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Independent Sets in Graphs with an Excluded Clique Minor

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Let G be a graph with n vertices, with independence number α , and with no K_{t+1} -minor for some $t \geq 5$. It is proved that $(2\alpha - 1)(2t - 5) \geq 2n - 5$. This improves upon the previous best bound whenever $n \geq \frac{2}{5}t^2$.

Keywords: graph, minor, independent set, Hadwiger's Conjecture.

Mathematics Subject Classification: 05C15 (Coloring of graphs and hypergraphs)

1 Introduction

In 1943, Hadwiger [7] made the following conjecture, which is widely considered to be one of the most important open problems in graph theory⁽ⁱ⁾; see [19] for a survey.

Hadwiger's Conjecture. For every integer $t \geq 1$, every graph with no K_{t+1} -minor is t -colourable. That is, $\chi(G) \leq \eta(G)$ for every graph G .

Hadwiger's Conjecture is trivial for $t \leq 2$, and is straightforward for $t = 3$; see [4, 7, 22]. In the cases $t = 4$ and $t = 5$, Wagner [20] and Robertson et al. [16] respectively proved that Hadwiger's Conjecture is equivalent to the Four-Colour Theorem [2, 3, 6, 15]. Hadwiger's Conjecture is open for all $t \geq 6$.

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⁽ⁱ⁾ All graphs considered in this note are undirected, simple and finite. Let G be a graph with vertex set $V(G)$. Let $X \subseteq V(G)$. X is *connected* if the subgraph of G induced by X is connected. X is *dominating* if every vertex of $G \setminus X$ has a neighbour in X . X is *independent* if no two vertices in X are adjacent. The *independence number* $\alpha(G)$ is the maximum cardinality of an independent set of G . X is a *clique* if every pair of vertices in X are adjacent. The *clique number* $\omega(G)$ is the maximum cardinality of a clique in G . A *k-colouring* of G is a function that assigns one of k colours to each vertex of G such that adjacent vertices receive distinct colours. The *chromatic number* $\chi(G)$ is the minimum integer k such that G is k -colourable. A *minor* of G is a graph that can be obtained from a subgraph of G by contracting edges. The *Hadwiger number* $\eta(G)$ is the maximum integer n such that the complete graph K_n is a minor of G .

Progress on the $t = 6$ case has been recently been obtained by Kawarabayashi and Toft [10] (without using the Four-Colour Theorem). The best known upper bound is $\chi(G) \leq c \cdot \eta(G) \sqrt{\log \eta(G)}$ for some constant c , independently due to Kostochka [11] and Thomason [17, 18].

Woodall [21] observed that since $\alpha(G) \cdot \chi(G) \geq |V(G)|$ for every graph G , Hadwiger's Conjecture implies that

$$\alpha(G) \cdot \eta(G) \geq |V(G)|. \quad (1)$$

Equation (1) holds for $\eta(G) \leq 5$ since Hadwiger's Conjecture holds for $t \leq 5$. For example, $\alpha(G) \geq \frac{1}{4}|V(G)|$ for every planar graph G . It is interesting that the only known proof of this result depends on the Four-Colour Theorem. The best bound not using the Four-Colour Theorem is $\alpha(G) \geq \frac{2}{9}|V(G)|$ due to Albertson [1].

Equation (1) is open for $\eta(G) \geq 6$. In general, (1) is weaker than Hadwiger's Conjecture, but for graphs with $\alpha(G) = 2$ (that is, graphs whose complements are triangle-free), Plummer et al. [13] proved that (1) is in fact equivalent to Hadwiger's Conjecture. The first significant progress towards (1) was made by Duchet and Meyniel [5] (also see [12]), who proved that

$$(2\alpha(G) - 1) \cdot \eta(G) \geq |V(G)|. \quad (2)$$

This result was improved by Kawarabayashi et al. [8] to

$$(2\alpha(G) - 1) \cdot \eta(G) \geq |V(G)| + \omega(G). \quad (3)$$

Assuming $\alpha(G) \geq 3$, Kawarabayashi et al. [8] proved that

$$(4\alpha(G) - 3) \cdot \eta(G) \geq 2|V(G)|, \quad (4)$$

which was further improved by Kawarabayashi and Song [9] to

$$(2\alpha(G) - 2) \cdot \eta(G) \geq |V(G)|. \quad (5)$$

The following theorem is the main contribution of this note.

Theorem 1 *Every graph G with $\eta(G) \geq 5$ satisfies*

$$(2\alpha(G) - 1)(2\eta(G) - 5) \geq 2|V(G)| - 5.$$

Observe that Theorem 1 represents an improvement over (2), (4) and (5) whenever $\eta(G) \geq 5$ and $|V(G)| \geq \frac{2}{5}\eta(G)^2$. For example, Theorem 1 implies that $\alpha(G) > \frac{1}{7}|V(G)|$ for every graph G with $\eta(G) \leq 6$, whereas each of (2), (4) and (5) imply that $\alpha(G) > \frac{1}{12}|V(G)|$.

2 Proof of Theorem 1

Theorem 1 employs the following lemma by Duchet and Meyniel [5]. The proof is included for completeness.

Lemma 1 ([5]) *Every connected graph G has a connected dominating set D and an independent set $S \subseteq D$ such that $|D| = 2|S| - 1$.*

Proof: Let D be a maximal connected set of vertices of G such that D contains an independent set S of G and $|D| = 2|S| - 1$. There is such a set since $D := S := \{v\}$ satisfies these conditions for each vertex v . We claim that D is dominating. Otherwise, since G is connected, there is a vertex v at distance 2 from D , and there is a neighbour w of v at distance 1 from D . Let $D' := D \cup \{v, w\}$ and $S' := S \cup \{v\}$. Thus D' is connected and contains an independent set S' such that $|D'| = 2|S'| - 1$. Hence D is not maximal. This contradiction proves that D is dominating. \square

The next lemma is the key to the proof of Theorem 1.

Lemma 2 *Suppose that for some integer $t \geq 1$ and for some real number $p \geq t$, every graph G with $\eta(G) \leq t$ satisfies $p \cdot \alpha(G) \geq |V(G)|$. Then every graph G with $\eta(G) \geq t$ satisfies*

$$\alpha(G) \geq \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2}.$$

Proof: We proceed by induction on $\eta(G) - t$. If $\eta(G) = t$ the result holds by assumption. Let G be a graph with $\eta(G) > t$. We can assume that G is connected. By Lemma 1, G has a connected dominating set D and an independent set $S \subseteq D$ such that $|D| = 2|S| - 1$. Now $\alpha(G) \geq |S| = \frac{|D|+1}{2}$. Thus we are done if

$$\frac{|D| + 1}{2} \geq \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2}. \quad (6)$$

Now assume that (6) does not hold. That is,

$$|D| \leq \frac{2|V(G)| - p}{2\eta(G) + p - 2t}.$$

Thus

$$|V(G \setminus D)| = |V(G)| - |D| \geq \frac{(2\eta(G) + p - 2t)|V(G)| + p}{2\eta(G) + p - 2t}.$$

Since D is dominating and connected, $\eta(G \setminus D) \leq \eta(G) - 1$. Thus by induction,

$$\begin{aligned} \alpha(G) &\geq \alpha(G \setminus D) \geq \frac{2|V(G \setminus D)| - p}{4\eta(G \setminus D) + 2p - 4t} + \frac{1}{2} \\ &\geq \frac{2(2\eta(G) + p - 2t)|V(G)| + 2p}{(2\eta(G) + p - 2t)(4\eta(G) - 4 + 2p - 4t)} - \frac{p}{4\eta(G) - 4 + 2p - 4t} + \frac{1}{2} \\ &= \frac{2|V(G)| - p}{4\eta(G) + 2p - 4t} + \frac{1}{2}. \end{aligned}$$

This completes the proof. \square

Lemma 3 *Suppose that Hadwiger's Conjecture is true for some integer t . Then every graph G with $\eta(G) \geq t$ satisfies*

$$(2\eta(G) - t)(2\alpha(G) - 1) \geq 2|V(G)| - t.$$

Proof: If Hadwiger's Conjecture is true for t then $t \cdot \alpha(G) \geq |V(G)|$ for every graph G with $\eta(G) \leq t$. Thus Lemma 2 with $p = t$ implies that every graph G with $\eta(G) \geq t$ satisfies

$$\alpha(G) \geq \frac{2|V(G)| - t}{4\eta(G) - 2t} + \frac{1}{2},$$

which implies the result. □

Theorem 1 follows from Lemma 3 with $t = 5$ since Hadwiger's Conjecture holds for $t = 5$ [16].

3 Concluding Remarks

The proof of Theorem 1 is substantially simpler than the proofs of (3)–(5), ignoring its dependence on the proof of Hadwiger's Conjecture with $t = 5$, which in turn is based on the Four-Colour Theorem. A bound that still improves upon (2), (4) and (5) but with a completely straightforward proof is obtained from Lemma 3 with $t = 3$: Every graph G with $\eta(G) \geq 3$ satisfies $(2\eta(G) - 3)(2\alpha(G) - 1) \geq 2|V(G)| - 3$.

We finish with an open problem. The method of Duchet and Meyniel [5] was generalised by Reed and Seymour [14] to prove that the fractional chromatic number $\chi_f(G) \leq 2\eta(G)$. For sufficiently large $\eta(G)$, is $\chi_f(G) \leq 2\eta(G) - c$ for some constant $c \geq 1$?

References

- [1] MICHAEL O. ALBERTSON. A lower bound for the independence number of a planar graph. *J. Combinatorial Theory Ser. B*, 20(1):84–93, 1976.
- [2] KENNETH APPEL AND WOLFGANG HAKEN. Every planar map is four colorable. I. Discharging. *Illinois J. Math.*, 21(3):429–490, 1977.
- [3] KENNETH APPEL, WOLFGANG HAKEN, AND JOHN KOCH. Every planar map is four colorable. II. Reducibility. *Illinois J. Math.*, 21(3):491–567, 1977.
- [4] GABRIEL A. DIRAC. A property of 4-chromatic graphs and some remarks on critical graphs. *J. London Math. Soc.*, 27:85–92, 1952.
- [5] PIERRE DUCHET AND HENRI MEYNIEL. On Hadwiger's number and the stability number. *Annals of Discrete Mathematics*, 13:71–73, 1982.
- [6] GEORGES GONTHIER. A computer-checked proof of the four colour theorem. 2005. Microsoft Research, Cambridge, U.S.A.
- [7] HUGO HADWIGER. Über eine Klassifikation der Streckenkomplexe. *Vierteljschr. Naturforsch. Ges. Zürich*, 88:133–142, 1943.
- [8] KEN-ICHI KAWARABAYASHI, MICHAEL D. PLUMMER, AND BJARNE TOFT. Improvements of the theorem of Duchet and Meyniel on Hadwiger's conjecture. *J. Combin. Theory Ser. B*, 95(1):152–167, 2005.
- [9] KEN-ICHI KAWARABAYASHI AND ZI-XIA SONG. Independence number and clique minors. *J. Graph Theory*, to appear.

- [10] KEN-ICHI KAWARABAYASHI AND BJARNE TOFT. Any 7-chromatic graph has K_7 or $K_{4,4}$ as a minor. *Combinatorica*, 25(3):327–353, 2005.
- [11] ALEXANDR V. KOSTOCHKA. The minimum Hadwiger number for graphs with a given mean degree of vertices. *Metody Diskret. Analiz.*, 38:37–58, 1982.
- [12] FRÉDÉRIC MAFFRAY AND HENRI MEYNIEL. On a relationship between Hadwiger and stability numbers. *Discrete Math.*, 64(1):39–42, 1987.
- [13] MICHAEL D. PLUMMER, MICHAEL STIEBITZ, AND BJARNE TOFT. On a special case of Hadwiger’s conjecture. *Discuss. Math. Graph Theory*, 23(2):333–363, 2003.
- [14] BRUCE REED AND PAUL SEYMOUR. Fractional colouring and Hadwiger’s conjecture. *J. Combin. Theory Ser. B*, 74(2):147–152, 1998.
- [15] NEIL ROBERTSON, DANIEL P. SANDERS, PAUL D. SEYMOUR, AND ROBIN THOMAS. The four-colour theorem. *J. Combin. Theory Ser. B*, 70(1):2–44, 1997.
- [16] NEIL ROBERTSON, PAUL D. SEYMOUR, AND ROBIN THOMAS. Hadwiger’s conjecture for K_6 -free graphs. *Combinatorica*, 13(3):279–361, 1993.
- [17] ANDREW THOMASON. An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.*, 95(2):261–265, 1984.
- [18] ANDREW THOMASON. The extremal function for complete minors. *J. Combin. Theory Ser. B*, 81(2):318–338, 2001.
- [19] BJARNE TOFT. A survey of Hadwiger’s conjecture. *Congr. Numer.*, 115:249–283, 1996.
- [20] KLAUS WAGNER. Über eine Eigenschaft der ebene Komplexe. *Math. Ann.*, 114:570–590, 1937.
- [21] DOUGLAS R. WOODALL. Subcontraction-equivalence and Hadwiger’s conjecture. *J. Graph Theory*, 11(2):197–204, 1987.
- [22] DOUGLAS R. WOODALL. A short proof of a theorem of Dirac’s about Hadwiger’s conjecture. *J. Graph Theory*, 16(1):79–80, 1992.

