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Abstract

A *facial parity edge colouring* of a connected bridgeless plane graph is such an edge colouring in which no two face-adjacent edges receive the same colour and, in addition, for each face f and each colour c, either no edge or an odd number of edges incident with f is coloured with c. Let $\chi'_p(G)$ denote the minimum number of colours used in such a colouring of G. In this paper we prove that $\chi'_p(G) \le 20$ for any 2-edge-connected plane graph G. In the case when G is a 3-edge-connected plane graph the upper bound for this parameter is 12. For G being 4-edge-connected plane graph we have $\chi'_p(G) \le 9$. On the other hand we prove that some bridgeless plane graphs require at least 10 colours for such a colouring.

Keywords: plane graph, facial walk, edge colouring
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1. Introduction

The famous Four Colour Problem has served as a motivation for many equivalent colouring problems, see e.g. the book of Saaty and Kainen [15]. The Four Colour Problem was solved in 1976 by Appel and Haken [1] (see also Robertson et al. [14] for another proof) and the result is presently known as the Four Colour Theorem (4CT). From the 4CT the following result follows, see [15].

Theorem 1. *The edges of a plane triangulation can be coloured with* 3 *colours so that the edges bounding every face are coloured distinctly.*

In 1965, Vizing [17] proved that simple planar graphs with maximum degree at least eight have the edge chromatic number equal to their maximum degree. He conjectured the same if the maximum degree is either seven or six. The first part of this conjecture was

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proved by Sanders and Zhao in 2001, see [16]. Note that (also by Vizing) every graph with maximum degree Δ has the edge chromatic number equal to Δ or $\Delta + 1$. These results of Sanders and Zhao and of Vizing can be reformulated in a sense of Theorem 1 in the following way:

Theorem 2. *Let G be a 3-edge-connected plane graph with maximum face size $\Delta^* \geq 7$. Then the edges of G can be coloured with Δ^* colours in such a way that the edges bounding every face of G are coloured distinctly.*

On the other hand, in 1997 Pyber [13] has shown that the edges of any simple graph can be coloured with at most 4 colours so that all the edges from the same colour class induce a graph with all vertices having odd degree. Mátrai [11] constructed an infinite sequence of finite simple graphs which require 4 colours in any such colouring. Pyber's result can be stated as follows:

Theorem 3. *Let G be a 3-edge-connected plane graph. Then the edges of G can be coloured with at most 4 colours so that for any colour c and any face f of G either no edge or an odd number of edges on the boundary of f is coloured with colour c.*

Recently Bunde, Milans, West, and Wu [4, 5] introduced a *strong parity edge colouring* of graphs. It is an edge colouring of a graph G such that each open walk in G uses at least one colour an odd number of times. Let $p(G)$ be the minimum number of colours in a strong parity edge colouring of a graph G. The exact value of $p(K_n)$ for complete graphs is determined in [4]. They also mention that computing $p(G)$ is NP-hard even when G is a tree.

We say that an edge colouring of a plane graph G is *facially proper* if no two face-adjacent edges of G receive the same colour. (Two edges are *face-adjacent* if they are consecutive edges of a facial walk of some face f of G.) Note that colourings in Theorems 1 and 2 are facially proper, but the colouring in Pyber's Theorem 3 need not be facially proper.

Motivated by the parity edge colouring concept introduced by Bunde et al. [5] and the above mentioned theorems we define a *facial parity edge colouring* of a plane graph G as a facially proper edge colouring with the following property: for each colour c and each face f of G either no edge or an odd number of edges incident with f is coloured with the colour c. The problem is to determine for a given bridgeless plane graph G the minimum possible number of colours, $\chi'_p(G)$, in such a colouring of G. The number $\chi'_p(G)$ is called the *facial parity chromatic index* of G.

Note that the facial parity chromatic index depends on the embedding of the graph. For example, the graph depicted in Figure 1 has different facial parity chromatic index depending on its embedding. With the embedding on the left, its facial parity chromatic index is 5; whereas with the embedding on the right, its parity chromatic index is 4.

The vertex version of this problem (*parity vertex colouring*) was introduced in [8]. The authors proved that every 2-connected plane graph G admits a proper vertex coloring with at most 118 colours such that for each face f and each colour c, either no vertex or an odd number of vertices incident with f is coloured with c. The constant 118 was recently

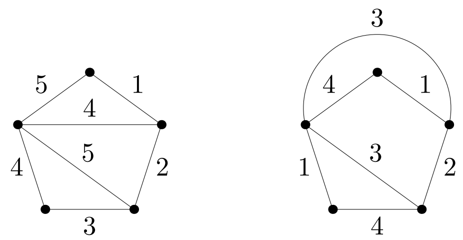

Figure 1: Two embeddings of the same graph with different facial parity chromatic index.

improved to 97 by Kaiser et al. [10]. Czap [6] proved that every 2-connected outerplane graph has a parity vertex colouring with at most 12 colours.

In this paper we prove that each connected bridgeless plane graph has a facial parity edge colouring using at most 20 colours, which improves the bound 92 published in [7]. The facial parity chromatic index is at most 12 for any 3-edge-connected plane graph. In the case when a plane graph is 4-edge-connected the upper bound is at most 9 for this parameter. We also present graphs which require 10 colours for such a colouring.

Throughout the paper, we mostly use the terminology from a recent book [2] of Bondy and Murty. All graphs considered are allowed to contain loops and multiedges, unless stated otherwise.

2. Results

2.1. 2-edge-connected plane graphs

Let φ be a facial parity edge colouring of a bridgeless plane graph G. Observe that in the dual graph G^*, the edges of G in each colour class correspond to a factor of G^* with the degrees of all the vertices odd or zero, i.e. it is an *odd* subgraph. Moreover, since φ is a facially proper edge colouring in G, it induces a facially proper edge colouring in G^* as well.

We say that an edge colouring of a plane graph is *odd*, if each colour class induces an odd subgraph.

Observation 1. *Let G be a plane graph. Then $\chi'_p(G) \leq k$ if and only if the dual graph G^* has a facially proper odd edge colouring using at most k colours.*

This observation will play a major role in proofs below. Instead of facial parity edge colouring of G we shall investigate facially proper odd edge colouring of G^*.

Let us recall the result of Pyber in its original form.

Theorem 4 (Pyber [13])**.** *The edge set of any simple graph H can be covered by at most 4 edge-disjoint odd subgraphs. Moreover, if H has an even number of vertices then it can be covered by at most 3 edge-disjoint odd subgraphs.*

We use this result to establish a general upper bound on facial parity chromatic index for the class of bridgeless plane graphs.

For a face f of a (connected) plane graph G let $E(f)$ denote the set of edges incident with f. Let φ be an edge colouring of a graph G and let c be a colour. Then $\varphi^{-1}(c) = \{e \in E(G) : \varphi(e) = c\}$ denotes the set of edges coloured with c.

Lemma 1. *Let G be a connected plane graph. Then there is a facially proper edge colouring φ of G using at most 5 colours such that for every two faces f_1 and f_2 of G and every colour c*

$$|\varphi^{-1}(c) \cap E(f_1) \cap E(f_2)| = 0 \quad or \quad |\varphi^{-1}(c) \cap E(f_1) \cap E(f_2)| \equiv 1 \pmod 2.$$

Proof. Let G be a counterexample with minimum number of edges. It is easy to see that G must be 2-connected.

Let $M(G)$ be the *medial graph* of G: vertices of $M(G)$ correspond to the edges of G; two vertices of $M(G)$ are adjacent if the corresponding edges of G are face-adjacent. Clearly, $M(G)$ is a plane graph, hence, it has a proper vertex colouring ψ_M using at most 4 colours.

If any two faces of G share at most one edge, then the colouring ψ of the edges of G, given by the colouring ψ_M of the vertices of $M(G)$, has the required property.

Assume that at least two faces, say f_1 and f_2, share at least two edges. Let $e_1, \ldots, e_k$ be the common edges of f_1 and f_2 ordered according to their appearance on the facial walk of f_1 (and f_2). Let $G_1, \ldots, G_k$ be the components of $G \setminus \{e_1, \ldots, e_k\}$ such that G_i is incident with e_i and not incident with e_{i+1} in G ($e_{k+1} = e_1$).

If all the graphs G_i are singletons ($i = 1, 2, \ldots, k$), then G is a cycle on k vertices and a required colouring can be found easily: Let $k = 4\ell + z$, where ℓ is a non-negative integer and $z \in \{2, 3, 4, 5\}$. We repeat ℓ times the pattern $1, 2, 1, 2$ and then use colours $1, 2 \ldots, z$. The colours 1 and 2 are thus used $2\ell + 1$ times, the remaining (at most three) colours are used once.

Assume that G_i has more than one vertex for at least one $i \in \{1, \ldots, k\}$. Suppose $k \geq 3$. Let H_0 be a cycle of length k. Let H_i be a graph obtained from G_i by pasting a path of length 2 to the endvertices of e_{i-1} and e_i ($e_0 = e_k$). The graphs $H_0, H_1, \ldots, H_k$ have less edges than G, hence, each of them has a required edge 5-colouring, say φ_i. The colouring φ_0 of H_0 can be extended to a colouring φ of G in the following way: For each $i \in \{1, \ldots, k\}$, find a permutation of colours used in the colouring φ_i of H_i such that the colours on the edges e_{i-1} and e_i in H_0 and on the corresponding edges in H_i coincide; use this colouring for the graph G_i. It is easy to see that the colouring φ of G obtained this way has the desired property.

Therefore, we may assume that whenever two faces of G share $k \geq 2$ edges, then k must be equal to 2. If for some faces f_1 and f_2 and their common edges e_1 and e_2 both the components G_1 and G_2 of $G \setminus \{e_1, e_2\}$ contain at least 2 vertices, we proceed in the same way as in the previous paragraph. Hence, for each two faces f_1 and f_2 that share two edges, the edges they share are adjacent. But then those two edges receive different colours in the colouring ψ, which implies ψ has the desired property. $\square$

Theorem 5. *Let G be a 2-edge-connected plane graph. Then*

$$\chi'_p(G) \leq 20.$$

Proof. Let φ be a facially proper edge colouring of G given by Lemma 1. This colouring induces an edge-decomposition of the dual graph G^* into five graphs, say G_i^*, $i = 1, 2, 3, 4, 5$. Observe that each edge has an odd multiplicity in every graph G_i^*, $i = 1, 2, 3, 4, 5$.

Let H_i^* be a graph obtained from G_i^* by simplifying the multiple edges (if two vertices are joined with more than one edge, then we remove all of them but one). The graph H_i^* is simple, hence, it can be edge-decomposed into at most 4 odd subgraphs (see Theorem 4). Colour each such subgraph of H_i^* with a distinct colour. To extend this colouring of H_i^* to a colouring of G_i^*, colour the multiedges with the same colour as has the corresponding edge in H_i^*. This way we obtain an edge-decomposition of each G_i^* into four odd subgraphs; altogether at most 20 colours are used in G^*. This colouring of the dual graph G^* induces a required colouring of the original graph G, see Observation 1. □

Note that there is a graph G such that $\chi'_p(G) = 10$ and there is a 2-connected graph G' such that $\chi'_p(G') = 9$. It is sufficient to consider the graphs depicted in Figure 2.

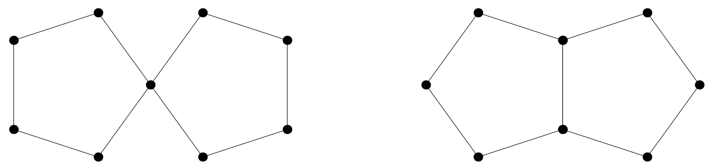

Figure 2: Examples of graphs with no facial parity edge colouring using less than 10 and 9 colours, respectively.

2.2. 3-edge-connected plane graphs

In the remaining parts of the paper we investigate facial parity edge colourings of 3-edge-connected plane graphs. Observe that if G is a 3-edge-connected plane graph, then its dual G^* is a simple plane graph. Therefore, we may apply structural properties of planar graphs on the dual graph; in particular, we will use the following one:

Theorem 6 (Gonçalves [9]). *Let $H = (V, E)$ be a simple planar graph. Then it has a bipartition of its edge set $E = E_1 \cup E_2$ such that the graphs induced by these subsets, $H[E_1]$ and $H[E_2]$, are outerplanar.*

Recall that a (planar) graph is *outerplanar* if it can be embedded in the plane in such a way that all the vertices are on the boundary of the outer face. Note that for a given plane embedding of a planar graph H, the two outerplanar graphs given by Theorem 6 need not be outerplanarly embedded.

In order to find bounds on the facial parity chromatic index of a 3-edge-connected plane graph, we first decompose its dual into two outerplanar graphs, and then bound the number of colors needed for a facially proper odd edge coloring of them. The structure of outerplanar graphs is given in the following theorem.

Theorem 7 (Borodin, Woodall [3]). *If H is a simple outerplanar graph, then at least one of the following cases holds.*

1. *There exists an edge uv such that* $\deg_H(u) = \deg_H(v) = 2$.
2. *There exists a 3-face uvx such that* $\deg_H(u) = 2$ *and* $\deg_H(v) = 3$.

3. *There exist two 3-faces xu_1v_1 and xu_2v_2 such that* $\deg_H(u_1) = \deg_H(u_2) = 2$*,* $\deg_H(x) = 4$*, and these five vertices are all distinct.*
4. $\delta(H) = 1$*, where* $\delta(H)$ *denotes the minimum vertex degree of* H*.*

Lemma 2. *Let H be an arbitrary plane embedding of a simple outerplanar graph. Then it has a facially proper odd edge colouring using at most 6 colours.*

Proof. Let H be a counterexample with minimum number of edges. Clearly, H is connected. First, we prove several structural properties of H.

Claim 1. *H is not a tree.*

Proof. If H is a tree, we can find a facially proper odd edge colouring using at most 5 colours as follows: Pick any vertex of H to be the root. We colour the edges of H starting from the root to the leaves. In each step it is sufficient to find a facially proper odd edge colouring of a star with (at most) one precoloured edge. Let S_n be a star on n edges $e_1, \ldots, e_n$ in a cyclic order. Let $\{1, 2, 3, 4, 5\}$ be a set of colours. We can assume that the edge e_1 has already been coloured with colour 1. Let $n = 4\ell + z$, where ℓ is a non-negative integer and $z \in \{2, 3, 4, 5\}$. We repeat ℓ times the pattern $1, 2, 1, 2$ and then use colours $1, 2, \ldots, z$. The colours 1 and 2 are thus used $2\ell + 1$ times, the remaining (at most three) colours are used once. ■

We say that a subgraph H_0 of a connected graph H is *hanging* on an edge uv if uv is a bridge in H and H_0 is a component of $H \setminus uv$.

Claim 2. *No tree of order at least two is hanging on any edge of H.*

Proof. Let T be a tree hanging on the edge uv, let $v \in T$. Let $H' = H \setminus (T \setminus \{v\})$ be a graph obtained from H by deleting the edges and vertices of T, except for the vertex v. Clearly, H' is outerplanar with less edges than H. Hence, it has a facially proper odd edge colouring using at most 6 colours. We use the same argument as in the proof of the previous claim to extend the colouring of H' to a required colouring of H. ■

Claim 3. *There is no edge uv in H such that* $\deg_H(u) = 2$ *and* $2 \leq \deg_H(v) \leq 4$*.*

Proof. Let $H' = H \setminus uv$ be a graph obtained from H by deleting the edge uv. The graph H' has a required colouring. We can easily extend this colouring to the colouring of H – it suffices to use any colour which does not appear on the edges incident with u or v. ■

Claim 4. *The minimum degree of H is one.*

Proof. It follows from Claim 3 and Theorem 7. ■

Let H' be a graph obtained from H by removing all the vertices of degree one. Clearly, H' is outerplanar and by Claim 2 the minimum vertex degree of H' is at least two. From Theorem 7 it follows that H' contains an edge uv such that $\deg_{H'}(u) = 2$ and $2 \leq \deg_{H'}(v) \leq 4$. There is no such edge in H (see Claim 3), hence, in H there are some vertices of degree one adjacent to u or v.

First assume that u is adjacent with some vertices of degree one in H.

If it is adjacent with at most three vertices of degree one, then the graph obtained by removing all these vertices is not a counterexample, hence, it has a required colouring. To extend it to a colouring of H, we colour the new edges with colours which do not appear on the edges incident with u.

If u is adjacent with at least four vertices of degree one, then we can find three vertices incident with u, say u_1, u_2, x, such that $\deg_H(u_1) = \deg_H(u_2) = 1$ and the edges uu_1, ux and uu_2, ux are face-adjacent. Let us call this configuration a *fork*. By induction, the graph $H \setminus \{uu_1, uu_2\}$ has a facially proper odd edge colouring using at most 6 colours. If all the edges incident with u in $H \setminus \{uu_1, uu_2\}$ are coloured with at most four colours then we colour the edges uu_1, uu_2 with two new colours, else we use the colour which appears on an edge incident with u not face-adjacent to uu_1 nor uu_2.

Now we assume that $\deg_H(u) = 2$ and the vertex v is incident with some vertices of degree one in H.

If $\deg_{H'}(v) \leq 3$, then we can use similar arguments as above. Assume that $\deg_{H'}(v) = 4$. If v is incident with one or two vertices of degree one in H, it suffices to delete these vertices, apply induction, and use the colour(s) which does not appear on the edges incident with v. If v is incident with at least five vertices of degree one in H, then we can always use the reduction using forks described above.

Suppose that v is incident with precisely three vertices of degree one in H. Let v_1 and v_2 be neighbours of v of degree one such that the edges vv_1 and vv_2 are not face-adjacent. Then, by induction, we find a facially proper odd edge colouring of $H \setminus \{v_1, v_2\}$ using at most 6 colours. The degree of v in $H \setminus \{v_1, v_2\}$ is five, therefore, on the five edges incident with v five different colours appear. To extend this colouring to a required colouring of H, we use the colour of the edge incident with v and not face-adjacent to vv_1 nor vv_2.

Finally, we can suppose that v is incident with precisely four vertices of degree one in H, and that there is no fork incident with v. Let $v_1, \ldots, v_8$ be the neighbours of v in the cyclic order. Since there is no fork, we may assume that v_1, v_2, v_5, and v_6 are vertices of degree one. Then, by induction, we find a facially proper odd edge colouring of $H \setminus \{v_1, v_2, v_5, v_6\}$ using at most 6 colours. The degree of v in $H \setminus \{v_1, v_2, v_5, v_6\}$ is four, therefore, on the four edges incident with v four different colours appear. To extend this colouring to a required colouring of H, we use the colour of the edge vv_3 to colour vv_1 and vv_6, and we use the colour of the edge vv_7 to colour vv_2 and vv_5. □

Combining Theorem 6 and Lemma 2 we obtain

Theorem 8. *Let G be a 3-edge-connected plane graph. Then*

$$\chi'_p(G) \leq 12.$$

Proof. Let G^* be the dual of G. 3-edge-connectedness of G implies that G^* is a simple (plane) graph. Let $E_1 \cup E_2 = E(G^*)$ be the bipartition of the edge set of G^* such that the graphs induced by these subsets are outerplanar (see Theorem 6).

Since the graph $G^*[E_i]$ induced by the edges from E_i is outerplanar, $i = 1, 2$, it has a facially proper odd edge coloring with at most six colours (see Lemma 2). In this way we obtain a facially proper odd edge colouring of G^* with at most 12 colours, therefore the claim follows from Observation 1. $\square$

However, this bound does not seem to be best possible.

Conjecture 1. *If H is an arbitrary plane embedding of a simple outerplanar graph then it has a facially proper odd edge colouring using at most* 5 *colours.*

It is easy to see that if this conjecture is true, then $\chi_p'(G) \leq 10$ for every 3-edge-connected plane graph G.

2.3. 4-edge-connected plane graphs

For the class of 4-edge-connected plane graphs we use a different approach. The *arboricity* of a graph is the minimum number of forests into which its edges can be decomposed.

Theorem 9 (Nash-Williams [12])**.** *Let G be a simple graph. Then the arboricity of G equals*

$$\max_{H \subseteq G, |V(H)| \geq 2} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil ,$$

where the maximum is taken over all connected subgraphs H on at least two vertices.

Corollary 1. *Let $G = (V, E, F)$ be a simple plane graph with girth at least* 4*. Then its arboricity is at most two.*

Proof. Observe that any connected subgraph H (on at least 3 vertices) of G also has girth at least 4. From the Euler's polyhedral formula $|V(H)|-|E(H)|+|F(H)| = 2$ and from the fact $2 \cdot |E(H)| = \sum_{f \in F(H)} \deg_H(f) \geq 4 \cdot |F(H)|$ we can easily derive that $|E(H)| \leq 2 \cdot |V(H)| - 4$. From this fact and from Theorem 9 it follows that the edges of G we can decomposed into two forests. $\square$

Lemma 3. *Let H be a simple plane graph. If its arboricity is* 2*, then there is a decomposition of its edge set into two forests A and B such that each vertex of H is incident with an edge from the forest B.*

Proof. Let A_0 and B_0 be two forests such that they form a decomposition of H and the number of vertices which are not incident with any edge of B_0 is the smallest possible. Assume there is a vertex v not incident with any edge of B_0 (it is incident only with A_0). Let e be an edge of A_0 incident with v. Let $A_1 = A_0 \setminus \{e\}$, $B_1 = B_0 \cup \{e\}$. Clearly, A_1 and B_1 are forests, $A_1 \cup B_1 = E(H)$, and the number of vertices not covered by B_1 is smaller than for B_0, a contradiction. $\square$

Theorem 10. *Let G be a 4-edge-connected plane graph. Then*

$$\chi_p'(G) \leq 9.$$

Proof. Let G^* be the dual of G. The graph G is 4-edge-connected, hence, the girth of G^* is at least 4.

Let A, B be a decomposition of G^* into two forests given by Lemma 3. First we find a facially proper odd edge colouring of B which uses at most 5 colours (see the proof of Claim 1).

For each component C of A we find a facially proper odd edge colouring using at most 4 colours similarly: We root C at any vertex. We proceed from the root to the leaves. It suffices to find facially proper odd edge colouring of stars with (at most) one precoloured edge. Let S_n be a star on n edges $e_1, \ldots, e_n$ in a cyclic order. Let $\{1, 2, 3, 4\}$ be a set of colours. We can assume that the edge e_1 has already been coloured with colour 1. Let $n = 4\ell + z$, where ℓ is a non-negative integer and $z \in \{2, 3, 4, 5\}$.

If $z \neq 5$, then we use the same colouring as in the proof of Claim 1.

If $n = 4\ell + 5, \ell \geq 1$, we repeat the pattern $1, 2, 3$ three times and then repeat $\ell - 1$ times the pattern $1, 2, 1, 2$.

Let $n = 5$. The central vertex of the star is incident with at least one edge from B, hence, there are two edges e_i, e_{i+1} that are not face-adjacent in G^*. Colour the edges $e_1, \ldots, e_5$ with three different colours such that the edges e_i, e_{i+1}, e_{i+3} (indices modulo 5) receive the same colour.

These colourings of A and B together induce a facially proper odd edge colouring of G^* using together at most 9 colours, hence, the claim follows from Observation 1. $\square$

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