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## GAUSS-COMPATIBLE GALERKIN SCHEMES FOR TIME-DEPENDENT MAXWELL EQUATIONS

MARTIN CAMPOS PINTO AND ERIC SONNENDRÜCKER

ABSTRACT. In this article we propose a unified analysis for conforming and non-conforming finite element methods that provides a partial answer to the problem of preserving discrete divergence constraints when computing numerical solutions to the time-dependent Maxwell system. In particular, we formulate a compatibility condition relative to the preservation of genuinely oscillating modes that takes the form of a generalized commuting diagram, and we show that compatible schemes satisfy convergence estimates leading to long-time stability near stationary solutions. We next apply these findings by specifying compatible formulations for several classes of Galerkin methods, such as the usual curl-conforming finite elements and the centered discontinuous Galerkin (DG) scheme. We also propose a new conforming/non-conforming Galerkin (Conga) method where fully discontinuous solutions are computed by embedding the general structure of curl-conforming finite elements into larger DG spaces. In addition to naturally preserving one of the Gauss laws in a strong sense, the Conga method is both spectrally correct and energy conserving, unlike existing DG discretizations where the introduction of a dissipative penalty term is necessary to avoid the presence of spurious modes.

### 1. INTRODUCTION

Preserving a discrete version of the Gauss laws has always been an important issue in the development of numerical schemes for the time-dependent Maxwell equations. At the continuous level indeed, an important property of the Ampere and Faraday equations

$$(1.1) \quad \begin{aligned} \partial_t \mathbf{E} - \mathbf{curl} \mathbf{B} &= -\mathbf{J} \\ \partial_t \mathbf{B} + \mathbf{curl} \mathbf{E} &= 0 \end{aligned}$$

is to preserve the divergence constraints on the fields

$$(1.2) \quad \begin{aligned} \operatorname{div} \mathbf{E} &= \rho \\ \operatorname{div} \mathbf{B} &= 0 \end{aligned}$$

provided they are satisfied at initial time and the sources  $\rho$  and  $\mathbf{J}$  satisfy the so-called continuity equation

$$(1.3) \quad \partial_t \rho + \operatorname{div} \mathbf{J} = 0.$$

However, when numerical approximations are involved this may not be true.

In that regard, the different classes of Galerkin approximations do not come with equal properties. For instance, curl-conforming finite element methods present a

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rather favorable situation, in that they usually preserve the Gauss laws in a natural finite element sense, see e.g., [34, 14]. This is not the case with non-conforming methods such as finite volume or discontinuous Galerkin schemes where it is known that on general meshes a proper discrete version of the divergence constraints cannot hold [20]. A practical consequence of that theoretical weakness is the development of small errors in the computed electromagnetic field which accumulate to large deviations for long simulation times [29, 35].

When the sources themselves are obtained by a numerical treatment, as is the case with particle simulations of the Vlasov-Maxwell system, such numerical artifacts are often designated as a lack of charge conservation. Indeed the growing inconsistencies that one observes in the fields may be related, through the Gauss laws, to a lack of charge separation: particles tend to stick together, which is normally prevented by Gauss' law. Nevertheless, part of the problem lies in the discretization of the Maxwell equations themselves, as can be seen in e.g. [35] where such unphysical behavior appears with analytical sources that do satisfy the balance equation (1.3).

In order to make DG – and other simulations lacking a proper discrete charge conservation – physically acceptable, one usually resorts to correction techniques such as projection methods or divergence cleaning methods based on generalized Lagrange multiplier formulations of the Maxwell equations [23, 28, 29]. This works well in many cases, in particular when the investigated problem is close to electrostatic. However it introduces an artificial non locality in the numerical scheme, which can have disastrous effects in some simulations, for example for laser plasma interaction problems, where these non localities can trigger an instability before the laser hits the plasma.

We note that the preservation of the Gauss law for time dependent problems is strongly related to the non existence of spurious eigenvalues for the discrete **curl curl** operator, which has triggered a lot of efforts in the applied mathematics community, see e.g. [15, 8, 5, 7]. The problem is that when Gauss' law is not explicitly applied, the eigenspace corresponding to the zero eigenvalue is infinite dimensional and not compact. Good numerical methods generally avoid mixing the eigenspaces corresponding to vanishing and non vanishing eigenvalues, but not all of them do.

Be it for the time dependent or the eigenvalue problem, solutions have been developed using appropriate exact sequences of discrete spaces along with commuting projections [9, 16, 17, 34, 1]. In addition to classical finite element spaces, this framework has been extended to spline finite elements in [13]. By construction, these solutions are restricted to conforming methods.

In this article we extend these results to non-conforming methods and propose a unified analysis. Our findings are twofold. First we formulate a compatibility condition for semi-discrete schemes with sources that takes the form of a generalized commuting diagram, and we show that methods verifying this condition are asymptotically stable relative to exact stationary solutions, which solves most of the large deviation problems described in the literature without resorting to divergence cleaning techniques. Specifically, by applying this analysis to conforming and non-conforming Galerkin methods we describe several approximation operators for

the sources that make the usual curl-conforming finite element method and the centered discontinuous Galerkin (DG) scheme compatible in the above sense.

Second, we propose a new conforming/non-conforming Galerkin (Conga) method that closely follows the structure of the curl-conforming finite element schemes but uses fully discontinuous spaces just as standard DG schemes. In contrast to existing DG discretizations where the introduction of a dissipative penalty term is necessary to avoid the presence of spurious modes [21, 36, 12] the Conga method is shown to be both spectrally correct and energy conserving, just as curl-conforming finite element schemes. Moreover, it naturally preserves one of the Gauss laws in a strong sense.

The outline is as follows. In Section 2 we consider an energy-preserving time-dependent Maxwell system  $\partial_t U - \mathcal{A}U = F$  expressed in the abstract setting of Hilbert complexes with exact sequences of differential operators and we ask ourselves in which sense a numerical method  $\partial_t U_h - \mathcal{A}_h U_h = F_h$  should be compatible with the Gauss laws. By studying first the homogeneous case  $F = 0$  we observe that the Gauss laws can be interpreted as a constraint relative to the existence of stationary modes, for which it is straightforward to derive a discrete analog that can be imposed on the numerical solutions.

In Section 3 we then extend this constraint to the case with sources by formulating a compatibility condition of the form  $F_h = \check{\Pi}_h F$  with  $\check{\Pi}_h \mathcal{A} = \mathcal{A}_h \hat{\Pi}_h$  and we show that compatible schemes are asymptotically stable with respect to exact stationary solutions. In this abstract setting we then provide compatible formulations for several conforming and non-conforming Galerkin methods, including the novel Conga method that is shown to be spectrally correct in Section 4. Finally, explicit schemes are described in Section 5 for the Maxwell equations in 3d.

## 2. AN ABSTRACT SETTING FOR TIME-DEPENDENT MAXWELL PROBLEMS

To highlight the generality of our approach we present our results in the abstract setting of Hilbert complexes, following the framework and notations from Ref. [2].

**2.1. Exact sequences in Hilbert complexes.** We consider a Hilbert complex  $(W, d) = (W^l, d^l)_{l=0, \dots, n}$  consisting of a finite sequence of Hilbert spaces  $W^l$  with closed and densely-defined linear operators  $d^l : W^l \rightarrow W^{l+1}$  that satisfy

$$(2.1) \quad d^l d^{l-1} = 0.$$

In particular, the range of  $d^{l-1}$  is contained in the domain of  $d^l$ , which we denote by  $V^l$ . We further assume that the sequence  $(V, d)$  is *exact*, in the sense that the range of  $d^{l-1}$  actually coincides with the null space of  $d^l$ , which we denote by  $\mathfrak{Z}^l$ . Thus,

$$(2.2) \quad \mathfrak{Z}^l := \ker d^l = d^{l-1} V^{l-1}.$$

In our applications the  $W^l$ 's will be  $L^2$  spaces, and the  $d^l$ 's will correspond to differential operators such as grad, curl or div. To define a version of the time-dependent Maxwell system in this abstract setting, following [2] we let  $d_{l+1}^*$  be the adjoint of  $d^l$ : it is a closed, densely-defined operator from  $W^{l+1}$  to  $W^l$  and its domain is denoted  $V_{l+1}^*$ . For instance if  $d^l$  is defined as the unbounded operator  $\mathbf{curl} : L^2(\Omega)^3 \rightarrow L^2(\Omega)^3$  with domain  $V^k = H(\mathbf{curl}; \Omega)$ , its adjoint  $d_{k+1}^*$  will also correspond to the  $\mathbf{curl}$  operator but with domain  $V_{k+1}^* = H_0(\mathbf{curl})$ , see e.g. [33,

Sec. 0]. And the reverse situation is also possible since the closed, densely-defined  $d^k$  coincides with  $(d_{k+1}^*)^*$ . In particular, we have

$$(2.3) \quad \langle d_{l+1}^* v, u \rangle = \langle v, d^l u \rangle, \quad v \in V_{l+1}^*, \quad u \in V^l.$$

Here we use the same notation  $\langle \cdot, \cdot \rangle$  for the scalar products in any of the spaces  $W^l$ , and accordingly we shall use  $\|\cdot\|$  for the corresponding norms. Applying (2.3) to  $u = d^{l-1}\varphi$  with  $\varphi \in V^{l-1}$ , we infer from (2.1) that  $d_{l+1}^*$  maps  $V_{l+1}^*$  to  $V_l^*$  and that

$$(2.4) \quad d_l^* d_{l+1}^* = 0.$$

Using the density of  $V^{l-1}$  we then see that  $d_l^* u = 0$  holds iff  $\langle u, d^{l-1}\varphi \rangle = 0$  for all  $\varphi \in V^{l-1}$ , hence (2.2) yields

$$(2.5) \quad \ker d_l^* = \mathfrak{Z}^{l \perp w}$$

where the  $\perp_W$  exponent denotes the orthogonal complement in the proper space from the complex  $W$ , here  $W^l$ . Let us verify the following estimates.

**Lemma 2.1** (Poincaré inequalities). *There exists a constant  $c_P = c_P(V, d)$  such that*

$$(2.6) \quad \|u\| \leq c_P \|d^l u\|, \quad u \in \mathfrak{Z}^{l \perp w} \cap V^l =: \mathfrak{Z}^{l \perp}$$

and

$$(2.7) \quad \|w\| \leq c_P \|d_{l+1}^* w\|, \quad w \in \mathfrak{Z}^{l+1} \cap V_{l+1}^*.$$

*Proof.* The first estimate is given in [2, Eq. (16)]. It is obtained by observing first that  $\mathfrak{Z}^{l \perp}$  is the orthogonal complement of  $\mathfrak{Z}^l$  in the Hilbert space  $V^l$  equipped with the scalar product  $\langle u, v \rangle_V := \langle u, v \rangle + \langle du, dv \rangle$ , and second that  $d^l$  defines a bounded bijection between the Hilbert spaces  $(\mathfrak{Z}^{l \perp}, \langle \cdot, \cdot \rangle_V)$  and  $(\mathfrak{Z}^{l+1}, \langle \cdot, \cdot \rangle)$ , the latter being closed according to (2.2). Estimate (2.6) then follows by Banach's bounded inverse theorem. To obtain estimate (2.7) we finally consider  $w \in \mathfrak{Z}^{l+1} \cap V_{l+1}^*$  and let  $\bar{u} \in \mathfrak{Z}^{l \perp}$  be such that  $d^l \bar{u} = w$ . Using the definition of the adjoint operator we write then

$$\|d_{l+1}^* w\| = \sup_{u \in V^l} \frac{\langle w, d^l u \rangle}{\|u\|} \geq \frac{\langle w, d^l \bar{u} \rangle}{\|\bar{u}\|} = \frac{\|w\| \|d^l \bar{u}\|}{\|\bar{u}\|} \geq c_P^{-1} \|w\|$$

where the last inequality is (2.6).  $\square$

Our analysis will use some of the properties of the inverse  $K$  of the abstract Hodge Laplacian operator  $L = d^{k-1}d_k^* + d_{k+1}^*d^k$  corresponding to a particular index  $k$  that will be chosen so that both  $d^k$  and its adjoint  $d_{k+1}^*$  correspond to curl operators. To do so we will assume that the inclusion from the dense intersection  $V^k \cap V_k^*$  in  $W^k$  is compact. According to [2, Sec. 3] we then know that  $K$  is compact and selfadjoint, as an operator from  $W^k$  to itself.

We next assume that  $(V, d)$  is approximated by a finite-dimensional subcomplex  $(V_h, d)$  (namely, a sequence of spaces satisfying  $V_h^l \subset V^l$  and  $d^l V_h^l \subset V_h^{l+1}$ ) for which there exists a bounded co-chain projection  $\pi_h$  from  $(V, d)$  to  $(V_h, d)$ , i.e., linear projections  $\pi_h^l : V^l \rightarrow V_h^l$  uniformly bounded with respect to  $h$ , that satisfy the commuting diagram property pictured in Fig. 1,

$$(2.8) \quad d^l \pi_h^l = \pi_h^{l+1} d^l.$$

$$\begin{array}{ccccccc}
\dots & \xrightarrow{d^{l-1}} & V^l & \xrightarrow{d^l} & V^{l+1} & \xrightarrow{d^{l+1}} & \dots \\
& & \downarrow \pi_h^l & & \downarrow \pi_h^{l+1} & & \\
\dots & \xrightarrow{d^{l-1}} & V_h^l & \xrightarrow{d^l} & V_h^{l+1} & \xrightarrow{d^{l+1}} & \dots
\end{array}$$

FIGURE 1. Commutative diagram for the abstract framework described in Section 2.1

Using (2.8) we verify that the discrete subcomplex inherits the exactness of  $(V, d)$ , in the sense that

$$(2.9) \quad \mathfrak{Z}_h^l := \mathfrak{Z}^l \cap V_h^l = \ker(d^l|_{V_h^l}) = d^{l-1}V_h^{l-1}.$$

An important tool in the study of Galerkin methods is the (bounded) adjoint  $d_{l+1,h}^* : V_h^{l+1} \rightarrow V_h^l$  of the operator  $d^l$  restricted to  $V_h^l$ . It is characterized by

$$(2.10) \quad \langle d_{l+1,h}^* v, u \rangle = \langle v, d^l u \rangle, \quad v \in V_h^{l+1}, \quad u \in V_h^l.$$

Using (2.1) we then find  $\langle d_{l,h}^* d_{l+1,h}^* v, \varphi \rangle = \langle d_{l+1,h}^* v, d^{l-1} \varphi \rangle = \langle v, d^l d^{l-1} \varphi \rangle = 0$  for all  $v \in V_h^{l+1}$  and  $\varphi \in V_h^{l-1}$ . That is,

$$(2.11) \quad d_{l,h}^* d_{l+1,h}^* = 0.$$

Finally, one easily verifies that

$$(2.12) \quad \mathfrak{Z}_h^{l\perp} := \mathfrak{Z}_h^{l\perp w} \cap V_h^l = \ker d_{l,h}^*.$$

**2.2. The abstract Maxwell evolution system.** In the above setting an abstract version of the time-dependent Maxwell system is given by

$$(2.13) \quad \begin{cases} \partial_t U - \mathcal{A}U = F \\ U(0) = U^0 \in \mathcal{V}, \end{cases}$$

where  $\mathcal{A}$  is the linear operator defined from  $\mathcal{W} := W^k \times W^{k+1}$  to itself by

$$(2.14) \quad \mathcal{A} = \begin{pmatrix} 0 & -d_{k+1}^* \\ d^k & 0 \end{pmatrix}, \quad \text{with dense domain } \mathcal{V} := V^k \times V_{k+1}^* = D(\mathcal{A}).$$

Indeed in the applications  $k$  will be chosen such that both  $d^k$  and its adjoint  $d_{k+1}^*$  correspond to curl operators. The source  $F = (f, g)^t$  corresponds to the current density. In Section 5 we will consider either terms of the form  $F = (0, -J)^t$  corresponding to a strong formulation of the Ampere equation (in which case  $U$  represents  $(B, E)^t$ ), or terms of the form  $F = (-J, 0)^t$  that correspond to a strong formulation of the Faraday equation (in which case  $U$  represents  $(E, -B)^t$ ). From the properties of  $d^k$  and  $d_{k+1}^*$  we readily see that  $\mathcal{A}$  is closed and skew-symmetric, indeed for any  $U = (u, v)^t$  and  $U' = (u', v')^t$  in  $\mathcal{V}$ , (2.3) gives

$$(2.15) \quad \langle \mathcal{A}U, U' \rangle = \langle d^k u, v' \rangle - \langle d_{k+1}^* v, u' \rangle = \langle u, d_{k+1}^* v' \rangle - \langle v, d^k u' \rangle = \langle U, -\mathcal{A}U' \rangle.$$

Since  $d^k$  is closed and densely-defined one can actually infer from the skew-symmetry that  $\mathcal{A}$  and its adjoint  $\mathcal{A}^*$  have the same domain, hence  $\mathcal{A}^* = -\mathcal{A}$ . In particular,  $\mathcal{A}$

generates a contraction semi-group of class  $\mathcal{C}_0$  ([38, Section IX.8]) and the following result is a direct application of Corollaries 2.2 p. 106 and 2.5 p. 107 in [32].

**Lemma 2.2.** *If  $F \in \mathcal{C}^1([0, T]; \mathcal{W})$ , then the Cauchy problem (2.13)-(2.14) has a unique solution  $U \in \mathcal{C}^1([0, T]; \mathcal{V})$ .*

In addition to the evolution equation (2.13), the Maxwell system consists of a two-component Gauss law

$$(2.16) \quad \mathcal{D}U(t) = R(t), \quad t \geq 0, \quad \text{with} \quad \mathcal{D} = \begin{pmatrix} d_k^* & 0 \\ 0 & d^{k+1} \end{pmatrix}.$$

Here  $\mathcal{D}$  corresponds to a two-components divergence operator and  $R(t) \in \mathcal{W}$  corresponds to the charge density. In particular, (2.16) implies that  $U(t)$  belongs to the domain of  $\mathcal{D}$ ,  $V_k^* \times V^{k+1}$ . From the cochain (2.1) and chain (2.4) relations one sees that  $\mathcal{D}\mathcal{A} = 0$ , hence (2.16) actually amounts to a property being satisfied by the data. Namely,

$$(2.17) \quad \begin{cases} F \in \mathcal{C}^0([0, T]; D(\mathcal{D})) \\ U^0 \in D(\mathcal{D}) = V_k^* \times V^{k+1} \end{cases} \quad \text{and} \quad \begin{cases} \partial_t R + \mathcal{D}F = 0 \\ \mathcal{D}U^0 = R(0). \end{cases}$$

Before studying the Galerkin approximations to (2.13), we verify the following result.

**Lemma 2.3.** *The kernel of  $\mathcal{A}$  reads*

$$(2.18) \quad \ker \mathcal{A} = \mathfrak{Z}^k \times \mathfrak{Z}^{k+1 \perp \mathcal{W}}$$

and its orthogonal complement in  $\mathcal{W} = W^k \times W^{k+1}$  coincides with the range of  $\mathcal{A}$ , namely

$$(2.19) \quad R(\mathcal{A}) = (\ker \mathcal{A})^{\perp \mathcal{W}} = \mathfrak{Z}^{k \perp \mathcal{W}} \times \mathfrak{Z}^{k+1}.$$

*Proof.* The identity (2.18) simply follows from (2.5) and the definition of  $\mathfrak{Z}^k$ . To prove (2.19) we may show that the range of  $\mathcal{A}$  is closed. The claimed identity will then follow from the closed range theorem and the fact that  $\mathcal{A}^* = -\mathcal{A}$ . Thus, let  $(u_n, v_n)^t$  be a sequence in  $\mathcal{V}$  such that  $(u'_n, v'_n)^t := \mathcal{A}(u_n, v_n)^t$  converges to some  $U'$  in  $\mathcal{W}$ . Setting  $\bar{u}_n := (I - P_{\mathfrak{Z}^k})u_n$  and  $\bar{v}_n := P_{\mathfrak{Z}^{k+1}}v_n$  we then infer from (2.2) and (2.5) that

$$\begin{pmatrix} -d_{k+1}^* \bar{v}_n \\ d^k \bar{u}_n \end{pmatrix} = \begin{pmatrix} -d_{k+1}^* v_n \\ d^k u_n \end{pmatrix} = \begin{pmatrix} u'_n \\ v'_n \end{pmatrix} \quad \text{and} \quad \begin{cases} \bar{u}_n \in \mathfrak{Z}^{k \perp} \\ \bar{v}_n \in V_{k+1}^* \cap \mathfrak{Z}^{k+1}. \end{cases}$$

The convergence of the sequence  $(\bar{u}_n, \bar{v}_n)^t$  to some  $\bar{U}$  (in  $\mathcal{W}$ ) follows then from the Poincaré inequalities (2.6)-(2.7), and since  $\mathcal{A}$  is closed  $(\bar{U}, U')$  is in the graph of  $\mathcal{A}$ . Hence  $R(\mathcal{A})$  is closed and the identity follows.  $\square$

**2.3. Conforming Galerkin approximations and the Gauss law.** To derive Galerkin approximations to the abstract Maxwell evolution system (2.13), we write it in variational form using (2.3), as

$$(2.20) \quad \begin{cases} \langle \partial_t u, z \rangle + \langle v, d^k z \rangle = \langle f, z \rangle & z \in V^k \\ \langle \partial_t v, w \rangle - \langle d^k u, w \rangle = \langle g, w \rangle & w \in V_{k+1}^*. \end{cases}$$

Here the standard Galerkin approach (see, e.g., [24, 34, 37]) defines  $(u_h, v_h)$  in  $\mathcal{C}^1([0, T]; V_h^k \times V_h^{k+1})$ , solution to

$$(2.21) \quad \begin{cases} \langle \partial_t u_h, z \rangle + \langle v_h, d^k z \rangle = \langle f_h, z \rangle & z \in V_h^k \\ \langle \partial_t v_h, w \rangle - \langle d^k u_h, w \rangle = \langle g_h, w \rangle & w \in V_h^{k+1}, \end{cases}$$

with proper approximations  $(f_h, g_h)$  and  $(u_h^0, v_h^0)$  to the source terms and initial data. Clearly, the embedding  $V_h^k \subset V^k$  is required for (2.21) to be well defined, and for this reason the method is said *conforming*. As for  $v_h$ , we note that other spaces could be chosen, and indeed in Refs. [25, 26] the authors consider a similar scheme where  $v_h$  is sought for in a fully discontinuous space. For additional examples and literature on conforming approximations, see e.g. [22, 10].

Here, the motivation for taking  $v_h$  (and  $w$ ) in  $V_h^{k+1}$  comes from the fact that the method preserves a strong discrete version of the second Gauss law in (2.16). Indeed, if the discrete data  $v_h^0$  and  $g_h$  are in  $V_h^{k+1}$  and if they satisfy a strong discrete version of the second equations in (2.17), namely

$$\begin{cases} \partial_t \rho_h + d^{k+1} g_h = 0 \\ d^{k+1} v_h^0 = \rho_h(0) \end{cases}$$

for some  $\rho_h$ , then  $\partial_t v_h$  is also in  $V_h^{k+1}$ , and using (2.1) we find that

$$(2.22) \quad d^{k+1} v_h(t) = \rho_h(t), \quad t \geq 0.$$

As for the first Gauss law, it can only be satisfied in a weak sense since  $\partial_t u_h$  has no reason to be in  $V_k^*$ . Specifically, we observe that the first equation from (2.21) reads

$$\partial_t u_h + d_{k+1,h}^* v_h = f_h$$

where  $d_{k+1,h}^* : V_h^{k+1} \rightarrow V_h^k$  is the adjoint of  $d^k|_{V_h^k}$ , see (2.10). Thus if the data  $u_h^0$  and  $f_h$  satisfy

$$\begin{cases} \partial_t \theta_h + d_{k,h}^* f_h = 0 \\ d_{k,h}^* u_h^0 = \theta_h(0) \end{cases}$$

for some  $\theta_h$ , then using (2.11) we obtain

$$(2.23) \quad d_{k,h}^* u_h(t) = \theta_h(t), \quad t \geq 0.$$

Now, as for any weak equation, the question arises as to whether (2.23) should be considered *too weak* to be numerically relevant. A partial answer to that question is obtained by observing that (2.23) amounts to satisfying

$$(2.24) \quad \langle u_h, d^{k-1} \varphi \rangle = \langle \theta_h, \varphi \rangle \quad \text{for all } \varphi \in V_h^{k-1}.$$

In the applications  $d^{k-1}$  corresponds to a grad operator, hence (2.23) is a standard finite element version of the Gauss law involving  $H^1$  test functions. In particular, if  $u_h = d^{k-1} \phi_h$  is an electric field deriving from a discrete potential  $\phi_h \in V_h^{k-1}$ , then (2.23) simply means that  $\phi_h$  solves a conforming Galerkin approximation of the abstract Poisson equation  $(d^{k-1})^* d^{k-1} \phi_h = \theta_h$ , which in itself can be considered as numerically relevant.

**2.4. The fundamental homogeneous Gauss law.** To give another argument that validates the relevance of the above discrete Gauss laws, we rewrite the conforming approximation (2.21) as an evolution equation

$$(2.25) \quad \begin{cases} \partial_t U_h - \mathcal{A}_h U_h = F_h \\ U_h(0) = U_h^0 \in \mathcal{V}_h, \end{cases}$$

where the skew-symmetric operator

$$(2.26) \quad \mathcal{A}_h := \begin{pmatrix} 0 & -d_{k+1,h}^* \\ d^k & 0 \end{pmatrix} \quad \text{maps} \quad \mathcal{V}_h := V_h^k \times V_h^{k+1} \quad \text{to itself.}$$

The discrete Gauss laws (2.22)-(2.23) read then

$$(2.27) \quad \mathcal{D}_h U_h = R_h \quad \text{with} \quad \mathcal{D}_h = \begin{pmatrix} d_{k,h}^* & 0 \\ 0 & d^{k+1} \end{pmatrix}$$

and here  $R_h = (\theta_h, \rho_h)^t$  corresponds to the discrete charge density. The preservation of this two-component Gauss law by the conforming scheme is then expressed by the relation  $\mathcal{D}_h \mathcal{A}_h = 0$ . Parallel to (2.25), one also finds *non-conforming* Galerkin approximations to (2.13), of the form

$$(2.28) \quad \begin{cases} \partial_t \tilde{U}_h - \tilde{\mathcal{A}}_h \tilde{U}_h = \tilde{F}_h \\ \tilde{U}_h(0) = \tilde{U}_h^0 \in \tilde{\mathcal{V}}_h, \end{cases}$$

where  $\tilde{\mathcal{A}}_h$  is a skew-symmetric operator approximating  $\mathcal{A}$  on a non-conforming space of the form  $\tilde{\mathcal{V}}_h = \tilde{V}_h^k \times \tilde{V}_h^{k+1}$  with  $\tilde{V}_h^k \not\subset V^k$ . For example in TE 2d centered-flux DG schemes [21, 18], the discontinuous scalar magnetic field  $\tilde{v}_h \equiv B_z$  is strongly divergence-free by construction and hence belongs to the conforming space  $V^{k+1}$ , whereas the electric field  $\tilde{u}_h \equiv (E_x, E_y)$  belongs to some non-conforming approximation of  $V^k$ . One can then show (see, e.g., [18, Prop. 3.7]) that  $\tilde{u}_h$  satisfies a weak Gauss law involving the same test functions as in (2.24). Thus, in the source-free case a two-component constraint of the form

$$(2.29) \quad \tilde{\mathcal{D}}_h \tilde{U}_h = 0$$

holds with essentially the same discrete divergence operator than in (2.27), extended to  $\tilde{\mathcal{V}}_h$ . Now, because  $\tilde{u}_h$  belongs to a larger space than in the conforming case, one intuitively feels that in order to be numerically relevant, a weak Gauss law should also involve a larger space of test functions.

An algebraic argument supporting the relevance of (2.27) compared to (2.29) can then be obtained by observing that in the homogeneous case ( $R, F = 0$ ), the Gauss law amounts to a constraint relative to the *oscillating modes* of the Maxwell equation  $\partial_t U = \mathcal{A}U$ .

**Definition 2.4** (oscillating modes). Since  $\mathcal{A}$  is skew-symmetric, all its eigenvalues are imaginary and correspond to solutions that oscillate. But not all are genuinely oscillating: the orthogonal decomposition

$$\mathcal{W} = \ker \mathcal{A} \oplus (\ker \mathcal{A})^{\perp \omega}$$

corresponds to a separation between the stationary solutions ( $\partial_t U = 0$ ) and those which really oscillate ( $\partial_t U = i\omega U$ ,  $\omega \neq 0$ ). Only the latter will be said *oscillating*.

Specifically, we observe that in addition to the inclusion  $R(\mathcal{A}) \subset \ker \mathcal{D}$  equivalent to  $\mathcal{D}\mathcal{A} = 0$  which expresses the preservation of the Gauss law  $\mathcal{D}U = 0$  by the Maxwell equation, what we actually have is an identity: using (2.5), (2.18) and (2.19) we find indeed  $R(\mathcal{A}) = (\ker \mathcal{A})^{\perp w} = \mathfrak{Z}^{k \perp w} \times \mathfrak{Z}^{k+1} = \ker \mathcal{D}$ . Thus in the absence of sources, the Gauss law (2.16) is equivalently rewritten as

$$U(t) \in (\ker \mathcal{A})^{\perp w}, \quad t \geq 0,$$

which states that  $U(t)$  can only be composed of oscillating modes. Since the discrete evolution operator is also assumed skew-symmetric, it is straightforward to formulate an analog property at the discrete level.

**Definition 2.5.** We say that a scheme of the form (2.28) satisfies the *fundamental homogeneous Gauss law* if the solutions to the homogeneous equation ( $F_h = 0$ ) are only composed of oscillating modes (see Def. 2.4) for the discrete evolution operator  $\tilde{\mathcal{A}}_h$ , i.e., if they verify

$$(2.30) \quad \tilde{U}_h(t) \in (\ker \tilde{\mathcal{A}}_h)^{\perp w}, \quad t \geq 0.$$

We note that in addition to being a natural property to expect from the discrete solutions, (2.30) is related to a classical condition in the study of spectrally correct methods for the Maxwell eigenvalue problem, see e.g., [15, 17]. Turning back to the relevance of the discrete Gauss laws, we observe that:

- (i) In the conforming case (2.25)-(2.26) the discrete operators satisfy a relation analog to the continuous one, namely

$$(2.31) \quad \ker \mathcal{D}_h = \mathfrak{Z}_h^{k \perp} \times \mathfrak{Z}_h^{k+1} = (\ker \mathcal{A}_h)^{\perp w} \cap \mathcal{V}_h = R(\mathcal{A}_h).$$

As a consequence, in the absence of sources the discrete Gauss law  $\mathcal{D}_h U_h = 0$  is *strong enough* to guarantee the fundamental Gauss law in the sense of Definition 2.5.

- (ii) In the non-conforming case (2.28) where the discontinuous space  $\tilde{\mathcal{V}}_h$  may be much larger, there is no reason why we should have  $\ker \tilde{\mathcal{D}}_h \subset (\ker \tilde{\mathcal{A}}_h)^{\perp w}$ . Therefore the discrete Gauss law  $\tilde{\mathcal{D}}_h \tilde{U}_h = 0$  is a priori *too weak* to guarantee the fundamental Gauss law in the sense of Definition 2.5.

In the remainder of the paper we will design non-conforming Galerkin methods of the form (2.28) that satisfy the fundamental homogeneous Gauss law, although no divergence constraint stronger than (2.29) is shown to hold. And when sources are present, we shall look for schemes that are *compatible* with the property stated in Definition 2.5, in the sense that they should satisfy

$$(2.32) \quad U(t) \in (\ker \mathcal{A})^{\perp w}, \quad t \geq 0 \quad \implies \quad \tilde{U}_h(t) \in \ker(\tilde{\mathcal{A}}_h)^{\perp w}, \quad t \geq 0.$$

In Section 3 below we shall give a somehow stronger definition for compatible schemes. But before doing so we observe that since  $\tilde{\mathcal{A}}_h$  is assumed skew-symmetric, solutions to (2.28) clearly satisfy

$$\partial_t \tilde{U}_h - \tilde{F}_h \in (\ker \tilde{\mathcal{A}}_h)^{\perp w}.$$

In particular, it is easily seen that satisfying property (2.32) (and (2.30)) essentially relies on the data approximation. An obvious sufficient condition reads indeed

$$(2.33) \quad U^0, F \in (\ker \mathcal{A})^{\perp w} \quad \implies \quad \tilde{U}_h^0, \tilde{F}_h \in (\ker \tilde{\mathcal{A}}_h)^{\perp w}.$$

## 3. COMPATIBLE GALERKIN APPROXIMATIONS

In this section we formulate and study an abstract compatibility property for generic Galerkin approximations to (2.13), based on non-conforming spaces.

Following our observation that for skew-symmetric Galerkin approximations, being compatible with the fundamental Gauss law in the sense of (2.32) is essentially a matter of proper data approximation, we make the latter part explicit and consider semi-discrete schemes of the form

$$(3.1) \quad \begin{cases} \partial_t \tilde{U}_h - \tilde{\mathcal{A}}_h \tilde{U}_h = \tilde{\Pi}_h F \\ \tilde{U}_h(0) = \tilde{U}_h^0 \in \tilde{\mathcal{V}}_h. \end{cases}$$

Here,

- $\tilde{\mathcal{V}}_h$  is a discrete subspace of  $\mathcal{W} = W^k \times W^{k+1}$  but not necessarily a subset of either  $V^k \times V^{k+1}$  or  $\mathcal{V} = V^k \times V_{k+1}^*$ , the domain of  $\mathcal{A}$  ;
- $\tilde{\mathcal{A}}_h : \tilde{\mathcal{V}}_h \rightarrow \tilde{\mathcal{V}}_h$  is a skew-symmetric bounded operator approximating  $\mathcal{A}$  ;
- $\tilde{\Pi}_h$  is a projection on  $\tilde{\mathcal{V}}_h$  that may not be  $\mathcal{W}$ -bounded but has a dense domain  $\hat{\mathcal{V}}$  in  $\mathcal{W}$ .

Thus, the conforming method corresponds to taking  $\tilde{\mathcal{V}}_h = \mathcal{V}_h$  and  $\tilde{\mathcal{A}}_h = \mathcal{A}_h$  as in (2.26), i.e.,

$$(3.2) \quad \mathcal{A}_h := \begin{pmatrix} 0 & -d_{k+1,h}^* \\ d^k & 0 \end{pmatrix} : \mathcal{V}_h \rightarrow \mathcal{V}_h \quad \text{with} \quad \mathcal{V}_h := V_h^k \times V_h^{k+1}.$$

Non-conforming methods are obtained with spaces  $\tilde{\mathcal{V}}_h$  that consist of totally discontinuous functions, typically piecewise polynomials with no continuity constraint between neighboring elements. Note that the semi-group arguments used in Section 2.2 guarantee the well-posedness of (3.1).

**3.1. The Gauss law meets commuting diagrams.** We now specify condition (2.33) for a scheme of the form (3.1). According to Lemma 2.3 and the identity  $R(\mathcal{A}) = (\ker \mathcal{A})^{\perp \mathcal{V}}$ , what we may require is essentially that the data approximation maps the range of the continuous evolution operator  $\mathcal{A}$  into that of the discrete one  $\tilde{\mathcal{A}}_h$ . In order to obtain convergent approximations we formulate the resulting property in terms of an auxiliary approximation operator  $\hat{\Pi}_h$ .

**Definition 3.1** (Compatible schemes). Let  $\hat{\Pi}_h : \hat{\mathcal{V}} \subset \mathcal{V} \rightarrow \tilde{\mathcal{V}}_h$  be a densely-defined projection operator. We say that the scheme (3.1) is *compatible with the Gauss law* (or compatible, for simplicity) on the set  $\hat{\mathcal{V}}$  if the diagram

$$(3.3) \quad \begin{array}{ccc} \hat{\mathcal{V}} & \xrightarrow{\mathcal{A}} & \check{\mathcal{V}} \\ \hat{\Pi}_h \downarrow & & \downarrow \tilde{\Pi}_h \\ \tilde{\mathcal{V}}_h & \xrightarrow{\tilde{\mathcal{A}}_h} & \tilde{\mathcal{V}}_h \end{array}$$

commutes, i.e., if  $\tilde{\Pi}_h \mathcal{A} = \tilde{\mathcal{A}}_h \hat{\Pi}_h$  holds on  $\hat{\mathcal{V}}$ .

Interestingly enough, we observe that although (3.3) has been primarily derived as a compatibility principle relative to the preservation of oscillating solutions at the discrete level (in the sense of Def. 2.4), it also expresses a compatibility property relative to stationary solutions. Indeed, if the continuous source is associated

to an exact steady state, i.e., if it is of the form  $F = -\mathcal{A}\bar{U}$  with  $\bar{U} \in \hat{\mathcal{V}}$ , then its approximation  $\check{\Pi}_h F$  is associated to the steady state  $\hat{\Pi}_h \bar{U} \approx \bar{U}$ . And in fact Property (3.3) implies two key embeddings for the involved projection operators, namely

$$(3.4) \quad \check{\Pi}_h((\ker \mathcal{A})^{\perp w} \cap \mathcal{A}\hat{\mathcal{V}}) \subset (\ker \tilde{\mathcal{A}}_h)^{\perp w} \quad \text{and} \quad \hat{\Pi}_h(\ker \mathcal{A} \cap \hat{\mathcal{V}}) \subset \ker \tilde{\mathcal{A}}_h.$$

It is then easily verified that compatible schemes enjoy the following important properties.

**Theorem 3.2.** *If the semi-discrete scheme (3.1) is compatible, then*

- (i) *when associated to the initial data*

$$\tilde{U}_h^0 = \check{\Pi}_h U^0,$$

*it satisfies Property (2.33) (on  $\mathcal{A}\hat{\mathcal{V}}$ ) and hence the fundamental homogeneous Gauss law ;*

- (ii) *if the solution  $U$  to the Cauchy problem (2.13) is in  $C^0([0, T]; \hat{\mathcal{V}})$ , we have a  $\mathcal{W}$ -error bound*

$$(3.5) \quad \|(\tilde{U}_h - \hat{\Pi}_h U)(t)\| \leq \|\tilde{U}_h^0 - \hat{\Pi}_h U^0\| + \int_0^t \|(\check{\Pi}_h - \hat{\Pi}_h)\partial_t U(s)\| ds, \quad t \geq 0$$

- (iii) *and an additional estimate for the discrete derivatives*

$$(3.6) \quad \|(\tilde{\mathcal{A}}_h \tilde{U}_h - \check{\Pi}_h \mathcal{A}U)(t)\| \leq \|\tilde{\mathcal{A}}_h \tilde{U}_h^0 - \check{\Pi}_h \mathcal{A}U^0\| + \int_0^t \|\tilde{\mathcal{A}}_h(\check{\Pi}_h - \hat{\Pi}_h)\partial_t U(s)\| ds, \quad t \geq 0.$$

*Proof.* Property (i) readily follows from the first embedding in (3.4) and the observation that (2.33) is a sufficient condition for (2.32). To show (ii) and (iii) we next observe that  $U$  satisfies

$$\check{\Pi}_h \partial_t U = \check{\Pi}_h \mathcal{A}U + \check{\Pi}_h F = \tilde{\mathcal{A}}_h \hat{\Pi}_h U + \check{\Pi}_h F,$$

hence for the discrete solution we have

$$\partial_t(U_h - \hat{\Pi}_h U) = (\partial_t U_h - \check{\Pi}_h \partial_t U) + (\check{\Pi}_h - \hat{\Pi}_h)\partial_t U = \tilde{\mathcal{A}}_h(U_h - \hat{\Pi}_h U) + (\check{\Pi}_h - \hat{\Pi}_h)\partial_t U.$$

Applying  $\tilde{\mathcal{A}}_h$  to the latter and using the commuting diagram property (3.3), we further obtain

$$\partial_t(\tilde{\mathcal{A}}_h U_h - \check{\Pi}_h \mathcal{A}U) = \tilde{\mathcal{A}}_h(\tilde{\mathcal{A}}_h U_h - \check{\Pi}_h \mathcal{A}U) + \tilde{\mathcal{A}}_h(\check{\Pi}_h - \hat{\Pi}_h)\partial_t U.$$

Estimates (3.5), (3.6) follow from the contraction properties of the semi-group generated by  $\tilde{\mathcal{A}}_h$ .  $\square$

From Theorem 3.2 we readily derive an asymptotical stability result.

**Corollary 3.3.** *If  $\bar{U} \in \hat{\mathcal{V}}$  is a steady state solution to (2.13), the discrete solution  $U_h$  satisfies (for all  $t \geq 0$ )*

$$\|\tilde{U}_h(t) - \hat{\Pi}_h \bar{U}\| \leq \|\tilde{U}_h^0 - \hat{\Pi}_h \bar{U}\| \quad \text{and} \quad \|\tilde{\mathcal{A}}_h \tilde{U}_h(t) - \check{\Pi}_h \mathcal{A}\bar{U}\| \leq \|\tilde{\mathcal{A}}_h \tilde{U}_h^0 - \check{\Pi}_h \mathcal{A}\bar{U}\|.$$

**3.2. The case of conforming Galerkin approximations.** Before designing non-conforming compatible schemes, we first show that the standard conforming Galerkin method (2.25)-(2.26) is compatible when the data approximation involves a projection operator that satisfies a commuting diagram property for sufficiently smooth functions. Specifically, we assume that we are given two densely-defined projection operators  $\bar{\pi}_h^l$  with  $l = k, k+1$ , that converge pointwise to the identity on their respective domains  $\bar{V}^l \subset V^l$ , and for which the diagram

$$(3.7) \quad \begin{array}{ccc} \bar{V}^k & \xrightarrow{d^k} & \bar{V}^{k+1} \\ \bar{\pi}_h^k \downarrow & & \bar{\pi}_h^{k+1} \downarrow \\ V_h^k & \xrightarrow{d^k} & V_h^{k+1} \end{array}$$

commutes, i.e.,  $\bar{\pi}_h^{k+1} d^k = d^k \bar{\pi}_h^k$  holds on  $\bar{V}^k$ .

Note that in principle it is possible to take the bounded co-chain projection operators  $\pi_h^l$  introduced in Section 2.1, in which case the domains simply correspond to  $\bar{V}^l = V^l$ . However, because these projection operators are usually not simple to implement, one may rather want to use standard finite element interpolation operators such as those recalled in Section 5.2 below. Indeed they verify a commuting diagram property, but they are defined on smaller spaces of smooth functions. In the sequel we denote by  $P_{W_h}$  the orthogonal projection on a closed subspace  $W_h$  of  $W$ , for the corresponding  $W$  scalar product.

**Theorem 3.4** (Compatibility of the conforming Galerkin method). *If  $\tilde{\mathcal{V}}_h = \mathcal{V}_h$  and  $\tilde{\mathcal{A}}_h = \mathcal{A}_h$  correspond to the conforming Galerkin approximation (3.2), the method (3.1) complemented with the projection*

$$(3.8) \quad \tilde{\Pi}_h = \begin{pmatrix} P_{V_h^k} & 0 \\ 0 & \bar{\pi}_h^{k+1} \end{pmatrix} : \tilde{\mathcal{V}} \rightarrow \mathcal{V}_h \quad \text{with} \quad \tilde{\mathcal{V}} := W^k \times \bar{V}^{k+1}$$

is compatible. In particular, it satisfies relation (3.3) with

$$(3.9) \quad \hat{\Pi}_h = \begin{pmatrix} \bar{\pi}_h^k & 0 \\ 0 & P_{V_h^{k+1}} \end{pmatrix} : \hat{\mathcal{V}} \rightarrow \mathcal{V}_h \quad \text{with} \quad \hat{\mathcal{V}} := \bar{V}^k \times V_{k+1}^*.$$

*Remark 3.5.* Here  $\hat{\Pi}_h$  is obviously defined on  $\bar{V}^k \times W^k$ , but for the diagram (3.3) to commute  $\mathcal{A}$  must be defined on  $\hat{\mathcal{V}}$ , i.e., we need  $\hat{\mathcal{V}} \subset \mathcal{V} = V^k \times V_{k+1}^*$ , hence (3.9).

*Proof.* With the above operators, the compatibility relation (3.3) reads

$$(3.10) \quad \begin{cases} \bar{\pi}_h^{k+1} d^k = d^k \bar{\pi}_h^k & \text{on } \bar{V}^k \\ P_{V_h^k} d_{k+1}^* = d_{k+1,h}^* P_{V_h^{k+1}} & \text{on } V_{k+1}^*. \end{cases}$$

The first relation is simply (3.7). Using the properties (2.3) and (2.10) of the continuous and discrete adjoints of  $d^k$ , we then compute for  $w \in V_{k+1}^*$  and  $z \in V_h^k$

$$\langle P_{V_h^k} d_{k+1}^* w, z \rangle = \langle d_{k+1}^* w, z \rangle = \langle w, d^k z \rangle = \langle P_{V_h^{k+1}} w, d^k z \rangle = \langle d_{k+1,h}^* P_{V_h^{k+1}} w, z \rangle.$$

□

We end this section with a classical observation. From the co-chain embedding  $d^k V_h^k \subset V_h^{k+1}$  and the fact that  $\bar{\pi}_h^{k+1}$  maps into  $\bar{V}_h^{k+1} = V_h^{k+1}$  according to (3.7),

we see that the second equation from  $\partial_t U_h - \mathcal{A}_h U_h = \check{\Pi}_h F$  defined with (3.2) and (3.8) holds in a strong sense in the space  $V_h^{k+1}$ . Thus, an equivalent formulation of the compatible conforming approximation is (writing  $U_h = (u_h, v_h)^t \in V_h^k \times V_h^{k+1}$ )

$$(3.11) \quad \begin{cases} \langle \partial_t u_h, \varphi_h \rangle + \langle v_h, d^k \varphi_h \rangle = \langle f, \varphi_h \rangle, & \text{for } \varphi_h \in V_h^k \\ \partial_t v_h - d^k u_h = \bar{\pi}_h^{k+1} g & \text{(in } V_h^{k+1}). \end{cases}$$

### 3.3. Conforming / non-conforming Galerkin (Conga) approximations.

We now turn to the problem which first motivated this work, namely the design of compatible schemes based on discontinuous spaces  $\tilde{\mathcal{V}}_h$  for which an explicit time discretization does not require to invert a global mass matrix. Although it is possible to make standard DG schemes Gauss-compatible by equipping them with proper approximation operators for the data, as will be seen in Sections 3.4 and 5.4 below, in this section we begin by describing an approximation method that aims at preserving the computational structure of the conforming scheme (3.11), in the framework of discontinuous function spaces. In this new scheme the unknown fields are sought in a product space made of conforming and non-conforming functions, namely

$$\tilde{\mathcal{V}}_h := \tilde{V}_h^k \times V_h^{k+1} \quad \text{with} \quad V_h^k \subset \tilde{V}_h^k \not\subset V^k.$$

Here the fact that the second space is conforming should not be a concern: indeed our new scheme will preserve the fact that, just as in (3.11), the second equation holds strongly in  $V_h^{k+1}$ . Hence it will involve no mass matrix in that space. Specifically, the Conga scheme is based on a projection operator

$$P_h^k : \tilde{V}_h^k \rightarrow V_h^k \subset \tilde{V}_h^k$$

seen as a bounded operator from  $\tilde{V}_h^k$  to itself. We shall ask that it satisfies a moment preserving property,

$$(3.12) \quad \langle (I - P_h^k)u, z \rangle = 0, \quad z \in M_h^k,$$

with spaces  $M_h^k \subset \tilde{V}_h^k$  that have a dense union for  $h \rightarrow 0$ , so that the adjoint operator satisfies

$$(3.13) \quad (P_h^k)^* P_{\tilde{V}_h^k} v \rightarrow v \quad \text{as } h \rightarrow 0$$

for all  $v \in W^k$ . We then define the Conga evolution operator as

$$(3.14) \quad \tilde{\mathcal{A}}_h := \begin{pmatrix} 0 & -(P_h^k)^* d_{k+1,h}^* \\ d^k P_h^k & 0 \end{pmatrix} : \tilde{\mathcal{V}}_h \rightarrow \tilde{\mathcal{V}}_h \quad \text{with} \quad \tilde{\mathcal{V}}_h := \tilde{V}_h^k \times V_h^{k+1},$$

indeed this will guarantee that the second equation of (3.1) holds strongly in  $V_h^{k+1}$ . Since  $(P_h^k)^* d_{k+1,h}^*$  is the adjoint of  $d^k P_h^k$  seen as an operator from  $\tilde{V}_h^k$  to  $V_h^{k+1}$ , we easily check that  $\tilde{\mathcal{A}}_h$  is skew-symmetric and bounded on  $\tilde{\mathcal{V}}_h$ .

As for the data approximation, we consider

$$(3.15) \quad \check{\Pi}_h := \begin{pmatrix} (P_h^k)^* P_{\tilde{V}_h^k} & 0 \\ 0 & \bar{\pi}_h^{k+1} \end{pmatrix} : \check{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}_h \quad \text{with} \quad \check{\mathcal{V}} := W^k \times \bar{V}^{k+1},$$

where  $\bar{\pi}_h^{k+1} : \bar{V}^{k+1} \rightarrow V_h^{k+1}$  is the projection operator involved the commuting diagram (3.7). We then have the following result.

**Theorem 3.6** (Compatibility of the Conga method). *The method (3.1) defined by (3.14) and (3.15) satisfies the compatibility relation (3.3) with the same projection operator  $\hat{\Pi}_h$  and domain  $\hat{\mathcal{V}} = \bar{V}^k \times V_{k+1}^*$  as in Theorem 3.4.*

*Proof.* Given (3.14) and (3.15), the claimed relation (3.3) now reads

$$(3.16) \quad \begin{cases} \bar{\pi}_h^{k+1} d^k = d^k \mathbf{P}_h^k \bar{\pi}_h^k & \text{on } \bar{V}^k \\ (\mathbf{P}_h^k)^* P_{\tilde{V}_h^k} d_{k+1}^* = (\mathbf{P}_h^k)^* d_{k+1,h}^* P_{V_h^{k+1}} & \text{on } V_{k+1}^*. \end{cases}$$

Here the first equality follows from the commuting diagram (3.7) and the fact that  $\bar{\pi}_h^k$  maps into  $V_h^k$ , where  $\mathbf{P}_h^k = I$ . We next infer from the embedding  $V_h^k \subset \tilde{V}_h^k$  that  $(P_{\tilde{V}_h^k} - P_{V_h^k}) \mathbf{P}_h^k = 0$ , as well as the adjoint identity

$$(\mathbf{P}_h^k)^* (P_{\tilde{V}_h^k} - P_{V_h^k}) = 0,$$

hence the second equality in (3.16) follows from the second one in (3.10).  $\square$

We may now give an explicit form for the Conga scheme. Using (3.15), the source  $F = (f, g)$  is approximated by  $\check{\Pi}_h F = ((\mathbf{P}_h^k)^* P_{\tilde{V}_h^k} f, \bar{\pi}_h^{k+1} g)$  and the first term gives

$$\langle (\mathbf{P}_h^k)^* P_{\tilde{V}_h^k} f, \tilde{\varphi}_h \rangle = \langle P_{\tilde{V}_h^k} f, \mathbf{P}_h^k \tilde{\varphi}_h \rangle = \langle f, \mathbf{P}_h^k \tilde{\varphi}_h \rangle \quad \text{for } \tilde{\varphi}_h \in \tilde{V}_h^k.$$

Next, from the co-chain embedding  $d^k V_h^k \subset V_h^{k+1}$  we infer that the second equation from  $\partial_t \tilde{U}_h - \tilde{\mathcal{A}}_h \tilde{U}_h = \check{\Pi}_h F$  holds strongly in  $V_h^{k+1}$ , as announced. It follows that the Conga scheme reads (with  $\tilde{U}_h = (\tilde{u}_h, \tilde{v}_h)^t \in \tilde{V}_h^k \times V_h^{k+1}$ )

$$(3.17) \quad \begin{cases} \langle \partial_t \tilde{u}_h, \tilde{\varphi}_h \rangle + \langle \tilde{v}_h, d^k \mathbf{P}_h^k \tilde{\varphi}_h \rangle = \langle f, \mathbf{P}_h^k \tilde{\varphi}_h \rangle & \text{for } \tilde{\varphi}_h \in \tilde{V}_h^k \\ \partial_t \tilde{v}_h - d^k \mathbf{P}_h^k \tilde{u}_h = \bar{\pi}_h^{k+1} g & \text{(in } V_h^{k+1}). \end{cases}$$

*Remark 3.7* (Conga as an intermediate method). Given a non-conforming space  $\tilde{V}_h^k \not\subset V^k$ , we observe that the Conga method is truly an intermediate approach in that it allows to *switch* from the conforming Galerkin method to non-conforming ones, just by changing the conforming projection  $\mathbf{P}_h^k$ . In particular if  $\mathbf{P}_h^k$  is defined as the orthogonal projection on  $V_h^k$ , then (3.17) is equivalent to the conforming scheme (3.1)-(3.2). Indeed, using the fact that  $(I - (\mathbf{P}_h^k)^*) \tilde{u}_h$  is always a constant in (3.17) we find that the Conga solution reads  $(\tilde{u}_h, \tilde{v}_h) = (u_h + \tilde{w}_h, v_h)$  where  $(u_h, v_h)$  is the conforming solution and  $\tilde{w}_h = (I - P_{V_h^k}) \tilde{u}_h = (I - P_{V_h^k}) \tilde{u}_h^0$ .

In Section 5.3 we will describe some operators  $\mathbf{P}_h^k$  that can be applied locally, so that the resulting Conga scheme can be implemented using only sparse matrices.

**3.4. The case of discontinuous Galerkin approximations.** Energy-conserving DG approximations to (2.13) can be cast into the form (3.1). In 3d they correspond to taking the same discontinuous space  $\tilde{V}_h$  for both  $\tilde{V}_h^k$  and  $\tilde{V}_h^{k+1}$ , and to setting

$$(3.18) \quad \tilde{\mathcal{A}}_h := \begin{pmatrix} 0 & -\tilde{d}_h^* \\ \tilde{d}_h & 0 \end{pmatrix} : \tilde{\mathcal{V}}_h \rightarrow \tilde{\mathcal{V}}_h \quad \text{with} \quad \tilde{\mathcal{V}}_h := \tilde{V}_h \times \tilde{V}_h$$

where  $\tilde{d}_h : \tilde{V}_h \rightarrow \tilde{V}_h$  is the corresponding approximation to the differential (**curl**) operator  $d^k$ . In Section 5.4 we will show that in the case of unpenalized centered DG schemes it is possible to rewrite this discrete operator as

$$(3.19) \quad \tilde{d}_h = P_{\tilde{V}_h} d^k \check{\mathbf{P}}_h$$

where  $\check{\mathbb{P}}_h$  is a projection mapping  $\tilde{V}_h$  on an *auxiliary conforming* space  $\check{V}_h^k \subset V^k$ , and a similar result holds for  $\tilde{d}_h^*$ . It will then be possible to build a data approximation operator  $\check{\mathbb{I}}_h$  that makes the semi-discrete DG scheme (3.1) Gauss-compatible. Indeed, if  $\check{\pi}_h^k$  and  $\check{\pi}_h^{k+1}$  are projection operators satisfying a commuting diagram property similar to (3.7) and such that

$$(3.20) \quad \check{\pi}_h^k \text{ maps into } \check{V}_h^k \cap \tilde{V}_h \quad \text{and} \quad \check{\pi}_h^{k+1} \text{ maps into } \tilde{V}_h,$$

then we have (on the domain of  $\check{\pi}_h^k$ )

$$(3.21) \quad \tilde{d}_h \check{\pi}_h^k = P_{\tilde{V}_h} d^k \check{\mathbb{P}}_h \check{\pi}_h^k = P_{\tilde{V}_h} d^k \check{\pi}_h^k = P_{\tilde{V}_h} \check{\pi}_h^{k+1} d^k = \check{\pi}_h^{k+1} d^k,$$

by using (3.19), the fact that  $\check{\mathbb{P}}_h$  is a projection on  $\check{V}_h^k$ , the commuting diagram property  $d^k \check{\pi}_h^k = \check{\pi}_h^{k+1} d^k$  and the fact that  $\check{\pi}_h^{k+1}$  maps into  $\tilde{V}_h$ . A similar identity holds for  $\tilde{d}_h^*$ , hence the following result (see Th. 5.4 below for a specific statement).

**Theorem 3.8** (Compatibility in the DG case). *Let  $\check{\pi}_h^k$  and  $\check{\pi}_h^{k+1}$  denote densely-defined projection operators with respective domains  $\check{V}^l \subset V^l$ ,  $l = k, k+1$ , such that (3.20) and the commuting diagram property  $d^k \check{\pi}_h^k = \check{\pi}_h^{k+1} d^k$  holds. Then the method (3.1) defined with the centered DG evolution operator (3.18) and complemented with the projection*

$$(3.22) \quad \check{\mathbb{I}}_h = \begin{pmatrix} \check{\pi}_h^{k+1} & 0 \\ 0 & \check{\pi}_h^{k+1} \end{pmatrix} : \check{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}_h \quad \text{where} \quad \check{\mathcal{V}} := \check{V}^{k+1} \times \check{V}^{k+1}$$

satisfies the compatibility relation (3.3) with

$$(3.23) \quad \hat{\mathbb{I}}_h = \begin{pmatrix} \check{\pi}_h^k & 0 \\ 0 & \check{\pi}_h^k \end{pmatrix} : \hat{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}_h \quad \text{where} \quad \hat{\mathcal{V}} := \check{V}^k \times \check{V}^k.$$

#### 4. SPECTRAL CORRECTNESS OF THE CONGA SCHEME

In this section we show that the discrete eigenmodes of the Conga operator  $\tilde{\mathcal{A}}_h$  defined in (3.14) converge towards the continuous ones, provided that the dense intersection  $V^k \cap V_k^*$  in  $W^k$  is compact. For the subsequent analysis we let  $\tilde{\mathfrak{Z}}_h^k$  be the null space of  $d^k \mathbb{P}_h^k : \tilde{V}_h^k \rightarrow V_h^{k+1}$ . Note that

$$(4.1) \quad \tilde{\mathfrak{Z}}_h^k := \ker(d^k \mathbb{P}_h^k) = \ker(d^k|_{V_h^k}) \oplus \ker \mathbb{P}_h^k = \mathfrak{Z}_h^k \oplus (I - \mathbb{P}_h^k) \tilde{V}_h^k.$$

We henceforth use short notations for the following orthogonal projection operators:

$$P^\perp := P_{\mathfrak{Z}^{k\perp}}, \quad P_h^\perp := P_{\mathfrak{Z}_h^{k\perp}} \quad \text{and} \quad \tilde{P}_h^\perp := P_{\tilde{\mathfrak{Z}}_h^{k\perp}}.$$

**4.1. Characterization of the continuous eigenmodes.** To begin our analysis we characterize the eigenvalues and eigenvectors of  $\mathcal{A}$  in terms of those of the compact operator  $K$  defined in [2, Sec. 3], see Section 2.1. Specifically, it will be convenient to restrict  $K$  to the orthogonal complement of  $\mathfrak{Z}^k$ . Thus we set

$$(4.2) \quad G := KP^\perp$$

and we observe from [2, Eq. (19)] that for any  $u \in W^k$ ,  $Gu$  is the unique element of  $\mathfrak{Z}^{k\perp}$  (see (2.6)) that satisfies

$$(4.3) \quad \langle d^k Gu, d^k z \rangle = \langle u, z \rangle, \quad z \in \mathfrak{Z}^{k\perp}.$$

In particular, we see that  $G = P^\perp K P^\perp$  is a compact and selfadjoint operator from  $W^k$  to itself. As such it has a countable set of nonnegative eigenvalues, each of finite multiplicity, that accumulate only at 0. We denote by

$$(4.4) \quad 0 < \lambda_1 \leq \lambda_2 \leq \dots$$

the inverses of its *positive* eigenvalues, each one being repeated according to its multiplicity. From the density of  $V^k$  we infer that  $\ker G = \mathfrak{Z}^k$ , hence the complement space  $\mathfrak{Z}^{k \perp w}$  admits an orthonormal basis  $(e_i)_{i \geq 1}$  of eigenvectors corresponding to the  $\lambda_i$ 's. We denote by  $\mathfrak{E}_i$  the one-dimensional space spanned by  $e_i$ .

**Proposition 4.1.** *We have the following results.*

- (i) *The eigenvalues of  $\mathcal{A}$  are of the form  $i\omega$  with  $\omega \in \mathbb{R}$ .*
- (ii) *The kernel of  $\mathcal{A}$  reads*

$$\ker \mathcal{A} = \mathfrak{Z}^k \times \mathfrak{Z}^{k+1 \perp w}.$$

- (iii) *Given  $\omega \neq 0$ ,  $U = (u, v)^t$  is an eigenvector of  $\mathcal{A}$  associated to  $i\omega$  iff*

$$U \in (\ker \mathcal{A})^{\perp w} \quad \text{and} \quad \begin{cases} Gu = \omega^{-2}u \\ v = (i\omega)^{-1}d^k u. \end{cases}$$

*Proof.* Assertions (i) and (ii) are direct consequences of the skew-symmetry (2.15) and of (2.5). Using again the skew-symmetry of  $\mathcal{A}$ , we then observe that  $U$  is an eigenvector associated to the eigenvalue  $i\omega \neq 0$  iff  $U \in (\ker \mathcal{A})^{\perp w} \cap \mathcal{V}$  satisfies  $i\omega U = \mathcal{A}U$ , which is equivalent (writing  $U = (u, v)^t$ ) to

$$\begin{cases} u \in \mathfrak{Z}^{k \perp w} \cap V^k = \mathfrak{Z}^{k \perp} \\ i\omega u = -d_{k+1}^* v \\ i\omega v = d^k u \end{cases} \iff \begin{cases} u \in \mathfrak{Z}^{k \perp} \\ \langle \omega^2 u, z \rangle = \langle d^k u, d^k z \rangle, \quad z \in \mathfrak{Z}^{k \perp} \\ i\omega v = d^k u \end{cases}$$

so that (iii) follows from the characterization (4.3) of  $G$ .  $\square$

**4.2. Characterization of the conforming discrete eigenmodes.** In Cor. 3.17 of [2], it is shown that the operator  $K_h : V_h^k \rightarrow V_h^k$  characterized by

$$(4.5) \quad \langle d_{k,h}^* K_h u, d_{k,h}^* z \rangle + \langle d^k K_h u, d^k z \rangle = \langle u, z \rangle, \quad z \in V_h^k,$$

is a convergent approximation to  $K$ , in the sense that

$$(4.6) \quad \|K - K_h P_{V_h^k}\|_{\mathcal{L}(W^k, W^k)} \rightarrow 0 \quad \text{as} \quad h \rightarrow 0.$$

Since  $K_h$  is selfadjoint and obviously compact, this norm convergence is equivalent to the convergence of the discrete eigenmodes towards the continuous ones, in a sense that will soon be recalled, and corresponds to [4, Def. 2.1]. We shall then characterize the eigenmodes of the conforming operator  $\mathcal{A}_h$  in terms of those of  $K_h$ . As above, what we are actually interested in is the restriction of  $K_h$  to the complement of  $\mathfrak{Z}_h^k$  in  $V_h^k$ . Thus we set

$$(4.7) \quad G_h := K_h P_h^\perp$$

and taking  $u \in \mathfrak{Z}_h^{k \perp}$  in (4.5) yields  $\langle d_{k,h}^* G_h u, d_{k,h}^* z \rangle = \langle d_{k,h}^* K_h u, d_{k,h}^* z \rangle = 0$  for all  $z \in \mathfrak{Z}_h^k$ . Now, since  $\ker d_{k,h}^* = \mathfrak{Z}_h^{k \perp}$  this also holds for all  $z \in V_h^k$ , hence we can take  $z = G_h u$  which shows that  $G_h u \in \mathfrak{Z}_h^{k \perp}$ . It follows that for any  $u \in W^k$ ,  $G_h u$  is the unique element of  $\mathfrak{Z}_h^{k \perp}$  that satisfies

$$(4.8) \quad \langle d^k G_h u, d^k z \rangle = \langle u, z \rangle, \quad z \in \mathfrak{Z}_h^{k \perp}.$$

Thus  $G_h = P_h^\perp K_h P_h^\perp$  is a compact and selfadjoint operator from  $W^k$  to itself, and as such its eigenvalues are nonnegative and of finite multiplicity. We denote by

$$(4.9) \quad 0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \dots \leq \lambda_{N_h^\perp,h},$$

$N_h^\perp = \dim \mathfrak{Z}_h^{k,\perp}$ , the inverses of the positive eigenvalues of  $G_h$ , each one being repeated according to its multiplicity. Similarly as for the continuous case we see that  $\mathfrak{Z}_h^{k,\perp}$  admits an orthonormal basis  $(e_{i,h})_{i \geq 1}$  of eigenvectors corresponding to the  $\lambda_{i,h}$ 's, and we denote by  $\mathfrak{E}_{i,h}$  the one-dimensional space spanned by  $u_{i,h}$ .

We then have the following proposition that characterizes the eigenvalues and eigenvectors of the conforming operator (2.26) in terms of those of  $G_h$ .

**Proposition 4.2.** *We have the following results.*

- (i) *The eigenvalues of  $\mathcal{A}_h$  are of the form  $i\omega$  with  $\omega \in \mathbb{R}$ .*
- (ii) *The kernel of  $\mathcal{A}_h$  reads*

$$(4.10) \quad \ker \mathcal{A}_h = \mathfrak{Z}_h^{k+1,\perp} \times \mathfrak{Z}_h^k.$$

- (iii) *Given  $\omega \neq 0$ ,  $U = (v, u)^t$  is an eigenvector of  $\mathcal{A}_h$  associated to  $i\omega$  iff*

$$\begin{cases} G_h u = \omega^{-2} u \\ v = (i\omega)^{-1} d^k u. \end{cases}$$

*Proof.* Assertions (i) and (ii) are direct consequences of the skew-symmetry of  $\mathcal{A}_h$  and of (2.9), (2.12). Using again the skew-symmetry of  $\mathcal{A}_h$ , we then observe that  $U = (v, u)^t$  is an eigenvector associated to the eigenvalue  $i\omega \neq 0$  iff

$$\begin{cases} u \in (\mathfrak{Z}_h^k)^\perp \cap V_h^k = \mathfrak{Z}_h^{k,\perp} \\ i\omega v = d^k u \\ i\omega u = -d_{k+1,h}^* v \end{cases} \iff \begin{cases} u \in \mathfrak{Z}_h^{k,\perp} \\ i\omega v = d^k u \\ \langle \omega^2 u, z \rangle = \langle d^k u, d^k z \rangle, \quad z \in \mathfrak{Z}_h^{k,\perp} \end{cases}$$

so that (iii) follows from the characterization (4.8) of  $G_h$ .  $\square$

**4.3. Characterization of the Conga eigenmodes.** Turning to the non-conforming case we let  $\tilde{G}_h : W^k \rightarrow \tilde{\mathfrak{Z}}_h^{k,\perp}$  be the operator characterized by

$$(4.11) \quad \langle d^k \mathbf{P}_h^k \tilde{G}_h u, d^k \mathbf{P}_h^k z \rangle = \langle u, z \rangle, \quad z \in \tilde{\mathfrak{Z}}_h^{k,\perp}.$$

It is easily verified that this operator is compact and selfadjoint. In fact, in Lemma 4.4 below we shall specify the link between  $\tilde{G}_h$  and its conforming counterpart  $G_h$ . Similarly as for the conforming case, we denote by

$$(4.12) \quad 0 < \tilde{\lambda}_{1,h} \leq \tilde{\lambda}_{2,h} \leq \dots \leq \tilde{\lambda}_{\tilde{N}_h^\perp,h},$$

$\tilde{N}_h^\perp = \dim \tilde{\mathfrak{Z}}_h^{k,\perp}$ , the inverses of its positive eigenvalues, each one being repeated according to its multiplicity. We let  $(\tilde{e}_{i,h})_{i \geq 1}$  denote an orthonormal basis of eigenvectors for  $\tilde{\mathfrak{Z}}_h^{k,\perp}$ , corresponding to the  $\tilde{\lambda}_{i,h}$ 's, and we denote by  $\tilde{\mathfrak{E}}_{i,h}$  the one-dimensional space spanned by  $\tilde{u}_{i,h}$ .

We may then characterize the eigenmodes of the non-conforming operator (3.14) in terms of those of  $\tilde{G}_h$ .

**Proposition 4.3.** *We have the following results.*

- (i) *The eigenvalues of  $\tilde{\mathcal{A}}_h$  are of the form  $i\omega$  with  $\omega \in \mathbb{R}$ .*
- (ii) *The kernel of  $\tilde{\mathcal{A}}_h$  reads*

$$(4.13) \quad \ker \tilde{\mathcal{A}}_h = \tilde{\mathfrak{Z}}_h^{k+1,\perp} \times \tilde{\mathfrak{Z}}_h^k.$$

(iii) Given  $\omega \neq 0$ ,  $U = (v, u)^t$  is an eigenvector of  $\tilde{\mathcal{A}}_h$  associated to  $i\omega$  iff

$$\begin{cases} \tilde{G}_h u = \omega^{-2} u \\ v = (i\omega)^{-1} d^k \mathbf{P}_h^k u. \end{cases}$$

*Proof.* Assertion (i) is a straightforward consequence of the skew-symmetry of  $\tilde{\mathcal{A}}_h$  and of (2.12), whereas (ii) follows from the observation that since  $(\mathbf{P}_h^k)^* d_{k+1,h}^*$  is the adjoint of  $d^k \mathbf{P}_h^k : \tilde{V}_h^k \rightarrow V_h^{k+1}$  which range is  $\mathfrak{Z}_h^{k+1}$ , one has

$$\ker((\mathbf{P}_h^k)^* d_{k+1,h}^*) = \mathfrak{Z}_h^{k+1 \perp w} \cap V_h^{k+1} = \mathfrak{Z}_h^{k+1 \perp},$$

moreover  $\tilde{\mathfrak{Z}}_h^k$  is the null space of  $d^k \mathbf{P}_h^k$ . Using again the skew-symmetry of  $\tilde{\mathcal{A}}_h$ , we then observe that  $U = (v, u)^t$  is an eigenvector associated to the eigenvalue  $i\omega \neq 0$  iff

$$\begin{cases} u \in (\tilde{\mathfrak{Z}}_h^k)^{\perp w} \cap \tilde{V}_h^k = \tilde{\mathfrak{Z}}_h^{k \perp} \\ i\omega v = d^k \mathbf{P}_h^k u \\ i\omega u = -(\mathbf{P}_h^k u)^* d_{k+1,h}^* v \end{cases} \iff \begin{cases} u \in \tilde{\mathfrak{Z}}_h^{k \perp} \\ i\omega v = d^k \mathbf{P}_h^k u \\ \langle \omega^2 u, z \rangle = \langle d^k \mathbf{P}_h^k u, d^k \mathbf{P}_h^k z \rangle, \quad z \in \tilde{\mathfrak{Z}}_h^{k \perp} \end{cases}$$

so that (iii) follows from the characterization (4.11) of  $\tilde{G}_h$ .  $\square$

We end this section by providing an expression of the non-conforming operator  $\tilde{G}_h$  in terms of the conforming one.

**Lemma 4.4.** *The operators  $G_h$  and  $\tilde{G}_h$ , characterized by (4.8) and (4.11), satisfy (on  $W^k$ )*

$$(4.14) \quad \tilde{G}_h = \tilde{P}_h^\perp G_h \tilde{P}_h^\perp.$$

*Proof.* The identity clearly holds on  $(\tilde{\mathfrak{Z}}_h^{k \perp})^{\perp w} = (I - \tilde{P}_h^\perp)W^k$ . We consider then  $v \in \tilde{\mathfrak{Z}}_h^{k \perp}$  and denote  $\tilde{v} := \tilde{P}_h^\perp G_h \tilde{P}_h^\perp v = \tilde{P}_h^\perp G_h v$ . Given  $w \in \tilde{\mathfrak{Z}}_h^{k \perp}$  (or even in  $\tilde{V}_h^k$ ), we compute

$$\begin{aligned} \langle d^k \mathbf{P}_h^k \tilde{v}, d^k \mathbf{P}_h^k w \rangle &= \langle \tilde{P}_h^\perp G_h v, (d^k \mathbf{P}_h^k)^* d^k \mathbf{P}_h^k w \rangle \\ &= \langle G_h v, (d^k \mathbf{P}_h^k)^* d^k \mathbf{P}_h^k w \rangle \\ &= \langle d^k \mathbf{P}_h^k G_h v, d^k \mathbf{P}_h^k w \rangle \\ &= \langle d^k G_h v, d^k \mathbf{P}_h^k w \rangle \\ &= \langle d^k G_h v, d^k P_h^\perp \mathbf{P}_h^k w \rangle \\ &= \langle v, P_h^\perp \mathbf{P}_h^k w \rangle \\ &= \langle v, \mathbf{P}_h^k w \rangle \\ &= \langle v, w \rangle. \end{aligned}$$

Here the respective equalities use (i) the definition of  $(d^k \mathbf{P}_h^k)^*$  as the adjoint of  $d^k \mathbf{P}_h^k : \tilde{V}_h^k \rightarrow V_h^{k+1}$ , (ii) the fact that its range is in  $\tilde{\mathfrak{Z}}_h^{k \perp}$ , (iii) the embeddings  $\mathfrak{Z}_h^{k \perp} \subset V_h^k \subset \tilde{V}_h^k$ , (iv) the identity  $\mathbf{P}_h^k = I$  on  $V_h^k$ , (v) the observation (used with  $\bar{w} = \mathbf{P}_h^k w$ ) that

$$(4.15) \quad d^k \bar{w} = d^k (P_h^\perp \bar{w} + (I - P_h^\perp) \bar{w}) = d^k P_h^\perp \bar{w}, \quad \bar{w} \in V_h^k,$$

(vi) the characterization (4.8) of  $G_h$ , (vii) the observation that  $(I - P_h^\perp) \bar{w}$  is then in  $\mathfrak{Z}_h^k \subset \tilde{\mathfrak{Z}}_h^k$  and hence orthogonal to  $v$ , and (viii) the fact that  $(I - \mathbf{P}_h^k)w$  is in  $\tilde{\mathfrak{Z}}_h^k$ ,

hence also orthogonal to  $v$ . The result  $\tilde{v} = \tilde{G}_h v$  then follows from the characterization (4.11) of  $\tilde{G}_h$ .  $\square$

**4.4. Convergence of the discrete eigenmodes.** We are now in position to show that both the conforming and non-conforming discrete eigenmodes converge towards the continuous ones in the sense of [4, Def. 2.1], that we now recall. For any positive integer  $N$  we let  $m(N)$  denote the dimension of the space generated by the eigenspaces of the first *distinct* eigenvalues (4.4). Thus  $\lambda_{m(1)}, \dots, \lambda_{m(N)}$  are the first  $N$  distinct eigenvalues and  $\mathfrak{E}_1 + \dots + \mathfrak{E}_{m(N)}$  is the space spanned by the associated eigenspaces: it does not depend on the choice of the eigenbasis  $(u_i)_{i \geq 1}$ . With these notations we say that the discrete eigenmodes  $(\lambda_{i,h}, e_{i,h})_{i \geq 1}$  converge to the continuous ones  $(\lambda_i, e_i)_{i \geq 1}$  if, for any given  $\varepsilon > 0$  and  $N \geq 1$ , there exists a mesh parameter  $h_0 > 0$  such that for all  $h \leq h_0$  we have

$$(4.16) \quad \max_{1 \leq i \leq m(N)} |\lambda_i - \lambda_{i,h}| \leq \varepsilon \quad \text{and} \quad \text{gap} \left( \sum_{1=i}^{m(N)} \mathfrak{E}_i, \sum_{1=i}^{m(N)} \mathfrak{E}_{i,h} \right) \leq \varepsilon,$$

where the gap between two spaces is classically defined as

$$(4.17) \quad \text{gap}(\mathfrak{E}, \mathfrak{F}) := \max \left( \sup_{\substack{u \in \mathfrak{E} \\ \|u\| \leq 1}} \inf_{v \in \mathfrak{F}} \|u - v\|, \sup_{\substack{v \in \mathfrak{F} \\ \|v\| \leq 1}} \inf_{u \in \mathfrak{E}} \|u - v\| \right).$$

Obviously the same applies to the non-conforming eigenmodes  $(\tilde{\lambda}_{i,h}, \tilde{e}_{i,h})_{i \geq 1}$  as well. A key result in the perturbation theory of linear operators (see, e.g. [4] or [2]) is that, for eigenmode problems corresponding to the compact selfadjoint operators  $G$  and  $G_h$  (resp.  $\tilde{G}_h$ ), the above convergence holds if the operators  $G_h$  (resp.  $\tilde{G}_h$ ) converge to  $G$  in  $\mathcal{L}(W^k, W^k)$ .

It is well known that this convergence holds in the conforming case, see e.g., [3, 27, 16]. In our abstract setting this is essentially a consequence of the uniform convergence (4.6). Specifically, the following result holds.

**Theorem 4.5.** *The operator  $G_h : W^k \rightarrow \mathfrak{Z}_h^{k \perp} \subset W^k$  defined by (4.8) satisfies*

$$(4.18) \quad \|G - G_h\|_{\mathcal{L}} \rightarrow 0$$

in the operator norm  $\|\cdot\|_{\mathcal{L}} = \|\cdot\|_{\mathcal{L}(W^k, W^k)}$ .

*Proof.* From the embedding  $\mathfrak{Z}_h^{k \perp} \subset V_h^k$  and the definition (4.7) we have  $G_h = K_h P_{V_h^k} P_h^\perp$ . Using next (4.2), i.e.  $G = K P^\perp$ , we decompose

$$\begin{aligned} \|G - G_h\|_{\mathcal{L}} &\leq \|(K - K_h P_{V_h^k}) P_h^\perp\|_{\mathcal{L}} + \|K(P^\perp - P_h^\perp)\|_{\mathcal{L}} \\ &\leq \|K - K_h P_{V_h^k}\|_{\mathcal{L}} + \|(P^\perp - P_h^\perp)K\|_{\mathcal{L}} \end{aligned}$$

where the second inequality uses the unit bound on orthogonal projections and the fact that a bounded operator and its adjoint have the same norm (here all the operators are selfadjoint). By right composition with the compact operator  $K$  the pointwise convergence of  $P^\perp - P_h^\perp$  to 0 that is proved in Lemma 4.7 yields the norm convergence of  $(P^\perp - P_h^\perp)K$  to 0. The limit (4.18) follows then from (4.6).  $\square$

**Corollary 4.6.** *The conforming discrete eigenmodes  $(\lambda_{i,h}, u_{i,h})_{i \geq 1}$  converge to the continuous ones  $(\lambda_i, u_i)_{i \geq 1}$  in the sense of (4.16).*

In Theorem 4.8 and Corollary 4.9 below we will show that a similar convergence holds true in the non-conforming case. We begin with an auxiliary lemma.

**Lemma 4.7.** *The distance between the closed subspaces  $\mathfrak{Z}_h^{k\perp}$  and  $\mathfrak{Z}^{k\perp w}$  is estimated by the bounds*

$$(4.19) \quad \|(I - P_h^\perp)P^\perp v\| \leq \|(I - P_{V_h^k})P^\perp v\|$$

and

$$(4.20) \quad \|P_h^\perp(I - P^\perp)v\| \leq \|(I - \pi_h^k)(I - P^\perp)v\|$$

for all  $v \in W^k$ . In particular, we have

$$(4.21) \quad (P^\perp - P_h^\perp)v \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

*Proof.* Let  $v^\perp \in \mathfrak{Z}^{k\perp w}$ : we have  $P_{V_h^k}v^\perp \in V_h^k$  and  $\langle P_{V_h^k}v^\perp, w \rangle = \langle v, w \rangle = 0$  for  $w \in \mathfrak{Z}_h^k \subset \mathfrak{Z}^k$ . Thus,

$$P_{V_h^k}\mathfrak{Z}^{k\perp w} \subset \mathfrak{Z}_h^{k\perp}$$

and estimate (4.19) follows, indeed

$$\|(I - P_h^\perp)v^\perp\| = \inf_{v_h^\perp \in \mathfrak{Z}_h^{k\perp}} \|v^\perp - v_h^\perp\| \leq \|(I - P_{V_h^k})v^\perp\|.$$

Next we observe from the commuting diagram (2.8) that  $\pi_h^k$  maps  $\mathfrak{Z}^k$  to  $\mathfrak{Z}_h^k$ . Since  $(I - P^\perp)W^k = (\mathfrak{Z}^{k\perp w})^\perp = \mathfrak{Z}^k$  this yields  $\pi_h^k(I - P^\perp)W^k \subset \mathfrak{Z}_h^k$ , hence we have

$$\|P_h^\perp(I - P^\perp)v\|^2 = \langle (I - P^\perp)v, P_h^\perp(I - P^\perp)v \rangle = \langle (I - \pi_h^k)(I - P^\perp)v, P_h^\perp(I - P^\perp)v \rangle$$

which gives (4.20) with a Cauchy-Schwarz inequality. The pointwise convergence (4.21) is then easily inferred from the fact that  $\pi_h^k$  and  $P_{V_h^k}$  are projections on the discrete spaces  $V_h^k$  which union  $\cup_{h \rightarrow 0} V_h^k$  is assumed dense in  $W^k$ .  $\square$

We are now in position to establish a uniform convergence result for the operator  $\tilde{G}_h$  characterizing the Conga eigenmodes.

**Theorem 4.8.** *The operator  $\tilde{G}_h : W^k \rightarrow \tilde{\mathfrak{Z}}_h^{k\perp} \subset W^k$  defined by (4.11) satisfies*

$$(4.22) \quad \|G - \tilde{G}_h\|_{\mathcal{L}(W^k, W^k)} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

*Proof.* Using Lemma 4.4 and the arguments in the proof of Theorem 4.5, we write

$$\|G_h - \tilde{G}_h\|_{\mathcal{L}} \leq \|(I - \tilde{P}_h^\perp)G_h\|_{\mathcal{L}} + \|\tilde{P}_h^\perp G_h(I - \tilde{P}_h^\perp)\|_{\mathcal{L}} \leq 2\|(I - \tilde{P}_h^\perp)G_h\|_{\mathcal{L}}$$

and using the fact that  $G_h$  maps into  $\mathfrak{Z}_h^{k\perp}$ , we continue with

$$\|(I - \tilde{P}_h^\perp)G_h\|_{\mathcal{L}} = \|(I - \tilde{P}_h^\perp)P_h^\perp G_h\|_{\mathcal{L}} \leq \|(I - \tilde{P}_h^\perp)P_h^\perp G\|_{\mathcal{L}} + \|G - G_h\|_{\mathcal{L}}.$$

We next observe that  $(\mathbf{P}_h^k)^*$  maps  $\mathfrak{Z}_h^{k\perp}$  into  $\tilde{\mathfrak{Z}}_h^{k\perp}$ . Hence

$$\|(I - \tilde{P}_h^\perp)P_h^\perp Gv\| = \inf_{\tilde{v}' \in \tilde{\mathfrak{Z}}_h^{k\perp}} \|P_h^\perp Gv - \tilde{v}'\| \leq \|(I - (\mathbf{P}_h^k)^*)P_h^\perp Gv\|$$

Here the operator  $(\mathbf{P}_h^k)^*$  can be replaced by  $(\mathbf{P}_h^k)^*P_{V_h^k}$  which is assumed to converge pointwise to  $I$ , see (3.13). Thus,  $(I - \tilde{P}_h^\perp)P_h^\perp Gv$  converges pointwise to 0, which by right composition with the compact operator  $G$  leads to a norm convergence,

$$\|(I - \tilde{P}_h^\perp)P_h^\perp G\|_{\mathcal{L}} \rightarrow 0 \quad \text{as } h \rightarrow 0.$$

The proof then follows from (4.18).  $\square$

**Corollary 4.9.** *The non-conforming discrete eigenmodes  $(\tilde{\lambda}_{i,h}, \tilde{u}_{i,h})_{i \geq 1}$  converge to the continuous ones  $(\lambda_i, u_i)_{i \geq 1}$  in the sense of (4.16).*

*Remark 4.10.* To our knowledge, the Conga method is the first non-conforming method that is shown to be energy-conserving and spectrally correct. For instance the centered DG method is known to have a large number of spurious eigenvalues [21, 36], and penalized DG schemes are spectrally correct [12, 11] but they dissipate energy.

## 5. APPLICATION TO THE MAXWELL EQUATIONS IN 3D

To apply the above analysis to the Maxwell equations in a bounded domain  $\Omega$  of  $\mathbb{R}^3$  with metallic boundary conditions, we can take

$$(5.1) \quad H^1(\Omega) \xrightarrow{\mathbf{grad}} H(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} H(\operatorname{div}; \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega)$$

for the primal domain complex  $V^0 \rightarrow V^1 \rightarrow V^2 \rightarrow V^3$  and

$$(5.2) \quad H_0^1(\Omega) \xrightarrow{\mathbf{grad}} H_0(\mathbf{curl}; \Omega) \xrightarrow{\mathbf{curl}} H_0(\operatorname{div}; \Omega) \xrightarrow{\operatorname{div}} L^2(\Omega)$$

for the dual one  $V_3^* \rightarrow V_2^* \rightarrow V_1^* \rightarrow V_0^*$ , or vice-versa. Here we assume that  $\Omega$  is a bounded and simply-connected Lipschitz domain of  $\mathbb{R}^3$ , so that the above de Rham complexes are exact sequences, see [39, Sec. 3.2]. In each case we define the Hilbert spaces as  $W^0 = W^3 = L^2(\Omega)$  and  $W^1 = W^2 = L^2(\Omega)^3$ , and we take  $k = 1$  so that both  $d^1$  and its adjoint  $d_2^*$  are  $\mathbf{curl}$  operators. The evolution problem (2.13)-(2.14) reads then

$$(5.3) \quad \begin{cases} \partial_t \mathbf{E} - \mathbf{curl} \mathbf{B} = -\mathbf{J} \\ \partial_t \mathbf{B} + \mathbf{curl} \mathbf{E} = 0 \end{cases} \quad \text{with} \quad \begin{cases} \mathbf{E}^0 \in H_0(\mathbf{curl}; \Omega) \\ \mathbf{B}^0 \in H(\mathbf{curl}; \Omega) \end{cases}$$

and Lemma 2.2 states that for  $\mathbf{J} \in \mathcal{C}^1([0, T]; L^2(\Omega)^3)$  there exists a unique solution  $(\mathbf{E}, \mathbf{B})$  in  $\mathcal{C}^1([0, T]; H_0(\mathbf{curl}; \Omega) \times H(\mathbf{curl}; \Omega))$ . As for the Gauss law (2.16), it reads

$$(5.4) \quad \begin{cases} \operatorname{div} \mathbf{E}(t) = -\rho \\ \operatorname{div} \mathbf{B}(t) = 0 \end{cases} \quad \text{with} \quad \begin{cases} \mathbf{E}(t) \in H(\operatorname{div}; \Omega) \\ \mathbf{B}(t) \in H_0(\operatorname{div}; \Omega) \end{cases}$$

and the continuity equation verified by the sources is just (1.3).

Although it makes no difference on the continuous problem whether one takes (5.1) or (5.2) for the primal complex, on the conforming Galerkin approximation (2.21) it leads to two different methods. As will be seen in Section 5.2, the first choice leads to a strong discretization of the Ampere equation with natural boundary conditions, whereas the second choice leads to a strong discretization of the Faraday equation with essential boundary conditions.

**5.1. Exact sequences of conforming finite element spaces.** To build conforming approximations of the above complexes, we assume that  $\Omega$  is partitioned by a regular family of conforming simplicial meshes  $(\mathcal{T}_h)_{h>0}$ . By  $\mathcal{F}_h$  and  $\mathcal{E}_h$  we denote the sets of faces and edges of the mesh, and we assume that the latter are oriented by some arbitrary choice of unit vectors  $\mathbf{n}_f$  and  $\boldsymbol{\tau}_e$ , respectively normal to the faces  $f \in \mathcal{F}_h$  and tangent to the edges  $e \in \mathcal{E}_h$ . Following [1, Section 3.5] we can then choose between several sequences of standard finite element spaces. These sequences are based on the piecewise polynomial spaces  $\mathcal{P}_r \Lambda^l(\mathcal{T}_h)$  and  $\mathcal{P}_r^- \Lambda^l(\mathcal{T}_h)$

that consist of differential  $l$ -forms of maximal degree  $r$ . For the sake of completeness we recall the correspondences given in [1, Table 5.2], and specify our notations. For conciseness we write  $X(\mathcal{T}_h) = \{u \in L^2(\Omega) : u|_T \in X(T), T \in \mathcal{T}_h\}$  to denote functions spaces with given piecewise structure  $X$ , as in (5.10) below. Thus,

•  $\mathcal{P}_r \Lambda^0(\mathcal{T}_h)$  corresponds to the continuous ‘‘Lagrange’’ elements of degree  $r$ ,

$$(5.5) \quad \mathcal{L}_r(\Omega, \mathcal{T}_h) := \mathcal{P}_r(\mathcal{T}_h) \cap \mathcal{C}(\Omega),$$

where  $\mathcal{P}_r(T)$  contains the polynomials of maximal degree  $\leq r$  on  $T \in \mathcal{T}_h$

•  $\mathcal{P}_r \Lambda^1(\mathcal{T}_h)$  corresponds to the second-kind Nédélec elements of degree  $r$ ,

$$(5.6) \quad \mathcal{N}_r^{\text{II}}(\Omega, \mathcal{T}_h) := \mathcal{P}_r(\mathcal{T}_h)^3 \cap H(\mathbf{curl}; \Omega)$$

•  $\mathcal{P}_r^- \Lambda^1(\mathcal{T}_h)$  corresponds to the first-kind Nédélec elements of order  $r - 1$

$$(5.7) \quad \begin{cases} \mathcal{N}_{r-1}^{\text{I}}(\Omega, \mathcal{T}_h) := \mathcal{N}_{r-1}(\mathcal{T}_h) \cap H(\mathbf{curl}; \Omega) \\ \text{with } \mathcal{N}_{r-1}(T) := \mathcal{P}_{r-1}(T)^3 \oplus (\mathbf{x} \wedge \mathcal{P}_{r-1}^{\text{hom}}(T)^3) \end{cases}$$

and where  $\mathcal{P}_{r-1}^{\text{hom}}$  is the space of homogeneous polynomials of degree  $r - 1$

•  $\mathcal{P}_r \Lambda^2(\mathcal{T}_h)$  corresponds to the Brezzi-Douglas-Marini elements of degree  $r$  (also called second-kind  $H(\text{div})$  Nédélec space),

$$(5.8) \quad \mathcal{BDM}_r(\Omega, \mathcal{T}_h) := \mathcal{P}_r(\mathcal{T}_h)^3 \cap H(\text{div}; \Omega)$$

•  $\mathcal{P}_r^- \Lambda^2(\mathcal{T}_h)$  corresponds to the Raviart-Thomas elements of order  $r - 1$  (also called first-kind  $H(\text{div})$  Nédélec spaces),

$$(5.9) \quad \begin{cases} \mathcal{RT}_{r-1}(\Omega, \mathcal{T}_h) := \mathcal{RT}_{r-1}(\mathcal{T}_h) \cap H(\text{div}; \Omega) \\ \text{with } \mathcal{RT}_{r-1}(T) := \mathcal{P}_{r-1}(T)^3 \oplus (\mathbf{x} \mathcal{P}_{r-1}^{\text{hom}}(T)) \end{cases}$$

•  $\mathcal{P}_r \Lambda^3(\mathcal{T}_h)$  corresponds to the discontinuous elements of degree  $r$ ,

$$(5.10) \quad \mathcal{P}_r(\mathcal{T}_h) := \{v \in L^2(\Omega) : v|_T \in \mathcal{P}_r(T), T \in \mathcal{T}_h\}.$$

Using the above spaces we can then invoke [1, Theorem 5.6] which says that for each of the following sequences of discrete spaces  $V_h^l$  there exists a co-chain projection  $\pi_h^l : V^l \rightarrow V_h^l$  that is uniformly bounded in the  $L^2$ -norm and for which Fig. 1 is a commuting diagram in the sense of (2.8):

$$(5.11) \quad \left\{ \begin{array}{l} \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\text{grad}} \mathcal{N}_{p-1}^{\text{II}}(\Omega, \mathcal{T}_h) \xrightarrow{\text{curl}} \mathcal{BDM}_{p-2}(\Omega, \mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}_{p-3}(\mathcal{T}_h) \\ \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\text{grad}} \mathcal{N}_{p-1}^{\text{II}}(\Omega, \mathcal{T}_h) \xrightarrow{\text{curl}} \mathcal{RT}_{p-2}(\Omega, \mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}_{p-2}(\mathcal{T}_h) \\ \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\text{grad}} \mathcal{N}_{p-1}^{\text{I}}(\Omega, \mathcal{T}_h) \xrightarrow{\text{curl}} \mathcal{BDM}_{p-1}(\Omega, \mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}_{p-2}(\mathcal{T}_h) \\ \mathcal{L}_p(\Omega, \mathcal{T}_h) \xrightarrow{\text{grad}} \mathcal{N}_{p-1}^{\text{I}}(\Omega, \mathcal{T}_h) \xrightarrow{\text{curl}} \mathcal{RT}_{p-1}(\Omega, \mathcal{T}_h) \xrightarrow{\text{div}} \mathcal{P}_{p-1}(\mathcal{T}_h). \end{array} \right.$$

Note that if one defines the primal spaces  $V^l$  with essential boundary conditions as in (5.2), then the boundary conditions need to be incorporated in the definition of

the discrete finite element spaces as well, by forcing the classical degrees of freedom associated to boundary faces to vanish.

**5.2. Compatible conforming finite elements.** To build conforming Galerkin methods that are compatible in the sense of Definition 3.1 we then need to choose the projection operators  $\bar{\pi}_h^1$  and  $\bar{\pi}_h^2$  involved in the commuting diagram (3.7). For clarity we denote them by

$$\bar{\pi}_h^{\mathbf{curl}} : \bar{V}^1 \rightarrow V_h^1 \quad \text{and} \quad \bar{\pi}_h^{\mathbf{div}} : \bar{V}^2 \rightarrow V_h^2.$$

Here classical finite element interpolation operators can be used. For instance if we take for

$$(5.12) \quad V_h^0 \xrightarrow{d^0 = \mathbf{grad}} V_h^1 \xrightarrow{d^1 = \mathbf{curl}} V_h^2 \xrightarrow{d^2 = \mathbf{div}} V_h^3$$

the last the sequence in (5.11) (with essential or natural boundary conditions), we can define

$$(5.13) \quad \bar{\pi}_h^{\mathbf{curl}} : H^{m+1}(\Omega)^3 \rightarrow \mathcal{N}_{p-1}^I(\Omega, \mathcal{T}_h) \quad \text{and} \quad \bar{\pi}_h^{\mathbf{div}} : H^m(\Omega)^3 \rightarrow \mathcal{RT}_{p-1}(\Omega, \mathcal{T}_h)$$

with  $m \geq 1$ , as the standard Nédélec and Raviart-Thomas interpolations. For a precise definition see, e.g., (2.5.49)-(2.5.32) and (2.5.26)-(2.5.10) in Ref. [6], where they are respectively denoted  $\Sigma_h$  and  $\Pi_h$ . Error estimates are available as well: for the operators just cited we have

$$(5.14) \quad \|(I - \bar{\pi}_h^{\mathbf{curl}})u\| \leq ch^m |u|_m \quad \text{and} \quad \|(I - \bar{\pi}_h^{\mathbf{div}})u\| \leq ch^m |u|_m, \quad 1 \leq m \leq p$$

where  $|\cdot|_m$  denotes the usual  $H^m(\Omega)$  semi-norm, see, e.g., Propositions 2.5.7 and 2.5.4 in [6]. Of course the same estimates hold for the orthogonal projection operators, namely

$$(5.15) \quad \|(I - P_{V_h^1})u\| \leq ch^m |u|_m \quad \text{and} \quad \|(I - P_{V_h^2})u\| \leq ch^m |u|_m, \quad 1 \leq m \leq p.$$

In particular, the projection operators corresponding to (3.8) and (3.9), i.e.,

$$\tilde{\Pi}_h = \begin{pmatrix} P_{V_h^1} & 0 \\ 0 & \bar{\pi}_h^{\mathbf{div}} \end{pmatrix} : \tilde{\mathcal{V}} \rightarrow V_h^1 \times V_h^2 \quad \text{with} \quad \tilde{\mathcal{V}} := L^2(\Omega)^3 \times H^m(\Omega)^3$$

and

$$\hat{\Pi}_h = \begin{pmatrix} \bar{\pi}_h^{\mathbf{curl}} & 0 \\ 0 & P_{V_h^2} \end{pmatrix} : \hat{\mathcal{V}} \rightarrow V_h^1 \times V_h^2 \quad \text{with} \quad \hat{\mathcal{V}} := H^{m+1}(\Omega)^3 \times H(\mathbf{curl}; \Omega)$$

satisfy the same a priori estimates,

$$(5.16) \quad \|(I - \tilde{\Pi}_h)U\| \leq ch^m |U|_m \quad \text{and} \quad \|(I - \hat{\Pi}_h)U\| \leq ch^m |U|_m, \quad 1 \leq m \leq p.$$

With the above material, Theorem 3.4 provides us with two compatible conforming schemes depending whether one takes (5.1) or (5.2) for the primal complex. The first choice corresponds to a strong discretization of the Ampere equation: the finite element spaces (5.12) are defined as one of the sequences from (5.11)

using natural boundary conditions, and the scheme computes the unique solution  $(\mathbf{B}_h, \mathbf{E}_h) \in \mathcal{C}^1([0, T]; V_h^1 \times V_h^2)$  to

$$(5.17) \quad \begin{cases} \langle \partial_t \mathbf{B}_h, \boldsymbol{\varphi}^\mu \rangle + \langle \mathbf{E}_h, \mathbf{curl} \boldsymbol{\varphi}^\mu \rangle = 0 & \boldsymbol{\varphi}^\mu \in V_h^1 \subset H(\mathbf{curl}; \Omega) \\ \langle \partial_t \mathbf{E}_h, \boldsymbol{\varphi}^\varepsilon \rangle - \langle \mathbf{curl} \mathbf{B}_h, \boldsymbol{\varphi}^\varepsilon \rangle = -\langle \bar{\pi}_h^{\text{div}} \mathbf{J}, \boldsymbol{\varphi}^\varepsilon \rangle & \boldsymbol{\varphi}^\varepsilon \in V_h^2 \subset H(\text{div}; \Omega). \end{cases}$$

Note that here the co-chain embedding  $\mathbf{curl} V_h^1 \subset V_h^2$  allows to rewrite the second equation as

$$\partial_t \mathbf{E}_h - \mathbf{curl} \mathbf{B}_h = -\bar{\pi}_h^{\text{div}} \mathbf{J} \quad (\text{in } V_h^2).$$

Moreover if one wants the scheme to be compatible with the fundamental homogeneous Gauss law, one may set  $(\mathbf{B}_h, \mathbf{E}_h)(0) := (P_{V_h^1} \mathbf{B}^0, \bar{\pi}_h^{\text{div}} \mathbf{E}^0)$  as specified in Theorem 3.2.

The second choice corresponds to a strong discretization of the Faraday equation: again the sequence of discrete spaces (5.12) is defined as one of those from (5.11), now with *essential* boundary conditions, and the scheme computes the unique solution  $(\mathbf{E}_h, \mathbf{B}_h) \in \mathcal{C}^1([0, T]; V_h^1 \times V_h^2)$  to

$$(5.18) \quad \begin{cases} \langle \partial_t \mathbf{E}_h, \boldsymbol{\varphi}^\varepsilon \rangle - \langle \mathbf{B}_h, \mathbf{curl} \boldsymbol{\varphi}^\varepsilon \rangle = -\langle \mathbf{J}, \boldsymbol{\varphi}^\varepsilon \rangle & \boldsymbol{\varphi}^\varepsilon \in V_h^1 \subset H_0(\mathbf{curl}; \Omega) \\ \langle \partial_t \mathbf{B}_h, \boldsymbol{\varphi}^\mu \rangle + \langle \mathbf{curl} \mathbf{E}_h, \boldsymbol{\varphi}^\mu \rangle = 0 & \boldsymbol{\varphi}^\mu \in V_h^2 \subset H_0(\text{div}; \Omega). \end{cases}$$

Here the co-chain embedding  $\mathbf{curl} V_h^1 \subset V_h^2$  allows to rewrite the second equation as

$$\partial_t \mathbf{B}_h + \mathbf{curl} \mathbf{E}_h = 0 \quad (\text{in } V_h^2).$$

And again if one wants the scheme to be compatible with the fundamental homogeneous Gauss law, one can follow the statement from Theorem 3.2 and set  $(\mathbf{E}_h, \mathbf{B}_h)(0) := (P_{V_h^1} \mathbf{E}^0, \bar{\pi}_h^{\text{div}} \mathbf{B}^0)$ .

Finally, for any of the above methods the error bound (3.5) and the stability result apply. For instance in the case where the finite element spaces are defined as the last sequence from (5.11), using the a priori estimate (5.16) one obtains

$$\|(\mathbf{E}_h - \mathbf{E}, \mathbf{B}_h - \mathbf{B})(t)\| \leq ch^m \left( |(\mathbf{E}^0, \mathbf{B}^0)|_m + |(\mathbf{E}, \mathbf{B})(t)|_m + \int_0^t |\partial_t(\mathbf{E}, \mathbf{B})(s)|_m ds \right)$$

for  $t \geq 0$ ,  $1 \leq m \leq p$  and a constant independent of  $h$ ,  $t$ .

### 5.3. Compatible conforming/non-conforming Galerkin (Conga) schemes.

We may now construct Conga schemes based on non-conforming spaces  $\tilde{V}_h^1 \not\subset V^1$  that are Gauss-compatible in the sense of Definition 3.1. Following Section 3.3, the main ingredients are:

- an exact sequence of *conforming* spaces (5.12) satisfying  $V_h^1 \subset \tilde{V}_h^1$
- a projection operator on the conforming space  $V_h^1$ ,

$$P_h^1 : \tilde{V}_h^1 \rightarrow V_h^1,$$

that preserves a sufficiently large space of moments  $M_h^1 \subset \tilde{V}_h^1$ , see (3.12).

Since every conforming space listed in (5.11) has the form  $V_h^1 = X^1(\mathcal{T}_h) \cap V^1$ , discontinuous spaces containing their conforming counterparts can be taken as  $\tilde{V}_h^1 := X^1(\mathcal{T}_h)$ , but standard spaces of piecewise polynomials with sufficiently large degrees could be used as well. We shall then construct the projection operators by averaging locally the standard finite element projections, as follows. To each of

the above spaces  $X^1(T)$  are classically associated  $d^1$  (i.e., **curl**)-conforming degrees of freedom, either of “volume” type which involve integrals over single elements  $T \in \mathcal{T}_h$ , or “interface” type which may involve integrals over faces and edges. Then, for some  $\mathbf{v} \in \tilde{V}_h^1$  we may define the projection  $\mathbf{P}_h^k \mathbf{v}$  by assigning the values of its degrees of freedom: either to those of  $\mathbf{v}$  if they are of volume type, or to those of an average trace of  $\mathbf{v}$  if they are of interface type. The resulting operator is simple to implement in a finite element code, and it is local.

To build a projection on the first-kind Nédélec space  $V_h^1 = \mathcal{N}_{p-1}^1(\Omega, \mathcal{T}_h)$  for instance, we may use the fact that the local element  $X^1(T) = \mathcal{N}_{p-1}(T)$  is equipped with degrees of freedom of volume, face and edge type, see [30] or [19, Sec. III-5.3],

$$(5.19) \quad \begin{cases} \mathcal{M}_{\text{vol}}(\mathbf{v}) := \{\int_T \mathbf{v} \cdot \boldsymbol{\pi} : \boldsymbol{\pi} \in \mathcal{P}_{p-3}(T)^3, T \in \mathcal{T}_h\} \\ \mathcal{M}_{\text{face}}(\mathbf{v}) := \{\int_f (\mathbf{v} \wedge \mathbf{n}_f) \cdot \boldsymbol{\pi} : \boldsymbol{\pi} \in \mathcal{P}_{p-2}(f)^2, f \in \mathcal{F}_h\} \\ \mathcal{M}_{\text{edge}}(\mathbf{v}) := \{\int_e (\mathbf{v} \cdot \boldsymbol{\tau}_e) \boldsymbol{\pi} : \boldsymbol{\pi} \in \mathcal{P}_{p-1}(e), e \in \mathcal{E}_h\}. \end{cases}$$

These linear forms are unisolvent in the sense that when restricted to some element  $T$  they characterize the functions of  $X^1(T)$ , and they are  $V^1$ -conforming in the sense that an element of  $X^1(\mathcal{T}_h)$  is in  $V^1 = H(\mathbf{curl}; \Omega)$  if and only if its one-sided traces over any mesh interface define the same degrees of freedom. Thus, given a smooth  $\mathbf{v}$  the relations  $\mathcal{M}_{\text{vol}}(\mathbf{v}_h - \mathbf{v}) = \{0\}$ ,  $\mathcal{M}_{\text{face}}(\mathbf{v}_h - \mathbf{v}) = \{0\}$  and  $\mathcal{M}_{\text{edge}}(\mathbf{v}_h - \mathbf{v}) = \{0\}$  define a unique interpolate  $\mathbf{v}_h \in V_h^1$ , and as previously said, for a piecewise smooth  $\mathbf{v}$  we can average the multivalued traces. Namely, we define  $\mathbf{P}_h^1 : \tilde{V}_h^1 \rightarrow V_h^1$  by

$$(5.20) \quad \mathcal{M}_{\text{vol}}(\mathbf{P}_h^1 \mathbf{v} - \mathbf{v}) = \{0\}, \quad \mathcal{M}_{\text{face}}(\mathbf{P}_h^1 \mathbf{v} - \{\mathbf{v}\}_f) = \{0\}, \quad \mathcal{M}_{\text{edge}}(\mathbf{P}_h^1 \mathbf{v} - \{\mathbf{v}\}_e) = \{0\}$$

with  $\{\mathbf{v}\}_f := \frac{1}{2}(\mathbf{v}|_{T^-} + \mathbf{v}|_{T^+})|_f$  and similarly for  $\{\mathbf{v}\}_e$  on the edges.

**Proposition 5.1.** *The projection (5.20) is uniformly bounded with respect to  $h$ ,*

$$(5.21) \quad \|\mathbf{P}_h^k \mathbf{v}\| \leq c \|\mathbf{v}\|, \quad \mathbf{v} \in \tilde{V}_h^1,$$

and it satisfies the moment preserving property (3.12) with

$$(5.22) \quad \mathbf{M}_h^1 = \mathcal{P}_{p-3}(\mathcal{T}_h)^3.$$

In particular its adjoint  $(\mathbf{P}_h^1)^* : \tilde{V}_h^1 \rightarrow \tilde{V}_h^1$  satisfies

$$(5.23) \quad \|(\mathbf{P}_h^1)^* P_{\tilde{V}_h^1} \mathbf{v}\| \leq ch^m |\mathbf{v}|_m, \quad 0 \leq m \leq p-2.$$

*Proof.* Relation (5.22) readily follows from the definition (5.19) of the volume dofs. As for Estimate (5.21), it is easily obtained with classical arguments. Denoting

$$F_T : \mathbf{x} \mapsto \mathbf{x}_T + B_T \mathbf{x}$$

the affine transformation that maps some fixed reference element  $\hat{T}$  onto  $T$ , we let

$$(5.24) \quad \Phi_T : \mathbf{v} \mapsto B_T^t(\mathbf{v} \circ F_T)$$

so that [19, Lemma 5.5] reads  $\Phi_T(\mathcal{N}_{p-1}(T)) = \mathcal{N}_{p-1}(\hat{T})$ . We next define a localized version of the projection  $\mathbf{P}_h^1$ , as follows. We let  $\mathcal{T}_h(T) := \{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset\}$  be the macro-element determined by  $T$  and define  $\bar{r}_T : \mathcal{C}^0(\mathcal{T}_h(T)) \rightarrow \mathcal{N}_{p-1}(T)$  by the following relations which, among the degrees of freedom from (5.19) only involve those that are defined on the element  $T$  and its boundary,

$$\mathcal{M}_{T, \text{vol}}(\bar{r}_T \mathbf{v} - \mathbf{v}) = \{0\}, \quad \mathcal{M}_{T, \text{face}}(\bar{r}_T \mathbf{v} - \{\mathbf{v}\}) = \{0\}, \quad \mathcal{M}_{T, \text{edge}}(\bar{r}_T \mathbf{v} - \{\mathbf{v}\}) = \{0\}.$$

Invoking next [19, Lemma 5.6] we verify that  $\Phi_T(\bar{r}_T \mathbf{v}) = \bar{r}_{\hat{T}}(\Phi_T \mathbf{v})$ . A straightforward computation using the shape regularity of  $\mathcal{T}_h$  and the expression (5.19) of the degrees of freedom gives then

$$(5.25) \quad \|\bar{r}_T \mathbf{v}\|_{L^2(T)} \sim \|\Phi_T(\bar{r}_T \mathbf{v})\|_{L^2(\hat{T})} = \|\bar{r}_{\hat{T}}(\Phi_T \mathbf{v})\|_{L^2(\hat{T})} \lesssim \max_{T' \in \mathcal{T}_h(T)} \|\Phi_T \mathbf{v}\|_{L^\infty(F_T^{-1}(T'))}$$

with a constant depending on  $p$ . Using the equivalence of norms over the finite dimensional space  $\mathcal{P}_p(F_T^{-1}(T'))$ ,  $T' \in \mathcal{T}_h(T)$ , and the same scaling argument as in (5.25) we compute next

$$\|\Phi_T \mathbf{v}\|_{L^\infty(F_T^{-1}(T'))} \sim \|\Phi_T \mathbf{v}\|_{L^2(F_T^{-1}(T'))} \sim \|\mathbf{v}\|_{L^2(T')}, \quad \mathbf{v} \in \tilde{V}_h^1.$$

Estimate (5.21) follows by summing over  $T \in \mathcal{T}_h$  and by using the fact that  $\|P_{\tilde{V}_h^1}\|_{\mathcal{L}(L^2(\Omega)^3)} \leq 1$ . Finally, the error estimate (5.23) follows from the fact that the operator

$$(P_h^1)^* P_{\tilde{V}_h^1} : L^2(\Omega)^3 \rightarrow \tilde{V}_h^1$$

preserves the polynomials of degree less than  $p - 2$  and that it is uniformly  $L^2$ -bounded with respect to  $h$ , as the adjoint of  $P_{\tilde{V}_h^1} P_h^1$ .  $\square$

In particular, if  $\bar{\pi}_h^{\text{div}}$  is as in (5.13) then the projection corresponding to (3.15),

$$\check{\Pi}_h = \begin{pmatrix} (P_h^1)^* P_{\tilde{V}_h^1} & 0 \\ 0 & \bar{\pi}_h^{\text{div}} \end{pmatrix} : \check{\mathcal{V}} \rightarrow \tilde{V}_h^1 \times V_h^2 \quad \text{with} \quad \check{\mathcal{V}} := L^2(\Omega)^3 \times H^m(\Omega)^3,$$

satisfies the a priori estimate

$$(5.26) \quad \|(I - \check{\Pi}_h)U\| \leq ch^m |U|_m, \quad 1 \leq m \leq p - 2.$$

Applying Theorem 3.6 we then obtain two compatible non-conforming Conga schemes, depending whether one takes (5.1) or (5.2) for the primal complex. As in the conforming case, the first choice corresponds to a strong discretization of the Ampere equation: again the conforming finite element spaces (5.12) are defined as one of the sequences from (5.11), using natural boundary conditions, and the non-conforming space  $\tilde{V}_h^1$  must contain its conforming counterpart  $V_h^1$ . The scheme computes then the unique solution  $(\tilde{\mathbf{B}}_h, \tilde{\mathbf{E}}_h) \in C^1([0, T]; \tilde{V}_h^1 \times V_h^2)$  to

$$(5.27) \quad \begin{cases} \langle \partial_t \tilde{\mathbf{B}}_h, \tilde{\varphi}^\mu \rangle + \langle \tilde{\mathbf{E}}_h, \mathbf{curl} P_h^1 \tilde{\varphi}^\mu \rangle = 0 & \tilde{\varphi}^\mu \in \tilde{V}_h^1 \not\subset H(\mathbf{curl}; \Omega) \\ \langle \partial_t \tilde{\mathbf{E}}_h, \varphi^\varepsilon \rangle - \langle \mathbf{curl} P_h^1 \tilde{\mathbf{B}}_h, \varphi^\varepsilon \rangle = -\langle \bar{\pi}_h^{\text{div}} \mathbf{J}, \varphi^\varepsilon \rangle & \varphi^\varepsilon \in V_h^2 \subset H(\text{div}; \Omega). \end{cases}$$

As in the conforming case, here the co-chain embedding  $\mathbf{curl} V_h^1 \subset V_h^2$  allows to rewrite the second equation as

$$\partial_t \tilde{\mathbf{E}}_h - \mathbf{curl} P_h^1 \tilde{\mathbf{B}}_h = -\bar{\pi}_h^{\text{div}} \mathbf{J} \quad (\text{in } V_h^2).$$

Moreover if one wants the scheme to be compatible with the fundamental homogeneous Gauss law, one may set  $(\tilde{\mathbf{B}}_h, \tilde{\mathbf{E}}_h)(0) := ((P_h^1)^* P_{\tilde{V}_h^1} \mathbf{B}^0, \bar{\pi}_h^{\text{div}} \mathbf{E}^0)$  as specified in Theorem 3.2.

The second choice corresponds to a strong discretization of the Faraday equation: again the sequence of conforming discrete spaces (5.12) is defined as one of those from (5.11), now with essential boundary conditions, and the non-conforming space

$\tilde{V}_h^1$  must contain its conforming counterpart  $V_h^1$ . The scheme computes the unique solution  $(\tilde{\mathbf{E}}_h, \tilde{\mathbf{B}}_h) \in \mathcal{C}^1([0, T]; \tilde{V}_h^1 \times V_h^2)$  to

$$(5.28) \quad \begin{cases} \langle \partial_t \tilde{\mathbf{E}}_h, \tilde{\varphi}^\varepsilon \rangle - \langle \tilde{\mathbf{B}}_h, \mathbf{curl} \mathbf{P}_h^1 \tilde{\varphi}^\varepsilon \rangle = -\langle \mathbf{J}, \mathbf{P}_h^1 \tilde{\varphi}^\varepsilon \rangle & \tilde{\varphi}^\varepsilon \in V_h^1 \not\subset H_0(\mathbf{curl}; \Omega) \\ \langle \partial_t \tilde{\mathbf{B}}_h, \varphi^\mu \rangle + \langle \mathbf{curl} \mathbf{P}_h^1 \tilde{\mathbf{E}}_h, \varphi^\mu \rangle = 0 & \varphi^\mu \in V_h^2 \subset H_0(\text{div}; \Omega) \end{cases}$$

where the form of the discrete source term follows from the discussion at the end of Section 3.3. Here the co-chain embedding  $\mathbf{curl} V_h^1 \subset V_h^2$  allows to rewrite the second equation as

$$\partial_t \tilde{\mathbf{B}}_h + \mathbf{curl} \mathbf{P}_h^1 \tilde{\mathbf{E}}_h = 0 \quad (\text{in } V_h^2).$$

And again if one wants the scheme to be compatible with the fundamental homogeneous Gauss law, one can follow the statement from Theorem 3.2 and set  $(\tilde{\mathbf{E}}_h, \tilde{\mathbf{B}}_h)(0) := ((\mathbf{P}_h^1)^* P_{V_h^1} \mathbf{E}^0, \bar{\pi}_h^{\text{div}} \mathbf{B}^0)$ .

*Remark 5.2* (Coupling with particle methods). It is interesting to note that the above scheme is particularly well suited for Maxwell solvers coupled with particle schemes. Indeed, since  $\mathbf{P}_h^1 \tilde{\varphi}^\varepsilon$  is an  $H(\mathbf{curl}; \Omega)$  function, Lemma 6 from Ref. [14] guarantees that the approximate source term  $\langle \mathbf{J}, \mathbf{P}_h^1 \tilde{\varphi}^\varepsilon \rangle$  involved in (5.28) is well-defined when the current density  $\mathbf{J}$  is defined from point particles, namely moving Dirac masses. And to compute this source term we can rely on Algorithm 9 from the same reference. On the contrary, a source term of the form  $\langle \mathbf{J}, \tilde{\varphi}^\varepsilon \rangle$  would be ill-defined because  $\tilde{\varphi}^\varepsilon$  is fully discontinuous at the mesh interfaces.

Finally, for any of the above methods the error bound (3.5) and the stability result apply. For instance in the case where the conforming finite element spaces are defined as the last sequence from (5.11) and the conforming projection operator  $\mathbf{P}_h^1$  is given by (5.20), using the a priori estimate (5.16) and (5.26) one obtains

$$\|(\tilde{\mathbf{E}}_h - \mathbf{E}, \tilde{\mathbf{B}}_h - \mathbf{B})(t)\| \leq ch^m \left( |(\mathbf{E}^0, \mathbf{B}^0)|_m + |(\mathbf{E}, \mathbf{B})(t)|_m + \int_0^t |\partial_t(\mathbf{E}, \mathbf{B})(s)|_m ds \right)$$

for  $t \geq 0$ ,  $1 \leq m \leq p - 2$  and a constant independent of  $h, t$ .

**5.4. Compatible DG schemes.** When built on conforming triangulations  $\mathcal{T}_h$  of  $\Omega$ , DG approximations to the time-dependent Maxwell equation (5.3) typically involve spaces of piecewise polynomial functions with no continuity requirements across inter-element faces, such as

$$(5.29) \quad \tilde{V}_h := \mathcal{P}_{p_{\text{dg}}}(\mathcal{T}_h) \quad \text{with} \quad p_{\text{dg}} \in \mathbb{N}.$$

The semi-discrete centered DG scheme reads then (see, e.g., [18, Eq. (22)]): find  $(\tilde{\mathbf{E}}_h, \tilde{\mathbf{B}}_h) \in \mathcal{C}^1([0, T]; \tilde{V}_h \times \tilde{V}_h)$  the unique solution to

$$(5.30) \quad \begin{cases} \langle \partial_t \tilde{\mathbf{E}}_h, \varphi^\varepsilon \rangle - \langle \tilde{\mathbf{B}}_h, \mathbf{curl}_h \varphi^\varepsilon \rangle - \langle \{\tilde{\mathbf{B}}_h\}, [\varphi^\varepsilon] \rangle_{\mathcal{F}_h} = -\langle \mathbf{J}_h, \varphi^\varepsilon \rangle & \varphi^\varepsilon \in \tilde{V}_h \\ \langle \partial_t \tilde{\mathbf{B}}_h, \varphi^\mu \rangle + \langle \tilde{\mathbf{E}}_h, \mathbf{curl}_h \varphi^\mu \rangle + \langle \{\tilde{\mathbf{E}}_h\}, [\varphi^\mu] \rangle_{\mathcal{F}_h^{\text{int}}} = 0 & \varphi^\mu \in \tilde{V}_h \end{cases}$$

with given initial data and approximated source  $\mathbf{J}_h \in \tilde{V}_h$ . Here  $\mathbf{curl}_h$  is the piecewise operator given by  $(\mathbf{curl}_h \mathbf{u})|_T := \mathbf{curl}(\mathbf{u}|_T)$  for all  $T \in \mathcal{T}_h$ , and

$$\{\mathbf{u}\}_f := \frac{1}{2}(\mathbf{u}|_{T^-} + \mathbf{u}|_{T^+})|_f \quad \text{and} \quad [\mathbf{u}]_f := (\mathbf{u}|_{T^-} \wedge \mathbf{n}^- + \mathbf{u}|_{T^+} \wedge \mathbf{n}^+)|_f, \quad f \in \mathcal{F}_h^{\text{int}},$$

denote the average and tangential jump on an internal face  $f$  shared by two elements  $T^\pm$  (for which  $\mathbf{n}^\pm$  are the outward unit vectors). On a boundary face they are defined as

$$\{\mathbf{u}\}_f := \mathbf{u} \quad \text{and} \quad [\mathbf{u}]_f := \mathbf{u} \wedge \mathbf{n}, \quad f \in \mathcal{F}_h \setminus \mathcal{F}_h^{\text{int}}.$$

To provide a compatible DG scheme we need to recast (5.30) as an evolution equation of the form (3.1) and identify proper projection operators  $\check{\Pi}_h, \hat{\Pi}_h$  following the lines of Section 3.4. Thus, we let  $\tilde{d}_h : \tilde{V}_h \rightarrow \tilde{V}_h$  be the DG approximation to the operator  $\mathbf{curl} : H(\mathbf{curl}; \Omega) \rightarrow L^2(\Omega)^3$ ,

$$\langle \tilde{d}_h \mathbf{u}, \mathbf{v} \rangle := \langle \mathbf{u}, \mathbf{curl}_h \mathbf{v} \rangle + \langle \{\mathbf{u}\}, [\mathbf{v}] \rangle_{\mathcal{F}_h}, \quad \mathbf{u}, \mathbf{v} \in \tilde{V}_h,$$

and similarly let  $\tilde{d}_{h,0} : \tilde{V}_h \rightarrow \tilde{V}_h$  be the approximation to the adjoint operator  $\mathbf{curl} : H_0(\mathbf{curl}; \Omega) \rightarrow L^2(\Omega)^3$ ,

$$\langle \tilde{d}_{h,0} \mathbf{u}, \mathbf{v} \rangle := \langle \mathbf{u}, \mathbf{curl}_h \mathbf{v} \rangle + \langle \{\mathbf{u}\}, [\mathbf{v}] \rangle_{\mathcal{F}_h^{\text{int}}}, \quad \mathbf{u}, \mathbf{v} \in \tilde{V}_h,$$

so that the DG evolution operator involved in (5.30) reads

$$(5.31) \quad \tilde{\mathcal{A}}_h := \begin{pmatrix} 0 & \tilde{d}_h \\ -\tilde{d}_{h,0} & 0 \end{pmatrix} : \tilde{\mathcal{V}}_h \rightarrow \tilde{\mathcal{V}}_h \quad \text{with} \quad \tilde{\mathcal{V}}_h := \tilde{V}_h \times \tilde{V}_h.$$

We may verify that  $\tilde{\mathcal{A}}_h$  is indeed skew-symmetric: using local Green formulas

$$\langle \mathbf{curl}(\mathbf{u}|_T), \mathbf{v} \rangle_T = \langle \mathbf{u}, \mathbf{curl}(\mathbf{v}|_T) \rangle_T + \langle \mathbf{u}|_T, \mathbf{v}|_T \wedge \mathbf{n}^T \rangle_{\partial T}, \quad T \in \mathcal{T}_h$$

and the identity

$$\langle [\mathbf{u}], \{\mathbf{v}\} \rangle_f + \langle \mathbf{u}|_{T^-}, \mathbf{v}|_{T^-} \wedge \mathbf{n}^- \rangle_f + \langle \mathbf{u}|_{T^+}, \mathbf{v}|_{T^+} \wedge \mathbf{n}^+ \rangle_f = \langle \{\mathbf{u}\}, [\mathbf{v}] \rangle_f, \quad f \in \mathcal{F}_h^{\text{int}},$$

we find that  $\langle \tilde{d}_h \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, \tilde{d}_{h,0} \mathbf{v} \rangle$  holds for all  $\mathbf{u}, \mathbf{v} \in \tilde{V}_h$ , hence

$$\tilde{d}_h^* = \tilde{d}_{h,0}.$$

In Section 3.4 the construction of a compatible DG scheme used an auxiliary  $\mathbf{curl}$ -conforming space  $\check{V}_h^1$  for which there exists a projection  $\check{P}_h : \tilde{V}_h \rightarrow \check{V}_h^1$  such that a relation like (3.19) holds for  $\tilde{d}_h$ , and similarly for  $\tilde{d}_h^*$ . Note that unlike in the Conga case, here the auxiliary conforming space does not need to be a subset of  $\tilde{V}_h$ . The following result specifies this property for the DG space (5.29).

**Lemma 5.3.** *Let*

$$\check{P}_h : \tilde{V}_h \rightarrow \check{V}_h^1 := \mathcal{N}_{p_{\text{dg}}+1}^1(\Omega; \mathcal{T}_h) \subset H(\mathbf{curl}; \Omega)$$

*be the averaged finite element interpolation operator defined as in (5.20), using the degrees of freedom from (5.19) with  $p = p_{\text{dg}} + 2$ . Similarly, we let*

$$\check{P}_{h,0} : \tilde{V}_h \rightarrow \check{V}_{h,0}^1 := \mathcal{N}_{p_{\text{dg}}+1}^1(\Omega; \mathcal{T}_h) \cap H_0(\mathbf{curl}; \Omega)$$

*be the projection operator obtained by forcing the boundary degrees of freedom to vanish in the Nédélec finite elements. We have*

$$(5.32) \quad \tilde{d}_h = P_{\tilde{V}_h}^- \mathbf{curl} \check{P}_h \quad \text{and} \quad \tilde{d}_{h,0} = P_{\tilde{V}_h}^- \mathbf{curl} \check{P}_{h,0}.$$

*Proof.* For  $\mathbf{u}, \mathbf{v} \in \tilde{V}_h$ , applying local Green formulas we compute

$$\begin{aligned} \langle P_{\tilde{V}_h} \mathbf{curl} \check{\mathbf{P}}_h \mathbf{u}, \mathbf{v} \rangle &= \langle \mathbf{curl} \check{\mathbf{P}}_h \mathbf{u}, \mathbf{v} \rangle = \sum_{T \in \mathcal{T}_h} \langle \check{\mathbf{P}}_h \mathbf{u}, \mathbf{curl}(\mathbf{v}|_T) \rangle_T + \langle \mathbf{n}^T \wedge \check{\mathbf{P}}_h \mathbf{u}, \mathbf{v}|_T \rangle_{\partial T} \\ &= \sum_{T \in \mathcal{T}_h} \langle \mathbf{u}, \mathbf{curl}(\mathbf{v}|_T) \rangle_T + \langle \mathbf{n}^T \wedge \{\mathbf{u}\}, \mathbf{v}|_T \rangle_{\partial T} \\ &= \langle \mathbf{u}, \mathbf{curl}_h \mathbf{v} \rangle + \langle \{\mathbf{u}\}, [\mathbf{v}] \rangle_{\mathcal{F}_h} = \langle \tilde{d}_h \mathbf{u}, \mathbf{v} \rangle \end{aligned}$$

where we have used the definition of  $\check{\mathbf{P}}_h$  in the third equality, in particular the fact that the polynomial pieces  $\mathbf{curl}(\mathbf{v}|_T)$  and  $\mathbf{v}|_f$  are of respective degrees  $p_{\text{dg}} - 1$  and  $p_{\text{dg}}$ , thus matching the degrees of freedom (5.19) with  $p = p_{\text{dg}} + 2$ . The identity  $\tilde{d}_h = P_{\tilde{V}_h} \mathbf{curl} \check{\mathbf{P}}_h$  easily follows, and the same computation gives the homogeneous relation  $\tilde{d}_{h,0} = P_{\tilde{V}_h} \mathbf{curl} \check{\mathbf{P}}_{h,0}$ . Note that here the orthogonal projection  $P_{\tilde{V}_h}$  is necessary, since  $\mathbf{curl}$  does not map  $\check{V}_h^1$  into  $\tilde{V}_h$ .  $\square$

To conclude the construction described in Section 3.4 we now let (for  $m \geq 1$ )

$$\tilde{\pi}_h^{\mathbf{curl}} : H^{m+1}(\Omega)^3 \rightarrow \mathcal{N}_{p_{\text{dg}}}^{\text{II}}(\Omega, \mathcal{T}_h) \quad \text{and} \quad \tilde{\pi}_h^{\text{div}} : H^m(\Omega)^3 \rightarrow \mathcal{BDM}_{p_{\text{dg}}-1}(\Omega, \mathcal{T}_h)$$

be the interpolation operators corresponding to the second-kind Nédélec elements defined in [31, Sec. 3.1 and 2.1]. According to Propositions 1 to 3 from [31], these operators satisfy the commuting diagram property  $\tilde{\pi}_h^{\text{div}} \mathbf{curl} = \mathbf{curl} \tilde{\pi}_h^{\mathbf{curl}}$  on  $H^{m+1}(\Omega)^3$ , as well as a priori error estimates

$$(5.33) \quad \|(I - \tilde{\pi}_h^{\mathbf{curl}}) \mathbf{u}\| \leq ch^m |\mathbf{u}|_m \quad \text{and} \quad \|(I - \tilde{\pi}_h^{\text{div}}) \mathbf{u}\| \leq ch^m |\mathbf{u}|_m, \quad 1 \leq m \leq p_{\text{dg}}.$$

Moreover, writing  $\check{V}^1 := H^{m+1}(\Omega)^3$  and  $\check{V}_0^1 := \bar{V}^1 \cap H_0(\mathbf{curl}; \Omega)$  we have

$$\begin{cases} \tilde{\pi}_h^{\mathbf{curl}} \check{V}^1 \subset \mathcal{P}_{p_{\text{dg}}}(\mathcal{T}_h) \cap H(\mathbf{curl}; \Omega) \subset \check{V}_h^1 \cap \tilde{V}_h \\ \tilde{\pi}_h^{\mathbf{curl}} \check{V}_0^1 \subset \mathcal{P}_{p_{\text{dg}}}(\mathcal{T}_h) \cap H_0(\mathbf{curl}; \Omega) \subset \check{V}_{h,0}^1 \cap \tilde{V}_h \end{cases}$$

and we also verify that  $\tilde{\pi}_h^{\text{div}}$  maps into  $\tilde{V}_h$  indeed. In particular, using (5.32) we can compute as in (3.21),

$$(5.34) \quad \tilde{d}_h \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \mathbf{curl} \check{\mathbf{P}}_h \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \mathbf{curl} \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \tilde{\pi}_h^{\text{div}} \mathbf{curl} = \tilde{\pi}_h^{\text{div}} \mathbf{curl}$$

on  $\check{V}^1$ , and similarly for the homogeneous operators,

$$(5.35) \quad \tilde{d}_{h,0} \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \mathbf{curl} \check{\mathbf{P}}_{h,0} \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \mathbf{curl} \tilde{\pi}_h^{\mathbf{curl}} = P_{\tilde{V}_h} \tilde{\pi}_h^{\text{div}} \mathbf{curl} = \tilde{\pi}_h^{\text{div}} \mathbf{curl}$$

on  $\check{V}_0^1$ . This leads to the following compatibility result.

**Theorem 5.4** (Compatible DG methods). *In the case where  $\tilde{\mathcal{A}}_h$  is the centered DG evolution operator (5.31), the scheme (3.1) complemented with the projection*

$$(5.36) \quad \check{\Pi}_h = \begin{pmatrix} \tilde{\pi}_h^{\text{div}} & 0 \\ 0 & \tilde{\pi}_h^{\mathbf{curl}} \end{pmatrix} : \check{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}_h \quad \text{with} \quad \check{\mathcal{V}} := H^m(\Omega)^3 \times H^m(\Omega)^3$$

satisfies the compatibility relation (3.3) with

$$(5.37) \quad \hat{\Pi}_h = \begin{pmatrix} \tilde{\pi}_h^{\mathbf{curl}} & 0 \\ 0 & \tilde{\pi}_h^{\mathbf{curl}} \end{pmatrix} : \hat{\mathcal{V}} \rightarrow \tilde{\mathcal{V}}_h \quad \text{with} \quad \hat{\mathcal{V}} := \check{V}_0^1 \times \check{V}^1.$$

*Proof.* Given (5.31) and (5.36)-(5.37), relation (3.3) reads (5.34)-(5.35).  $\square$

In more classical terms, what the above result states is that the DG scheme

$$\begin{cases} \langle \partial_t \tilde{\mathbf{E}}_h, \boldsymbol{\varphi}^\varepsilon \rangle - \langle \tilde{\mathbf{B}}_h, \mathbf{curl}_h \boldsymbol{\varphi}^\varepsilon \rangle - \langle \{\tilde{\mathbf{B}}_h\}, [\boldsymbol{\varphi}^\varepsilon] \rangle_{\mathcal{F}_h} = -\langle \bar{\pi}^{\text{div}} \mathbf{J}, \boldsymbol{\varphi}^\varepsilon \rangle & \boldsymbol{\varphi}^\varepsilon \in \tilde{V}_h \\ \langle \partial_t \tilde{\mathbf{B}}_h, \boldsymbol{\varphi}^\mu \rangle + \langle \tilde{\mathbf{E}}_h, \mathbf{curl}_h \boldsymbol{\varphi}^\mu \rangle + \langle \{\tilde{\mathbf{E}}_h\}, [\boldsymbol{\varphi}^\mu] \rangle_{\mathcal{F}_h^{\text{int}}} = 0 & \boldsymbol{\varphi}^\mu \in \tilde{V}_h \end{cases}$$

is compatible in the sense of Definition 3.1. In particular, the  $L^2$  error bound (3.5) applies and using the a priori estimate (5.33) we obtain (5.38)

$$\|(\tilde{\mathbf{E}}_h - \mathbf{E}, \tilde{\mathbf{B}}_h - \mathbf{B})(t)\| \leq ch^m \left( |(\mathbf{E}^0, \mathbf{B}^0)|_m + |(\mathbf{E}, \mathbf{B})(t)|_m + \int_0^t |\partial_t(\mathbf{E}, \mathbf{B})(s)|_m ds \right)$$

for  $t \geq 0$ ,  $1 \leq m \leq p_{\text{dg}}$  and a constant independent of  $h$ ,  $t$ . Here we observe that the convergence order is the same than [18, Th. 3.5], but the time dependence is improved. Indeed in (5.38) the upper bound is a constant for steady state solutions.

## 6. CONCLUSION

In this work we have formulated an abstract compatibility property for energy-preserving approximations to the time-dependent Maxwell equations with sources. This property takes the form of a two-component commuting diagram and derives from a fundamental interpretation of the homogeneous Gauss law that is also related to the study of spurious modes in the numerical approximation to the Maxwell eigenvalue problem.

We have shown that semi-discrete schemes satisfying this compatibility property are asymptotically stable with respect to steady state solutions in both  $L^2$  and energy norm, which solves the problem of the large deviations developed by certain classes of numerical schemes on long simulation times.

In addition, we have introduced a new Galerkin method called conforming/non-conforming Galerkin (Conga), defined as a relaxation of the standard conforming approximation. Like DG schemes this method can be implemented using only local discrete operators, and it preserves some of the structural benefits of conforming approximations such as the spectral correctness of its discrete evolution operator, and the ability to preserve one Gauss law in a strong sense.

Finally, we have described several approximation operators for the sources that make Galerkin methods Gauss-compatible in the above sense, be it for the standard conforming Galerkin method, for centered DG schemes or for this new Conga method.

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