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THE RIEMANN PROBLEM FOR KERR EQUATIONS AND NON-UNIQUENESS OF SELFSIMILAR ENTROPY SOLUTIONS

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Abstract. We solve the Riemann problem for a nonlinear full wave Maxwell system arising in nonlinear optics. This system is hyperbolic, some eigenvalues have non-constant multiplicity and are neither genuinely nonlinear, nor linearly degenerate. In a particular 2×2 reduced case, we are able to exhibit two distinct selfsimilar entropy solutions. We compute the amounts of entropy dissipation and compare them.

1. Introduction.

In nonlinear optics, the propagation of electromagnetic waves in a crystal can be modelized by the so-called Kerr model, which consists of Maxwell's equations

$$\left\{\begin{array}{l} \partial_t D - \mathrm{curl} H = 0, \\ \partial_t B + \mathrm{curl} E = 0, \end{array}\right.$$

with $\mathrm{div} D = \mathrm{div} B = 0$ and the constitutive relations

$$\left\{\begin{array}{lcl} B & = & \mu_0 H \\ D & = & \mathbf{D}(E) = \varepsilon_0(1+\varepsilon_r |E|^2)E. \end{array}\right.$$

Here μ_0, ε_0 are the free space permeability and permittivity and ε_r is the relative permittivity, see [12] for further details.

The model is a 6×6 system of conservation laws in the unknown $u = (D, H)$:

$$\left\{\begin{array}{l} \partial_t D - \mathrm{curl} H = 0, \\ \partial_t H + \mu_0^{-1} \mathrm{curl}(\mathbf{P}(D)) = 0 \end{array}\right. \tag{1}$$

where $\mathbf{P}$ is the reciprocal function of $\mathbf{D}$. Denoting

$$q(e) = \varepsilon_0(e + \varepsilon_r e^3), \qquad e \in \mathbb{R}, \qquad p = q^{-1},$$

we have

$$E = \mathbf{P}(D) = \frac{D}{\varepsilon_0(1+\varepsilon_r p^2(|D|))}\,.$$

As proposed in [6] we also introduce the one dimensional model satisfied by solutions $D(x,t) = (0, d(x,t), 0)$, $H(x,t) = (0, 0, h(x,t))$ and $x = x_1 \in \mathbb{R}$. In that framework the solutions of Kerr model 1 satisfy the following p-system:

$$\left\{\begin{array}{lcl} \partial_t d + \partial_x h & = & 0, \\ \partial_t h + \mu_0^{-1} \partial_x p(d) & = & 0. \end{array}\right. \tag{2}$$

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As $p' > 0$ it is strictly hyperbolic but the properties of the function p differ from the ones which appear in the general framework of gas dynamics or viscoelasticity. Here:

$$p(0) = 0, \quad p' > 0,$$

and p is strictly convex on $]-\infty, 0]$, strictly concave on $[0, +\infty[$.

Known existence results for system 1 are related to strong solutions, see [10], [7] and references therein. A first insight into weak solutions, which is also useful in the aim of designing numerical schemes, is the study of the Riemann problem: a direction $\omega \in \mathbb{R}^3$, $|\omega| = 1$, and $u_\pm \in \mathbb{R}^6$ being fixed, one looks for the solution of system 1 with initial data

$$u(x,0) = \left| \begin{array}{ll} u_- & \text{if } x\cdot\omega < 0, \\ u_+ & \text{if } x\cdot\omega > 0. \end{array} \right. \tag{3}$$

This work is devoted to the resolution of this problem and to the link between the solutions for the full model 1 and the ones for the reduced system 2. We point out the fact that we do not suppose that the initial data are divergence free (*ie* that $(D_+ - D_-)\cdot\omega = 0$ and $(H_+ - H_-)\cdot\omega = 0$) because in numerical applications, this condition is not exactly satisfied in general.

The electromagnetic energy is a mathematical entropy [6]. In the reduced 2×2 case it reduces to the classical entropy of the p-system.

The characteristic fields of system 1 have been described in [3], the following proposition summarizes the results:

Proposition 1. [3] *The Kerr system 1 is hyperbolic diagonalizable: for all $\omega \in \mathbb{R}^3$, $|\omega| = 1$, the eigenvalues are given by*

$$\lambda_1 \le \lambda_2 = -\lambda < \lambda_3 = \lambda_4 = 0 < \lambda_5 = \lambda \le \lambda_6 = -\lambda_1 \tag{4}$$

where, denoting $c = (\varepsilon_0\mu_0)^{-\frac{1}{2}}$ the light velocity:

$$\lambda_1^2 = \frac{c^2}{1+\varepsilon_r|E|^2}, \quad \lambda^2 = c^2\,\frac{1+\varepsilon_r(|E|^2 + 2(E\cdot\omega)^2)}{(1+\varepsilon_r|E|^2)(1+3\varepsilon_r|E|^2)}. \tag{5}$$

and the inequalities in 4 are strict if and only if $\omega \times D \neq 0$.

The characteristic fields 1,3,4,6 are linearly degenerate.

The characteristic fields 2 and 5 are genuinely nonlinear in the open set

$$\Omega(\omega) = \{(D,H) \in \mathbb{R}^6 \; ; \; \omega \times D \neq 0\}.$$

The system is not strictly hyperbolic and the second and fifth characteristic fields are neither genuinely nonlinear, nor linearly degenerate. Hence, Lax' wellknown result [8] does not apply. Here for $|u_+ - u_-|$ small enough, we construct a "Lax' solution" and prove that such a construction is unique.

In the 2×2 case, the system 2 is strictly hyperbolic but similarly to the 6×6 case, the characteristic fields are genuinely nonlinear only in the domain $\Omega_1 = \{(d,h) \in \mathbb{R}^2 \; ; \; d \neq 0\}$. Nevertheless in that case we can construct the solution by using multiple waves, following Wendroff [13] and Liu [9].

System 2 being a particular case of 1, we find Lax' solutions of 1 which are also weak solutions of 2, but they are different from the "Liu's solutions". Moreover, we shall prove that the electromagnetic energy is dissipated by both solutions, so that there exists (at least) two selfsimilar entropy solutions of the Riemann problem for system 2.

A first study of the Riemann problem can be found in [4]: the problem is solved for a reduced 4×4 system and it is assumed that $D \cdot \omega$ is identically zero. Related numerical schemes are constructed in 1D and 2D transverse electric configurations.

In [3] we studied Kerr shocks and related shock profiles provided by the Kerr-Debye model, which is a hyperbolic quasilinear relaxation approximation of Kerr model. We studied Lax and Liu's admissibility criteria and proved that only Lax shocks give rise to Kerr-Debye shock profiles.

In [7], Godunov's scheme, which requires the solution of the Riemann problem, was implemented for a a two-dimensionnal transverse electric configuration. Actually this case reduces to the one of the p-system 2. Liu's solution was implemented. The results were found to coincide with the ones obtained by a Kerr-Debye relaxation scheme.

The plan of the paper is the following. In section 2 we determine the simple waves and the wave functions. In section 3 we construct the solution of the Riemann problem. Section 4 is devoted to the 2×2 case.

2. Wave functions.

If $u_- \neq u_+$ are connected by a k-Lax shock or a k-rarefaction wave or a k-contact discontinuity, u_- and u_+ are said to be connected by a k-wave. A plane discontinuity σ, u_+, u_- is a weak solution u of 1 such that $u(x,t) = u_-$ if $x \cdot \omega < \sigma t$, $u(x,t) = u_+$ else. All the plane discontinuities have been studied in [3]. The centred rarefaction waves are computed in [1]. The results are the following.

Proposition 2. Contact discontinuities. *Stationary contact discontinuities are characterized by*

$$\omega \times [H] = 0, \quad \omega \times [E] = 0. \tag{6}$$

The divergence free ones are constant.

A discontinuity σ, u_+, u_- is a contact discontinuity associated to λ_1 or λ_6 if and only if

$$\begin{cases} |E_+| = |E_-| \\ \sigma^2 = c^2(1+\epsilon_r|E_+|^2)^{-1} = c^2(1+\epsilon_r|E_-|^2)^{-1} \end{cases}$$

and

$$\begin{cases} \omega \cdot [D] = 0 \\ [H] = \sigma\omega \times [D]. \end{cases}$$

Moreover the only discontinuities satisfying Rankine-Hugoniot conditions and such that $|E_-| = |E_+|$ are the above contact discontinuities.

The shocks and rarefactions are related to the second and sixth characteristic fields. We recall that a discontinuity σ, u_-, u_+ is a Lax' k-shock if

$$\lambda_k(u_+) < \sigma < \lambda_{k+1}(u_+), \quad \lambda_{k-1}(u_-) < \sigma < \lambda_k(u_-).$$

Those inequalities are entropy conditions and also ensure that one can construct the solution of the Riemann problem as a superposition of simple waves. In our case, defining

$$f(d, d_0) = \frac{c^2\, d}{1 + \epsilon_r p^2\left(\sqrt{d_0^2 + d^2}\right)}, \qquad (d, d_0) \in \mathbb{R}^2,$$

we express the shocks by using the function S defined as

$$S(d_1, d_2, d_0) = \left((f(d_2, d_0) - f(d_1, d_0))(d_2 - d_1)\right)^{\frac{1}{2}}.$$

The rarefaction waves are obtained *via* the integral curves of the eigenvectors of the system. ζ being a unitary vector orthogonal to ω, we use the function R_ζ defined by

$$R_\zeta(d_1, d_2, d_0) = \int_{d_1}^{d_2} \lambda(d_0\omega + s\zeta)ds, \quad d_1 \le d_2\,, \quad d_0 \in \mathbb{R}.$$

Finally let ϕ_ζ be the function defined for $d_1 \ge 0$, $d_2 \ge 0$ and $d_0 \in \mathbb{R}$ by

$$\phi_\zeta(d_1, d_2, d_0) = \begin{cases} S(d_2, d_1, d_0) & if \quad d_1 \ge d_2\,, \\ -R_\zeta(d_1, d_2, d_0) & if \quad d_1 < d_2\,. \end{cases}$$

Proposition 3. *ϕ_ζ is a decreasing C^1 function with respect to d_2 and for all $d \ge 0$, $d_0 \in \mathbb{R}$:*

$$\phi_\zeta(d, 0, d_0) = \frac{cd}{\sqrt{1 + \epsilon_r p^2(\sqrt{d_0^2 + d^2})}}\,, \quad \lim_{d_2 \to +\infty} \phi_\zeta(d, d_2, d_0) = -\infty.$$

The 2 and 5 waves are characterized as follows, see [1], [3] for the proof:

Proposition 4. *If $u_- \ne u_+$ are connected by a 2 or a 5 wave, then $D_- \ne D_+$ and $D_- \cdot \omega = D_+ \cdot \omega$. Moreover $\omega \times (\omega \times D_-)$ and $\omega \times (\omega \times D_+)$ are colinear.*

Reciprocally, let us consider u_- and u_+ such that $D_- \ne D_+$ and $D_- \cdot \omega = D_+ \cdot \omega = d_0$. If $\omega \times D_+ \ne 0$, we set $\overline{D} = D_+$. Else, $\omega \times D_- \ne 0$ and we set $\overline{D} = D_-$. We define ζ by

$$\zeta = -\frac{\omega \times (\omega \times \overline{D})}{|\omega \times (\omega \times \overline{D})|}\,.$$

u_- and u_+ are connected by a 2-wave if there exist two distinct nonnegative real numbers d_-, d_+ such that

$$D_\pm = d_0\omega + d_\pm\zeta, \quad H_+ = H_- + \phi_\zeta(d_-, d_+, d_0)\omega \times \zeta\,.$$

u_- and u_+ are connected by a 5-wave if there exist two distinct nonnegative real numbers d_-, d_+ such that

$$D_\pm = d_0\omega + d_\pm\zeta, \quad H_+ = H_- + \phi_\zeta(d_+, d_-, d_0)\omega \times \zeta\,.$$

3. **Solution of the full wave Riemann problem.** Suppose that $u_\pm = (D_\pm, H_\pm)$ and $\omega \in R^3$, $|\omega| = 1$, are given. We look for intermediate states u_1, u_*, u_{**}, u_2 such that:

- u_- and u_1 are connected by a 1-contact discontinuity,
- u_1 and u_* are connected by a 2-wave,
- u_* and u_{**} are connected by a stationary contact discontinuity,
- u_{**} and u_2 are connected by a 5-wave,
- u_2 and u_+ are connected by a 6-contact discontinuity.

In the following we shall denote $d_0^\pm = D_\pm \cdot \omega$.

Suppose that a solution exists. For the 1 and 6 contact discontinuities, the following conditions have to be fulfilled:

$$\begin{cases} D_1 \cdot \omega = D_- \cdot \omega = d_0^-\,, \quad |D_1| = |D_-|, \\ D_2 \cdot \omega = D_+ \cdot \omega = d_0^+\,, \quad |D_2| = |D_+|, \end{cases} \tag{7}$$

$$\begin{cases} H_1 - H_- = \sigma_-\, \omega \times (D_1 - D_-), \\ H_+ - H_2 = \sigma_+\, \omega \times (D_+ - D_2). \end{cases} \tag{8}$$

where

$$\sigma_\pm = \pm c\,(1 + \epsilon_r|E_\pm|)^{-\frac{1}{2}}\,.$$

For the 2 and 5 waves we know that D_1, D_*, ω are coplanar and D_2, D_{**}, ω are coplanar. Moreover $[D]\cdot\omega = 0$. There exist unitary vectors ζ_1, ζ_2, orthogonal to ω such that

$$D_1 = d_0^-\omega + d_1\zeta_1, \quad D_* = d_0^-\omega + d_*\zeta_1$$

and

$$D_2 = d_0^+\omega + d_2\zeta_2, \quad D_{**} = d_0^+\omega + d_{**}\zeta_2$$

and d_1, d_*, d_{**}, d_2 are non negative.

The stationary contact discontinuity is defined by conditions 6. One has

$$E_* = e_0^*\omega + e_*\zeta_1, \quad E_{**} = e_0^{**}\omega + e_{**}\zeta_2\,,$$

where

$$e_* = \frac{d_*}{\epsilon_0(1+\epsilon_r p^2(|D_*|))}, \quad e_{**} = \frac{d_{**}}{\epsilon_0(1+\epsilon_r p^2(|D_{**}|))}\,.$$

Therefore $e_*\,\omega\times\zeta_1 = e_{**}\,\omega\times\zeta_2$. Hence either $e_* = e_{**} = 0$ or those quantities are both positive and $\zeta_1 = \zeta_2$. The first case occurs if and only if $\omega\times D_* = \omega\times D_{**} = 0$. In the second case we have $e_* = e_{**}$, which also reads as

$$f(d_*, d_0^-) = f(d_{**}, d_0^+). \tag{9}$$

First case: $\omega\times D_* = \omega\times D_{**} = 0$. In that case, $D_* = d_0^-\omega$, $D_{**} = d_0^+\omega$. u_1 and u_* are the left and right states of a 2-shock propagating with speed

$$\sigma_2 = -\sqrt{\frac{f(d_1,d_0^-) - f(0,d_0^-)}{d_1}} = \sigma_-\,.$$

In the same way, u_{**} and u_2 are the left and right states of a 5-shock propagating with speed σ_+. Consequently the contact discontinuities merge with the shocks. Let us denote

$$V = \omega\times(H_+ - H_- - \omega\times(\sigma_+ D_+ - \sigma_- D_-))\,. \tag{10}$$

Conditions 6, 8 and Rankine-Hugoniot conditions on the shocks imply that $V = 0$ and

$$H_* = H_- - \omega\times\sigma_- D_-\,, \quad H_{**} = H_+ - \omega\times\sigma_- D_+\,. \tag{11}$$

If $D_-\times\omega = 0$ then $u_- = u_*$. Else u_- and u_* are connected by a 2-Lax shock.

In the same way, if $D_+\times\omega = 0$ then $u_+ = u_{**}$, else u_+ and u_{**} are connected by a 5-Lax shock.

Second case: $D_*\times\omega \neq 0$ and $D_{**}\times\omega \neq 0$. In this case, $\zeta_1 = \zeta_2 = \zeta$ and

$$D_1 = d_0^-\omega + d_1\zeta, \quad D_2 = d_0^+\omega + d_2\zeta, \tag{12}$$

$$D_* = d_0^-\omega + d_*\zeta, \quad D_{**} = d_0^+\omega + d_{**}\zeta, \tag{13}$$

with $d_1 \geq 0$, $d_* > 0$, $d_{**} > 0$, $d_2 \geq 0$. Let us denote

$$d = D\cdot\zeta, \quad h = H\cdot(\omega\times\zeta).$$

By 7-8:

$$\begin{cases} d_1 = |\omega\times(\omega\times D_-)|, & d_2 = |\omega\times(\omega\times D_+)|, \\ h_1 = h_- + \sigma_-(d_1 - d_-) & h_2 = h_+ + \sigma_+(d_2 - d_+). \end{cases} \tag{14}$$

As $u_*, u_{**} \in \Omega(\omega)$, one can define the 2 and 5-wave curves:

$$H_* - H_1 = \phi_\zeta(d_1, d_*, d_0^-)\omega\times\zeta\,, \quad H_2 - H_{**} = \phi_\zeta(d_2, d_{**}, d_0^+)\omega\times\zeta\,. \tag{15}$$

By 6, $h_* = h_{**}$ and

$$h_* = h_1 + \phi_\zeta(d_1, d_*, d_0^-) = h_2 - \phi_\zeta(d_2, d_{**}, d_0^+).$$

Therefore, using 9, we see that d_* and d_{**} are solution of the two by two system

$$\begin{cases} f(d_*, d_0^-) = f(d_{**}, d_0^+)\,, \\ \\ h_1 + \phi_\zeta(d_1, d_*, d_0^-) = h_2 - \phi_\zeta(d_2, d_{**}, d_0^+). \end{cases} \tag{16}$$

As ϕ_ζ is decreasing and d_*, d_{**} are positive:

$$\phi_\zeta(d_1, d_*, d_0) + \phi_\zeta(d_2, d_{**}, d_0) < \phi_\zeta(d_1, 0, d_0^-) + \phi_\zeta(d_2, 0, d_0^+) = \sigma_+ d_2 - \sigma_- d_1.$$

This inequality is useful to determine ζ. As a matter of fact, using 8, we have also

$$\begin{cases} H_* = H_- + \sigma_- \omega \times (D_1 - D_-) + \phi_\zeta(d_1, d_*, d_0^-)\omega \times \zeta, \\ H_{**} = H_+ - \sigma_+ \omega \times (D_+ - D_2) - \phi_\zeta(d_2, d_{**}, d_0^+)\omega \times \zeta\,. \end{cases}$$

Again by 6:

$$V = \left(\sigma_+ d_2 - \sigma_- d_1 - \phi_\zeta(d_1, d_*, d_0^-) - \phi_\zeta(d_2, d_{**}, d_0^+)\right)\zeta.$$

Therefore $V \neq 0$ and

$$\zeta = \frac{V}{|V|}. \tag{17}$$

Those results can be summarized as follows.

Proposition 5. *Consider u_-, u_+ such that the Riemann problem for system 1 has a solution which is a superposition of simple waves. Let V be the vector defined in 10. Only the following two cases occur:*

1) $V = 0$, u_- and u_* are connected by a 2-Lax shock propagating with velocity σ_-, u_+ and u_{**} are connected by a 5-Lax shock propagating with velocity σ_+, $D_* = (D_- \cdot \omega)\omega$, $D_{**} = (D_+ \cdot \omega)\omega$, H_* and H_{**} are given by 11.

2) $V \neq 0$, ζ is defined by 17, u_1 and u_2 are determined by conditions 12, 14 and u_*, u_{**} are determined by 13, 15 and the solution of system 16.

Sufficient conditions are as follows.

Theorem 3.1. *Let u_-, u_+ be a Riemann data for system 1 in the direction ω. There exists $\eta > 0$ such that if $|(D_- - D_+) \cdot \omega| < \eta$ then the Riemann problem has a unique solution in the class of the functions which are superpositions of simple waves. Let V be the vector defined in 10.*

If $V = 0$, then the solution is the superposition of a 2-Lax' shock, a stationary contact discontinuity and a 5-Lax' shock.

If $V \neq 0$, then the solution is the superposition of a 1-contact discontinuity, a 2-wave (Lax' shock or rarefaction), a stationary contact discontinuity, a 5-wave (Lax' shock or rarefaction) and a 6-contact discontinuity.

Moreover we can construct the solution in every case.

To prove this theorem, we remark that if $d_0^+ = d_0^-$, then system 16 can be solved. For the general case we use the implicit function theorem. We refer to [1] for the detailed proof.

4. **Non-uniqueness of selfsimilar entropy solutions.** Here we consider the Riemann problem 1-3 with $\omega = (1,0,0)^T$, $D_\pm = (0, D_{2,\pm}, 0)$, $H_\pm = (0,0,H_{3,\pm})$. As $(D_+ - D_-)\cdot\omega = (H_+ - H_-)\cdot\omega = 0$, the stationary contact discontinuity is trivial. We have $V = (0,v,0)$ and if $v \neq 0$ then $\zeta = (0, \mathrm{sgn}(v), 0)$. Hence Lax' solution is of the form $(0, D_2, 0)$, $(0,0,H_3)$ and (D_2, H_3) is a weak solution of the Riemann problem for the p-system 2.

Reciprocally, weak solutions (d,h) of 2 give solutions $(0,d,0)$, $(0,0,h)$ of 1. Following [9] (see also [13]) one can compute a weak solution of the Riemann problem for the p-system by using multiple waves. In that case the shocks satisfy Liu's (E) condition, which generalize Lax' conditions when p has inflexion points. This solution differs from the first one.

As an example, we computed both solutions in a particular case. The results for a fixed time t are depicted in Figure 1. The Lax' solution of 1-3 consists of a 1-contact discontinuity, a 2-shock, a 5-shock and a 6-contact discontinuity. Liu's solution consists of a shock propagating to the left and a shock propagating to the right. Observe that in that case the sign of d can change through the shock, while Lax' shock condition applied to 1 imply that the sign of $d = D_2$ is constant.

To be more precise, we denote $-\mu_1 = \mu_2 = \sqrt{\mu_0^{-1} p'(d)}$ the eigenvalues of 2. Let (d_-, h_-) a fixed left state. The set of right states (d_+, h_+) connected to (d_-, h_-) by a Liu's one-shock is parametrized by $d \in [d_-, d_*(d_-)]$ if $d_- < 0$, by $d \in [d_*(d_-), d_-]$ if $d_- > 0$, where d_* is defined by

$$d_*(d) = q(-\frac{1}{2}p(d)).$$

Symmetrically, the set of left states (d_-, h_-) connected to a given right state (d_+, h_+) by a Liu's two-shock is parametrized by $d \in [d_+, d_*(d_+)]$ if $d_+ < 0$, and by $d \in [d_*(d_+), d_+]$ if $d_+ > 0$.

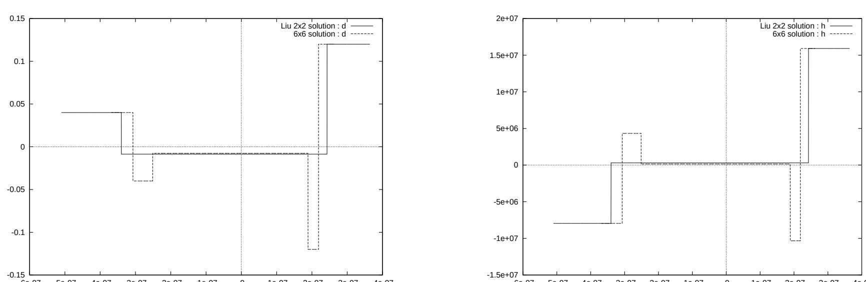

FIGURE 1. Two solutions of the p-system for the same initial data. Left: d, right: h.

At this point, we have two distinct solutions of the Riemann problem for 2, or equivalently for 1. Moreover, we can prove that both dissipate the physical entropy, namely the electromagnetic energy. For those solutions, this entropy is just the wellknown entropy of the p-system:

$$\eta(d,h) = E(d) + \frac{1}{2}\mu_0 h^2, \quad E(d) = \frac{1}{2}\epsilon_0(e^2 + \frac{3\epsilon_r}{2}e^4), \quad e = p(d)$$

with entropy flux $Q(d,h) = eh$. Contact discontinuities and rarefactions conserve the entropy. For the shocks a straightforward calculation leads to the following, see [1] for details.

Proposition 6. Entropy dissipation for the 2×2 system

- *Lax' and Liu's shocks satisfy the entropy dissipation property:*

$$[Q(d,h)] - \sigma[\eta(d,h)] = -\frac{\epsilon_0 \epsilon_r}{4}\sigma[e]^2\ [e^2] \leq 0.$$

- *Let (d_-, h_-) be a fixed left state for a Liu's 1-shock. The entropy dissipation rate is a decreasing function of $|d_+ - d_-|$ over the interval $[0, |d^*(d_-) - d_-|]$.*
- *Let (d_+, h_+) be a fixed right state for a Liu's 2-shock. The entropy dissipation rate is a decreasing function of $|d_+ - d_-|$ over the interval $[0, |d^*(d_+) - d_+|]$.*

We have solved the Riemann problem for the full wave Kerr system and we have proved the non-uniqueness of selfsimilar entropy solutions. Which is the physical solution? Liu's solution is more dissipative than the other one. 1D and 2D numerical experiments with a physically relevant relaxation model, the Kerr-Debye system, lead to Liu's solutions see [2], [7], but this is possibly due to numerical viscosity: it is wellknown that contact discontinuities are not easy to catch numerically. On another hand the full wave model should be the more realistic one. Further investigations are in progress.

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