Toward nonlinear tracking and rejection using LPV control

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***Abstract*— (Quasi) LPV control methods and more generally $\mathcal{L}_2$ gain control methods, referred to as nonlinear $\mathcal{H}_\infty$ control methods, are usually applied in order to ensure reference tracking and disturbance rejection. In this paper, we exhibit a counterexample that reveals that these specifications can not be ensured by these methods. We then propose a new LPV based approach in order to a priori ensure these specifications by combining the LPV method with the incremental $\mathcal{L}_2$ gain analysis of nonlinear performance. Its benefit is illustrated on the counterexample.**

I. INTRODUCTION

The gain-scheduling approach is a very classical and widespread but heuristic nonlinear control approach. The underlying idea of the traditional gain-scheduling is to design at one or more operating points LTI controllers using the linearised plant models associated to the operating points. The nonlinear control law is then obtained by interpolating (or scheduling) these LTI controllers as a function of the operating point, see *e.g.* [1] and references therein. Despite its widespread application, up to recently, there was no systematic gain scheduling controller design method which a priori ensures the desired specifications to the closed loop system.

Two alternative approaches to the traditional gain-scheduling approach are based on the Linear Parameter-Varying (LPV) methods. They were initially developed as an extension of the H_∞ control problem to the case of LPV systems [2]–[4], that is, linear systems whose state space matrices depend on time-varying parameters. The first alternative is actually an improvement of the traditional gain-scheduling approach where the operating points Linear Time Invariant (LTI) controllers are computed in one shot as a single LPV controller. As pointed out in [1], "Typically, stability can be assured only locally and in a "slow-variation" setting, and typically there are no performance guarantees." In order to offer the potential of both stability and performance guarantees [1], an alternative approach, referred to as the *quasi-LPV* approach, proposes to embed the nonlinear system in an LPV model. An LPV controller can be computed using convex optimization involving Linear Matrix Inequality (LMI) constraints in order to ensure an upper bound on the $\mathcal{L}_2$ gain of the closed loop system, see [3]–[7] to cite a few. A nonlinear controller is then deduced from this LPV controller which ensures $\mathcal{L}_2$ gain stability and performance for the closed loop nonlinear system.

Since the quasi-LPV approach seems seducing, it was largely applied for tracking and disturbance rejection, see e.g. [8], [9]. Nevertheless, beyond the $\mathcal{L}_2$ gain performance, it is necessary to question the actual rigorous guarantees on usual tracking and rejection specifications. In section III, we reveal that *there are none* using an illustrative example. This example emphasizes that in contrast with the LTI closed loop systems, it is not possible to ensure tracking and rejection specifications using the $\mathcal{L}_2$ gain stability and performance concepts, in the case of LPV or nonlinear closed loop systems. For these usual specifications, the quasi-LPV approach does not actually introduce performance guarantees, as in the gain-scheduling one.

The theoretical explanation of this fact can be found in our longstanding investigations on the nonlinear system performance, see e.g. [10]–[12]. As the final objective is to ensure tracking and rejection specifications for a nonlinear system, the question is how to test these specifications on a nonlinear system. In [11], [12], it was pointed out that $\mathcal{L}_2$ gain stability and performance fail to ensure these specifications. It was proved that these properties can be ensured by incremental $\mathcal{L}_2$ gain stability and performance. As the two LPV approaches fail to ensure incremental $\mathcal{L}_2$ gain stability and performance, in Section IV, we pave the way to a third LPV approach whose objective is to compute a nonlinear controller which ensures incremental $\mathcal{L}_2$ gain stability and performance and which is able to ensure tracking and rejection specifications. A first solution in a special but important case is discussed here based on the results proposed in [13].

Notations The identity matrix of $\mathbb{R}^{n\times n}$ is denoted I_n and the zero matrix of $\mathbb{R}^{n\times m}$ is denoted $0_{n\times m}$. The subscripts are omitted when obvious from the context. For two operators A and B, $\mathbf{diag}(A,B)$ denotes the operator $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. For a full-rank matrix U, $U^\perp$ denotes an orthogonal complement of U, *i.e.*, $UU^\perp = 0$ and $\left[\, U^T \; U^\perp \,\right]$ is of maximal rank. For a square matrix M, $M > 0$ and $M \geq 0$ mean respectively positive and semi-positive definiteness. The symbol $\star$ denotes the Redheffer star product [14]. $\mathcal{L}_2$ is the space of $\mathbb{R}^n$ square integrable valued functions defined on $\mathbb{R}$, where the norm is defined by $\|f\|_2 = (\int \|f(t)\|^2 dt)^{1/2}$. The causal truncation $P_T f$ is defined by $P_T f(t) = f(t)$ for $t \leq T$ and 0 otherwise. The *extended space* $\mathcal{L}_2^e$ is the space of $\mathbb{R}^n$ valued functions defined on $\mathbb{R}$ whose causal truncations belong to $\mathcal{L}_2$. For a system G, respectively the $\mathcal{H}_\infty$ norm, the $\mathcal{L}_2$ gain and the incremental norm, if they exist, are respectively denoted $||G||_\infty$, $||G||_{i,2}$ and $||G||_\Delta$, see e.g. [11], [12].

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II. ILLUSTRATIVE CONTROL PROBLEM

Let G_{NL} be the nonlinear plant defined by $y = G_{NL}(u)$ with

$$\begin{cases} \dot{x}_1(t) &= -100\varphi(x_1(t)) - 70x_2(t) + 300u(t) \\ \dot{x}_2(t) &= 70x_1(t) - 14x_2(t) \\ y(t) &= x_1(t) \\ x(t_0) &= x_0 \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ the input, $y(t) \in \mathbb{R}^{n_y}$ the measured to-be-controlled output. The function φ is defined as follows (see Fig. 1):

$$\begin{cases} \text{for } |x| < 2 & \varphi(x) = 0.9x^3 - 2|x|x + 1.2x \\ \text{for } x \geq 2 & \varphi(x) = 2x - 2.4 \\ \text{for } x \geq 2 & \varphi(x) = 2x + 2.4 \end{cases}$$

The purpose is to design an output feedback controller $u = K_{NL}(r, y)$ where r denotes the reference signal such that the closed loop system satisfies the following specifications:

- tracking of step reference with a null static error and a response time less than 0.1 s;
- rejection of step disturbance at the plant input;
- limited control energy

that is, typical control specifications.

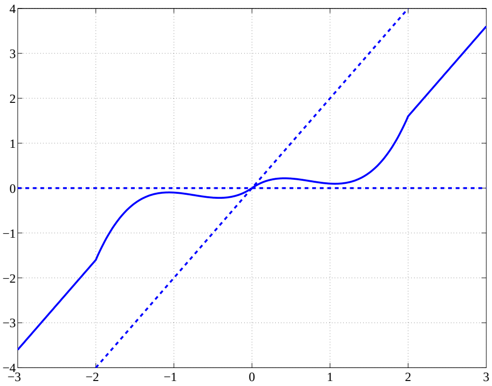

Fig. 1. Plot of the function φ

III. NONLINEAR CONTROL USING LPV CONTROL

A. LPV control

A Linear Parameter Varying (LPV) system is a linear system whose dynamics (*e.g.* defined by a state space representation) depend on time-varying exogenous parameters whose trajectories are *a priori* unknown [2]–[4], [6]. Nevertheless, some information is available, in particular the intervals the parameters belong to. An LPV system G_{LPV} is usually defined by state-space equations of the form: $y = G_{LPV}(u)$ with

$$\begin{cases} \dot{x}(t) &= A(\theta(t))x(t) + B(\theta(t))w(t) \\ y(t) &= C(\theta(t))x(t) + D(\theta(t))w(t), \qquad \exists\theta(.) \in \mathbf{\Theta} \\ x(0) &= x_0 \end{cases} \quad (2)$$

where

- $x(t) \in \mathbb{R}^n$ is the state vector, $w(t) \in \mathbb{R}^{n_w}$ the input, $z(t) \in \mathbb{R}^{n_z}$ the output;
- $\mathbf{\Theta}$ is a set of measurable functions from $\mathbb{R}^+$ to $\mathbb{R}^r$ such that for all $\theta(.) \in \mathbf{\Theta}$, for all $t \geq 0$, $\theta(t)$ belongs to an hyper rectangle $\mathbf{\Theta_t}$, usually $[-1,\ 1] \times \cdots \times [-1,\ 1]$;
- the matrix function
$$\begin{bmatrix} A(\theta) & B(\theta) \\ C(\theta) & D(\theta) \end{bmatrix} \quad (3)$$
is a continuous matrix of real rational functions defined from $\mathbf{\Theta_t}$ to $\mathbb{R}^{(n+n_z)\times(n+n_w)}$.

The different LPV control problem can be expressed as follows. Given the (generalized) plant P_{LPV} defined as:

$$\begin{bmatrix} z \\ y \end{bmatrix} = P_{LPV}\left(\begin{bmatrix} w \\ u \end{bmatrix}\right)$$

with

$$\begin{cases} \dot{x}(t) &= A(\theta(t))x(t) + B_w(\theta(t))w(t) + B_u(\theta(t))u(t) \\ z(t) &= C_z(\theta(t))x(t) + D_{zw}(\theta(t))w(t) + D_{zu}(\theta(t))u(t) \\ y(t) &= C_y(\theta(t))x(t) + D_{yw}(\theta(t))w(t) \\ x(0) &= x_0 \qquad \exists\theta(.) \in \mathbf{\Theta} \end{cases}, \quad (4)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ the command input, $y(t) \in \mathbb{R}^{n_y}$ the measured output, $z(t) \in \mathbb{R}^{n_z}$ the controlled output, $w(t) \in \mathbb{R}^{n_w}$ the disturbance input, with the time-varying exogenous parameters $\theta(t)$ measured on-line, find an controller defined as: $u = K_{LPV}(y)$ with

$$\begin{cases} \dot{x}_K(t) &= A_K(\theta(t))x_K(t) + B_K(\theta(t))y(t) \\ u(t) &= C_K(\theta(t))x_K(t) + D_K(\theta(t))y(t) \\ x_K(0) &= 0 \end{cases} \quad (5)$$

where $x_K \in \mathbb{R}^n$, such that the closed-loop LPV system $P_{LPV} \star K_{LPV}$ is asymptotically stable (for null input) with an $\mathcal{L}_2$ gain less than a given γ. In the case when the rate of variation of $\theta(t)$ is possibly unbounded, solutions to the LPV control problem are based on a (parameter independent) quadratic Lyapunov function $V(x) = x^T P x$, see *e.g.* [3], [4], [6], [7], [15]. The solutions usually rely on convex optimization involving Linear Matrix Inequality constraints, an important class of problems for which there exist efficient solvers [16], [17].

B. Application of the LPV approach to nonlinear control

Roughly speaking, from a technical point of view, the LPV $\mathcal{L}_2$ gain control problem was developed as an extension of the LTI H_∞ control problem to LPV systems. On the other hand, a strong motivation for the introduction of LPV systems was the development of computationally efficient methods for the control of nonlinear systems. Different approaches exist. In the most popular approach, referred to as *quasi-LPV*, the purpose is to compute a non linear controller K_{NL} such that the closed loop system $P_{NL} \star K_{NL}$ has an $\mathcal{L}_2$ gain less than a given γ where the (augmented) nonlinear plant $\begin{bmatrix} z \\ y \end{bmatrix} = P_{NL}\left(\begin{bmatrix} w \\ u \end{bmatrix}\right)$ is defined by the state-space representation:

$$\begin{cases} \dot{x}(t) &= f(x(t), w(t), u(t)) \\ z(t) &= g(x(t), w(t), u(t)) \\ y(t) &= h(x(t), w(t)) \\ x(0) &= x_0 \end{cases} \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ the command input, $y(t) \in \mathbb{R}^{n_y}$ the measured output, $z(t) \in \mathbb{R}^{n_z}$ the controlled output, $w(t) \in \mathbb{R}^{n_w}$ the disturbance input. The functions f and h are assumed uniformly Lipschitz and C^1 and such that $f(x_0, 0) = 0$ and $h(x_0, 0) = 0$. When P_{NL} and K_{NL} are LTI, this problem reduces to the H_∞ control problem.

To this purpose, an LPV augmented plant is obtained from the nonlinear plant by including the nonlinear terms in newly defined time-varying parameters. It is an embedding approach, that is, the LPV system is selected such that the trajectories of the nonlinear system are trajectories of the LPV system, that is, with the following sets defined on $\mathcal{L}_2$, $\Omega_{NL} = \left\{ \begin{pmatrix} x & z & y & w & u \end{pmatrix} \mid (6) \text{ is satisfied} \right\}$ and $\Omega_{LPV} = \left\{ \begin{pmatrix} x & z & y & w & u \end{pmatrix} \mid (4) \text{ is satisfied} \right\}$, we have

$$\Omega_{NL} \subset \Omega_{LPV}. \tag{7}$$

In this approach, the components of the $\theta(t)$ can usually be computed from the measurement of components of the state space vector, see *e.g.* [18]. As a consequence, the objective of the quasi-LPV approach is to compute a nonlinear controller on the LPV model using a quadratic Lyapunov function, in order to ensure the stability of the closed loop nonlinear system and performance evaluated by an upper bound on the $\mathcal{L}_2$ gain of the closed loop system.

C. Application to the illustrative control problem

The quasi LPV approach is applied to the illustrative control problem introduced section II using a quite typical approach. To this purpose, an augmented plant presented Fig. 2 which corresponds to the four- block $\mathcal{L}_2$ gain criterion is proposed. This is the nonlinear counterpart of the usual four-block $\mathcal{H}_\infty$ criterion [19], with a small modification in order to introduce a pure integrator in the controller. It is a well-known fact that for LTI closed loop systems, integral control ensures the tracking of step reference signals and the rejection of step disturbances, with a null static error.

The first step is to derive an LPV model G_{LPV} associated to the plant G_{NL} defined by (1) such that the inclusion (7) holds. A simple LPV model is straightforwardly defined as: $y = G_{LPV}(u)$ with

$$\begin{cases} \dot{x}(t) &= A_G(\theta(t))x(t) + \begin{bmatrix} 300 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases}, \quad \theta(t) \in [0,\ 2]$$

where

$$\theta(t) = \frac{\varphi(x_1(t))}{x_1(t)} \tag{8}$$

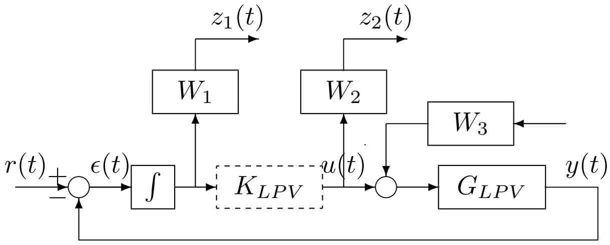

Fig. 2. Augmented plant corresponding to a modified four-block $\mathcal{L}_2$ gain criterion

and

$$A_G(\theta(t)) = \begin{bmatrix} 0 & -70 \\ 70 & -14 \end{bmatrix} + \theta(t) \begin{bmatrix} -100 & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

Since the measured signal y is equal to x_1, the parameter θ can be computed on-line using (8). The augmented LPV plant P_{LPV} is then obtained from Fig. 2 where the weighting functions are defined by

- $W_1 = 50$ in order to ensure a time response of the reference tracking less than 0.1 s;
- $W_3 = 0.1$ in order to ensure a time response of the disturbance rejection larger than for the reference tracking one;
- $W_2(s) = 0.1 \dfrac{\frac{s}{10} + 1}{\frac{s}{1000} + 1}$ in order to limit the controller bandwith.

The weighting functions are selected following the usual $\mathcal{H}_\infty$ approach as in [4]. The LPV controller is then computed using the approach presented in [6] in order to minimize an upper bound γ on the $\mathcal{L}_2$ gain of the augmented closed loop system $P_{LPV} \star K_{LPV}$. The obtained value of γ is 0.89. In order to evaluate the possible conservatism of this upper bound, $\theta(t)$ is set to a constant $\theta_i \in [0,\ 2]$: in this case, the augmented closed loop system is LTI and its $\mathcal{L}_2$ gain boils down to the $\mathcal{H}_\infty$ norm which can be easily computed. By computing the maximum value of the $\mathcal{H}_\infty$ norm when θ_i goes from 0 to 2 by step of 0.01, a lower bound on the actual $\mathcal{L}_2$ gain is obtained: 0.85. The true value of the $\mathcal{L}_2$ gain is between 0.85 and 0.89: in this example, the use of a quadratic Lyapunov function is not very conservative. Let us now evaluate the time domain performance with the following scenario: (i) tracking of square periodic reference whose mean value is 0.4 with amplitude 0.35 and frequency 0.1 Hz, see Fig. 3; (ii) rejection of a step disturbance of amplitude 1.75 at time 4s.

The first simulation is performed on $P_{LPV} \star K_{LPV}$ with $\theta(t)$ which is set to a constant, 0 (see Fig. 4) and 2 (see Fig. 5). In both cases, the closed loop system is LTI. We observe that the specifications are satisfied.

Now, let us simulate the LPV closed loop system with $\theta(t) \in [0,\ 2]$ defined Fig. 6 (top): the output y is represented Fig. 6 (middle and bottom). Note that even if the reference input is constant, the output is oscillating and the oscillations

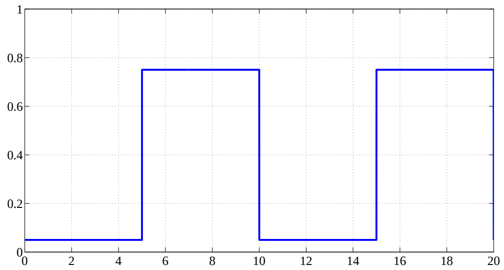

Fig. 3. Square periodic reference signal r

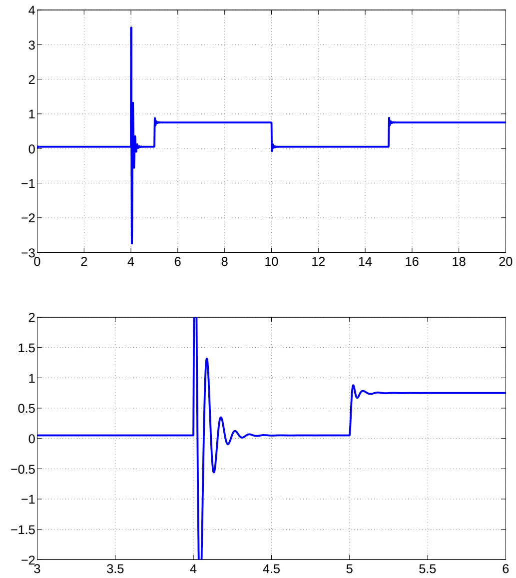

Fig. 4. Simulation of the LTI closed loop system with $\theta = 0$

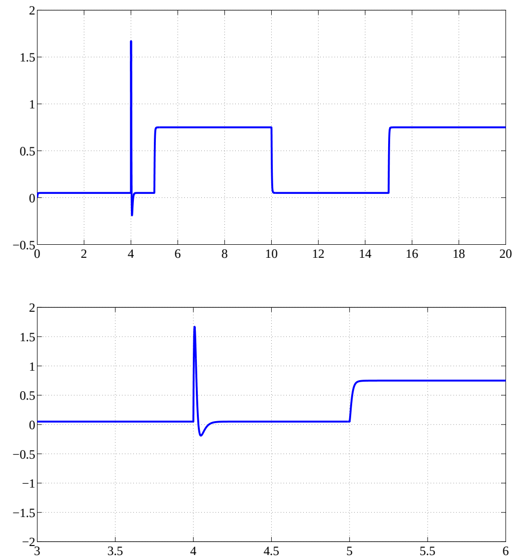

Fig. 5. Simulation of the LTI closed loop system with $\theta = 2$

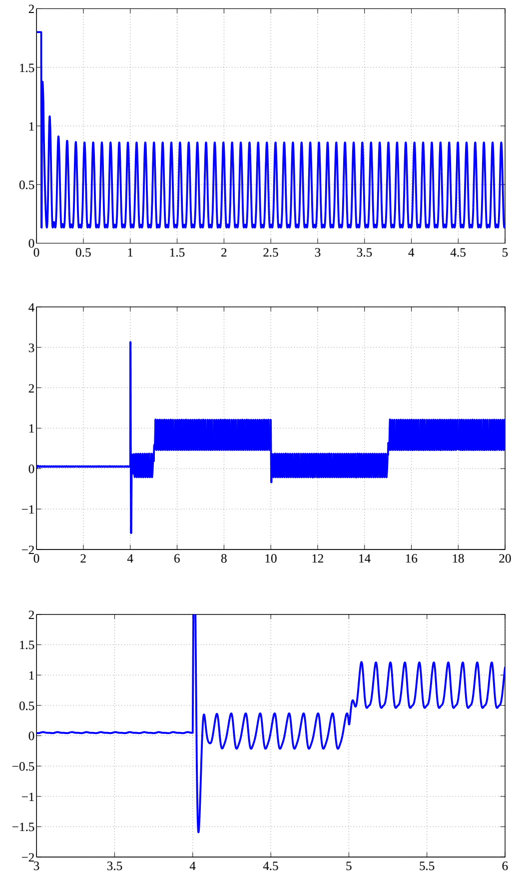

Fig. 6. Simulation of the LPV closed loop system with time varying $\theta(t)$ (top), output $y(t)$ (middle), zoom on the output (bottom)

are not damped. This result seems to be quite surprising since the closed loop system is ($\mathcal{L}_2$ gain) stable. It reveals that, in contrast with the case of LTI systems, in the case of Linear Time Varying (LTV) systems, (exponential or asymptotic) stability does not ensure that for constant inputs, the internal signals and the output signals of the system tend to a constant. As a consequence, even if the closed loop system is ($\mathcal{L}_2$ gain) stable, integral control is unable to ensure the tracking of step reference with a null static error. Nevertheless, if the control specifications has not satisfied, it does not mean that the closed loop system behaviour does not have nice properties. Due to asymptotic stability, with the same time function θ, zero inputs but different initial conditions, the system output converges to the same steady state, see Fig. 7.

From the LPV controller K_{LPV}, a nonlinear controller is obtained by replacing $\theta(t)$ by $\frac{\varphi(y(t))}{y(t)}$ in (5), thanks to (8). Let us simulate the behavior of this controller on the nonlinear plant G_{NL} defined by (1), see Fig. 8. Note that if the disturbance is rejected, the tracking of step references depends strongly on the actual value of the step. If the tracking is satisfying for reference close to 0, it is no longer true for constant input close to 0.75. This fact is in accordance with the result proved, in [20], for the zero input, the $\mathcal{L}_2$ gain

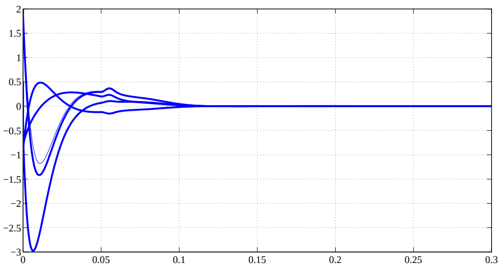

Fig. 7. Simulation of the LPV closed loop system with time varying $\theta(t)$, zero input and different initial conditions

control solution reduces to the $\mathcal{H}_\infty$ control solution. As in the LTV case, in the case of nonlinear systems, in contrast with the case of LTI systems, even if the closed loop system is ($\mathcal{L}_2$ gain) stable, integral control is unable to ensure the tracking of step reference with a null static error. The behaviour is

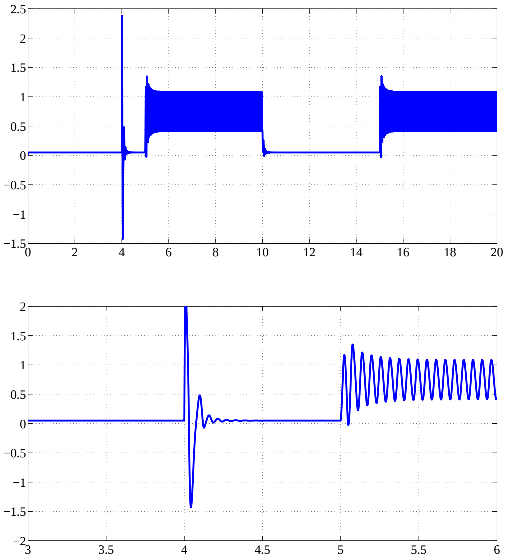

Fig. 8. Simulation of the non linear closed loop system, ouput $y(t)$ (top), zoom on the output (bottom)

actually worse than it seems. Actually, the linearisations of the closed-loop system defined by constant inputs could be not unique with some of them unstable. This is another drawback of the general $\mathcal{L}_2$ gain approach. For $\mathcal{L}_2$ gain stable systems, the stability of the linearisations defined by constant input is not sufficient for ensuring the good behaviour of the system: resonance due to jumps can occur, see [21].

IV. Nonlinear Incremental Control using LPV

In [11], [12], it was revealed that reference tracking and disturbance rejections can be achieved for a nonlinear closed-loop system if the nonlinear controller is computed such that a suitable augmented plant (6) is incrementally $\mathcal{L}_2$ stable with its incremental $\mathcal{L}_2$ gain less than 1.

In order to compute such a controller, we propose a new approach, based on the LPV control. One of the key ingredients of this approach is the generalized version of the mean value theorem [22] which is now recalled.

Theorem 4.1: Let $\tilde{G}$ be a dynamical system defined from $\mathcal{L}_2^e$ into $\mathcal{L}_2^e$ and let us assume that its Gâteaux derivative (time varying linearisation), $D\widetilde{G}[u_0]$ exists for any u_0 belonging to $\mathcal{L}_2^e$. Then $\|\widetilde{G}\|_\Delta \leq \eta$ if and only if $\|D\widetilde{G}[u_0]\|_{i2} \leq \eta$ for any u_0 belonging to $\mathcal{L}_2$.

This theorem is applied as follows. For any input u_0 belonging to $\mathcal{L}_2^e$, the Gâteaux derivative of P_{NL} at u_0 exists and is defined by $\begin{bmatrix} \overline{z} \\ \overline{y} \end{bmatrix} = DP_{NL}[w_0, u_0]\left(\begin{bmatrix} \overline{w} \\ \overline{u} \end{bmatrix}\right)$ with

$$\begin{cases} \dot{\overline{x}}(t) &= A(t)\overline{x}(t) + B_w(t)\overline{w}(t) + B_u(t)\overline{u}(t) \\ \overline{z}(t) &= C_z(t)\overline{x}(t) + D_{zw}(t)\overline{w}(t) + D_{zu}(t)\overline{u}(t) \\ \overline{y}(t) &= C_y(t)\overline{x}(t) + D_{yw}(t)\overline{w}(t) + D_{yu}(t)\overline{u}(t) \\ \overline{x}(0) &= 0, \end{cases} \tag{10}$$

with $A(t) = \frac{\partial f}{\partial x}(x_0(t), w_0(t), u_0(t)), \; \cdots \; D_{yu}(t) = \frac{\partial h}{\partial u}(x_0(t), w_0(t), u_0(t))$ where $x_0(t)$ is the solution of (6) under inputs $w_0(t)$, $u_0(t)$ and the initial condition $x(0) = x_0$.

In order to compute a nonlinear controller K_{NL} such that the closed loop system $P_{NL} \star K_{NL}$ has an incremental $\mathcal{L}_2$ gain less than a given γ where P_{NL} is defined by (6), we first compute the time varying linearisations (Gâteaux derivative) $DK_{NL}[y_0]$ of the nonlinear controller K_{NL} such that the closed loop system $DP_{NL}[u_0, w_0] \star DK_{NL}[y_0]$ has an $\mathcal{L}_2$ gain less than a given γ for any $w_0 \in \mathcal{L}_2^e$. The second step is to compute K_{NL} such that $DK_{NL}[y_0]$ are the time varying linearisation of K_{NL} at y_0.

We propose to realize the first step (computation of $DK_{NL}[y_0]$ for any y_0) by application of the LPV control method. To this purpose, an LPV (augmented) plant (4) is obtained such that the trajectories of the time varying linearisations (10) of the nonlinear plant (6) are trajectories of the LPV system, that is, with the sets of $\mathcal{L}_2\Omega_{DNL} = \left\{\begin{pmatrix} \overline{x} & \overline{z} & \overline{y} & \overline{w} & \overline{u} \end{pmatrix} \mid (10) \text{ is satisfied}\right\}$ and $\Omega_{LPV} = \left\{\begin{pmatrix} x & z & y & w & u \end{pmatrix} \mid (4) \text{ is satisfied}\right\}$, we have

$$\Omega_{DNL} \subset \Omega_{LPV}. \tag{11}$$

The LPV controller is then computed such that the $\mathcal{L}_2$ gain of the closed loop LPV system is less than γ. Since the obtained LPV controller has to correspond to the linearisations of the nonlinear controller, the question is how to enforce this property during the synthesis, that is, how to compute the LPV controller such that there exists a nonlinear controller which can be obtained such that the trajectories of its time-varying linearisations are trajectories of the LPV controller. To our best knowledge, this problem is still open in the general case. In the sequel, we nevertheless exhibit a special class of nonlinear control problems, referred to as "filtered cancellation control" problems for which this problem can be solved.

A. Filtered cancellation control problem

Let us consider the particular class of (generalized) nonlinear plants (6) denoted in the sequel $\widetilde{P}_{NL}$ and defined by the state-space representation:

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 w(t) + B_2 u(t) + \widetilde{f}(x(t)) \\ z(t) = C_1 x(t) + D_{11} w(t) + D_{12} u(t) + \widetilde{g}(x(t)) \\ y(t) = C_2 x(t) + D_{21} w(t) + D_{22} u(t) + \widetilde{h}(x(t)) \\ x(0) = x_0 \end{cases} \tag{12}$$

with

$$\begin{bmatrix} \widetilde{f}(x(t)) \\ \widetilde{g}(x(t)) \\ \widetilde{h}(x(t)) \end{bmatrix} = \begin{bmatrix} B_0 \\ D_{10} \\ D_{20} \end{bmatrix} p(t) \tag{13}$$

where

- $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{n_u}$ the command input, $y(t) \in \mathbb{R}^{n_y}$ the measured output, $z(t) \in \mathbb{R}^{n_z}$ the controlled output, $w(t) \in \mathbb{R}^{n_w}$ the disturbance input;
- $p(t)$ is either measured on-line or constructed when the components of $x(t)$, $w(t)$ and $u(t)$ necessary for the computation of $p(t)$ are measured, that is, there exists a function β such that $p(t) = \beta(x(t), w(t), u(t))$.

The nonlinear controller will be defined as follows: $u = \widetilde{K}_p \left(\begin{bmatrix} y \\ p \end{bmatrix} \right)$ with

$$\begin{cases} \dot{x}_K(t) = A_K x_K(t) + B_{K_y} y(t) + B_{K_p} p(t) \\ u(t) = C_K x_K(t) + D_{K_y} y(t) + D_{K_p} p(t) \\ x_K(0) = x_0 \end{cases} \tag{14}$$

In the case when $p(t)$ is computed from the measure of components of $x(t)$, $w(t)$ and $u(t)$ a nonlinear state space representation is obtained: $u = \widetilde{K}_{xwu} \left(\begin{bmatrix} y \\ x \\ w \end{bmatrix} \right)$ with

$$\begin{cases} \dot{x}_K(t) = A_K x_K(t) + B_{K_y} y(t) + B_{K_p} \beta(x(t), w(t), u(t)) \\ u(t) = C_K x_K(t) + D_{K_y} y(t) + D_{K_p} \beta(x(t), w(t), u(t)) \\ x(0) = x_0 \end{cases} \tag{15}$$

The incremental gain control problem Given $\gamma > 0$ and the generalized plant $\widetilde{P}_{NL}$ defined by (6), find a controller of the form (14) or (15) such that the closed loop system has an incremental gain less than γ.

In the next subsection, a solution to this problem is proposed.

B. LPV filtered cancellation control

When the matrices of the state space representation of the (generalized) LPV plant (4) are rational functions of $\theta(t)$, the latter can be modeled by a Linear Fractional Transformation (LFT) on a parameter block diagonal structure [6]:

$$\begin{cases} \left[\begin{array}{c} \dot{x}(t) \\ z(t) \\ \hline y(t) \end{array}\right] = \left[\begin{array}{ccc|c} A & B_0 & B_1 & B_2 \\ C_1 & D_{10} & D_{11} & D_{12} \\ \hline C_2 & D_{20} & D_{21} & 0 \end{array}\right] \left[\begin{array}{c} x(t) \\ p(t) \\ w(t) \\ \hline u(t) \end{array}\right] \\ \\ p(t) = \Theta(t) \left(I - D_{00}\Theta(t) \right)^{-1} \left(C_0 x(t) + D_{01} w(t) + D_{02} u(t) \right) \end{cases} \tag{16}$$

where the parameter block $\Theta(t)$ is defined as $\Theta(t) = \mathbf{diag}(\theta_1(t) I_{m_1}, \cdots, \theta_r(t) I_{m_r})$. We now assume that the signal $p(t)$ is measured on-line, instead of the parameters $\theta(t)$. The problem of the design of a cancellation controller for an LPV (generalized) system can be defined as the following.

The LPV filtered cancellation control problem Given $\gamma > 0$ and the LPV generalized plant (16), find a filtered cancellation controller $u = K_{cancel}(y, p)$ of the form:

$$\begin{cases} \dot{x}_K(t) = A_K x_k(t) + B_{K_y} y(t) + B_{K_p} p(t) \\ u(t) = C_K x_K(t) + D_{K_{uy}} y(t) + D_{K_{up}} p(t) \\ x_K(0) = 0 \end{cases} \tag{17}$$

such that the closed loop system is asymptotically stable (for null input) with an $\mathcal{L}_2$ gain less than γ.

In the papers [4], [6], convex optimization approaches, involving LMI constraints, were proposed in order to compute an LPV controller of the form (5) in the case when the generalized plant is defined by an LFT representation (16). The LPV controller has also an LFT representation with the same $\Theta(t)$ than in (16). In [13], we reveal that in the case when in addition to $y(t)$, some components of the vector $p(t)$ are measured on-line, the size of $\Theta(t)$ block of the controller can be reduced. If these results are applied to the case when all the components of $p(t)$ are measured then the state space matrices of the "LPV" controller can be chosen independent of $\theta(t)$, that is, the controller has a state-space representation (17). In this case, a similar result can be found in [23]. This result is very nice since, in the general case (that is, no component of $p(t)$ is measured), if a reduced size is enforced for the $\Theta(t)$ block of the controller then a non convex constraint is introduced in the optimization problem.

Theorem 4.2: The LPV filtered cancellation control problem has a solution when $\mathbf{\Theta_t} = [-1,\ 1] \times \cdots \times [-1,\ 1]$ if there exist matrices $P = P^T,\ Q = Q^T \in \mathbb{R}^{n \times n}$, $T \in \mathbb{R}^{k \times k}$ such that:

$$\mathcal{N}^T \left[\begin{array}{cc|c} A^T P + PA & PB_1 & \begin{bmatrix} C_0^T & C_1^T \end{bmatrix} \\ B_1^T P & -\gamma I & \begin{bmatrix} D_{01}^T & D_{11}^T \end{bmatrix} \\ \hline \begin{bmatrix} C_0 \\ C_1 \end{bmatrix} & \begin{bmatrix} D_{01} \\ D_{11} \end{bmatrix} & -\begin{bmatrix} T & 0 \\ 0 & \gamma I \end{bmatrix} \end{array}\right] \mathcal{N} < 0 \tag{18}$$

$$\mathcal{M}^T \left[\begin{array}{c|c} \begin{matrix} AQ + QA^T + \cdots \\ +B_0 T B_0^T + \gamma^{-1} B_1 B_1^T \end{matrix} & \begin{matrix} QC_0^T + D_{00} T B_0^T + \\ \cdots + \gamma^{-1} D_{01} B_1^T \end{matrix} \\ \hline \begin{matrix} C_0 Q + B_0 T D_{00}^T + \\ \cdots + \gamma^{-1} B_1 D_{01}^T \end{matrix} & \begin{matrix} D_{00} T D_{00}^T + \\ +\gamma^{-1} D_{01} D_{01}^T + \\ \cdots - \begin{bmatrix} T & 0 \\ 0 & \gamma I \end{bmatrix} \end{matrix} \end{array}\right] \mathcal{M} < 0 \tag{19}$$

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} > 0 \text{ and } T > 0. \tag{20}$$

with $\mathcal{N} = \left[\begin{array}{c|c} \begin{bmatrix} C_2 & D_{21} \end{bmatrix}^{\perp} & 0 \\ 0 & I_{n_z+k} \end{array}\right]$ and $\mathcal{M} = \begin{bmatrix} B_2^T & D_{02}^T \end{bmatrix}^{\perp}$

Proof: The proof is a direct application of Theorem 3.2 presented in [13]. ∎

For a given $\gamma > 0$, find P, Q, T such that (18), (19) and (20) is a (feasibility) convex optimization problem involving LMI constraints. If this optimization problem has a solution then the state space matrices of the cancellation controller can be computed, see [13] for the details.

V. Application to the nonlinear performance control

A. A filtered cancellation control solution to the incremental gain control problem

The major interest of the cancellation controller is that it can be readily obtained from its time-varying linearizations. The time-varying linearization $D\widetilde{P}_{NL}$ of (12) are defined by:

$$\begin{cases} \dot{\bar{x}}(t) = A\bar{x}(t) + B_1\bar{w}(t) + B_2\bar{u}(t) + \dfrac{\partial \widetilde{f}}{\partial x}(x_0(t))\bar{x}(t) \\ \bar{z}(t) = C_1\bar{x}(t) + D_{11}\bar{w}(t) + D_{12}\bar{u}(t) + \dfrac{\partial \widetilde{g}}{\partial x}(x_0(t))\bar{x}(t) \\ \bar{y}(t) = C_2\bar{x}(t) + D_{21}\bar{w}(t) + D_{22}\bar{u}(t) + \dfrac{\partial \widetilde{h}}{\partial x}(x_0(t))\bar{x}(t) \\ \bar{x}(0) = 0 \end{cases} \tag{21}$$

Let us introduce the vector $\bar{p}(t)$ such as

$$\begin{bmatrix} \dfrac{\partial \widetilde{f}}{\partial x}(x_0(t)) \\ \dfrac{\partial \widetilde{g}}{\partial x}(x_0(t)) \\ \dfrac{\partial \widetilde{h}}{\partial x}(x_0(t)) \end{bmatrix} = \begin{bmatrix} B_0 \\ D_{10} \\ D_{20} \end{bmatrix} \bar{p}(t)$$

The solution is obtained in several steps.

1) Compute an LPV system for the time-varying linearization of the nonlinear generalized plant (21) such that for any $\bar{p}(t)$, there exists a $\theta \in \boldsymbol{\Theta}$, matrices C_0, D_{00}, D_{01} and D_{02} such that $\forall t$ $\bar{p}(t)$ is defined by

$$\Theta(t)\left(I - D_{00}\Theta(t)\right)^{-1}\left(C_0\bar{x}(t) + D_{01}\bar{w}(t) + D_{02}\bar{u}(t)\right).$$

2) Solve the following (feasibility) convex optimization problem involving LMI constraints: find P, Q, T such that (18), (19) and (20).

3) If the problem is feasible, compute the state space matrices of the linearization of the controller (14) using [13], that is: $\bar{u} = D\widetilde{K}_p \begin{bmatrix} y_0 \\ p_0 \end{bmatrix} \left(\begin{bmatrix} \bar{y} \\ \bar{p} \end{bmatrix} \right)$ with

$$\begin{cases} \dot{\bar{x}}_K(t) = A_K\bar{x}_K(t) + B_{K_y}\bar{y}(t) + B_{K_p}\bar{p}(t) \\ \bar{u}(t) = C_K\bar{x}_K(t) + D_{K_y}\bar{y}(t) + D_{K_p}\bar{p}(t) \\ \bar{x}_K(0) = 0 \end{cases}$$

The corresponding nonlinear controller is then given by: $u = \widetilde{K}_p\left(\begin{bmatrix} y \\ p \end{bmatrix}\right)$ with

$$\begin{cases} \dot{x}_K(t) = A_K x_K(t) + B_{K_y} y(t) + B_{K_p} p(t) \\ u(t) = C_K x_K(t) + D_{K_y} y(t) + D_{K_p} p(t) \\ x_K(0) = 0 \end{cases}$$

where $p(t)$ is such that (13).

B. Application to the illustrative control problem

In order to apply the previous method, we first compute an LPV system which embeds the time-varying linearization $DG_{NL}[u_0]$: with A_G defined by (9),

$$\begin{cases} \dot{\bar{x}}(t) = A_G(\theta(t))\bar{x}(t) + \begin{bmatrix} 300 \\ 0 \end{bmatrix} \bar{u}(t) \\ \bar{y}(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \bar{x}(t) \end{cases}, \quad \theta(t) \in [-0.3,\ 4].$$

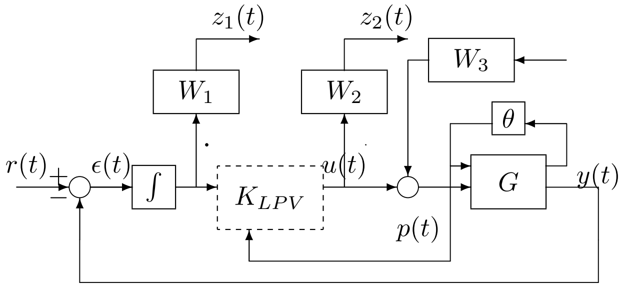

Fig. 9. Augmented plant corresponding to a modified four-block $\mathcal{L}_2$ gain criterion

The time-varying linearization of the nonlinear generalized plant (21) is then obtained from Fig. 9. The resolution of the LMI optimization problem allows to obtain a controller (that is, the matrices A_K, B_K, C_K and D_K) for $\gamma \approx 1$. The simulation is performed on the nonlinear plant using the nonlinear controller of the form (15), see Fig. 10. Note that the tracking and disturbance rejection specifications are satisfied. Since for any constant input, the internal signal of an incremental $\mathcal{L}_2$ gain stable system goes to a constant [10] when t goes to ∞, necessarily the input $\epsilon(t)$ of the integrator Fig. 9 goes to zero when t goes to ∞, in order to ensure that the integrator output goes to a constant. The static error is then well-defined and equal to 0.

The damping can be improved by computing a controller which uses y as an extra input, see Fig 11 and Fig 12.

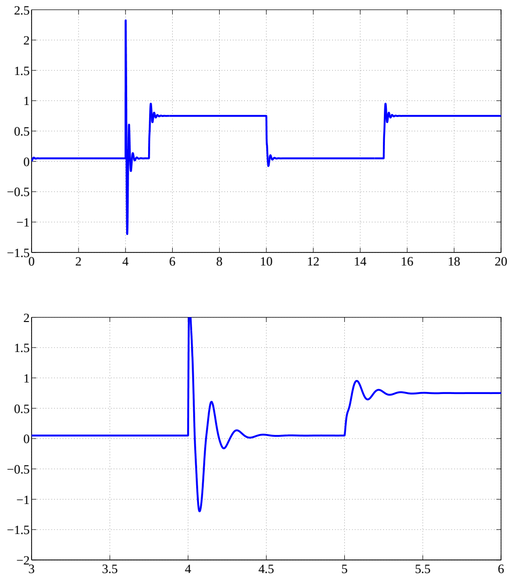

Fig. 10. Simulation of the non linear closed loop system with incremental controller 2, ouput $y(t)$ (top), zoom on the output (bottom)

VI. Conclusion

In this paper, we exhibit a counterexample which reveals that a control approach whose purpose is to ensure for a

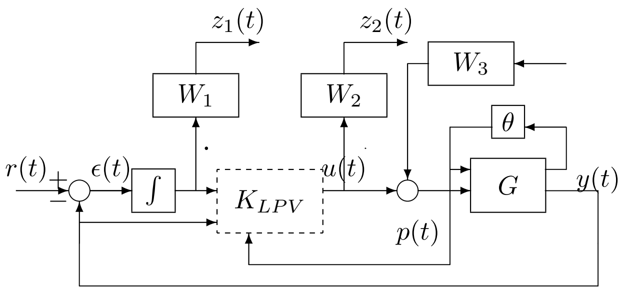

Fig. 11. Augmented plant corresponding to a modified four-block $\mathcal{L}_2$ gain criterion

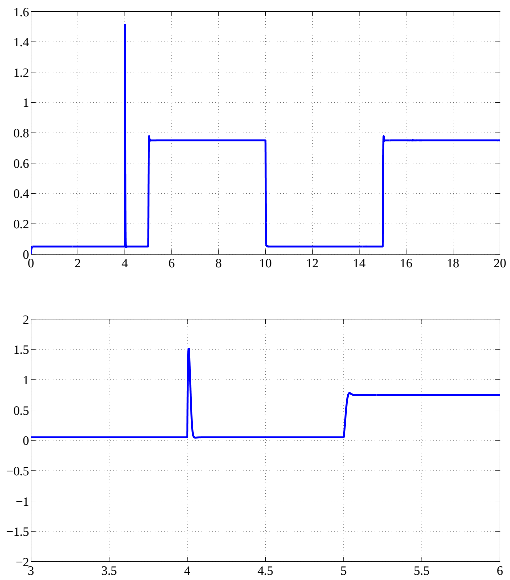

Fig. 12. Simulation of the non linear closed loop system with incremental controller, ouput $y(t)$ (top), zoom on the output $y(t)$ (bottom)

nonlinear closed loop system its $\mathcal{L}_2$ gain stability with the $\mathcal{L}_2$ gain less than a given γ fails to ensure usual tracking and rejection specifications and that the quasi LPV approach does not ensure such specifications, even if they are largely applied for such specifications. We emphasize that in order to ensure tracking and rejection specifications when applying LPV control, it is necessary to associate to the LPV approach a nonlinear framework in order to test such specifications on a nonlinear system. This framework is the incremental gain approach introduced in [12]. We propose a new LPV approach in order to compute nonlinear controllers with a priori guarantees on the tracking and rejection specifications for a class of nonlinear systems. We point out that for the generalization of this approach, it is necessary to solve the "integrability problem", that is, the computation of a nonlinear controller from its time-varying linearisations. To our best knowledge, such a problem is still an open problem.

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