$

We introduce a smooth cut-off function χ which is 1 near 0. Finally, we define the auxiliary function $\tilde{g}_s := g_s - \chi m$ and we obtain:

$$\big(l_s^{\mathsf{mod}} - \kappa_n(s)\big)\tilde{g}_s = f_s + \omega(s)\mathsf{T}_s^n + \big((\partial_\alpha^2 \chi)m + 2\theta_s(\partial_\alpha \chi)\mathbb{1}_{\mathbb{R}_+} - v_s^{\mathsf{mod}}(\alpha)\chi m + \kappa_n(s)\chi m\big). \tag{3.7}$$

By definition, $\tilde{g}_s$ belongs to the form domain of the operator l_s^{mod}. The right hand side of equation (3.7) being in $L^2(\mathbb{R})$, $\tilde{g}_s$ is also in the domain of the operator l_s^{mod}. As a consequence, equation (3.7) is also true in $L^2(\mathbb{R})$ and we can apply the Fredholm alternative to find a solution $\tilde{g}_s$ and $\omega(s)$, this latest satisfying:

$$\omega(s) = \big\langle (v_s^{\mathsf{mod}} - \kappa_n(s))\chi m - (\partial_\alpha^2)m - 2\theta_s(\partial_\alpha \chi)\mathbb{1}_{\mathbb{R}_+} - f_s, \mathsf{T}_s^n \big\rangle.$$

It concludes the proof of Lemma 3.4. $\diamond$

In the following construction we use a version of this lemma up to some normalization constants.

Determination of the profiles

Now we can start the construction of the Ansatz.

Terms of order h^0 Let us write the equations inside the strip:

$$\mathsf{H}_{\pm,\alpha}: \quad -\partial_t^2 \Psi_{s,0}^{\pm} = \gamma_0(s)\Psi_{s,0}^{\pm} \quad \mathsf{H}_{\pm,\beta}: \quad -(\partial_t^2 + \partial_\beta^2)\Phi_{s,0}^{\pm} = \gamma_0(s)\Phi_{s,0}^{\pm}$$

The transmission conditions are:

$$(\Psi_{s,0}^- + \Phi_{s,0}^-)(0,t) = (\Psi_{s,0}^+ + \Phi_{s,0}^+)(0,t),$$

$$(\partial_\beta \Phi_{s,0}^- - \partial_\beta \Phi_{s,0}^+)(0,t) = 0.$$

With the Dirichlet boundary conditions (3.6), we get:

$$\gamma_0(s) = \pi^2, \quad \Psi_{s,0}^{\pm}(\alpha,t) = g_{s,0}^{\pm}(\alpha)\mathfrak{s}_1(t).$$

We now apply Lemma 3.2 with $F_s^- \equiv 0$, $F_s^+ \equiv 0$, $G_s \equiv 0$ and $H_s \equiv 0$ to get:

$$\xi_s = 0 \quad \text{and} \quad \delta_s = 0.$$

We deduce $\Phi_{s,0}^- \equiv 0$ and $\Phi_{s,0}^+ \equiv 0$ and, since $\xi_s = g_{s,0}^+(0) - g_{s,0}^-(0)$, $g_{s,0}^+(0) = g_{s,0}^-(0)$. At this step we do not have determined $g_{s,0}^{\pm}$ yet.

Terms of order $h^{1/3}$ The equations inside the strip read:

$$\mathsf{H}_{\pm,\alpha}: \quad (-\partial_t^2 - \pi^2)\Psi_{s,1}^{\pm} = \gamma_1(s)\Psi_{s,0}^{\pm} \quad \mathsf{H}_{\pm,\beta}: \quad (-\partial_t^2 - \partial_\beta^2 - \pi^2)\Phi_{s,1}^{\pm} = 0$$

The transmission conditions are:

$$(\Phi_{s,1}^- + \Psi_{s,1}^-)(0,t) = (\Phi_{s,1}^+ + \Psi_{s,1}^+)(0,t),$$

$$(\partial_\alpha \Psi_{s,0}^- + \partial_\beta \Phi_{s,1}^-)(0,t) = (\partial_\alpha \Psi_{s,0}^+ + \partial_\beta \Phi_{s,1}^+)(0,t).$$

We also take the Dirichlet boundary conditions (3.6) into account. Lemma 3.3 implies:

$$\gamma_1(s) = 0, \quad \Psi_{s,1}^{\pm}(\alpha,t) = g_{s,1}^{\pm}(\alpha)\mathfrak{s}_1(t).$$

We now apply Lemma 3.2 with $F_s^- \equiv 0$, $F_s^+ \equiv 0$, $G_s \equiv 0$ and $H_s \equiv 0$, we get:

$$\xi_s = 0, \quad \delta_s = 0.$$

Since $\xi_s = g_{s,1}^+(0) - g_{s,1}^-(0)$ and $\delta_s = (g_{s,0}^+)'(0) - (g_{s,0}^-)'(0)$ we have:

$$g_{s,1}^+(0) = g_{s,1}^-(0), \quad (g_{s,0}^+)'(0) = (g_{s,0}^-)'(0).$$

We also deduce that $\Phi_{s,1}^- \equiv 0$ and $\Phi_{s,1}^+ \equiv 0$.

Terms of order $h^{2/3}$ The equations inside the strip read:

$$\mathsf{H}_{\pm,\alpha}: \quad (-\partial_t^2 - \pi^2)\Psi_{s,2}^{\pm} = -\mathcal{L}_{s,2}^{\pm}\Psi_{s,0}^{\pm} + \gamma_2(s)\Psi_{s,0}^{\pm} \quad \mathsf{H}_{\pm,\beta}: \quad (-\partial_t^2 - \partial_\beta^2 - \pi^2)\Phi_{s,2}^{\pm} = 0$$

where we have:

$$\mathcal{L}_{s,2}^{\pm} := \frac{2\alpha}{s_\pm}\partial_t^2 - \partial_\alpha^2.$$

The transmission conditions are:

$$(\Psi_{s,2}^- + \Phi_{s,2}^-)(0,t) = (\Psi_{s,2}^+ + \Phi_{s,2}^+)(0,t),$$

$$\partial_\alpha \Psi_{s,1}^-(0,t) + \partial_\beta \Phi_{s,2}^-(0,t) = \partial_\alpha \Psi_{s,1}^+(0,t) + \partial_\beta \Phi_{s,2}^+(0,t),$$

We also have to take the Dirichlet boundary conditions (3.6) into account. Then, we apply a renormalized version of Lemma 3.3. Consequently, there exists a solution $(\Psi_{s,2}^-, \Psi_{s,2}^+)$ if and only if the following system is verified:

$$\begin{cases} \Big(-\partial_\alpha^2 - \dfrac{2\alpha\pi^2}{1+s} - \gamma_2(s)\Big)g_{s,0}^-(\alpha) = 0 \quad \text{in } \mathbb{R}_-, \quad g_{s,0}^+(0) - g_0^-(0) = 0 \\ \Big(-\partial_\alpha^2 + \dfrac{2\alpha\pi^2}{1-s} - \gamma_2(s)\Big)g_{s,0}^+(\alpha) = 0 \quad \text{in } \mathbb{R}_+, \quad (g_{s,0}^+)'(0) - (g_{s,0}^-)'(0) = 0. \end{cases}$$

This leads to the choice:

$$\gamma_2(s) = (2\pi^2)^{2/3}\kappa_n(s); \quad g_{s,0}(\alpha) = \mathsf{T}_s^n((2\pi^2)^{2/3}\alpha),$$

with $g^{\pm}_{s,0}(\alpha) = \mathsf{T}^n_s((2\pi^2)^{2/3}\alpha)\mathbb{1}_{\mathbb{R}_\pm}(\alpha)$. Particularly, this determines the unknown functions of the previous steps (this choice of $g_{s,0}$ gives an explicit expression of $\Psi^{\pm}_{s,0}$). We are led to take:

$$\Psi^{\pm}_{s,2}(\alpha,t) = \Psi^{\pm,\perp}_{s,2}(\alpha,t) + g^{\pm}_{s,2}(\alpha)\mathfrak{s}_1(t),$$

with $\Psi^{+,\perp}_{s,2} \equiv 0$ and $\Psi^{-,\perp}_{s,2} \equiv 0$. Finally we have to solve the system:

$$\begin{cases} (-\partial_t^2 - \partial_\beta^2 - \pi^2)\Phi^{\pm}_{s,2} = 0 \text{ in } \mathsf{H}_\pm, \\ \Phi^{\pm}_{s,2}(\cdot, 0 \text{ and } 1) = 0, \\ \Phi^{-}_{s,2}(0,t) - \Phi^{+}_{s,2}(0,t) = (g^{+}_{s,2}(0) - g^{-}_{s,2}(0))\mathfrak{s}_1(t) \\ \partial_\beta\Phi^{-}_{s,2}(0,t) - \partial_\beta\Phi^{+}_{s,2}(0,t) = ((g^{+}_{s,1})'(0) - (g^{-}_{s,1})'(0))\mathfrak{s}_1(t). \end{cases}$$

Then, we apply Lemma 3.2 with $F^-_s \equiv 0$, $F^+_s \equiv 0$, $G_s \equiv 0$ and $H_s \equiv 0 = ((g^{+}_{s,1})'(0) - (g^{-}_{s,1})'(0))\mathfrak{s}_1(t)$ and we get:

$$\xi_s = g^{+}_{s,2}(0) - g^{-}_{s,2}(0) = 0, \quad \delta_s = (g^{+}_{s,1})'(0) - (g^{-}_{s,1})'(0) = 0.$$

This gives $\Phi^{+}_{s,2} \equiv 0$, $\Phi^{-}_{s,2} \equiv 0$.

Terms of order h The equations inside the strip read:

$$\mathsf{H}_{\pm,\alpha}: \ (-\partial_t^2 - \pi^2)\Psi^{\pm}_{s,3} = \gamma_3(s)\Psi^{\pm}_{s,0} \quad \mathsf{H}_{\pm,\beta}: \ (-\partial_t^2 - \partial_\beta^2 - \pi^2)\Phi^{\pm}_{s,3} = 0$$

The transmission conditions are:

$$(\Psi^{-}_{s,3} + \Phi^{-}_{s,3})(0,t) = (\Psi^{+}_{s,3} + \Phi^{+}_{s,3})(0,t),$$

$$\partial_\alpha\Psi^{-}_{s,2}(0,t) + \partial_\beta\Phi^{-}_{s,3}(0,t) - \frac{t}{s_-}\partial_t\Psi^{-}_{s,0}(0,t) = \partial_\alpha\Psi^{+}_{s,2}(0,t) + \partial_\beta\Phi^{+}_{s,3}(0,t) - \frac{t}{s_+}\partial_t\Psi^{+}_{s,0}(0,t).$$

We also take the Dirichlet boundary conditions (3.6) into account. Lemma 3.3 gives:

$$\gamma_3(s) = 0, \quad \Psi^{\pm}_{s,3}(\alpha,t) = g^{\pm}_{s,3}(\alpha)\mathfrak{s}_1(t).$$

We have to solve the system:

$$\begin{cases} (-\partial_t^2 - \partial_\beta^2 - \pi^2)\Phi^{\pm}_{s,3} = 0 \quad \text{in } \mathsf{H}_\pm, \\ \Phi^{\pm}_{s,3}(\cdot, 0 \text{ and } 1) = 0 \\ \Phi^{-}_{s,3}(0,t) - \Phi^{+}_{s,3}(0,t) = (g^{+}_{s,3}(0) - g^{-}_{s,3}(0))\mathfrak{s}_1(t) \\ (\partial_\beta\Phi^{-}_{s,3} - \partial_\beta\Phi^{+}_{s,3})(0,t) = ((g^{+}_{s,2})'(0) - (g^{-}_{s,2})'(0))\mathfrak{s}_1(t) + g^{+}_{s,0}(0)\Big(\dfrac{t\pi}{1-s} + \dfrac{t\pi}{1+s}\Big)\cos(\pi t). \end{cases}$$

Hence, we apply Lemma 3.2 with $F^-_s \equiv 0$, $F^+_s \equiv 0$, $G^0_s \equiv 0$ and $H_s(t) = g^{+}_{s,0}(0)\Big(\frac{t\pi}{1-s} + \frac{t\pi}{1+s}\Big)\cos(\pi t)$ and we get:

$$\xi_s = g^{+}_{s,3}(0) - g^{-}_{s,3}(0) = 0, \quad \delta_s = (g^{+}_{s,2})'(0) - (g^{-}_{s,2})'(0) = -\frac{2\sqrt{2}\pi}{1-s^2}\mathsf{T}^n_s(0)\langle t\cos(\pi t), \mathfrak{s}_1\rangle_t.$$

This determines $\Phi^{-}_{s,3}$ and $\Phi^{+}_{s,3}$ which are not necessarily zero.

Continuation Let us assume that we can write $\Psi^{\pm}_{s,k}(\alpha,t) = \Psi^{\pm,\perp}_{s,k}(\alpha,t) + g^{\pm}_{s,k}(\alpha)\mathfrak{s}_1(t)$ for all $0 \le k \le n$ and that $(g^{\pm}_{s,k})_{0\le k\le n-3}$, $(\Psi^{\pm,\perp}_{s,k})_{0\le k\le n-1}$ are determined. Let us also assume that $g^{-}_{s,n-2}(0) - g^{+}_{s,n-2}(0)$, $(g^{-}_{s,n-2})'(0) - (g^{+}_{s,n-2})'(0)$, $(\gamma_k(s))_{0\le k\le n}$ and $(\Phi^{\pm}_{s,k})_{0\le k\le n-2}$ are already known. Finally, we assume that $g^{-}_{s,n-1}(0) - g^{+}_{s,n-1}(0)$, $\Phi^{\pm}_{s,n-1}$ are known once $g^{-}_{s,n-2}$ and $g^{+}_{s,n-2}$ are determined and all that functions are exponentially decaying and analytic in the s variable for $s \in (-1,1)$.
The equations inside the strip read:

$$\begin{aligned}
\mathsf{H}_{\pm,\alpha}: \qquad (-\partial_t^2 - \pi^2)\Psi^{\pm}_{s,n} &= \gamma_n(s)\Psi^{\pm}_{s,0} - \mathcal{L}^{\pm}_{s,n}\Psi^{\pm}_{s,0} - \sum_{j=2}^{n-1}(\mathcal{L}^{\pm}_{s,j} - \gamma_j(s))\Psi^{\pm}_{s,n-j}\\
\mathsf{H}_{\pm,\beta}: \quad (-\partial_t^2 - \partial_\gamma^2 - \pi^2)\Phi^{\pm}_{s,n} &= -\sum_{j=1}^{n-1}(\mathcal{N}^{\pm}_{s,j} - \gamma_j(s))\Phi^{\pm}_{s,n-j}.
\end{aligned}$$

The transmission conditions are:

$$(\Psi^{-}_{s,n} + \Phi^{-}_{s,n}(0,t) = (\Psi^{+}_{s,n} + \Phi^{+}_{s,n})(0,t)$$

$$\begin{aligned}
(\partial_\beta\Phi^{-}_{s,n} - \partial_\beta\Phi^{+}_{s,n})(0,t) = \; &((g^{+}_{s,n-1})'(0) - (g^{-}_{s,n-1})'(0))\mathfrak{s}_1(t) + (\partial_\alpha\Psi^{+,\perp}_{s,n-1} - \partial_\alpha\Psi^{-,\perp}_{s,n-1})(0,t)\\
&+\frac{t}{1+s}(\partial_t\Psi^{-,\perp}_{s,n-3} + \partial_t\Phi^{-}_{s,n-3})(0,t) + \frac{t}{1-s}(\partial_t\Psi^{+,\perp}_{s,n-3} + \partial_t\Phi^{+}_{s,n-3})(0,t)\\
&+\sqrt{2}\pi\Big(\frac{1}{1+s}g^{-}_{s,n-3}(0) + \frac{1}{1-s}g^{+}_{s,n-3}(0)\Big)t\cos(\pi t).
\end{aligned}$$

In order to apply Lemma 3.3 we need to solve equations in the form:

$$\begin{aligned}
\Big(-\partial_\alpha^2 - \frac{2\alpha}{1+s}\pi^2 - \gamma_2(s)\Big)g^{-}_{s,n-2}(\alpha) &= \gamma_n(s)g^{-}_{s,0}(\alpha) + f^{-}_s(\alpha),\\
\Big(-\partial_\alpha^2 + \frac{2\alpha}{1-s}\pi^2 - \gamma_2(s)\Big)g^{+}_{s,n-2}(\alpha) &= \gamma_n(s)g^{+}_{s,0}(\alpha) + f^{+}_s(\alpha).
\end{aligned}$$

We can apply Lemma 3.4 because $f^{\pm}_s$, $g^{-}_{s,n-2}(0) - g^{+}_{s,n-2}(0)$ and $(g^{-}_{s,n-2})'(0) - (g^{+}_{s,n-2})'(0)$ are known. It provides an unique $\gamma_n(s)$, moreover $g^{\pm}_{s,n-2}$ are now determined. From the recursion assumption, we deduce that $\Phi^{\pm}_{s,n-1}$ are now determined. Lemma 3.3 uniquely determines $\Psi^{\pm,\perp}_{s,n}$ such that:

$$\Psi^{\pm}_{s,n}(\alpha,t) = \Psi^{\pm,\perp}_{s,n}(\alpha,t) + g^{\pm}_{s,n}(\alpha)\mathfrak{s}_1(t).$$

We can now write the system in the form:

$$\begin{cases}
(\mathcal{N}^{\pm}_{s,0} - \pi^2)\Phi^{\pm}_{s,n} = F^{\pm}_s, \text{in } \mathsf{H}_{\pm},\\
\Phi^{\pm}_{s,n}(\cdot, 0 \text{ and } 1) = 0,\\
\Phi^{-}_{s,n}(0,t) - \Phi^{+}_{s,n}(0,t) = (\Psi^{+,\perp}_{s,n} - \Psi^{-,\perp}_{s,n})(0,t) + (g^{-}_{s,n}(0) - g^{+}_{s,n}(0))\mathfrak{s}_1(t).\\
\partial_\beta\Phi^{-}_{s,n}(0,t) - \partial_\beta\Phi^{+}_{s,n}(0,t) = H_s(t) + ((g^{+}_{s,n-1})'(0) - (g^{-}_{s,n-1})'(0))\mathfrak{s}_1(t),
\end{cases}$$

where H_s is known. We can apply Lemma 3.2 which determines $g^{-}_{s,n}(0) - g^{+}_{s,n}(0)$, $(g^{+}_{s,n-1})'(0) - (g^{-}_{s,n-1})'(0)$, $\Phi^{-}_{s,n}$ and $\Phi^{+}_{s,n}$.

Quasimodes The previous construction leads to introduce:

$$\widehat{\psi}^{[J]}_{s,h}(u,t) := \sum_{j=0}^{J+2}(\Psi^{\pm}_{s,j}(uh^{-2/3},t) + \Phi^{\pm}_{s,j}(uh^{-1},t))h^{j/3} - u\chi^{\pm}(uh^{-1})R^{\pm}_{J,s,h}(p), \text{when } u \in \mathbb{R}_{\pm}. \quad (3.8)$$

where the correctors are

$$R^{\pm}_{J,s,h}(t) = \partial_\alpha \Psi^{\pm}_{s,J+2}(0,t)h^{J/3} - \frac{t}{s_\pm}\sum_{j=J}^{J+2}(\partial_t \Psi^{\pm}_{s,j}(0,t) + \partial_t \Phi^{\pm}_{s,j}(0,t))$$

are added to make $\widehat{\psi}^{[J]}_{s,h}$ satisfy the second transmission condition. Here $\chi^\pm$ are two smooth cut-off functions being 1 near 0. Then, by construction, $\psi^{[J]}_{s,h}$ defined by

$$\psi^{[J]}_{s,h}(x,y) = \chi(u)\widehat{\psi}^{[J]}_{s,h}(u,t)$$

belongs to the domain of $\mathcal{L}_s(h)$. Using the exponential decay, for all $s_0 \in (-1,1)$, $J \in \mathbb{N}$ we get the existence of $h_0 > 0$, $C(J, s_0, h_0) > 0$ such that for all $s \in [-s_0, s_0]$ and all $h \in (0, h_0)$:

$$\left\|(\mathcal{L}_s(h) - \sum_{j=0}^{J}\gamma_j(s)h^{j/3})\psi^{[J]}_{s,h}\right\| \leq C(J, s_0, h_0)h^{(J+1)/3}.$$

□

3.2 Agmon estimates

In order to prove Theorem 1.2, we need Agmon localization estimates about $\mathcal{L}_s(h)$ (see the work [1, 2] and in the semiclassical context [7, Chap. 6]) and [12]). We remark that thanks to Propositions 1.1 and 3.1, for all $s_0 \in (0,1)$ the N_0 lowest eigenvalues λ_s of $\mathcal{L}_s(h)$ satisfy for all $s \in [-s_0, s_0]$:

$$|\lambda_s - \pi^2| \leq \Gamma_0 h^{2/3}, \tag{3.9}$$

for some positive constant Γ_0 depending on N_0 and s_0. We define $\mathsf{Tri}^\pm(s) := \mathsf{Tri}(s) \cap \{x \in \mathbb{R}_\pm\}$. We have the following two Agmon localization estimates for the true eigenfunctions of $\mathcal{L}_s(h)$:

Proposition 3.5 *Let $s_0 \in [0,1)$. Let $\Gamma_0 > 0$ and $\rho_0 \in (0,\pi)$. There exist $h_0 > 0$, $C_0 > 0$ and $\eta_0 > 0$ such that for all $s \in [-s_0, s_0]$, $h \in (0, h_0)$ and all eigenpair (λ_s, ψ_s) of $\mathcal{L}_s(h)$ satisfying $|\lambda_s - \pi^2| \leq \Gamma_0 h^{2/3}$, we have:*

$$\int_{\mathsf{Tri}^\pm(s)} e^{\Phi_1^\pm(x)/h}(|\psi_s|^2 + |h^{2/3}\partial_x\psi_s|^2)\mathsf{d}x\mathsf{d}y \leq C_0\|\psi_s\|^2 \text{ and } \int_{\mathsf{Tri}^\pm(s)} e^{\Phi_2^\pm(x)/h}(|\psi_s|^2 + |h\partial_x\psi_s|^2)\mathsf{d}x\mathsf{d}y \leq C_0\|\psi_s\|^2,$$

with

$$\Phi_1^\pm(x) := \frac{\eta_0}{\sqrt{|s_\pm|}}|x|^{3/2} \text{ and } \Phi_2^\pm(x) := -\rho_0|s_\pm|\ln\left(1 + \frac{x}{s_\pm}\right).$$

In the regime $h \to 0$, Proposition 3.5 localize the eigenfunctions of $\mathcal{L}_s(h)$ in a neighborhood of $\mathsf{Tri}(s) \cap \{x = 0\}$ and gives decay estimates away from this set. These estimates justify that the Feshbach-Grushin projections of the true eigenfunctions of $\mathcal{L}_s(h)$ are good quasimodes for the one dimensional operator $(u \mapsto h^2 l_s^{\mathsf{tan}}(u + s; \partial_u))$, cf. (1.1), that is the tangent approximation of the Born-Oppenheimer approximation of $\mathcal{L}_s(h)$.

Because the shape of the effective potential v_s defined in (1.4) is close to the one of [6], the proof of Proposition 3.5 uses the technical background of Propositions 5.5. and 5.6 of [6].

3.3 Approximation of the first eigenfunctions by tensor products

In this subsection we will work with the operator $\mathcal{L}_{\mathsf{Rec}(s)}(h)$ defined on the left side $u \leq 0$ by $\mathcal{L}_s^{\mathsf{lef}}(h)$ and on the right side $u \geq 0$ by $\mathcal{L}_s^{\mathsf{rig}}(h)$. Let us consider the N_0 first eigenvalues of $\mathcal{L}_{\mathsf{Rec}(s)}(h)$ (shortly denoted by $\lambda_{s,n}(h)$). In each corresponding eigenspace we choose a normalized eigenfunction $\hat{\psi}_{s,n}$ so that $\langle \hat{\psi}_{s,n}, \hat{\psi}_{s,p} \rangle = 0$ if $n \neq p$. We introduce:

$$\widehat{\mathcal{S}}_{s,N_0}(h) = \mathsf{span}\big(\hat{\psi}_{s,1}, \dots, \hat{\psi}_{s,N_0}\big).$$

Then, we follow the same lines as in [6, Sec. 4.3] and define the following quadratic form:

$$\begin{aligned} Q^0_{\mathsf{Rec}(s)}(\hat{\psi}_s) \quad = \quad & \int_{\mathsf{R}_-(s)} (|\partial_t \hat{\psi}_s|^2 - \pi^2 |\hat{\psi}_s|^2)\Big(1 + \frac{u}{s_-}\Big)\mathsf{d}u\mathsf{d}t \, + \\ & \int_{\mathsf{R}_+(s)} (|\partial_t \hat{\psi}_s|^2 - \pi^2 |\hat{\psi}_s|^2)\Big(1 + \frac{u}{s_+}\Big)\mathsf{d}u\mathsf{d}t, \end{aligned}$$

where $\mathsf{R}_-(s) = \mathsf{Rec}(s) \,\cap\, \{u \leq 0\}$ and $\mathsf{R}_+(s) = \mathsf{Rec}(s) \,\cap\, \{u \geq 0\}$. We consider the projection:

$$\Pi_0 \hat{\psi}_s(u,t) = \langle \hat{\psi}_s(u,\cdot), \mathfrak{s}_1 \rangle_t \mathfrak{s}_1(t).$$

We can now state a first approximation result:

Proposition 3.6 *Let $s_0 \in (0,1)$, there exists $h_0 > 0$ and $C > 0$ such that for all $s \in [-s_0, s_0]$, all $h \in (0,h_0)$ and all $\hat{\psi}_s \in \widehat{\mathcal{S}}_{s,N_0}(h)$:*

$$0 \leq Q_{\mathsf{Rec}(s)}(\hat{\psi}_s) \leq C h^{2/3} \|\hat{\psi}_s\|^2$$

and

$$\big\|(\mathrm{Id} - \Pi_0)\hat{\psi}_s\big\| + \big\|\partial_t\big((\mathrm{Id} - \Pi_0)\hat{\psi}_s\big)\big\| \leq C h^{1/3} \|\hat{\psi}_s\|.$$

Moreover we have, $\Pi_0 : \widehat{\mathcal{S}}_{s,N_0} \to \Pi_0\big(\widehat{\mathcal{S}}_{s,N_0}\big)$ is an isomorphism.

Proof: We use the same reasoning as in [6, Sec 4.3] and [20, Sec 4.3]. □

3.4 Reduction to the model operator

The aim of this subsection is to prove Theorem 1.2 using the projections of the true eigenfunctions $(\Pi_0 \widehat{\psi}_{s,n})$ as test functions for the quadratic form of the model operator. We apply the technical background of [20, Sec. 4.4]. Let us consider $\widehat{\psi} \in \widehat{S}_{s,N_0}(h)$. We will need the two following lemmas to estimate the quadratic form of the model operator tested on $(\Pi_0 \widehat{\psi})$. The key in their proof is the use of Proposition 3.5.

Lemma 3.7 *Let $s_0 \in (0,1)$. For all $\widehat{\psi} \in \widehat{\mathcal{S}}_{s,N_0}$ there exist $h_0 > 0$ and $C > 0$ such that for all $s \in [-s_0, s_0]$ and all $h \in (0,h_0)$:*

$$\big|h^2 \int_{\mathsf{R}_\pm(s)} \partial_u \widehat{\psi}_s \partial_t \widehat{\psi}_s t \Big(1 + \frac{u}{s_\pm}\Big)\mathsf{d}u\mathsf{d}t\big|^2 \leq C h^{4/3} \big\|\widehat{\psi}_s\big\|^2,$$

Lemma 3.8 *Let $s_0 \in (0,1)$. Let $\widehat{\psi}_s \in \widehat{\mathcal{S}}_{s,N_0}(h)$. There exist $h_0 > 0$ and $C > 0$ such that for all $s \in [-s_0, s_0]$ and all $h \in (0, h_0)$:*

$$\Big|\frac{h^2}{s_\pm}\int_{\mathsf{R}_\pm(s)} |\partial_u \widehat{\psi}_s|^2 |u| \mathsf{d}u\mathsf{d}t\Big| \le Ch^{4/3}\big\|\widehat{\psi}_s\big\|^2\,, \quad \Big|\frac{1}{s_\pm^2}\int_{\mathsf{R}_\pm(s)} |\widehat{\psi}_s|^2 |u|^2 \mathsf{d}u\mathsf{d}t\Big| \le Ch^{4/3}\big\|\hat{\psi}_s\big\|^2\,.$$

We can now prove the

Proposition 3.9 *Let $s_0 \in (0,1)$, $\hat{\psi}_s \in \widehat{\mathcal{S}}_{s,N_0}(h)$. There exist $h_0 > 0$ and $C > 0$ such that for all $s \in [-s_0, s_0]$ and $h \in (0, h_0)$ we have:*

$$Q^{\mathsf{mod}}_{s,h}(\hat{\psi}_s) \le (\lambda_{s,N_0}(h) - \pi^2)\|\hat{\psi}_s\|^2 + Ch^{4/3}\|\hat{\psi}_s\|^2,$$

where

$$Q^{\mathsf{mod}}_{s,h}(\hat{\psi}_s) := \int_{\mathsf{R}_-(s)} h^2|\partial_u \hat{\psi}_s|^2 + \frac{2\pi^2}{1+s}|u||\hat{\psi}_s|^2 \mathsf{d}u\mathsf{d}t + \int_{\mathsf{R}_+(s)} h^2|\partial_u \hat{\psi}_s| + \frac{2\pi^2}{1-s}|u||\psi_s|^2 \mathsf{d}u\mathsf{d}t.$$

We remark that $Q^{\mathsf{mod}}_{s,h}$ is, up to h^2 in front of the derivative term and a factor $2\pi^2$ for the potential, the quadratic form of the model operator in two dimension.
Proof: Let us take $\psi_s \in \mathcal{S}_{s,N_0}(h)$. As the $(\psi_{s,j})_{j\in\{1,\ldots,N_0\}}$ are orthogonal, we have:

$$Q_{s,h}(\psi_s) \le \lambda_{s,N_0}(h)\|\psi_s\|^2.$$

By definition, for all $\psi \in \mathsf{Dom}(Q_{s,h})$ we have:

$$Q_{s,h}(\psi) \ge \int_{\mathsf{Tri}(s)} h^2|\partial_x \psi|^2 + v_s(x)|\psi|^2 \mathsf{d}x\mathsf{d}y.$$

The last inequality combined with the convexity of the effective potential v_s yields:

$$\int_{\mathsf{Tri}(s)} h^2|\partial_x \psi_s|^2 + 2\pi^2\Big(\frac{\mathbb{1}_{\mathbb{R}_-}(x)}{1+s} + \frac{\mathbb{1}_{\mathbb{R}_-}(x)}{1-s}\Big)|x||\psi_s|^2 \mathsf{d}x\mathsf{d}y \le (\lambda_{s,N_0}(h) - \pi^2)\|\psi_s\|^2.$$

Then, we perform the change of variables (3.3) to get:

$$\begin{aligned} Q^{\mathsf{mod}}_{s,h}(\hat{\psi}_s) \;\le\; & (\lambda_{s,N_0}(h) - \pi^2)\|\hat{\psi}_s\|^2 + \frac{h^2}{s_-}\int_{\mathsf{R}_-(s)} |\hat{\psi}_s|^2|u|\mathsf{d}u\mathsf{d}t + \frac{h^2}{|s_+|}\int_{\mathsf{R}_+(s)} |\hat{\psi}_s|^2|u|\mathsf{d}u\mathsf{d}t + \\ & \frac{2\pi^2}{s_-^2}\int_{\mathsf{R}_-(s)} |\hat{\psi}_s|^2|u|^2\mathsf{d}u\mathsf{d}t + \frac{2\pi^2}{s_+^2}\int_{\mathsf{R}_+(s)} |\hat{\psi}_s|^2|u|^2\mathsf{d}u\mathsf{d}t + \\ & 2h^2\int_{\mathsf{R}_-(s)} t\partial_u\hat{\psi}_s\partial_t\hat{\psi}_s\Big(1 + \frac{u}{s_-}\Big)\mathsf{d}u\mathsf{d}t - 2h^2\int_{\mathsf{R}_+(s)} t\partial_u\hat{\psi}_s\partial_t\hat{\psi}_s\Big(1 + \frac{u}{s_+}\Big)\mathsf{d}u\mathsf{d}t \end{aligned}$$

To obtain Proposition 3.9 we apply Lemmas 3.7 and 3.8 taking into account (3.9). □

Proof of Theorem 1.2 We apply Proposition 3.6 to the result of Proposition 3.9 and we obtain:

$$Q^{\mathsf{mod}}_{s,h}(\hat{\psi}_s) \leq (\lambda_{s,N_0}(h) - \pi^2)\big\|\Pi_0\hat{\psi}_s\big\|^2 + Ch^{4/3}\big\|\Pi_0\hat{\psi}_s\big\|^2.$$

Then, equation (3.9) and Lemma 3.6 yield:

$$Q^{\mathsf{mod}}_{s,h}(\hat{\psi}_s) \leq (\lambda_{s,N_0}(h) - \pi^2)\big\|\Pi_0\hat{\psi}_s\big\|^2_{L^2(\mathsf{Rec}(s))} + Ch^{4/3}\big\|\Pi_0\hat{\psi}_s\big\|^2_{L^2(\mathsf{Rec}(s))},$$

where

$$\|\hat{\psi}_s\|^2_{L^2(\mathsf{Rec}(s))} = \|\hat{\psi}_s\|^2_{L^2(\mathsf{R}_-(s),\mathsf{d}u\mathsf{d}t)} + \|\hat{\psi}_s\|^2_{L^2(\mathsf{R}_+(s),\mathsf{d}u\mathsf{d}t)}.$$

Moreover, we have

$$Q^{\mathsf{mod}}_{s,h}(\hat{\psi}_s) = Q^{\mathsf{mod}}_{s,h}(\Pi_0\hat{\psi}_s) + Q^{\mathsf{mod}}_{s,h}((\mathrm{Id} - \Pi_0)\hat{\psi}_s) + 2b^{\mathsf{mod}}_{s,h}\big(\Pi_0\hat{\psi}_s, (\mathrm{Id} - \Pi_0)\hat{\psi}_s\big),$$

where $b^{\mathsf{mod}}_{s,h}$ is the bilinear form associated with $Q^{\mathsf{mod}}_{s,h}$. We remark that

$$\begin{aligned} b^{\mathsf{mod}}_{s,h}\big(\Pi_0\hat{\psi}_s, (\mathrm{Id} - \Pi_0)\hat{\psi}_s\big) \quad = \quad & \int_{\mathsf{R}_-(s)} \langle \Pi_0\Big(\big(-h^2\partial_u^2 + \frac{2\pi^2}{1+s}|u|\big)\psi_s\Big), (\mathrm{Id} - \Pi_0)\hat{\psi}_s\rangle_t \mathsf{d}u\mathsf{d}t \, + \\ & \int_{\mathsf{R}_+(s)} \langle \Pi_0\Big(\big(-h^2\partial_u^2 + \frac{2\pi^2}{1-s}|u|\big)\psi_s\Big), (\mathrm{Id} - \Pi_0)\hat{\psi}_s\rangle_t \mathsf{d}u\mathsf{d}t = 0. \end{aligned}$$

Finally we have

$$Q^{\mathsf{mod}}_{s,h}(\Pi_0\hat{\psi}_s) \leq (\lambda_{s,N_0}(h) - \pi^2)\big\|\Pi_0\hat{\psi}_s\big\|^2_{L^2(\mathsf{Rec}(s))} + Ch^{4/3}\big\|\Pi_0\hat{\psi}_s\big\|^2_{L^2(\mathsf{Rec}(s))}.$$

Now the conclusion is standard. We denote by $\pi_0\hat{\psi}_s := \langle\hat{\psi}_s(u,\cdot), \mathfrak{s}_1\rangle_t$. Let us consider the smooth cut-off function χ being 1 for $|u| \leq \dfrac{1}{4}$ and 0 for $|u| \geq \dfrac{1}{2}$. We define $\chi_s(u) := \chi\Big(\dfrac{u}{1-s}\Big)$ it holds:

$$q^{\mathsf{mod}}_{s,h}(\chi_s\pi_0\hat{\psi}_s) \leq (\lambda_{s,N_0}(h) - \pi^2)\big\|\pi_0\hat{\psi}_s\big\|^2 + Ch^{4/3}\big\|\pi_0\hat{\psi}_s\big\|^2,$$

where

$$q^{\mathsf{mod}}_{s,h}(\varphi) := \int_{-1-s}^{1-s} h^2|\partial_u\varphi|^2 + 2\pi^2\Big(\frac{\mathbb{1}_{\mathbb{R}_-}(u)}{1+s} + \frac{\mathbb{1}_{\mathbb{R}_-}(u)}{1-s}\Big)|u||\varphi|^2\mathsf{d}u.$$

Then, we consider $\widehat{s}_{s,N_0}(h) := \mathsf{span}\big(\pi_0\hat{\psi}_{s,1}, \ldots, \pi_0\hat{\psi}_{s,N_0}\big)$ and we apply the min-max principle to the N_0 dimensional space $\chi_s\widehat{s}_{s,N_0}(h)$ wich yields

$$\pi^2 + (2\pi^2)^{2/3}\kappa_{N_0}(s)h^{2/3} \leq \lambda_{s,N_0}(h).$$

Jointly with Proposition 3.1, this finishes the proof of Theorem 1.2.

4 An application to tunneling: a symmetric mountain

The aim of this section is to prove Theorem 1.4. We follow the philosophy of [12] about tunneling. After the proof of some Agmon localization estimates in Subsection 4.1 we study in Subsection 4.2

the splitting of eigenvalues for this problem. To avoid the dependence on h of the domain $\Omega(h)$ we perform the scaling:

$$x = x_1; \quad y = \frac{1}{h}x_2.$$

The domain $\Omega(h)$ is now $\Omega := \Omega(1)$ and we respectively denote by $\mathcal{L}(h), \mathcal{L}^{\mathsf{lef}}(h), \mathcal{L}^{\mathsf{rig}}(h)$ and $\mathfrak{L}(h)$ the operators $-\Delta^{\mathsf{Dir}}_{\Omega(h)}, -\Delta^{\mathsf{Dir}}_{\Omega^{\mathsf{lef}}(h)}, -\Delta^{\mathsf{Dir}}_{\Omega^{\mathsf{rig}}(h)}$ and $\mathfrak{D}(h)$ in this new variables, up to a multiplication by h^2. $\mathcal{L}(h), \mathcal{L}^{\mathsf{lef}}(h)$ and $\mathcal{L}^{\mathsf{rig}}(h)$ are the Dirichlet realization of $-h^2\partial_x^2 + \partial_y^2$ on each associated geometric domain (denoted $\Omega, \Omega^{\mathsf{lef}}$ and Ω^{rig}). We denote by $(\zeta_n(h))_{n\geq 1}$ the eigenvalues of $\mathcal{L}(h)$ and $(\lambda_n(h))_{n\geq 1}$ the eigenvalues of the two iso-spectral operators $\mathcal{L}^{\mathsf{lef}}(h)$ and $\mathcal{L}^{\mathsf{rig}}(h)$. In terms of physical variables, we obtain:

$$\nu_n(h) = h^{-2}\zeta_n(h); \quad \mu_n(h) = h^{-2}\lambda_n(h).$$

For the operator $\mathcal{L}(h)$ we construct an effective potential in the spirit of the Born-Oppenheimer approximation as in Subsection 1.4, we obtain the operator:

$$-h^2\partial_x^2 + v(x),$$

where the effective potential v is:

$$v(x) = \pi^2\Big(\frac{(1-s)^2}{(x+1)^2}\mathbb{1}_{(-1,-s)}(x) + \frac{4s^2}{(x-s)^2}\mathbb{1}_{(-s,0)}(x) + \frac{4s^2}{(x+s)^2}\mathbb{1}_{(0,s)}(x) + \frac{(1-s)^2}{(x-1)^2}\mathbb{1}_{(s,1)}(x)\Big)$$

The shape of the potential helps us to get the Agmon estimates of Subsection 4.1.

4.1 Agmon estimates

To enlighten tunneling we need Agmon localization estimates on Ω for the true eigenfunctions of $\mathcal{L}(h)$, $\mathcal{L}^{\mathsf{lef}}(h)$ and $\mathcal{L}^{\mathsf{rig}}(h)$.

Agmon estimates for $\mathcal{L}(h)$ between the two peaks

Proposition 4.1 *Let $\Gamma_0 > 0$. There exist $h_0 > 0$, $C_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and all eigenpair (ζ, ψ) of $\mathcal{L}(h)$ satisfying $|\zeta - \pi^2| \leq \Gamma_0 h^{2/3}$, we have:*

$$\int_{\Omega\,\cap\,\{(-s/2,s/2)\times\mathbb{R}\}} |\psi|^2 + |h^{2/3}\partial_x\psi|^2 \mathsf{d}x\mathsf{d}y \leq C_0 e^{-C/h}\|\psi\|^2.$$

Proof: For Φ a Lipschitz function to be dertermined, if (ζ, ψ) is an eigenpair of $\mathcal{L}(h)$, the IMS formula reads:

$$\int_\Omega h^2|\partial_x(e^{\Phi/h}\psi)|^2 + |e^{\Phi/h}\partial_y\Psi|^2 - |\Phi' e^{\Phi/h}\psi|^2 - \zeta|e^{\Phi/h}\psi|^2 \mathsf{d}x\mathsf{d}y = 0.$$

Consequently we have:

$$\int_\Omega h^2|\partial_x(e^{\Phi/h}\psi)|^2 + v(x)|e^{\Phi/h}\Psi|^2 - |\Phi' e^{\Phi/h}\psi|^2 - \zeta|e^{\Phi/h}\psi|^2 \mathsf{d}x\mathsf{d}y \leq 0.$$

We denote by:

$$\mathcal{I}_1 = (-1,-s), \quad \mathcal{I}_2 = (-s,0), \quad \mathcal{I}_3 = (0,s), \quad \mathcal{I}_4 = (s,1),$$

these intervals. Let $\Omega_j = \Omega \cap \{\mathcal{I}_j \times \mathbb{R}\}$ Then, by convexity near each minimum, and for symmetry reasons we have:

$$\int_\Omega h^2 |\partial_x (e^{\Phi/h}\psi)|^2 \mathsf{d}x\mathsf{d}y + \sum_{j=1}^{4} \int_{\Omega_j} (\pi^2 + t_j(x) - \zeta - \Phi'(x)^2)|e^\Phi \psi|^2 \mathsf{d}x\mathsf{d}y \leq 0,$$

where we have:

$$t_1(x) = -\frac{2\pi^2}{1-s}(x+s), \quad t_2(x) = \frac{\pi^2}{s}(x+s), \quad t_3(x) = -\frac{\pi^2}{s}(x-s), \quad t_4(x) = \frac{2\pi^2}{1-s}(x-s).$$

Then, we apply the same reasoning as in the proof of Proposition 3.5 and, using the reflection symmetry, we obtain a $C_0 > 0$ such that:

$$\int_\Omega e^{2\Phi(x)/h}\big(|\psi|^2 + |h^{2/3}\partial_x \psi|^2\big)\mathsf{d}x\mathsf{d}y \leq C_0 \|\psi\|^2, \tag{4.1}$$

with, for some $\eta_1, \eta_2 > 0$:

$$\Phi(x) = \eta_1 |x+s|^{3/2}\, \mathbb{1}(-1,-s)(x) + \eta_2 |x+s|^{3/2}\, \mathbb{1}(-s,0)(x) + \eta_2 |x-s|^{3/2}\, \mathbb{1}(0,s)(x) + \eta_1 |x-s|^{3/2}\, \mathbb{1}(s,1)(x).$$

We have $\mathbb{1}_{(-s/2,s/2)}\Phi \leq \Phi$. moreover there exists a constant $\tilde{C} > 0$ such that:

$$\tilde{C} \leq \Phi(x), \forall x \in (-s/2, s/2).$$

Combined with (4.1), it yields for some constant $C > 0$:

$$\int_{\Omega \,\cap\, \{(-s/2,s/2)\times\mathbb{R}\}} |\psi|^2 + |h^{2/3}\partial_x \psi|^2 \mathsf{d}x\mathsf{d}y \leq C_0 e^{-C/h}\|\psi\|^2.$$

□

Agmon estimates for $\mathcal{L}^{\mathsf{lef}}(h)$ and $\mathcal{L}^{\mathsf{rig}}(h)$ Using the same philosophy as in the proof of Proposition 4.1 and adapting the proof of Proposition 3.5 we have:

Proposition 4.2 *Let $\Gamma_0 > 0$. There exist $h_0 > 0$, $C_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}^{\mathsf{lef}}(h)$ satisfying $|\lambda - \pi^2| \leq \Gamma_0 h^{2/3}$, we have:*

$$\int_{\Omega^{\mathsf{lef}} \,\cap\, \{(-s/2,0)\times\mathbb{R}\}} |\psi|^2 + |h^{2/3}\partial_x \psi|^2 \mathsf{d}x\mathsf{d}y \leq C_0 e^{-C/h}\|\psi\|^2.$$

The same proposition holds for $\mathcal{L}^{\mathsf{rig}}(h)$.

4.2 Spectrum of $\mathcal{L}(h)$

In our way to prove Theorem 1.4 we need the

Proposition 4.3 *For all $N \in \mathbb{N}^*$ there exist $h_0 > 0$, $C_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and all $j \in \{1, \ldots, N\}$:*

$$\zeta_{2j-1}(h) \leq C_0 e^{-C/h} + \lambda_j(h), \quad \zeta_{2j}(h) \leq C_0 e^{-C/h} + \lambda_j(h).$$

Proof: Let χ^{lef} and χ^{rig} be a smooth cut-off functions such that $\chi^{\mathsf{lef}}(x) = 1$ for all $x \in (-1, -s/2)$ and $\chi^{\mathsf{lef}}(x) = 0$ for all $x \in (-s/4, 0)$ and $\chi^{\mathsf{rig}}(x) = \chi^{\mathsf{lef}}(-x)$ for all $x \in (-1, 1)$. We take ψ_j^{lef} and ψ_j^{rig} two eigenfunctions associated with $\lambda_j(h)$, respectively for $\mathcal{L}^{\mathsf{lef}}(h)$ and $\mathcal{L}^{\mathsf{rig}}(h)$, and we define the two dimensional space $E_j := \mathsf{span}\big(\chi^{\mathsf{lef}}\psi_j^{\mathsf{lef}}, \chi^{\mathsf{rig}}\psi_j^{\mathsf{rig}}\big) \subset \mathsf{Dom}(\mathcal{L}(h))$. Let $\psi \in E_j$, thanks to Proposition 4.2 there exist $h_0 > 0$, $C_0 > 0$ and $C > 0$ such that, for all $h \in (0, h_0)$:

$$\|(\mathcal{L}(h) - \lambda_j(h))\psi\| \leq C_0 e^{-C/h}\|\psi\|.$$

The spectral theorem yields the existence of $k_1(j,h) \in \mathbb{N}^*$ such that:

$$|\zeta_{k_1(j,h)}(h) - \lambda_j(h)| \leq C_0 e^{-C/h}.$$

Nevertheless, because $\mathsf{dim}(E_j) = 2$ we can find another $k_2(j) \in \mathbb{N}^*$ such that:

$$|\zeta_{k_2(j,h)}(h) - \lambda_j(h)| \leq C_0 e^{-C/h}.$$

Without loss of generality we can assume that $k_1(j,h) < k_2(j,h)$. Now, we prove in three steps that $k_1(j,h) \geq 2j$.

$(j \mapsto k_1(j,h))$ **and** $(j \mapsto k_2(j,h))$ **are injective functions** Assume that there exists such that $k_1(j_1,h) = k_1(j_2,h)$. We have

$$|\lambda_{j_1}(h) - \zeta_{k_1(j_1,h)}(h)| \leq C_0 e^{-C/h}, \quad |\lambda_{j_2}(h) - \zeta_{k_1(j_2,h)}(h)| \leq C_0 e^{-C/h},$$

which gives

$$|\lambda_{j_1}(h) - \lambda_{j_2}(h)| \leq 2C_0 e^{-C/h}.$$

Nevertheless, two distinct eigenvalues in the spectrum of $\mathfrak{L}(h)$ satisfy $|\lambda_{j_1}(h) - \lambda_{j_2}(h)| \geq \tilde{C}h^{2/3}$, for some $\tilde{C} > 0$. Consequently $j_1 = j_2$. The proof is the same for $k_2(j,h)$.

$(j \mapsto k_1(j,h))$ **and** $(j \mapsto k_2(j,h))$ **are non-decreasing functions** Let $j_2 > j_1$, we have:

$$\zeta_{k_1(j_2,h)}(h) \geq \lambda_{j_2}(h) - C_0 e^{-C/h}, \quad \zeta_{k_1(j_1,h)}(h) \leq \lambda_{j_1}(h) + C_0 e^{-C/h}.$$

We obtain:

$$\zeta_{k_1(j_2,h)}(h) - \zeta_{k_1(j_1,h)}(h) \geq \lambda_{j_2}(h) - \lambda_{j_1}(h) - 2C_0 e^{-C/h},$$

where the right-hand side of the inequality is positive for h small enough. Necessarily $k_1(j_2,h) \geq k_1(j_1,h)$ and thanks to the first step $k_1(j_1,h) > k_1(j_2,h)$. The same reasoning hold for $k_2(j_1,h)$ and $k_2(j_2,h)$. Moreover we can prove with the same technical background that $\zeta_{k_2(j_1,h)}(h) < \zeta_{k_1(j_2,h)}(h)$.

Induction For $j = 1$ we have $k_2(j,h) > k_1(j,h) \geq 1$. Let $j \in \mathbb{N}^*$ such that $k_2(j,h) > k_1(j,h) > 2j$. Thanks to the previous steps we have:

$$k_1(j+1,h) > k_2(j,h) > k_1(j,h) \geq 2j.$$

We deduce that $k_1(j+1,h) \geq 2j+2$. Because $k_2(j+1,h) > k_1(j+1,h)$, we have $k_2(j+1,h) \geq 2j+3$, which concludes the induction and the proof of Proposition 4.3. $\square$

Now, we prove that there is at most two eigenvalues of $\mathcal{L}(h)$ at a distance of order $\mathcal{O}(e^{-C/h})$ from an eigenvalue of $\mathfrak{L}(h)$.

Proposition 4.4 *For all $N \in \mathbb{N}^*$, $C_0 > 0$, $C > 0$, there exist $h_0 > 0$ such that for all $h \in (0, h_0)$ and all $j \in \{1, \ldots, N\}$:*

$$\#\{\zeta(h) \in \mathfrak{S}(\mathcal{L}(h)); |\lambda_j(h) - \zeta(h)| \leq C_0 e^{-C/h}\} \leq 2.$$

Proof: Let us assume that there exist $\zeta_{j_1}(h) < \zeta_{j_2}(h) < \zeta_{j_3}(h)$ in $\mathfrak{S}(\mathcal{L}(h))$ such that, for all $p \in \{1, 2, 3\}$, we have:

$$|\lambda_j(h) - \zeta_{j_p}(h)| \leq C_0 e^{-C/h}.$$

We consider the family of orthonormal functions $(\varphi_{j_p})_{p\in\{1,2,3\}}$ where, for $p \in \{1, 2, 3\}$, φ_{j_p} is an eigenfunction of $\mathcal{L}(h)$ associated with $\lambda_{j_p}(h)$. Now we define $\psi_{j_p} = (\chi^{\mathsf{lef}}\varphi_{j_p}, \chi^{\mathsf{rig}}\varphi_{j_p})$ and the space $\mathcal{E}_j = \mathsf{span}(\psi_{j_p})_{p\in\{1,2,3\}}$. We will also be lead to investigate the eigenspace $\mathcal{F}_j$ of $\mathfrak{L}(h)$ associated with $\lambda_j(h)$: we denote by $\Phi_{j,1}$ and $\Phi_{j,2}$ an orthonormal basis of $\mathcal{F}_j$. Let us take $\psi \neq 0 \in \mathcal{E}_j$ such that, for $p = 1, 2$, $\langle \psi, \Phi_{j,p} \rangle = 0$. Such a function exists because $\mathsf{dim}(\mathcal{E}_j) = 3$ and, thanks to Proposition 4.1, we have:

$$\|(\mathfrak{L}(h) - \lambda_j(h))\psi\| \leq C_0 e^{-C/h}\|\psi\|.$$

The spectral theorem yields $n(j, h) \in \mathbb{N}^*$, with $n(j, h) \neq j$, such that

$$|\lambda_{n(j,h)}(h) - \lambda_j(h)| \leq C_0 e^{-C/h}.$$

Finally, we know that there exists $\tilde{C} > 0$ such that $|\lambda_{n(j,h)}(h) - \lambda_j(h)| \geq \tilde{C}h^{2/3}$. For h small enough we have a contradiction. □

Proposition 4.5 *For all $N \in \mathbb{N}^*$ there exist h_0, $C_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ and all $j \in \{1, \ldots, 2N\}$:*

$$|\zeta_j(h) - \lambda_{n(j,h)}(h)| \leq C_0 e^{-C/h},$$

for some $n(j, h) \in \mathbb{N}^*$.

Proof: Let φ_j be an eigenfunction of $\mathcal{L}(h)$ associated with the eigenvalue $\zeta_j(h)$. Let us take χ^{lef} and χ^{rig} like in the proof of Proposition 4.3. We define $\Phi_j = (\chi^{\mathsf{lef}}\varphi_j, \chi^{\mathsf{rig}}\varphi_j) \in \mathsf{Dom}(\mathfrak{L}(h))$ and we have, thanks to Proposition 4.1:

$$\|(\mathfrak{L}(h) - \zeta_j(h))\Phi_j\| \leq C_0 e^{-C/h}\|\Phi_j\|.$$

The spectral theorem yields the conclusion. □

Proof of Theorem 1.4 We prove that in Proposition 4.5 we have:

$$n(2j - 1, h) \geq j, \quad n(2j, h) \geq j.$$

The proof is organized in three steps, as in Proposition 4.3.

$(j \mapsto n(2j-1,h))$ and $(j \mapsto n(2j-1,h))$ are injective functions Let $j_1, j_2 \in \mathbb{N}^*$ such that $j_1 < j_2$ and

$$n(2j_1 - 1, h) = n(2j_2 - 1, h).$$

Consequently we have:

$$\zeta_{2j_2}(h) - \zeta_{2j_1}(h) \leq 2C_0 e^{-C/h}.$$

There exists $\tilde{j} \in \mathbb{N}^*$ such that $2j_1 - 1 < \tilde{j} < 2j_2 - 1$ and we have:

$$\zeta_{\tilde{j}}(h) - \zeta_{2j_1}(h) \leq \zeta_{2j_2}(h) - \zeta_{2j_1}(h) \leq 2C_0 e^{-C/h}.$$

Finally, we get:

$$|\lambda_{n(2j_1-1,h)}(h) - \zeta_{\tilde{j}}(h)| = |\lambda_{n(2j_1-1,h)}(h) - \zeta_{2j_1-1}(h) + \zeta_{2j_1-1}(h) - \zeta_{\tilde{j}}(h)| \leq 3C_0 e^{-C/h}.$$

Then, we apply Proposition 4.4 and we have a contradiction because we found three eigenvalues exponentially close to $\lambda_{n(2j_1-1,h)}(h)$. The proof is the same for $(j \mapsto n(2j-1,h))$.

$(j \mapsto n(2j-1,h))$ and $(j \mapsto n(2j-1,h))$ are non-decreasing functions Let $j_1, j_2 \in \mathbb{N}^*$ such that $j_1 < j_2$, we have:

$$\lambda_{n(2j_2-1,h)}(h) - \lambda_{n(2j_1-1,h)}(h) + 2C_0 e^{-C/h} \geq \zeta_{2j_2-1}(h) - \zeta_{2j_1-1}(h) > 0.$$

Thanks to the injectivity, for some $\tilde{C} > 0$, we get:

$$\mathsf{sgn}(n(2j_2 - 1, h) - n(2j_1 - 1, h))\tilde{C}h^{2/3} > 0,$$

where sgn denotes the function sign. Necessarily we get $n(2j_2 - 1, h) > n(2j_1 - 1, h)$.

Induction For $j = 1$ we have $n(1,h) \geq 1$ and $n(2,h) \geq 1$. Let $j \in \mathbb{N}^*$ such that $n(2j-1,h) \geq j$ and $n(2j,h) \geq j$. Thanks to the previous step we have:

$$n(2j+1,h) > n(2j-1,h) \geq j, \quad n(2j+2,h) > n(2j,h) \geq j.$$

which achieves the induction. Finally we get:

$$\lambda_j(h) \leq C_0 e^{-C/h} + \zeta_{2j-1}(h), \quad \lambda_j \leq C_0 e^{-C/h} + \zeta_{2j}(h).$$

Combined with Proposition 4.3 it gives Theorem 1.4.

Figure 4 illustrates Theorem 1.4 and enlighten the localization of the spectrum of $\mathcal{L}(h)$.

Remark 4.6 Due to symmetry reason, we can be more accurate about the spectrum $\mathfrak{S}(\mathcal{L}(h))$. In fact, for all $j \geq 1$, we necessarily have $\zeta_{2j-1}(h) < \lambda_j(h)$ and $\zeta_{2j}(h) = \lambda_j(h)$. △

A Shape of the eigenfunctions in the semiclassical limit

Let us now illustrate some theoretical properties of the eigenfunctions of $\mathcal{L}_s(h)$ with numerical simulations. These computations are performed in the domain $\mathsf{Tri}(s)$ with the finite element library Melina++ [17]. The mesh is constituted of triangles with 4 subdivisions on $\mathsf{Tri}^-(s)$ and $\mathsf{Tri}^+(s)$ with 6 as interpolation degree. Figure 5 pictures the dominant term in the construction (3.8): It is almost a tensor product of the eigenfunction of the toy model operator and the sinus (respectively along the X-axis and the Y-axis). The eigenfunctions are localized near the altitude of the triangle. This matches with the Agmon estimates of Proposition 3.5.

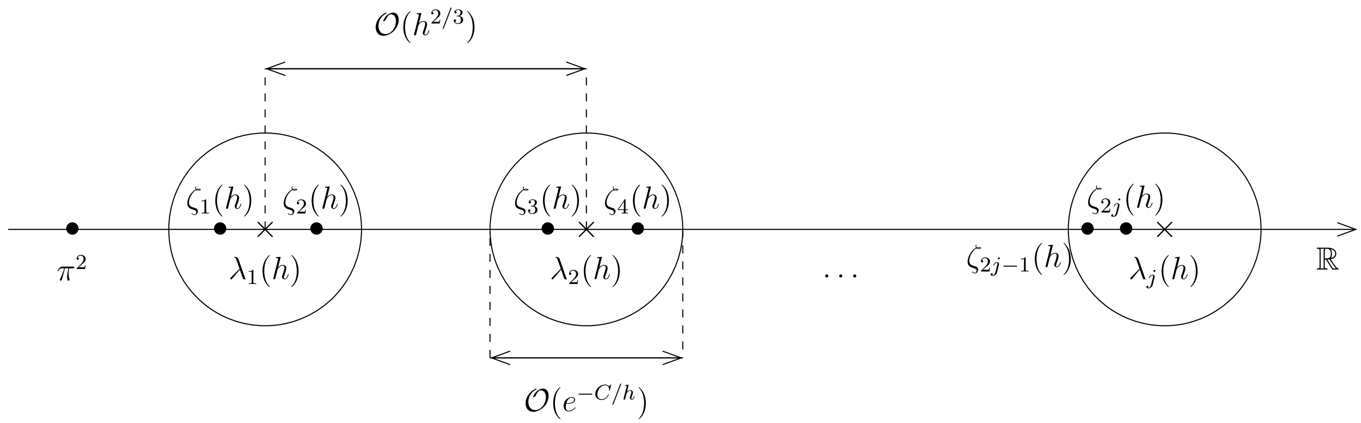


Figure 4: Repartition of $\mathfrak{S}(\mathcal{L}(h))$ around $\mathfrak{S}(\mathfrak{L}(h))$

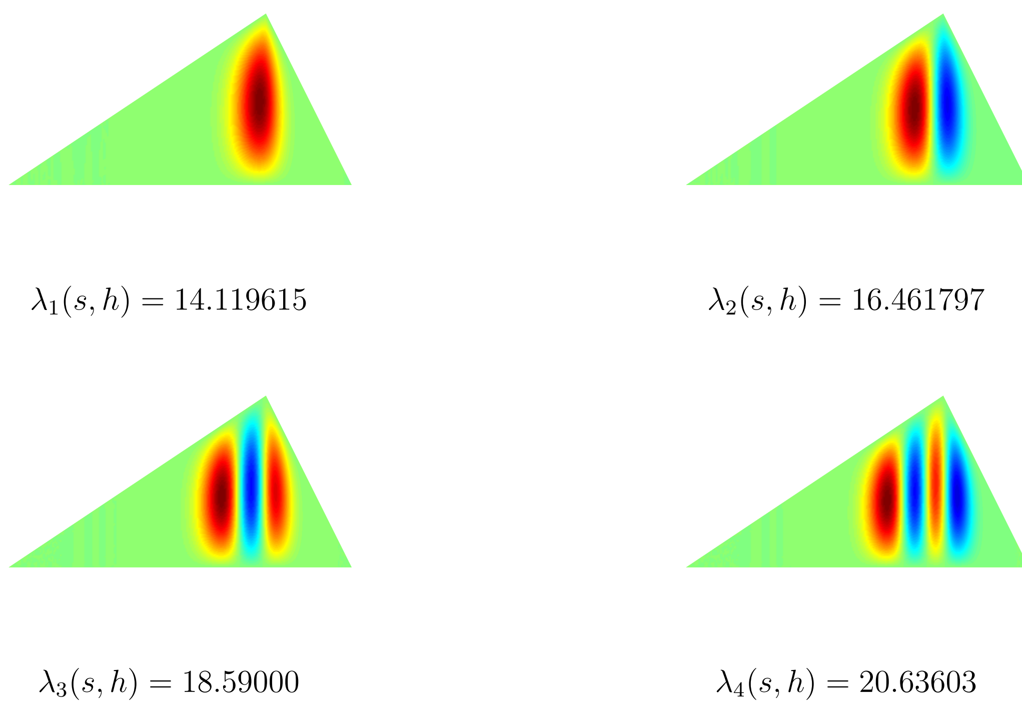


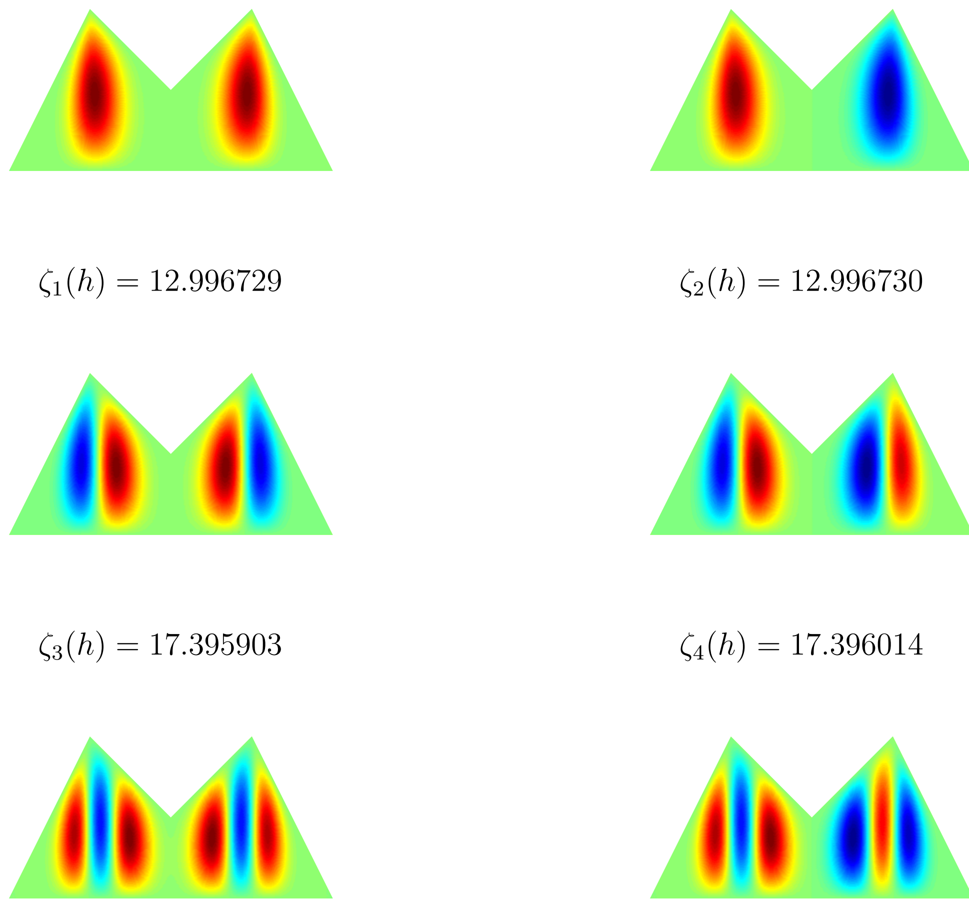
Figure 5: This figure represents the four first eigenfunctions of $\mathcal{L}_s(h)$ and their corresponding eigenvalue for $s = 0.5$ and $h = 0.1$.

B Illustrations of tunneling

To illustrate some properties of the eigenfunctions of a symmetric mountain we compute some of them. These computations are performed in the domain Ω for the operator $\mathcal{L}(h)$. The mesh is constituted of triangles with 3 subdivisions on the Ω_j ($j = 1, \dots, 4$) with 6 as interpolation degree. Figure 6 depicts the phenomenon of tunneling discussed in Section 4. We remark that the localization of the eigenfunctions matches with the Agmon estimates of Subsection 4.1. Moreover a pair of exponentially close eigenvalues is associated with eigenfunctions even or odd along the $X-$axis.

Figure 7 pictures Theorem 1.4 and the exponentially close difference between $\zeta_{2j-1}(h)$ and $\zeta_{2j}(h)$

(for $j = 1 \ldots 3$). The interaction is of order $e^{-C/h}$ (for some $C > 0$). Nevertheless for small h there is a stalling in the computations: we can not do the computation correctly because the difference between the eigenvalues is two close to zero to compute them correctly. Consequently we did a linear approximations for small h but before the stalling. These linear approximations go to -1 as h goes to 0.



$\zeta_1(h) = 12.996729$ $\zeta_2(h) = 12.996730$

$\zeta_3(h) = 17.395903$ $\zeta_4(h) = 17.396014$

$\zeta_5(h) = 21.411375$ $\zeta_6(h) = 21.413785$

Figure 6: Computation for $s = 0.5$ and $h = 0.15$. Numerical values of the first six eigenvalues. Plots of the associated eigenfunctions in the domain Ω.

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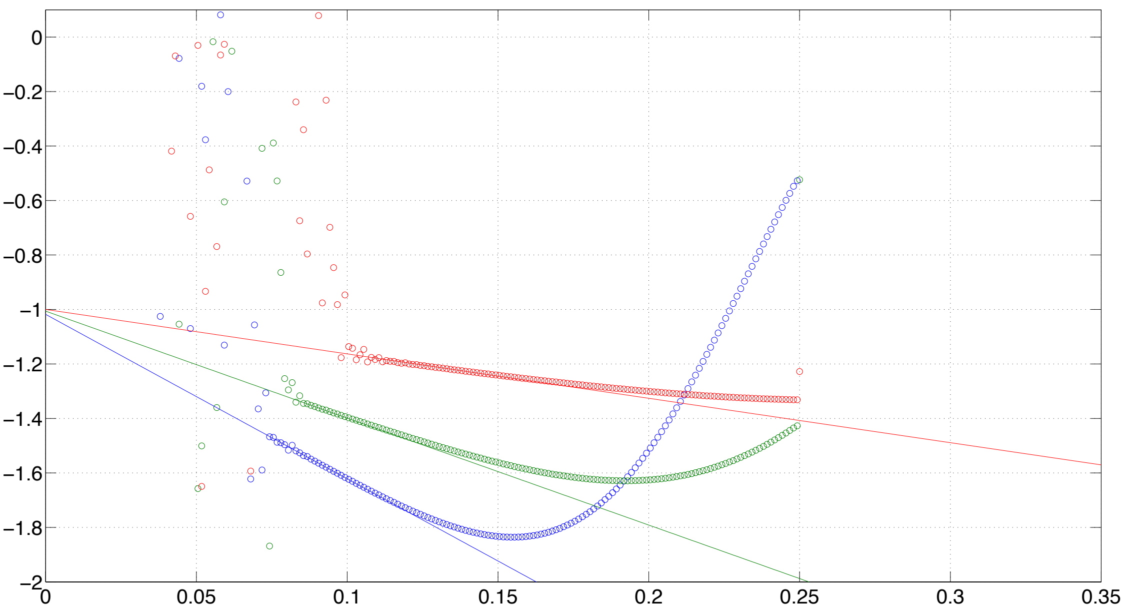

Figure 7: Divided differences of $\log_{10}(-\ln(\zeta_{2j}(h) - \zeta_{2j-1}(h)))$ (j = 1 … 3) on h.

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