Little Magnetic Book. Geometry and Bound States of the Magnetic Schrodinger Operator

Nicolas Raymond

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Geometry and Bound States of the Magnetic Schrödinger Operator

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τὸ αὐτὸ νοεῖν ἔστιν τε καὶ εἶναι

Παρμενίδης

Contents

Prolégomènes

Toute oeuvre qui se destine aux hommes ne devrait jamais être écrite que sous le nom de Οὖτίς. C'est le nom par lequel Όδυσσεύς (Ulysse) s'est présenté au cyclope Polyphème dont il venait de crever l'oeil. Rares sont les moments de l'Odyssée où Όδυσσεύς communique son véritable nom ; il est le voyageur anonyme par excellence et ne sera reconnu qu'à la fin de son périple par ceux qui ont fidèlement préservé sa mémoire. Mais que vient faire un tel commentaire au début d'un livre de mathématiques ? Toutes les activités de pensée nous amènent, un jour ou l'autre, à nous demander si nous sommes bien les propriétaires de nos pensées. Peut-on seulement les enfermer dans un livre et y associer notre nom ? N'en va-t-il pas pour elles comme il en va de l'amour ? Aussitôt possédées, elles perdent leur attrait, aussitôt enfermées elles perdent vie. Plus on touche à l'universel, moins la possession n'a de sens. Les Idées n'appartiennent à personne et la vérité est ingrate : elle n'a que faire de ceux qui la disent. Ô lecteur ! Fuis la renommée ! Car, aussitôt une reconnaissance obtenue, tu craindras de la perdre et, tel Don Quichotte, tu t'agiteras à nouveau pour te placer dans une vaine lumière. C'est un plaisir tellement plus délicat de laisser aller et venir les Idées, de constater que les plus belles d'entre elles trouvent leur profondeur dans l'éphémère et que, à peine saisies, elles ne sont déjà plus tout à fait ce qu'on croit. Le doute est essentiel à toute activité de recherche. Il s'agit non seulement de vérifier nos affirmations, mais aussi de s'étonner devant ce qui se présente. Sans le doute, nous nous contenterions d'arguments d'autorité et nous passerions devant les problèmes les plus profonds avec indifférence. On écrit rarement toutes les interrogations qui ont jalonné la preuve d'un théorème. Une fois une preuve correcte établie, pourquoi se souviendrait-on de nos errements ? Il est si reposant de passer d'une cause à une conséquence, de voir dans le présent l'expression mécanique du passé et de se libérer ainsi du fardeau de la mémoire. Dans la vie morale, personne n'oserait pourtant penser ainsi et cette paresse démonstrative passerait pour une terrible insouciance. Ce Petit Livre Magnétique présente une oeuvre continue et tissée par la mémoire de son auteur au cours de trois années de méditation. Au lieu de se disperser dans la multitude des représentations et des problèmes qui recouvrent la surface du monde de l'analyse, il fait le pari qu'une profonde singularité peut parler au plus grand nombre. Pourquoi courir après les modes, si nous voulons durer ? Pourquoi vouloir changer, puisque la réalité elle-même est changement ? Ô lecteur, prends le temps de juger des articulations et du développement des concepts pour t'en forger une idée vivante ! Si ce livre fait naître le doute et l'étonnement, c'est qu'il aura rempli son oeuvre.

Acknowledgments

This little book was born in September 2012 during a summer school in Tunisia organized by H. Najar. I thank him very much for this exciting invitation! It also contains my lecture notes for a master's degree. I would like to thank my collaborators, colleagues or students for all our magnetic discusssions: V. Bonnaillie-Noël, M. Dauge, N. Dombrowski, V. Duchêne, F. Faure, S. Fournais, B. Helffer, F. Hérau, Y. Kordyukov, D. Krejčiřík, Y. Lafranche, J-P. Miqueu, T. Ourmières, M. Persson, N. Popoff, M. Tušek and S. Vũ Ngọc. This book is the story of our discussions.

Part 1

Ideas

CHAPTER 1

A magnetic story

Γνῶθι σεαυτόν.

1. The realm of $\lambda_1(h)$

1.1. Once upon a time... Let us present two reasons which lead to the analysis of the magnetic Laplacian.

The first motivation arises in the mathematical theory of superconductivity. A model for this theory (see [**160**]) is given by the Ginzburg-Landau functional:

$$\mathcal{G}(\psi, \mathbf{A}) = \int_\Omega |(-i\nabla + \kappa\sigma\mathbf{A})\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \,\mathrm{d}x + \kappa^2 \int_\Omega |\sigma\nabla \times \mathbf{A} - \sigma\mathbf{B}|^2 \,\mathrm{d}x,$$

where $\Omega \subset \mathbb{R}^d$ is the place occupied by the superconductor, ψ is the so-called order parameter ($|\psi|^2$ is the density of Cooper pairs), $\mathbf{A}$ is a magnetic potential and $\mathbf{B}$ the applied magnetic field. The parameter κ is characteristic of the sample (the superconductors of type II are such that $\kappa >> 1$) and σ corresponds to the intensity of the applied magnetic field. Roughly speaking, the question is to determine the nature of the minimizers. Are they normal, that is $(\psi, \mathbf{A}) = (0, \mathbf{F})$ with $\nabla \times \mathbf{F} = \mathbf{B}$ (and $\nabla \cdot \mathbf{F} = 0$), or not? We can mention the important result of Giorgi-Phillips [**76**] which states that, if the applied magnetic field does not vanish, then, for σ large enough, the normal state is the unique minimizer of $\mathcal{G}$ (with the divergence free condition). When analyzing the local minimality of $(0, \mathbf{F})$, we are led to compute the Hessian of $\mathcal{G}$ at $(0, \mathbf{F})$ and to analyze the positivity of:

$$(-i\nabla + \kappa\sigma\mathbf{A})^2 - \kappa^2.$$

For further details, we refer to the book by Fournais and Helffer [**68**] and to the papers by Lu and Pan [**123, 124**]. Therefore the theory of superconductivity leads to investigate the lowest eigenvalue $\lambda_1(h)$ of the Neumann realization of the *magnetic Laplacian*, that is $(-ih\nabla + \mathbf{A})^2$, where $h > 0$ is small (κ is assumed to be large).

The second motivation is to understand at which point there is an analogy between the electric Laplacian $-h^2\Delta + V(x)$ and the magnetic Laplacian $(-ih\nabla + \mathbf{A})^2$. For instance, in the electric case, when V admits a unique and non-degenerate minimum at 0 and satisfies $\liminf_{|x|\to+\infty} V(x) > V(0)$, we know that the n-th eigenvalue $\lambda_n(h)$ exists and satisfies:

$$\lambda_n(h) = V(0) + (2n-1)\gamma h + O(h^2), \tag{1.1.1}$$

where γ is related to the Hessian matrix of V at 0. Therefore a natural question arises:

"Are there similar results to (1.1.1) in pure magnetic cases?"

In order to answer this question this book develops a theory of the *Magnetic Harmonic Approximation*. Concerning the Schrödinger equation in presence of magnetic field the reader may consult [**7**] (see also [**38**]) and the surveys [**134**], [**60**] and [**90**].

Jointly with (1.1.1) it is also well-known that we can perform WKB constructions for the electric Laplacian (see the book of Dimassi and Sjöstrand [**49**, Chapter 3]). Unfortunately such constructions do not seem to be possible *in full generality* for the pure magnetic case (see the course of Helffer [**83**, Section 6] and the paper by Martinez and Sordoni [**130**]) and the naive localization estimates of Agmon are no more optimal (see [**100**], the paper by Erdős [**58**] or the papers by Nakamura [**137, 138**]). For the magnetic situation, such accurate expansions of the eigenvalues (and eigenfunctions) are difficult to obtain. In fact, the more we know about the expansion of the eigenpairs, the more we can estimate the tunnel effect in the spirit of the electric tunnel effect of Helffer and Sjöstrand (see for instance [**98, 99**] and the papers by Simon [**162, 163**]) when there are symmetries. Estimating the magnetic tunnel effect is still a widely open question directly related to the approximation of the eigenfunctions (see [**100**] and [**30**] for electric tunneling in presence of magnetic field and [**14**] in the case with corners). Hopefully the main philosophy living throughout this book will prepare the future investigations on this fascinating subject. In particular we will provide the first examples of magnetic WKB constructions inspired by the recent work [**18**]. Anyway this book proposes a change of perspective in the study of the magnetic Laplacian. In fact, during the past decades, the philosophy behind the spectral analysis was essentially variational. Many papers dealt with the construction of *quasimodes* used as test functions for the quadratic form associated with the magnetic Laplacian. In any case the attention was focused on the functions of the domain more than on the operator itself. In this book we systematically try to inverse the point of view: the main problem is no more to find *appropriate quasimodes* but an *appropriate (and sometimes microlocal) representation of the operator*. By doing this we will partially leave the min-max principle and the variational theory for the spectral theorem and the microlocal and hypoelliptic spirit.

1.2. Definitions. Let Ω be a Lipschitzian domain in $\mathbb{R}^d$. Let us denote $\mathbf{A} = (A_1, \cdots, A_d)$ a smooth vector potential on $\overline{\Omega}$. We consider the 1-form (see [**5**, Chapter 7] for a brief introduction to differential forms):

$$\omega_{\mathbf{A}} = \sum_{k=1}^{d} A_k \, \mathrm{d}x_k.$$

We introduce the exterior derivative of $\omega_{\mathbf{A}}$:

$$\sigma_{\mathbf{B}} = \mathrm{d}\omega_{\mathbf{A}} = \sum_{j<k} B_{j,k} \, \mathrm{d}x_j \wedge \mathrm{d}x_k.$$

In dimension two, the only coefficient is $B_{12} = \partial_{x_1} A_2 - \partial_{x_2} A_1$. In dimension three, the magnetic field is defined as:

$$\mathbf{B} = (B_1, B_2, B_3) = (B_{23}, -B_{13}, B_{12}) = \nabla \times \mathbf{A}.$$

We will discuss in this book the spectral properties of some self-adjoint realizations of the magnetic operator:

$$\mathfrak{L}_{h,\mathbf{A},\Omega} = \sum_{k=1}^{d} (-ih\partial_k + A_k)^2,$$

where $h > 0$ is a parameter (related to the Planck constant). We notice the fundamental property, called gauge invariance:

$$e^{-i\phi}(-i\nabla + \mathbf{A})e^{i\phi} = -i\nabla + \mathbf{A} + \nabla\phi$$

so that:

$$e^{-i\phi}(-i\nabla + \mathbf{A})^2 e^{i\phi} = (-i\nabla + \mathbf{A} + \nabla\phi)^2,$$

where $\phi \in \mathsf{H}^1(\Omega, \mathbb{R})$.

1.3. A fascination for $\lambda_1(h)$.

1.3.1. *Constant magnetic field.* In dimension two the constant magnetic field case is treated when Ω is a disk (with Neumann condition) by Bauman, Phillips and Tang in [**9**] (see [**11**] and [**59**] for the Dirichlet case). In particular, they prove a two terms expansion in the form:

$$\lambda_1(h) = \Theta_0 h - \frac{C_1}{R} h^{3/2} + o(h^{3/2}),$$

where $\Theta_0 \in (0, 1)$ and $C_1 > 0$ are universal constants. This result, which was conjectured in [**10, 48**], is generalized to smooth and bounded domains by Helffer and Morame in [**92**] where it is proved that:

$$\lambda_1(h) = \Theta_0 h - C_1 \kappa_{max} h^{3/2} + o(h^{3/2}), \tag{1.1.2}$$

where κ_{max} is the maximal curvature of the boundary. Let us emphasize that, in these papers, the authors are only concerned by the first terms of the asymptotic expansion of $\lambda_1(h)$. In the case of smooth domains the complete asymptotic expansion of all the eigenvalues is done by Fournais and Helffer in [**67**]. When the boundary is not smooth, we may mention the papers of Jadallah and Pan [**106, 142**]. In the semiclassical regime, we refer to the papers of Bonnaillie-Noël, Dauge and Fournais [**12, 13, 17**] where perturbation theory is used in relation with the estimates of Agmon. For numerical investigations the reader may consider the paper [**14**].

In dimension three the constant magnetic field case (with intensity 1) is treated by Helffer and Morame in [**94**] under generic assumptions on the (smooth) boundary of Ω:

$$\lambda_1(h) = \Theta_0 h + \hat{\gamma}_0 h^{4/3} + o(h^{4/3}),$$

where the constant $\hat{\gamma}_0$ is related to the magnetic curvature of a curve in the boundary along which the magnetic field is tangent to the boundary. The case of the ball is analyzed in details by Fournais and Persson in [**69**].

1.3.2. *Variable magnetic field.* The case when the magnetic field is not constant can be motivated by anisotropic superconductors (see for instance [**33, 3**]) or the liquid crystal theory (see [**95, 96, 151, 149**]). For the case with a non vanishing variable magnetic field, we refer to [**123, 148**] for the first terms of the lowest eigenvalue. In particular the paper [**148**] provides (under a generic condition) an asymptotic expansion with two terms in the form:

$$\lambda_1(h) = \Theta_0 b' h + C_1^{\mathsf{2D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega) h^{3/2} + o(h^{3/2}),$$

where $C_1^{\mathsf{2D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega)$ depends on the geometry of the boundary and on the magnetic field at $\mathbf{x}_0$ and where $b' = \min_{\partial\Omega} B = B(\mathbf{x}_0)$. When the magnetic field vanishes, the first analysis of the lowest eigenvalue is due to Montgomery in [**135**] followed by Helffer and Morame in [**91**] (see also [**143, 85, 87**]).

In dimension three (with Neumann condition on a smooth boundary), the first term of $\lambda_1(h)$ is given by Lu and Pan in [**124**]. The next terms in the expansion are investigated in [**150**] where we can find in particular an upper bound in the form

$$\lambda_1(h) \leq \|\mathbf{B}(\mathbf{x}_0)\| \mathfrak{s}(\theta(\mathbf{x}_0)) h + C_1^{\mathsf{3D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega) h^{3/2} + C_2^{\mathsf{3D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega) h^2 + C h^{5/2},$$

where $\mathfrak{s}$ is a spectral invariant defined in the next section, $\theta(\mathbf{x}_0)$ is the angle of $\mathbf{B}(\mathbf{x}_0)$ with the boundary at $\mathbf{x}_0$ and the constants $C_j^{\mathsf{3D}}(\mathbf{x}_0, \mathbf{B}, \partial\Omega)$ are related to the geometry and the magnetic field at $\mathbf{x}_0 \in \partial\Omega$. Let us finally mention the recent paper by Bonnaillie-Noël-Dauge-Popoff [**15**] which establishes a one term asymptotics in the case of Neumann boundaries with corners.

1.4. Some model operators. It turns out that the results recalled in Section 1.3 are related to many model operators. Let us introduce some of them.

1.4.1. *De Gennes operator.* The analysis of the magnetic Laplacian with Neumann condition on $\mathbb{R}^2_+$ makes the so-called de Gennes operator to appear. We refer to [**44**] where this model is studied in details (see also [**68**]). For $\zeta \in \mathbb{R}$, we consider the Neumann realization on $\mathsf{L}^2(\mathbb{R}_+)$ of

$$\mathfrak{L}_\zeta^{[0]} = D_t^2 + (\zeta - t)^2. \tag{1.1.3}$$

We denote by $\nu_1^{[0]}(\zeta)$ the lowest eigenvalue of $\mathfrak{L}^{[0]}(\zeta)$. It is possible to prove that the function $\zeta \mapsto \nu_1^{[0]}(\zeta)$ admits a unique and non-degenerate minimum at a point $\zeta_0^{[0]} > 0$, shortly denoted by ζ_0, and that we have

$$\Theta_0 := \min_{\xi \in \mathbb{R}} \nu_1^{[0]}(\zeta) \in (0, 1). \tag{1.1.4}$$

1.4.2. *Montgomery operator.* Let us now introduce another important model. This one was introduced by Montgomery in [**135**] to study the case of vanishing magnetic fields in dimension two (see also [**143**] and [**94**, Section 2.4]). This model was revisited by Helffer in [**84**], generalized by Helffer and Persson in [**97**] and Fournais and Persson in [**70**]. The Montgomery operator with parameter $\zeta \in \mathbb{R}$ is the self-adjoint realization on $\mathbb{R}$ of:

$$\mathfrak{L}_{\zeta}^{[1]} = D_t^2 + \left(\zeta - \frac{t^2}{2}\right)^2. \tag{1.1.5}$$

1.4.3. *Popoff operator.* The investigation of the magnetic Laplacian on dihedral domains (see [**145**]) leads to the analysis of the Neumann realization on $\mathsf{L}^2(\mathcal{S}_\alpha, \mathrm{d}t\,\mathrm{d}z)$ of:

$$\mathfrak{L}_{\alpha,\zeta}^{\mathsf{e}} = D_t^2 + D_z^2 + (t-\zeta)^2, \tag{1.1.6}$$

where $\mathcal{S}_\alpha$ is the sector with angle α,

$$\mathcal{S}_\alpha = \left\{(t,z) \in \mathbb{R}^2 : |z| < t\tan\left(\frac{\alpha}{2}\right)\right\}.$$

1.4.4. *Lu-Pan operator.* Let us present a last model operator appearing in dimension three in the case of smooth Neumann boundary (see [**124, 93, 16**]). We denote by (s,t) the coordinates in $\mathbb{R}^2$ and by $\mathbb{R}^2_+$ the half-plane:

$$\mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2,\ t > 0\}.$$

We introduce the self-adjoint Neumann realization on the half-plane $\mathbb{R}^2_+$ of the Schrödinger operator $\mathfrak{L}_\theta^{\mathsf{LP}}$ with potential V_θ:

$$\mathfrak{L}_\theta^{\mathsf{LP}} = -\Delta + V_\theta = D_s^2 + D_t^2 + V_\theta, \tag{1.1.7}$$

where V_θ is defined for any $\theta \in (0, \frac{\pi}{2})$ by

$$V_\theta : (s,t) \in \mathbb{R}^2_+ \longmapsto (t\cos\theta - s\sin\theta)^2.$$

We can notice that V_θ reaches its minimum 0 all along the line $t\cos\theta = s\sin\theta$, which makes the angle θ with $\partial\mathbb{R}^2_+$. We denote by $\mathfrak{s}_1(\theta)$ or simply $\mathfrak{s}(\theta)$ the infimum of the spectrum of $\mathfrak{L}_\theta^{\mathsf{LP}}$. In [**68**] (and [**93, 124**]), it is proved that $\mathfrak{s}$ is analytic and strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

2. A connection with waveguides

2.1. Existence of a bound state of $\mathfrak{L}_\theta^{\mathsf{LP}}$. Among other things one can prove (cf. [**93, 124**]):

Lemma 1.1. *For all $\theta \in \left(0, \frac{\pi}{2}\right)$ there exists an eigenvalue of $\mathfrak{L}_\theta^{\mathsf{LP}}$ below the essential spectrum which equals* $[1, +\infty)$.

A classical result combining an estimate of Agmon (cf. [**1**]) and a theorem due to Persson (cf. [**144**]) implies that the corresponding eigenfunctions are localized near $(0,0)$.

This result is slightly surprising since the existence of the discrete spectrum is related to the association between the Neumann condition and the partial confinement of V_θ. After translation and rescaling, we are led to a new operator:

$$hD_s^2 + D_t^2 + (t - \zeta_0 - sh^{1/2})^2 - \Theta_0,$$

where $h = \tan\theta$. Then one can reduce the (semiclassical) analysis to the so-called *Born-Oppenheimer* approximation (see for instance [**126**]):

$$hD_s^2 + \nu_1^{[0]}(\zeta_0 + sh^{1/2}) - \Theta_0.$$

This last operator is very easy to analyze with the classical theory of the harmonic approximation and we get (see [**16**]):

Theorem 1.2. *The lowest eigenvalues of $\mathfrak{L}_\theta^{\mathsf{LP}}$ admit the following expansions:*

(1.2.1)
$$\mathfrak{s}_n(\theta) \underset{\theta\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n}\theta^j$$

with $\gamma_{0,n} = \Theta_0$ et $\gamma_{1,n} = (2n-1)\sqrt{\frac{(\nu_1^{[0]})''(\zeta_0)}{2}}$.

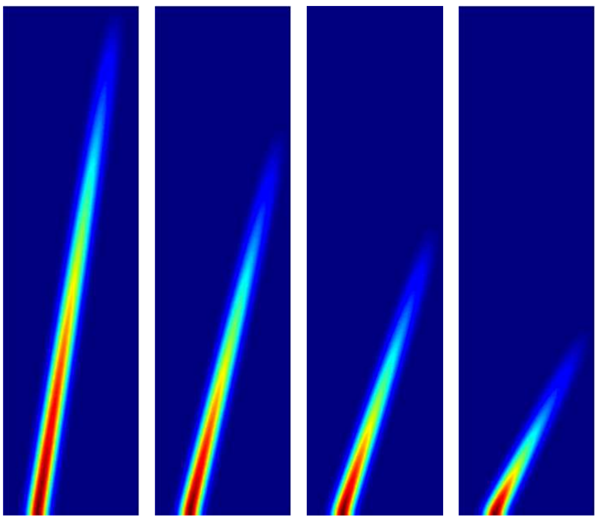


FIGURE 1. First eigenfunction of $\mathfrak{L}_\theta^{\mathsf{LP}}$ for $\theta = \vartheta\pi/2$ with $\vartheta = 0.9$, 0.85, 0.8 et 0.7.

2.2. A result of Duclos and Exner. Figure 1 can make us think to a *broken waveguide*. Indeed, if one uses the Neumann condition to symmetrize $\mathfrak{L}_\theta^{\mathsf{LP}}$ and if one replaces the confinement property of V_θ by a Dirichlet condition, we are led to the situation described in Figure 2. This heuristic comparison reminds the famous paper [**53**] where Duclos and Exner introduce a definition of standard (and smooth) waveguides and perform a spectral analysis. For example, in dimension two (see Figure 3), a waveguide of width ε is determined by a smooth curve $s \mapsto c(s) \in \mathbb{R}^2$ as the subset of $\mathbb{R}^2$ given by:

$$\{c(s) + t\boldsymbol{n}(s), \quad (s,t) \in \mathbb{R} \times (-\varepsilon, \varepsilon)\},$$

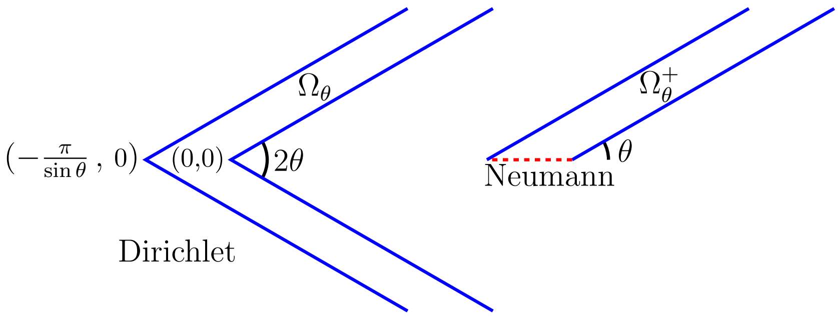


FIGURE 2. Waveguide with corner Ω_θ and half-waceguide Ω_θ^+.

where $\boldsymbol{n}(s)$ is the normal to the curve $c(\mathbb{R})$ at the point $c(s)$.

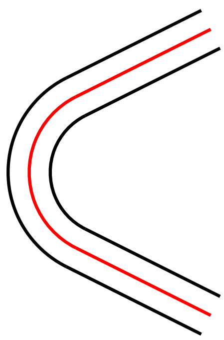


FIGURE 3. Waveguide

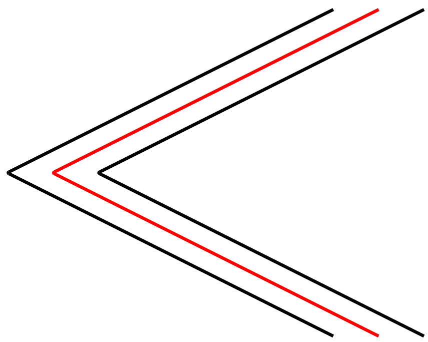


FIGURE 4. Broken guide

Assuming that the waveguide is straight at infinity but not everywhere, Duclos and Exner prove that there is always an eigenvalue below the essential spectrum (in the case of a circular cross section in dimensions two and three). Let us notice that the essential spectrum is $[\lambda, +\infty)$ where λ is the lowest eigenvalue of the Dirichlet Laplacian on the cross section. The proof of the existence of discrete spectrum is elementary and relies on the min-max principle. Letting $\psi \in H_0^1(\Omega)$:

$$q(\psi) = \int_\Omega |\nabla\psi|^2 \,\mathrm{d}\mathbf{x},$$

it is enough to find ψ_0 such that $q(\psi_0) < \lambda\|\psi_0\|_{\mathsf{L}^2(\Omega)}$. Such a function can be constructed by considering a perturbed Weyl sequence associated with λ.

2.3. Waveguides and magnetic fields. Bending a waveguide induces discrete spectrum below the essential spectrum, but what about twisting a waveguide? This question arises for instance in the papers [**112, 116, 57**] where it is proved that twisting a waveguide plays against the existence of the discrete spectrum. In the case without curvature, the quadratic form is defined for $\psi \in \mathsf{H}_0^1(\mathbb{R}\times\omega)$ by:

$$q(\psi) = \|\partial_1\psi - \rho(s)(t_3\partial_2 - t_2\partial_3)\psi\|^2 + \|\partial_2\psi\|^2 + \|\partial_3\psi\|^2,$$

where $s \mapsto \rho(s)$ represents the effect of twisting the cross section ω and (t_2, t_3) are coordinates in ω. From a heuristic point of view, the twisting perturbation seems to act "as" a magnetic field. This leads to the natural question:

"Is the spectral effect of a torsion the same as the effect of a magnetic field?"

If the geometry of a waveguide can formally generate a magnetic field, we can conversely wonder if a magnetic field can generate a waveguide. This remark partially appears in [**50**] where the discontinuity of a magnetic field along a line plays the role of a waveguide. More generally it appears that, when the magnetic field cancels along a curve, this curve becomes an effective waveguide.

3. Organization of the book

3.1. Spectral analysis of model operators and spectral reductions. Chapter 2 deals with model operators. This notion of model operators is fundamental in the theory of the magnetic Laplacian. We have already introduced some important and historical examples. There are essentially two natural ways to meet reductions to model operators. The first approach can be done thanks to a (space) partition of unity which reduces the spectral analysis to the one of localized and simplified models (we straighten the geometry and freeze the magnetic field). The second approach, which involves an analysis in the phase space, is to identify the possible different scales of the problem, that is the fast and slow variables. This often involves an investigation in the microlocal spirit: we shall analyze the properties of symbols and deduce a microlocal reduction to a spectral problem in lower dimension. In Chapter 2 we provide explicit examples of models and provide their spectral analysis. In Chapter 2, Section 1 we introduce a model which is fundamental to describe the effect of conical singularities of the boundary on the magnetic eigenvalues. This is an example which is provided by the first kind of approach (freeze the geometry and the magnetic field). It will turn out that part of the spectral analysis of this model can be done in the spirit of the second approach in the small aperture limit (different scales and dimensional reduction). In Chapter 2, Section 2 we present a model related to vanishing magnetic fields in dimension two. Due to an inhomogeneity of the magnetic operator, this other model leads to a microlocal reduction and therefore to the investigation of an effective symbol. In fact, the example of Section 2 can lead to a more general framework. In Chapter 2, Section 3 we provide a general and elementary theory of the "magnetic Born-Oppenheimer approximation" which is a systematic semiclassical reduction to model operators (under generic assumptions on some effective symbols). We also provide the first known examples of pure magnetic WKB constructions.

The model operators are studied in Chapters 7 and 8 (see also basic arguments and examples in Chapters 5 and 6) and the Born-Oppenheimer approximation is discussed in Chapters 9 and 10, whereas elementary WKB constructions are analyzed in Chapter 11.

3.2. Normal forms philosophy and the magnetic semi-excited states. As we have seen there is a non trivial connection between the discrete spectrum, the possible magnetic field and the possible boundary. In fact *normal form* procedures are often deeply rooted in the different proofs, not only in the semiclassical framework. We present in Chapter 3 the results of four studies [**51**], [**152**], [**147**], [**154**] which are respectively detailed in Chapters 12, 13, 14, 15. These studies are concerned by the semiclassical asymptotics of the magnetic eigenvalues and eigenfunctions. Nevertheless, the philosophy which is developed there may apply to more general situations.

3.2.1. *Towards the magnetic semi-excited states.* We now describe the philosophy of the proofs of asymptotic expansions for the magnetic Laplacian with respect to a parameter α (which tends to zero and which might be for example the semiclassical parameter). Let us distinguish between the different conceptual levels of the analysis. Our analysis uses the standard construction of quasimodes, localization techniques ("IMS" formula) and *a priori* estimates of Agmon type satisfied by the eigenfunctions. These "standard" tools, which are used in most of the papers dealing with $\lambda_1(\alpha)$, are not enough to investigate $\lambda_n(\alpha)$ due to the spectral splitting arising sometimes in the subprincipal terms. In fact such a fine behavior is the sign of a microlocal effect. In order to investigate this effect, we use normal form procedures *in the spirit of the Egorov theorem.* It turns out that this normal form strategy also strongly simplifies the construction of quasimodes. Once the behavior of the eigenfunctions in the phase space is established, we use the Feshbach-Grushin approach to reduce our operator to an electric Laplacian. Let us comment more in details the whole strategy.

The first step to analyze such problems is to perform an accurate construction of quasimodes and to apply the spectral theorem. In other words we look for pairs (λ, ψ) such that we have $\|(\mathcal{L}_\alpha - \lambda)\psi\| \leq \varepsilon\|\psi\|$. Such pairs are constructed through an homogenization procedure involving different scales with respect to the different variables. In particular the construction uses a formal power series expansion of the operator and an Ansatz in the same form for (λ, ψ). The main difficulty in order to succeed is to choose the appropriate scalings.

The second step aims at giving *a priori* estimates satisfied by the eigenfunctions.These are localization estimates *à la Agmon* (see [**1**]). To prove them one generally needs to have *a priori* estimates for the eigenvalues which can be obtained with a partition of unity and local comparisons with model operators. Then such *a priori* estimates, which are in general not optimal, involve an improvement in the asymptotic expansion of the eigenvalues. If we are just interested in the first terms of $\lambda_1(\alpha)$, these classical tools are enough.

In fact, the major difference with the electric Laplacian arises precisely in the analysis of the spectral splitting between the lowest eigenvalues. Let us describe what is done in [**67**] (dimension two, constant magnetic field, $\alpha = h$) and in [**153**] (non constant magnetic field). In [**67, 153**] quasimodes are constructed and the usual localization estimates are proved. Then the behavior with respect to a phase variable needs to be determined to allow a dimensional reduction. Let us underline here that this phenomenon of phase localization is characteristic of the magnetic Laplacian and is intimately related to the structure of the low lying spectrum. In [**67**] Fournais and Helffer are led to use the pseudo-differential calculus and the Grushin formalism. In [**153**] the approach is structurally not the same. In [**153**], in the spirit of the Egorov theorem (see [**55, 158, 128**]), we use successive canonical transforms of the symbol of the operator corresponding to unitary transforms (change of gauge, change of variable, Fourier transform) and we reduce the operator, modulo remainders which are controlled thanks to the *a priori* estimates, to an electric Laplacian being in the Born-Oppenheimer form (see [**37, 126**] and more recently [**16**]). This reduction enlightens the crucial idea that the inhomogeneity of the magnetic operator is responsible for its spectral structure.

3.2.2. *... which leads to the Birkhoff procedure.* As we suggested above, our magnetic normal forms are close to the Birkhoff procedure and it is rather surprising that it has never been implemented to enlighten the effect of magnetic fields on the low lying eigenvalues of the magnetic Laplacian. A reason might be that, compared to the case of a Schrödinger operator with an electric potential, the pure magnetic case presents a specific feature: the symbol "itself" is not enough to generate a localization of the eigenfunctions. This difficulty can be seen in the recent papers by Helffer and Kordyukov [**86**] (dimension two) and [**88**] (dimension three) which treat cases without boundary. In dimension three they provide accurate constructions of quasimodes, but do not establish the semiclassical asymptotic expansions of the eigenvalues which is still an open problem. In dimension two, they prove that if the magnetic field has a unique and non-degenerate minimum, the j-th eigenvalue admits an expansion in powers of $h^{1/2}$ of the form:

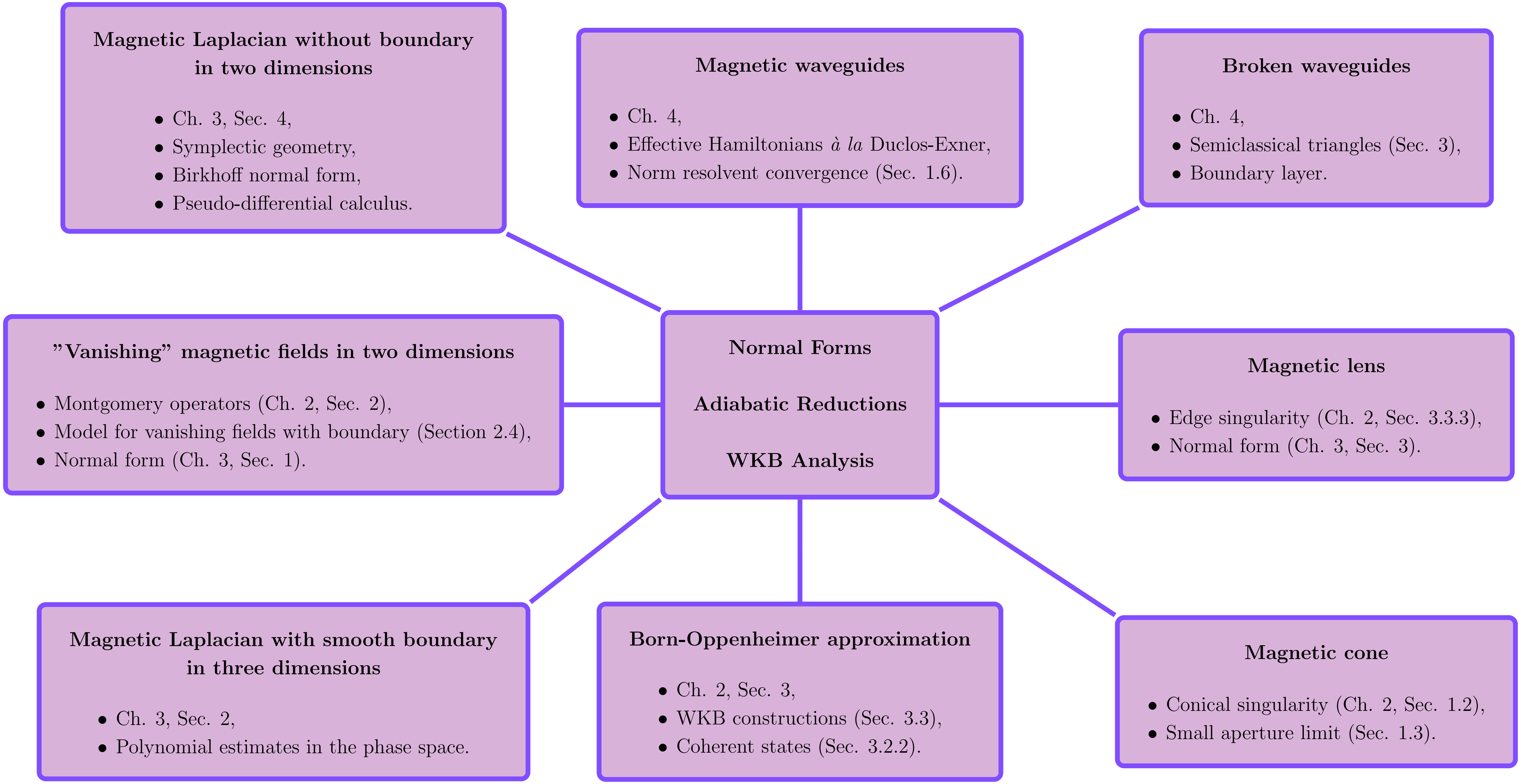
$$\lambda_j(h) \sim h \min_{q\in\mathbb{R}^2} B(q) + h^2(c_1(2j-1)+c_0) + O(h^{5/2}),$$

where c_0 and c_1 are constants depending on the magnetic field. In Chapter 15, we extend their result by obtaining a complete asymptotic expansion which actually applies to more general magnetic wells and allows to describe larger eigenvalues.

3.3. The spectrum of waveguides. In Chapter 4 we present some results occurring in the theory of waveguides. In particular we consider the question:

"What is the spectral influence of a magnetic field on a waveguide ?"

We answer this question in Chapter 16. Then, when there is no magnetic field, we would also like to analyze the effect of a corner on the spectrum and present a non smooth version of the result of Duclos and Exner (see Chapter 18). For that purpose we also present some results concerning the *semiclassical triangles* in Chapter 17.


Magnetic Laplacian without boundary in two dimensions
• Ch. 3, Sec. 4,
• Symplectic geometry,
• Birkhoff normal form,
• Pseudo-differential calculus.
Magnetic waveguides
• Ch. 4,
• Effective Hamiltonians *à la* Duclos-Exner,
• Norm resolvent convergence (Sec. 1.6).
Broken waveguides
• Ch. 4,
• Semiclassical triangles (Sec. 3),
• Boundary layer.
"Vanishing" magnetic fields in two dimensions
• Montgomery operators (Ch. 2, Sec. 2),
• Model for vanishing fields with boundary (Section 2.4),
• Normal form (Ch. 3, Sec. 1).
Normal Forms
Adiabatic Reductions
WKB Analysis
Magnetic lens
• Edge singularity (Ch. 2, Sec. 3.3.3),
• Normal form (Ch. 3, Sec. 3).
Magnetic Laplacian with smooth boundary in three dimensions
• Ch. 3, Sec. 2,
• Polynomial estimates in the phase space.
Born-Oppenheimer approximation
• Ch. 2, Sec. 3,
• WKB constructions (Sec. 3.3),
• Coherent states (Sec. 3.2.2).
Magnetic cone
• Conical singularity (Ch. 2, Sec. 1.2),
• Small aperture limit (Sec. 1.3).

CHAPTER 2

Models and spectral reductions

The soul unfolds itself, like a lotus of countless petals.
The Prophet, Self-Knowledge, Khalil Gibran

In this chapter we introduce two model operators (depending on parameters). The first one is the Neumann Laplacian on a circular cone of aperture α with a constant magnetic field. This model is quite important in the study of problems with non smooth boundaries in dimension three. The second one appears in dimension two when studying vanishing magnetic fields in the case when the cancellation line of the field intersects the boundary. These models will already give a flavor of the techniques which travel through this book.

1. The power of the peaks

We are interested in the low-lying eigenvalues of the magnetic Neumann Laplacian with a constant magnetic field applied to a " peak ", i.e. a right circular cone $\mathcal{C}_\alpha$. The right circular cone $\mathcal{C}_\alpha$ of angular opening $\alpha \in (0,\pi)$ (see Figure 1) is defined in the Cartesian coordinates (x,y,z) by

$$\mathcal{C}_\alpha = \{(x,y,z) \in \mathbb{R}^3,\ z>0,\ x^2+y^2 < z^2\tan^2\tfrac{\alpha}{2}\}.$$

Let $\mathbf{B}$ be the constant magnetic field

$$\mathbf{B}(x,y,z) = (0, \sin\beta, \cos\beta)^\mathsf{T},$$

where $\beta \in \left[0, \frac{\pi}{2}\right]$. We choose the following magnetic potential $\mathbf{A}$:

$$\mathbf{A}(x,y,z) = \frac{1}{2}\mathbf{B}\times\mathbf{x} = \frac{1}{2}(z\sin\beta - y\cos\beta, x\cos\beta, -x\sin\beta)^\mathsf{T}.$$

We consider $\mathfrak{L}_{\alpha,\beta}$ the Friedrichs extension associated with the quadratic form

$$\mathcal{Q}_\mathbf{A}(\psi) = \|(-i\nabla+\mathbf{A})\psi\|^2_{\mathsf{L}^2(\mathcal{C}_\alpha)},$$

defined for $\psi \in \mathsf{H}^1_\mathbf{A}(\mathcal{C}_\alpha)$ with

$$\mathsf{H}^1_\mathbf{A}(\mathcal{C}_\alpha) = \{u \in \mathsf{L}^2(\mathcal{C}_\alpha), (-i\nabla+\mathbf{A})u \in \mathsf{L}^2(\mathcal{C}_\alpha)\}.$$

The operator $\mathfrak{L}_\alpha$ is $(-i\nabla+\mathbf{A})^2$ with domain:

$$\mathsf{H}^2_\mathbf{A}(\mathcal{C}_\alpha) = \{u \in \mathsf{H}^1_\mathbf{A}(\mathcal{C}_\alpha), (-i\nabla+\mathbf{A})^2 u \in \mathsf{L}^2(\mathcal{C}_\alpha), (-i\nabla+\mathbf{A})u\cdot\nu = 0 \text{ on } \partial\mathcal{C}_\alpha\}.$$

We define the n-th eigenvalue $\lambda_n(\alpha,\beta)$ of $\mathfrak{L}_{\alpha,\beta}$ as the n-th Rayleigh quotient (see Chapter 5). Let $\psi_n(\alpha,\beta)$ be a normalized associated eigenvector (if it exists).

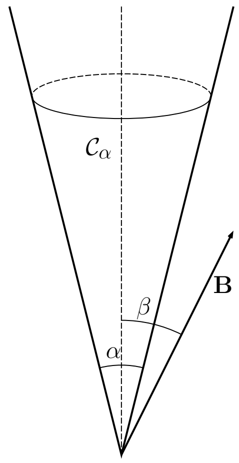


FIGURE 1. Geometric setting.

1.1. Why studying magnetic cones? One of the most interesting results of the last fifteen years is provided by Helffer and Morame in [**92**] where they prove that the magnetic eigenfunctions, in 2D, concentrates near the points of the boundary where the (algebraic) curvature is maximal, see (1.1.2). This property aroused interest in domains with corners, which somehow correspond to points of the boundary where the curvature becomes infinite (see [**106, 142**] for the quarter plane and [**12, 13**] for more general domains). Denoting by $\mathcal{S}_\alpha$ the sector in $\mathbb{R}^2$ with angle α and considering the magnetic Neumann Laplacian with constant magnetic field of intensity 1, it is proved in [**12**] that, as soon as α is small enough, a bound state exists. Its energy is denoted by $\mu(\alpha)$. An asymptotic expansion at any order is even provided (see [**12**, Theorem 1.1]):

$$\mu(\alpha) \sim \alpha \sum_{j\geq 0} m_j \alpha^{2j}, \qquad \text{with} \quad m_0 = \frac{1}{\sqrt{3}}. \tag{2.1.1}$$

In particular, this proves that $\mu(\alpha)$ becomes smaller than the lowest eigenvalue of the magnetic Neumann Laplacian in the half-plane with constant magnetic field (of intensity 1), that is:

$$\mu(\alpha) < \Theta_0, \qquad \alpha \in (0, \alpha_0),$$

where Θ_0 is defined in (1.1.4).This motivates the study of dihedral domains (see [**145, 146**]). Another possibility of investigation, in dimension three, is the case of a conical singularity of the boundary. We would especially like to answer the following questions: Can we go below $\mu(\alpha)$ and can we describe the structure of the spectrum when the aperture of the cone goes to zero?

1.2. The magnetic Laplacian in spherical coordinates. Since the spherical coordinates are naturally adapted to the geometry, we consider the change of variable:

$$\Phi(\tau,\theta,\varphi) := (x,y,z) = \alpha^{-1/2}(\tau\cos\theta\sin\alpha\varphi,\ \tau\sin\theta\sin\alpha\varphi,\ \tau\cos\alpha\varphi).$$

This change of coordinates is nothing but a first normal form. We denote by $\mathcal{P}$ the semi-infinite rectangular parallelepiped

$$\mathcal{P} := \{(\tau,\theta,\varphi)\in\mathbb{R}^3,\ \tau>0,\ \theta\in[0,2\pi),\ \varphi\in(0,\tfrac{1}{2})\}.$$

Let $\psi\in\mathsf{H}^1_{\mathbf{A}}(\mathcal{C}_\alpha)$. We write $\psi(\Phi(\tau,\theta,\varphi)) = \alpha^{1/4}\tilde{\psi}(\tau,\theta,\varphi)$ for any $(\tau,\theta,\varphi)\in\mathcal{P}$ in these new coordinates and we have

$$\|\psi\|^2_{\mathsf{L}^2(\mathcal{C}_\alpha)} = \int_{\mathcal{P}} |\tilde{\psi}(\tau,\theta,\varphi)|^2\,\tau^2\sin\alpha\varphi\,\mathrm{d}\tau\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

and:

$$\mathfrak{Q}_{\mathbf{A}}(\psi) = \alpha\mathcal{Q}_{\alpha,\beta}(\tilde{\psi}),$$

where the quadratic form $\mathcal{Q}_{\alpha,\beta}$ is defined on the transformed form domain $\mathsf{H}^1_{\tilde{\mathbf{A}}}(\mathcal{P})$ by

$$\mathcal{Q}_{\alpha,\beta}(\psi) := \int_{\mathcal{P}} \left(|P_1\psi|^2 + |P_2\psi|^2 + |P_3\psi|^2\right)\,\mathrm{d}\tilde{\mu}, \tag{2.1.2}$$

with the measure

$$\mathrm{d}\tilde{\mu} = \tau^2\sin\alpha\varphi\,\mathrm{d}t\,\mathrm{d}\theta\,\mathrm{d}\varphi,$$

and:

$$\mathsf{H}^1_{\tilde{\mathbf{A}}}(\mathcal{P}) = \{\psi\in\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu}), (-i\nabla+\tilde{\mathbf{A}})\psi\in\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})\}.$$

We also have:

$$\begin{aligned}
&P_1 = D_\tau - \tau\varphi\cos\theta\sin\beta)\tau^2(D_\tau - \tau\varphi\cos\theta\sin\beta,\\
&P_2 = (\tau\sin(\alpha\varphi))^{-1}\left(D_\theta + \frac{\tau^2}{2\alpha}\sin^2(\alpha\varphi)\cos\beta + \frac{\tau^2\varphi}{2}\left(1-\frac{\sin(2\alpha\varphi)}{2\alpha\varphi}\right)\sin\beta\sin\theta\right),\\
&P_3 = (\tau\sin(\alpha\varphi))^{-1}D_\varphi.
\end{aligned}$$

We consider $\mathcal{L}_{\alpha,\beta}$ the Friedrichs extension associated with the quadratic form $\mathcal{Q}_{\alpha,\beta}$:

$$\begin{aligned}
\mathcal{L}_{\alpha,\beta} \;=\; & \tau^{-2}(D_\tau - \tau\varphi\cos\theta\sin\beta)\tau^2(D_\tau - \tau\varphi\cos\theta\sin\beta)\\
&+\frac{1}{\tau^2\sin^2(\alpha\varphi)}\left(D_\theta + \frac{\tau^2}{2\alpha}\sin^2(\alpha\varphi)\cos\beta + \frac{\tau^2\varphi}{2}\left(1-\frac{\sin(2\alpha\varphi)}{2\alpha\varphi}\right)\sin\beta\sin\theta\right)^2\\
&+\frac{1}{\alpha^2\tau^2\sin(\alpha\varphi)}D_\varphi\sin(\alpha\varphi)D_\varphi.
\end{aligned}$$

We define $\tilde{\lambda}_n(\alpha,\beta)$ the n-th Rayleigh quotient of $\mathcal{L}_{\alpha,\beta}$.

1.3. Spectrum of the magnetic cone in the small angle limit.

1.3.1. *Eigenvalues in the small angle limit.* We aim at estimating the discrete spectrum, if it exists, of $\mathfrak{L}_{\alpha,\beta}$. For that purpose, we shall first determine the bottom of its essential spectrum. From Persson's characterization of the infimum of the essential spectrum, it is enough to consider the behavior at infinity and it is possible to establish:

Proposition 2.1. *Let us denote by* $\mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_{\alpha,\beta})$ *the essential spectrum of* $\mathfrak{L}_{\alpha,\beta}$*. We have:*

$$\inf \mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_{\alpha,\beta}) \in [\Theta_0, 1],$$

where $\Theta_0 > 0$ *is defined in* (1.1.4)*.*

At this stage we still do not know that discrete spectrum exists. As it is the case in dimension two (see [**12**]) or in the case on the infinite wedge (see [**145**]), there is hope to prove such an existence in the limit $\alpha \to 0$.

Theorem 2.2. *For all* $n \geq 1$*, there exist* $\alpha_0(n) > 0$ *and a sequence* $(\gamma_{j,n})_{j\geq 0}$ *such that, for all* $\alpha \in (0, \alpha_0(n))$*, the* n*-th eigenvalue exists and satisfies:*

$$\lambda_n(\alpha,\beta) \underset{\alpha\to 0}{\sim} \alpha \sum_{j\geq 0} \gamma_{j,n}\alpha^j, \qquad \text{with} \quad \gamma_{0,n} = \frac{\sqrt{1+\sin^2\beta}}{2^{5/2}}(4n-1).$$

Remark 2.3. *In particular the main term is minimum when* $\beta = 0$ *and in this case Theorem 2.2 states that* $\lambda_1(\alpha) \sim \frac{3}{2^{5/2}}\alpha$*. We have* $\frac{3}{2^{5/2}} < \frac{1}{\sqrt{3}}$ *so that the lowest eigenvalue of the magnetic cone goes below the lowest eigenvalue of the two dimensional magnetic sector (see* (2.1.1)*).*

Remark 2.4. *As a consequence of Theorem 2.2, we deduce that the lowest eigenvalues are simple as soon as* α *is small enough. Therefore, the spectral theorem implies that the quasimodes constructed in the proof are approximations of the eigenfunctions of* $\mathcal{L}_{\alpha,\beta}$*. In particular, using the rescaled spherical coordinates, for all* $n \geq 1$*, there exist* $\alpha_n > 0$ *and* C_n *such that, for* $\alpha \in (0, \alpha_n)$*:*

$$\|\tilde{\psi}_n(\alpha,\beta) - \mathfrak{f}_n\|_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \leq C_n\alpha^2,$$

where $\mathfrak{f}_n$ *(which is* β *dependent) is related to the* n*-th Laguerre's function and* $\tilde{\psi}_n(\alpha,\beta)$ *is the* n*-th normalized eigenfunction.*

Let us now sketch the proof of Theorem 2.2 (see the papers [**20, 21**]).

1.3.2. *Axissymmetric case:* $\beta = 0$. We apply the strategy presented in Chapter 1, Section 3. In this situation, the phase variable that we should understand is the dual variable of θ given by a Fourier series decomposition and denoted by $m \in \mathbb{Z}$. In other words, we make a Fourier decomposition of $\mathcal{L}_{\alpha,0}$ with respect to θ and we introduce the family of 2D-operators $(\mathcal{L}_{\alpha,0,m})_{m\in\mathbb{Z}}$ acting on $\mathsf{L}^2(\mathcal{R}, \mathrm{d}\mu)$:

$$\mathcal{L}_{\alpha,0,m} = -\frac{1}{\tau^2}\partial_\tau \tau^2 \partial_\tau + \frac{1}{\tau^2\sin^2(\alpha\varphi)}\left(m + \frac{\sin^2(\alpha\varphi)}{2\alpha}\tau^2\right)^2 - \frac{1}{\alpha^2\ \tau^2\sin(\alpha\varphi)}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi,$$

with

$$\mathcal{R} = \{(\tau,\varphi) \in \mathbb{R}^2,\ \tau > 0,\ \varphi \in (0, \tfrac{1}{2})\},$$

and

$$\mathrm{d}\mu = \tau^2 \sin(\alpha\varphi)\,\mathrm{d}\tau\,\mathrm{d}\varphi.$$

We denote by $\mathcal{Q}_{\alpha,0,m}$ the quadratic form associated with $\mathcal{L}_{\alpha,0,m}$. This normal form is also the suitable form to construct quasimodes. Then an integrability argument proves that the eigenfunctions are microlocalized in $m = 0$, i.e. they are axisymmetric. Thus this allows a first reduction of dimension. It remains to notice that the last term in $\mathcal{L}_{\alpha,0,0}$ is penalized by α^{-2} so that the Feshbach-Grushin projection on the groundstate of $-\alpha^{-2}(\sin(\alpha\varphi))^{-1}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi$ (the constant function) acts as an approximation of the identity on the eigenfunctions. Therefore the spectrum of $\mathcal{L}_{\alpha,0,0}$ is described modulo lower order terms by the spectrum of the average of $\mathcal{L}_{\alpha,0}$ with respect to φ which involves the so-called Laguerre operator (radial harmonic oscillator).

A detailed proof is given in Chapter 8.

1.3.3. *Case* $\beta \in \left[0, \frac{\pi}{2}\right]$. In this case we cannot use the axisymmetry, but we are still able to construct formal series and prove localization estimates of Agmon type. Moreover we notice that the magnetic momentum with respect to θ is strongly penalized by $(\tau^2 \sin^2(\alpha\varphi))^{-1}$ so that, jointly with the localization estimates it is possible to prove that the eigenfunctions are asymptotically independent from θ and we are reduced to the situation $\beta = 0$.

2. Vanishing magnetic fields and boundary

2.1. Why considering vanishing magnetic fields? A motivation is related to the papers of R. Montgomery [**135**], X-B. Pan and K-H. Kwek [**143**] and B. Helffer and Y. Kordyukov [**85**] (see also [**91**], [**83**] and the thesis of Miqueu [**132**]) where the authors analyze the spectral influence of the cancellation of the magnetic field in the semiclassical limit. Another motivation appears in the paper [**50**] where the authors are concerned with the "magnetic waveguides" and inspired by the physical considerations [**157, 81**] (see also [**103**]). In any case the case of vanishing magnetic fields can inspire the analysis of non trivial examples of magnetic normal forms, as we will see later.

2.2. Montgomery operator. Without going into the details let us describe the model operator introduced in [**135**]. Montgomery was concerned by the magnetic Laplacian $(-ih\nabla + \mathbf{A})^2$ on $\mathsf{L}^2(\mathbb{R}^2)$ in the case when the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ vanishes along a smooth curve Γ. Assuming that the magnetic field non degenerately vanishes, he was led to consider the self-adjoint realization on $\mathsf{L}^2(\mathbb{R}^2)$ of:

$$\mathfrak{L} = D_t^2 + (D_s - st)^2.$$

In this case the magnetic field is given by $\beta(s,t) = s$ so that the zero locus of β is the line $s = 0$. Let us write the following change of gauge:

$$\mathfrak{L}^{\mathsf{Mo}} = e^{-i\frac{s^2 t}{2}}\,\mathfrak{L}\,e^{i\frac{s^2 t}{2}} = D_s^2 + \left(D_t + \frac{s^2}{2}\right)^2.$$

The Fourier transform (after changing ζ in $-\zeta$) with respect to t gives the direct integral:

$$\mathfrak{L}^{\mathsf{Mo}} = \int^{\oplus} \mathfrak{L}^{[1]}_{\zeta}\, \mathrm{d}\zeta, \quad \text{where} \quad \mathfrak{L}^{[1]}_{\zeta} = D_s^2 + \left(\zeta - \frac{s^2}{2}\right)^2 .$$

Note that this family of model operators will be seen as special case of a more general family in Section 3.2. Let us recall a few important properties of the lowest eigenvalue $\nu_1^{[1]}(\zeta)$ of $\mathfrak{L}^{[1]}_{\zeta}$ (for the proofs, see [**143, 84, 97**]).

Proposition 2.5. *The following properties hold:*

(1) For all $\zeta \in \mathbb{R}$, $\nu_1^{[1]}(\zeta)$ is simple.
(2) The function $\zeta \mapsto \nu_1^{[1]}(\zeta)$ is analytic.
(3) We have: $\lim_{|\zeta|\to+\infty} \nu_1^{[1]}(\zeta) = +\infty$.
(4) The function $\zeta \mapsto \nu_1^{[1]}(\zeta)$ admits a unique minimum at a point $\zeta_0^{[1]}$ and it is non degenerate.

We have:

$$\mathsf{sp}(\mathfrak{L}) = \mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}) = [\nu_{\mathsf{Mo}}, +\infty)\,, \tag{2.2.1}$$

with $\nu_{\mathsf{Mo}} = \nu_1^{[1]}(\zeta_0^{[1]})$. With a finite element method and Dirichlet condition on the artificial boundary, a upper-bound of the minimum is obtained in [**97**, Table 1] and the numerical simulations provide $\nu_{\mathsf{Mo}} \simeq 0.5698$ reached for $\zeta_0^{[1]} \simeq 0.3467$ with a discretization step at 10^{-4} for the parameter ζ. This numerical estimate is already mentioned in [**135**]. In fact we can prove the following lower bound (see [**19**] for a proof using the Temple inequality).

Proposition 2.6. *We have: $\nu_{\mathsf{Mo}} \geq 0.5$.*

If we consider the Neumann realization $\mathfrak{L}^{[1],+}_{\zeta}$ of $D_s^2 + \left(\zeta - \frac{s^2}{2}\right)^2$ on $\mathbb{R}^+$, then, by symmetry, the bottom of the spectrum of this operator is linked to the Montgomery operator:

Proposition 2.7. *If we denote by $\nu_1^{[1],+}(\zeta)$ the bottom of the spectrum of $\mathfrak{L}^{[1],+}_{\zeta}$, then*

$$\nu_1^{[1],+}(\zeta) = \nu_1^{[1]}(\zeta).$$

2.3. Generalized Montgomery operators. It turns out that we can generalize the Montgomery operator by allowing an higher order of degeneracy of the magnetic field. Let k be a positive integer. The generalized Montgomery operator of order k is the self-adjoint realization on $\mathbb{R}$ defined by:

$$\mathfrak{L}^{[k]}_{\zeta} = D_t^2 + \left(\zeta - \frac{t^{k+1}}{k+1}\right)^2 .$$

The following theorem (which generalizes Proposition 2.5) is proved in [**70**, Theorem 1.3].

Theorem 2.8. $\zeta \mapsto \nu_1^{[k]}(\zeta)$ *admits a unique and non-degenerate minimum at $\zeta = \zeta_0^{[k]}$.*

Notation 2.9. *For real ζ, the lowest eigenvalue of $\mathfrak{L}_\zeta^{[k]}$ is denoted by $\nu_1^{[k]}(\zeta)$ and we denote by $u_\zeta^{[k]}$ the positive and L^2-normalized eigenfunction associated with $\nu_1^{[k]}(\zeta)$. We denote in the same way its holomorphic extension near $\zeta_0^{[k]}$.*

2.4. A broken Montgomery operator.

2.4.1. *Heuristics and motivation.* As mentioned above, the bottom of the spectrum of $\mathfrak{L}$ is essential. This fact is due to the translation invariance along the zero locus of $\mathbf{B}$. This situation reminds what happens in the waveguides framework. Guided by the ideas developed for the waveguides, we aim at analyzing the effect of breaking the zero locus of $\mathbf{B}$. Introducing the "breaking parameter" $\theta \in (-\pi, \pi]$, we will break the invariance of the zero locus in three different ways:

(1) Case with Dirichlet boundary: $\mathfrak{L}_\theta^{\mathsf{Dir}}$. We let $\mathbb{R}^2_+ = \{(s,t) \in \mathbb{R}^2, t > 0\}$ and consider $\mathfrak{L}_\theta^{\mathsf{Dir}}$ the Dirichlet realization, defined as a Friedrichs extension, on $\mathsf{L}^2(\mathbb{R}^2_+)$ of:
$$D_t^2 + \left(D_s + \frac{t^2}{2}\cos\theta - st\sin\theta\right)^2.$$
(2) Case with Neumann boundary: $\mathfrak{L}_\theta^{\mathsf{Neu}}$. We consider $\mathfrak{L}_\theta^{\mathsf{Neu}}$ the Neumann realization, defined as a Friedrichs extension, on $\mathsf{L}^2(\mathbb{R}^2_+)$ of:
$$D_t^2 + \left(D_s + \frac{t^2}{2}\cos\theta - st\sin\theta\right)^2.$$
The corresponding magnetic field is $\mathbf{B}(s,t) = t\cos\theta - s\sin\theta$. It cancels along the half-line $t = s\tan\theta$.
(3) Magnetic broken line: $\mathfrak{L}_\theta$. We consider $\mathfrak{L}_\theta$ the Friedrichs extension on $\mathsf{L}^2(\mathbb{R}^2)$ of:
$$D_t^2 + \left(D_s + \mathsf{sgn}(t)\frac{t^2}{2}\cos\theta - st\sin\theta\right)^2.$$
The corresponding magnetic field is $\beta(s,t) = |t|\cos\theta - s\sin\theta$; it is a continuous function which cancels along the broken line $|t| = s\tan\theta$.

Notation 2.10. *We use the notation $\mathfrak{L}_\theta^\bullet$ where $\bullet$ can be* Dir*,* Neu *or $\emptyset$.*

2.4.2. *Properties of the spectra.* Let us analyze the dependence of the spectra of $\mathfrak{L}_\theta^\bullet$ on the parameter θ. Denoting by S the axial symmetry $(s,t) \mapsto (-s,t)$, we get:
$$\mathfrak{L}_{-\theta}^\bullet = S\overline{\mathfrak{L}_\theta^\bullet}S,$$
where the line denotes the complex conjugation. Then, we notice that $\mathfrak{L}_\theta^\bullet$ and $\overline{\mathfrak{L}_\theta^\bullet}$ are isospectral. Therefore, the analysis is reduced to $\theta \in [0,\pi)$. Moreover, we get:
$$S\mathfrak{L}_\theta^\bullet S = \mathfrak{L}_{\pi-\theta}^\bullet.$$
The study is reduced to $\theta \in \left[0, \frac{\pi}{2}\right]$.

We observe that at $\theta = 0$ and $\theta = \frac{\pi}{2}$ the domain of $\mathfrak{L}_\theta^\bullet$ is not continuous.

Lemma 2.11. *The family* $(\mathfrak{L}_\theta^\bullet)_{\theta\in(0,\frac{\pi}{2})}$ *is analytic of type (A).*

The following proposition states that the infimum of the essential spectrum is the same for $\mathfrak{L}_\theta^{\mathsf{Dir}}$, $\mathfrak{L}_\theta^{\mathsf{Neu}}$ and $\mathfrak{L}_\theta$.

Proposition 2.12. *For* $\theta \in \left(0, \frac{\pi}{2}\right)$*, we have* $\inf \mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_\theta^\bullet) = \nu_{\mathsf{Mo}}$.

In the Dirichlet case, the spectrum is essential:

Proposition 2.13. *For all* $\theta \in \left(0, \frac{\pi}{2}\right)$*, we have* $\mathsf{sp}(\mathfrak{L}_\theta^{\mathsf{Dir}}) = [\nu_{\mathsf{Mo}}, +\infty)$.

From now on we assume that $\bullet = \mathsf{Neu}, \emptyset$.

Notation 2.14. *Let us denote by* $\lambda_n^\bullet(\theta)$ *the* n*-th number in the sense of the Rayleigh variational formula for* $\mathfrak{L}_\theta^\bullet$.

The two following propositions are Agmon type estimates and give the exponential decay of the eigenfunctions (a proof is given in [**19**]). $\mathbb{R}^2_\bullet$ denotes $\mathbb{R}^2_+$, $\mathbb{R}^2$ when $\bullet = \mathsf{Neu}, \emptyset$ respectively.

Proposition 2.15. *There exist* $\varepsilon_0, C > 0$ *such that for all* $\theta \in \left(0, \frac{\pi}{2}\right)$ *and all eigenpair* (λ, ψ) *of* $\mathfrak{L}_\theta^\bullet$ *such that* $\lambda < \nu_{\mathsf{Mo}}$*, we have:*

$$\int_{\mathbb{R}^2_\bullet} e^{2\varepsilon_0|t|\sqrt{\nu_{\mathsf{Mo}}-\lambda}}|\psi|^2 \, \mathrm{d}s\,\mathrm{d}t \leq C(\nu_{\mathsf{Mo}} - \lambda)^{-1}\|\psi\|^2.$$

Proposition 2.16. *There exist* $\varepsilon_0, C > 0$ *such that for all* $\theta \in \left(0, \frac{\pi}{2}\right)$ *and all eigenpair* (λ, ψ) *of* $\mathfrak{L}_\theta^\bullet$ *such that* $\lambda < \nu_{\mathsf{Mo}}$*, we have:*

$$\int_{\mathbb{R}^2_\bullet} e^{2\varepsilon_0|s|\sin\theta\sqrt{\nu_{\mathsf{Mo}}-\lambda}}|\psi|^2 \, \mathrm{d}s\,\mathrm{d}t \leq C(\nu_{\mathsf{Mo}} - \lambda)^{-1}\|\psi\|^2.$$

The following proposition (the proof of which can be found in [**143**, Lemma 5.2]) states that $\mathfrak{L}_\theta^{\mathsf{Neu}}$ admits an eigenvalue below its essential spectrum when $\theta \in \left(0, \frac{\pi}{2}\right]$.

Proposition 2.17. *For all* $\theta \in \left(0, \frac{\pi}{2}\right]$, $\lambda_1^{\mathsf{Neu}}(\theta) < \nu_{\mathsf{Mo}}$.

Remark 2.18. *The situation seems to be different for* $\mathfrak{L}_\theta$*. According to numerical simulations with finite element method, there exists* $\theta_0 \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ *such that* $\lambda_1(\theta) < \nu_{\mathsf{Mo}}$ *for all* $\theta \in (0, \theta_0)$ *and* $\lambda_1(\theta) = \nu_{\mathsf{Mo}}$ *for all* $\theta \in \left[\theta_0, \frac{\pi}{2}\right)$.

2.5. Singular limit $\theta \to 0$.

2.5.1. *Renormalization.* Thanks to Proposition 2.17, one knows that breaking the invariance of the zero locus of the magnetic field with a Neumann boundary creates a bound state. We also would like to tackle this question for $\mathfrak{L}_\theta$ and in any case to estimate more quantitatively this effect. A way to do this is to consider the limit $\theta \to 0$ which reveals new model operators. First, we perform a scaling:

$$s = (\cos\theta)^{-1/3}\hat{s}, \quad t = (\cos\theta)^{-1/3}\hat{t}. \tag{2.2.2}$$

The operator $\mathfrak{L}^\bullet_\theta$ is thus unitarily equivalent to $(\cos\theta)^{2/3}\hat{\mathfrak{L}}^\bullet_{\tan\theta}$, where the expression of $\hat{\mathfrak{L}}^\bullet_{\tan\theta}$ is given by:

$$D_{\hat{t}}^2+\left(D_{\hat{s}}+\mathsf{sgn}(\hat{t})\frac{\hat{t}^2}{2}-\hat{s}\hat{t}\tan\theta\right)^2.$$

Notation 2.19. *We let* $\varepsilon=\tan\theta$.

For $(x,\xi)\in\mathbb{R}^2$ and $\varepsilon>0$, we introduce the unitary transform:

$$V_{\varepsilon,x,\xi}\psi(\hat{s},\hat{t})=e^{-i\xi\hat{s}}\psi\left(\hat{s}-\frac{x}{\varepsilon},\hat{t}\right),$$

and the conjugate operator:

$$\hat{\mathfrak{L}}^\bullet_{\varepsilon,x,\xi}=V^{-1}_{\varepsilon,x,\xi}\hat{\mathfrak{L}}^\bullet_\varepsilon V_{\varepsilon,x,\xi}.$$

Its expression is given by:

$$\hat{\mathfrak{L}}^\bullet_{\varepsilon,x,\xi}=D_{\hat{t}}^2+\left(-\xi-x\hat{t}+\mathsf{sgn}(\hat{t})\frac{\hat{t}^2}{2}+D_{\hat{s}}-\varepsilon\hat{s}\hat{t}\right)^2. \tag{2.2.3}$$

Let us introduce the new variables:

$$\hat{s}=\varepsilon^{-1/2}\sigma,\quad \hat{t}=\tau \tag{2.2.4}$$

Therefore $\hat{\mathfrak{L}}^\bullet_{\varepsilon,x,\xi}$ is unitarily equivalent to $\mathfrak{L}^\bullet_{\varepsilon,x,\xi}$ whose expression is given by:

$$\mathfrak{L}^\bullet_{\varepsilon,x,\xi}=D_\tau^2+\left(-\xi-x\tau+\mathsf{sgn}(\tau)\frac{\tau^2}{2}+\varepsilon^{1/2}D_\sigma-\varepsilon^{1/2}\sigma\tau\right)^2. \tag{2.2.5}$$

2.5.2. *New model operators.* By taking formally $\varepsilon=0$ in (2.2.5) we are led to two families of one dimensional operators on $\mathsf{L}^2(\mathbb{R}^2_\bullet)$ with two parameters $(x,\xi)\in\mathbb{R}^2$:

$$\mathcal{M}^\bullet_{x,\xi}=D_\tau^2+\left(-\xi-x\tau+\mathsf{sgn}(\tau)\frac{\tau^2}{2}\right)^2.$$

These operators have compact resolvents and are analytic families with respect to the variables $(x,\xi)\in\mathbb{R}^2$.

Notation 2.20. *We denote by* $\mu^\bullet_n(x,\xi)$ *the* n*-th eigenvalue of* $\mathcal{M}^\bullet_{x,\xi}$.

Roughly speaking $\mathcal{M}^\bullet_{x,\xi}$ is the operator valued symbol of (2.2.5), so that we expect that the behavior of the so-called "band function" $(x,\xi)\mapsto\mu^\bullet_1(x,\xi)$ determines the structure of the low lying spectrum of $\mathfrak{M}^\bullet_{\varepsilon,x,\xi}$ in the limit $\varepsilon\to0$.

The following two theorems, the proof of which can be found in Chapter 7, state that the band functions admit a minimum.

Theorem 2.21. *The function* $\mathbb{R}\times\mathbb{R}\ni(x,\xi)\mapsto\mu_1^{\mathsf{Neu}}(x,\xi)$ *admits a minimum denoted by* $\underline{\mu}_1^{\mathsf{Neu}}$. *Moreover we have:*

$$\liminf_{|x|+|\xi|\to+\infty}\mu_1^{\mathsf{Neu}}(x,\xi)\geq\nu_{\mathsf{Mo}}>\min_{(x,\xi)\in\mathbb{R}^2}\mu_1^{\mathsf{Neu}}(x,\xi)=\underline{\mu}_1^{\mathsf{Neu}}.$$

Theorem 2.22. *The function* $\mathbb{R} \times \mathbb{R} \ni (x, \xi) \mapsto \mu_1(x, \xi)$ *admits a minimum denoted by* $\underline{\mu}_1$. *Moreover we have:*

$$\liminf_{|x|+|\xi|\to+\infty} \mu_1(x,\xi) \geq \nu_{\mathsf{Mo}} > \min_{(x,\xi)\in\mathbb{R}^2} \mu_1(x,\xi) = \underline{\mu}_1.$$

Remark 2.23. *We have:*

$$\underline{\mu}_1^{\mathsf{Neu}} \leq \underline{\mu}_1. \tag{2.2.6}$$

Numerical experiments lead to the following conjecture.

Conjecture 2.24.
- *The inequality* (2.2.6) *is strict.*
- *The minimum* $\underline{\mu}_1^{\bullet}$ *is unique and non-degenerate.*

Remark 2.25. *Under Conjecture 2.24, it is possible to prove complete asymptotic expansions of the first eigenvalues of* $\mathfrak{L}_\theta$. *In fact, this can be done by using the magnetic Born-Oppenheimer approximation (see Section 3).*

3. Magnetic Born-Oppenheimer approximation

This section is devoted to the analysis of the operator on $\mathsf{L}^2(\mathbb{R}^m_s \times \mathbb{R}^n_t, \, \mathrm{d}s\,\mathrm{d}t)$:

$$\mathfrak{L}_h = (-ih\nabla_s + A_1(s,t))^2 + (-i\nabla_t + A_2(s,t))^2, \tag{2.3.1}$$

Note that (2.2.3) can easily be put in this form. We will assume that A_1 and A_2 are real analytic. We would like to describe the lowest eigenvalues of this operator in the limit $h \to 0$ under elementary confining assumptions. The problem of considering partial semiclassical problems appears for instance in the context of [**126, 110**] where the main issue is to approximate the eigenvalues and eigenfunctions of operators in the form:

$$-h^2\Delta_s - \Delta_t + V(s,t). \tag{2.3.2}$$

The main idea, due to Born and Oppenheimer in [**25**], is to replace, for fixed s, the operator $-\Delta_t + V(s,t)$ by its eigenvalues $\mu_k(s)$. Then we are led to consider for instance the reduced operator (called Born-Oppenheimer approximation):

$$-h^2\Delta_s + \mu_1(s)$$

and to apply the semiclassical techniques *à la* Helffer-Sjöstrand [**98, 99**] to analyze in particular the tunnel effect when the potential μ_1 admits symmetries. The main point it to make the reduction of dimension rigorous. Note that we have always the following lower bound:

$$-h^2\Delta_s - \Delta_t + V(s,t) \geq -h^2\Delta_s + \mu_1(s), \tag{2.3.3}$$

which involves accurate estimates of Agmon with respect to s.

3.1. Electric Born-Oppenheimer approximation and low lying spectrum. Before dealing with the so-called Born-Oppenheimer approximation in presence of magnetic fields, we shall recall the philosophy in a simplified electric case.

3.1.1. *Electric result.* Let us explain the question in which we are interested. We shall study operators in $\mathsf{L}^2(\mathbb{R}\times\Omega)$ (with $\Omega\subset\mathbb{R}^n$) in the form:

$$\mathfrak{H}_h = h^2 D_s^2 + \mathcal{V}(s),$$

where $\mathcal{V}(s) = -\Delta_t + P(t,s)$ is a family of semi-bounded self-adjoint operators, analytic of type (B), with P polynomial for simplicity. We will denote by $\mathfrak{Q}_h$ the corresponding quadratic form.

We want to analyze the low lying eigenvalues of this operator. We will assume that the lowest eigenvalue $\nu(s)$ of $\mathcal{V}(s)$ (which is simple) admits, as a function of s, a unique and non degenerate minimum at s_0.

We now try to understand the heuristics. We hope that $\mathfrak{H}_h$ can be described by its "Born-Oppenheimer" approximation:

$$\mathfrak{H}_h^{\mathsf{BO}} = h^2 D_s^2 + \mu(s),$$

which is an electric Laplacian in dimension one. Then, we guess that $\mathfrak{H}_h^{\mathsf{BO}}$ is well approximated by its Taylor expansion:

$$h^2 D_s^2 + \mu(s_0) + \frac{\nu''(s_0)}{2}(s-s_0)^2.$$

In fact this heuristics can be made rigorous.

Assumption 2.26. *Let us assume that* $\liminf\limits_{s\to\pm\infty}\nu(s) > \nu(s_0)$ *and that*

$$\inf_s \mathsf{sp}_{\mathsf{ess}}(\mathcal{V}(s)) > \nu(s_0).$$

Theorem 2.27. *Let us assume that* $\nu(s)$ *admits a unique and non degenerate minimum at* s_0 *and that Assumption 2.26 is satisfied then the* n*-th eigenvalue of* $\mathfrak{H}_h$ *has the expansion*

$$\lambda_n(h) = \nu(s_0) + h(2n-1)\left(\frac{\nu''(s_0)}{2}\right)^{1/2} + o(h).$$

3.1.2. *A non example: the broken* δ*-interactions.* In the last theorem we were only interested in the low lying spectrum. It turns out that the so-called Born-Oppenheimer reduction is a slightly more general procedure (see [**126, 110**]) which provides in general an effective Hamiltonian which describes the spectrum below some fixed energy level (and allows for instance to estimate the counting function). Let us discuss it with the example of broken δ-interactions for which, due to the singularity of the δ interaction, the standard technique needs to be adapted (one may consult [**62, 63, 29, 61**] for perspectives and motivation). Le us consider $\mathfrak{H}_h$ the Friedrichs extension (see [**27**]) of the rescaled quadratic form:

$$\mathfrak{Q}_h(\psi) = \int_{\mathbb{R}^2} h^2|\partial_x\psi|^2 + |\partial_y\psi|^2 \,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}} |\psi(|s|,s)|^2\,\mathrm{d}s, \quad \forall\psi\in H^1(\mathbb{R}^2). \tag{2.3.4}$$

Formally we may write

$$\mathfrak{H}_h = -h^2\partial_x^2 - \partial_y^2 - \delta_{\Sigma_{\frac{\pi}{4}}}, \tag{2.3.5}$$

where

$$\Sigma_{\frac{\pi}{4}} = \{(|s|, s), \quad s \in \mathbb{R}\}.$$

In particular, we notice that:

$$\mathsf{sp}_{\mathsf{ess}}(\mathfrak{H}_h) = \left[-\frac{1}{4(1+h^2)}, +\infty\right).$$

Let us introduce some notation.

Notation 2.28. *We denote by $W : [-e^{-1}, +\infty) \to [-1, +\infty)$ the Lambert function defined as the inverse of $[-1, +\infty) \ni w \mapsto we^w \in [-e^{-1}, +\infty)$.*

Notation 2.29. *Given $\mathfrak{H}$ a semi-bounded self-adjoint operator and $a < \inf \mathsf{sp}_{\mathsf{ess}}(\mathfrak{H})$, we denote*

$$\mathsf{N}(\mathfrak{H}, a) \;=\; \#\{\lambda \in \mathsf{sp}(\mathfrak{H}) \;:\; \lambda \leq a\} < +\infty.$$

The eigenvalues are counted with multiplicity.

The following theorem provides the asymptotics of the number of bound states.

Theorem 2.30. *There exists $M_0 > 0$ such that for all $C(h) \geq M_0 h$ with $C(h) \underset{h\to 0}{\to} C_0 \geq 0$:*

$$\mathsf{N}\left(\mathfrak{H}_h, -\frac{1}{4} - C(h)\right) \underset{h\to 0}{\sim} \frac{1}{\pi h} \int_{x=0}^{+\infty} \sqrt{-\frac{1}{4} - C_0 + \left(\frac{1}{2} + \frac{1}{2x} W(xe^{-x})\right)^2}\, \mathrm{d}x.$$

Remark 2.31. *It is important to notice that in the above result, we estimate the counting function below a potentially moving (w.r.t. h) threshold. In particular, the distance between $-\frac{1}{4} - C(h)$ and the bottom of the essential spectrum is allowed to vanish in the semiclassical limit. Therefore our statement is slightly unusual as customary results would typically concern $\mathsf{N}(\mathfrak{H}_h, E)$ with E fixed and satisfying $E < -\frac{1}{4}$, so as to insure a fixed security distance to the bottom of the essential spectrum (see for instance the related works* [**8, 136**]*).*

The next theorem is the analogous of Theorem 2.27.

Theorem 2.32. *For all $n \geq 1$, we have:*

$$\lambda_n(h) \underset{h\to 0}{=} -1 + 2^{2/3} z_{\mathsf{Ai}^{\mathsf{rev}}}(n) h^{2/3} + O(h),$$

where $z_{\mathsf{Ai}^{\mathsf{rev}}}(n)$ is the n-th zero of the reversed Airy function.

3.2. Magnetic case. We would like to understand the analogy between (2.3.1) and (2.3.2). In particular even the formal dimensional reduction does not seem to be as clear as in the electric case. Let us write the operator valued symbol of $\mathfrak{L}_h$. For $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, we introduce the electro-magnetic Laplacian acting on $\mathsf{L}^2(\mathbb{R}^n, \mathrm{d}t)$:

$$\mathcal{M}_{x,\xi} = (-i\nabla_t + A_2(x,t))^2 + (\xi + A_1(x,t))^2.$$

Denoting by $\mu_1(x,\xi) = \mu(x,\xi)$ its lowest eigenvalue we would like to replace $\mathfrak{L}_h$ by the m-dimensional pseudo-differential operator:

$$\mu(s, -ih\nabla_s).$$

This can be done modulo $\mathcal{O}(h)$ (see [**129**]). Nevertheless we do not have an obvious comparison as in (2.3.3) so that the microlocal behavior of the eigenfunctions with respect to s is not directly reachable (we can not directly apply the exponential estimates of [**127**] due to the possible essential spectrum, see Asumption 2.35). In particular we shall prove that the remainder $\mathcal{O}(h)$ is indeed small when acting on the eigenfunctions and then estimate it precisely.

3.2.1. *Eigenvalue asymptotics in the magnetic Born-Oppenheimer approximation.* We will work under the following assumptions. The first assumption states that the lowest eigenvalue of the operator symbol of $\mathfrak{L}_h$ admits a unique and non-degenerate minimum.

Assumption 2.33.
- *The family $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ is analytic of type (B) in the sense of Kato* [**108**, Chapter VII].
- *For all $(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m$, the bottom of the spectrum of $\mathcal{M}_{x,\xi}$ is a simple eigenvalue denoted by $\mu(x,\xi)$ (in particular it is an analytic function) and associated with a L^2-normalized eigenfunction $u_{x,\xi}\in\mathcal{S}(\mathbb{R}^n)$ which also analytically depends on (x,ξ).*
- *The function μ admits a unique and non degenerate minimum μ_0 at point denoted by (x_0,ξ_0) and such that $\liminf_{|x|+|\xi|\to+\infty}\mu(x,\xi) > \mu_0$.*
- *The family $(\mathcal{M}_{x,\xi})_{(x,\xi)\in\mathbb{R}^m\times\mathbb{R}^m}$ can be analytically extended in a complex neighborhood of (x_0,ξ_0).*

Assumption 2.34. *Under Assumption 2.33, let us denote by* $\mathsf{Hess}\,\mu_1(x_0,\xi_0)$ *the Hessian matrix of μ_1 at (x_0,ξ_0). We assume that the spectrum of the operator* $\mathsf{Hess}\,\mu_1(x_0,\xi_0)(\sigma, D_\sigma)$ *is simple.*

The next assumption is a spectral confinement.

Assumption 2.35. *For $R\geq 0$, we let $\Omega_R = \mathbb{R}^{m+n}\setminus\overline{B(0,R)}$. We denote by $\mathfrak{L}_h^{\mathsf{Dir},\Omega_R}$ the Dirichlet realization on Ω_R of $(-i\nabla_t + A_2(s,t))^2 + (-ih\nabla_s + A_1(s,t))^2$. We assume that there exist $R_0\geq 0$, $h_0>0$ and $\mu_0^*>\mu_0$ such that for all $h\in(0,h_0)$:*

$$\lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h)\geq\mu_0^*.$$

Remark 2.36. *In particular, due to the monotonicity of the Dirichlet realization with respect to the domain, Assumption 2.35 implies that there exist $R_0>0$ and $h_0>0$ such that for all $R\geq R_0$ and $h\in(0,h_0)$:*

$$\lambda_1^{\mathsf{Dir},\Omega_R}(h)\geq\lambda_1^{\mathsf{Dir},\Omega_{R_0}}(h)\geq\mu_0^*.$$

By using the Persson's theorem (see Chapter 5, Proposition 5.8), we have the following proposition.

Proposition 2.37. *Let us assume Assumption 2.35. There exists $h_0 > 0$ such that for all $h \in (0, h_0)$:*

$$\inf \mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_h) \geq \mu_0^*.$$

We can now state the theorem concerning the spectral asymptotics.

Theorem 2.38. *We assume that A_1 and A_2 are polynomials and Assumptions 2.33, 2.34 and 2.35. For all $n \geq 1$, there exist a sequence $(\gamma_{j,n})_{j\geq 0}$ and $h_0 > 0$ such that for all $h \in (0, h_0)$ the n-th eigenvalue of $\mathfrak{L}_h$ exists and satifies:*

$$\lambda_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n} h^{j/2},$$

where $\gamma_{0,n} = \mu_0, \quad \gamma_{1,n} = 0$ and $\mu_{2,n}$ is the n-th eigenvalue of $\frac{1}{2}\mathsf{Hess}_{x_0,\xi_0}\,\mu_1(\sigma, D_\sigma)$.

3.2.2. *Coherent states.* Let us recall the formalism of coherent states which play a central role in the proof of Theorem 2.38. We refer to the books [**66**] and [**39**] for details. We let:

$$g_0(\sigma) = \pi^{-1/4} e^{-|\sigma|^2/2}$$

and the usual creation and annihilation operators:

$$\mathfrak{a}_j = \frac{1}{\sqrt{2}}(\sigma_j + \partial_{\sigma_j}), \qquad \mathfrak{a}_j^* = \frac{1}{\sqrt{2}}(\sigma_j - \partial_{\sigma_j})$$

which satisfy the commutator identities:

$$[\mathfrak{a}_j, \mathfrak{a}_j^*] = 1, \qquad [\mathfrak{a}_j, \mathfrak{a}_k^*] = 0 \text{ if } k \neq j.$$

We notice that:

$$\sigma_j = \frac{\mathfrak{a}_j + \mathfrak{a}_j^*}{\sqrt{2}}, \quad \partial_{\sigma_j} = \frac{\mathfrak{a}_j - \mathfrak{a}_j^*}{\sqrt{2}}, \quad \mathfrak{a}_j \mathfrak{a}_j^* = \frac{1}{2}(D_{\sigma_j}^2 + \sigma_j^2 + 1).$$

For $(u, p) \in \mathbb{R}^m \times \mathbb{R}^m$, we introduce the coherent state

$$f_{u,p}(\sigma) = e^{ip\cdot\sigma} g_0(\sigma - u),$$

and the associated projection

$$\Pi_{u,p}\psi = \langle \psi, f_{u,p}\rangle_{\mathsf{L}^2(\mathbb{R}^m)} f_{u,p} = \psi_{u,p} f_{u,p},$$

which satisfies

$$\psi = \int_{\mathbb{R}^{2m}} \Pi_{u,p}\psi \,\mathrm{d}u\,\mathrm{d}p,$$

and the Parseval formula

$$\|\psi\|^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{2m}} |\psi_{u,p}|^2 \,\mathrm{d}u\,\mathrm{d}p\,\mathrm{d}\tau.$$

We recall that:

$$\mathfrak{a}_j f_{u,p} = \frac{u_j + ip_j}{\sqrt{2}} f_{u,p}$$

and
$$(\mathfrak{a}_j)^\ell(\mathfrak{a}_k^*)^q\psi = \int_{\mathbb{R}^{2m}} \left(\frac{u_j+ip_j}{\sqrt{2}}\right)^\ell \left(\frac{u_k-ip_k}{\sqrt{2}}\right)^q \Pi_{u,p}\psi \,\mathrm{d}u\,\mathrm{d}p.$$
We recall that (see (10.1.1)):
$$\mathcal{L}_h = (-i\nabla_\tau + A_2(x_0+h^{1/2}\sigma,\tau))^2 + (\xi_0 - ih^{1/2}\nabla_\sigma + A_1(x_0+h^{1/2}\sigma,\tau))^2$$
and, assuming that A_1 and A_2 are polynomial:
$$\mathcal{L}_h = \mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2 + \ldots + (h^{1/2})^M\mathcal{L}_M.$$
If we write the Wick ordered operator, we get:
$$\mathcal{L}_h = \underbrace{\mathcal{L}_0 + h^{1/2}\mathcal{L}_1 + h\mathcal{L}_2^{\mathsf{W}} + \ldots + (h^{1/2})^M\mathcal{L}_M^{\mathsf{W}}}_{\mathcal{L}_h^{\mathsf{W}}} + \underbrace{hR_2 + \ldots + (h^{1/2})^M R_M}_{\mathcal{R}_h}, \tag{2.3.6}$$
where the R_j satisfy, for $j \geq 2$:
$$h^{j/2}R_j = h^{j/2}\mathcal{O}_{j-2}(\sigma, D_\sigma) \tag{2.3.7}$$
and are the remainders in the so-called Wick ordering. In the last formula the notation $\mathcal{O}_k(\sigma, D_\sigma)$ stands for a polynomial operator with total degree in (σ, D_σ) less than k. We recall that
$$\mathcal{L}_h^{\mathsf{W}} = \int_{\mathbb{R}^{2m}} \mathcal{M}_{x_0+h^{1/2}u,\xi_0+h^{1/2}p}\,\mathrm{d}u\,\mathrm{d}p.$$

3.2.3. *A family of examples.* In order to make our Assumptions 2.33 and 2.35 more concrete, let us provide a family of examples in dimension two which is related to [**97**] and the more recent result by Fournais and Persson [**70**]. Our examples are strongly connected with [**85**, Conjecture 1.1 and below].

For $k \in \mathbb{N}\setminus\{0\}$, we consider the operator the following magnetic Laplacian on $\mathsf{L}^2(\mathbb{R}^2, \mathrm{d}x\,\mathrm{d}s)$:
$$\mathfrak{L}_{h,\mathbf{A}^{[k]}} = h^2D_{\mathsf{t}}^2 + \left(hD_{\mathsf{s}} - \gamma(\mathsf{s})\frac{\mathsf{t}^{k+1}}{k+1}\right)^2.$$
Let us perform the rescaling:
$$\mathsf{s} = s, \quad \mathsf{t} = h^{\frac{1}{1+k}}t.$$
The operator becomes:
$$h^{\frac{2k+2}{k+2}}\left(D_t^2 + \left(h^{\frac{1}{k+2}}D_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2\right).$$
and the investigation is reduced to the one of:
$$\mathfrak{L}_h^{\mathsf{vf},[k]} = D_t^2 + \left(h^{\frac{1}{k+2}}D_s - \gamma(s)\frac{t^{k+1}}{k+1}\right)^2.$$

Proposition 2.39. *Let us assume that either γ is polynomial and admits a unique minimum $\gamma_0 > 0$ at $s_0 = 0$ which is non degenerate, or γ is analytic and such that*

$\liminf_{x\to\pm\infty}\gamma=\gamma_\infty\in(\gamma_0,+\infty)$. *For* $k\in\mathbb{N}\setminus\{0\}$, *the operator* $\mathfrak{L}_h^{[k]}$ *satisfies Assumptions 2.33, 2.34 and 2.35. Moreover we can choose* $\mu_0^*>\mu_0$.

PROOF. Let us verify Assumption 2.33. The $h^{\frac{1}{k+2}}$-symbol of $\mathfrak{L}_h^{[k]}$ with respect to s is:

$$\mathcal{M}_{x,\xi}^{[k]}=D_t^2+\left(\xi-\gamma(x)\frac{t^{k+1}}{k+1}\right)^2.$$

The lowest eigenvalue of $\mathcal{M}_{x,\xi}^{[k]}$, denoted by $\mu_1^{[k]}(x,\xi)$, satisfies:

$$\mu_1^{[k]}(x,\xi)=(\gamma(x))^{\frac{2}{k+2}}\nu_1^{[k]}\left((\gamma(x))^{-\frac{1}{k+2}}\xi\right),$$

where $\nu_1^{[k]}(\zeta)$ denotes the first eigenvalue of:

$$\mathfrak{L}_\zeta^{[k]}=D_t^2+\left(\zeta-\frac{t^{k+1}}{k+1}\right)^2.$$

We recall that $\zeta\mapsto\nu_1^{[k]}(\zeta)$ admits a unique and non-degenerate minimum at $\zeta=\zeta_0^{[k]}$ (see Theorem 2.8). Therefore Assumption 2.33 is satisfied. This is much more delicate (and beyond the scope of this book) to verify Assumption 2.35 and this relies on a basic normal form procedure that we will use for our magnetic WKB constructions. □

3.3. The magnetic WKB expansions: examples.

3.3.1. *WKB analysis and estimates of Agmon.* As we explained in Chapter 1, Section 3.2.1, in many papers about asymptotic expansions of the magnetic eigenfunctions, one of the methods consists in using a formal power series expansion. It turns out that these constructions are never in the famous WKB form, but in a weaker and somehow more flexible one. When there is an additional electric potential, the WKB expansions are possible as we can see in [**100**] and [**130**]. The reason for which we would like to have a WKB description of the eigenfunctions is to get a precise estimate of the magnetic tunnel effect in the case of symmetries. Until now, such estimates are only investigated in two dimensional corner domains in [**13**] and [**14**] for the numerical counterpart. It turns out that the crucial point to get an accurate estimate of the exponentially small splitting of the eigenvalues is to establish exponential decay estimates of Agmon type. These localization estimates are rather easy to obtain (at least to get the good scale in the exponential decay) in the corner cases due to the fact that the operator is "more elliptic" than in the regular case in the following sense: the spectral asymptotics is completely drifted by the principal symbol. Nevertheless, let us notice here that, on the one hand, the numerics suggests that the eigenvalues do not seem to be simple and, on the other hand, that establishing the optimal estimates of Agmon is still an open problem. In smooth cases, due to a lack of ellipticity and to the multiple scales, the localization estimates obtained in the literature are in general not optimal or rely on the presence of an electric potential (see [**137, 138**]): the principal symbol provides only a partial confinement whereas the precise localization of the eigenfunctions seems to be determined by the subprincipal terms. Our WKB analysis (inspired by our paper [**18**]), in the explicit cases discussed in this book, will give some hints for the optimal candidate to be the effective Agmon distance.

3.3.2. *WKB expansions for* $\mathfrak{L}_h^{\mathsf{vf},[k]}$. The following theorem states that the first eigenfunctions of $\mathfrak{L}_h^{\mathsf{vf},[k]}$ are in the WKB form. It turns out that this property is very general and verified for the general $\mathfrak{L}_h$ under our generic assumptions. Nevertheless this general and fundamental result is beyond the scope of this book. We will only give the flavor of such constructions for our explicit model. As far as we know such a result was not even known on an example. Let us state one of the main results of this book concerning the WKB expansions (see [**18**] for a more general statement about $\mathfrak{L}_h$).

Theorem 2.40. *Let us assume that either* γ *is polynomial and admits a unique minimum* $\gamma_0 > 0$ *at* $s_0 = 0$ *which is non degenerate, either* γ *is analytic and such that* $\liminf_{x\to\pm\infty} \gamma = \gamma_\infty \in (\gamma_0, +\infty)$. *There exist a function* $\Phi = \Phi(s)$ *defined in a neighborhood* $\mathcal{V}$ *of* 0 *with* $\mathsf{Re}\,\Phi''(0) > 0$ *and a sequence of real numbers* $(\lambda_{n,j}^{\mathsf{vf}})_{j\geq 0}$ *such that the* n*-th eigenvalue of* $\mathfrak{L}_h^{\mathsf{vf},[k]}$ *satisfies*

$$\lambda_n^{\mathsf{vf}}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j}^{\mathsf{vf}} h^{\frac{j}{k+2}}$$

in the sense of formal series, with $\lambda_{n,0}^{\mathsf{vf}} = \mu_0 = \nu_1^{[k]}(\zeta_0^{[k]})$. *Besides there exists a formal series of smooth functions on* $\mathcal{V} \times \mathbb{R}_t^n$

$$a_n^{\mathsf{vf}}(.,h) \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}^{\mathsf{vf}} h^{\frac{j}{k+2}}$$

with $a_{n,0}^{\mathsf{vf}} \neq 0$ *such that*

$$\left(\mathfrak{L}_h^{\mathsf{vf},[k]} - \lambda_n(h)\right)\left(a_n^{\mathsf{vf}}(\cdot,h)e^{-\Phi/h^{\frac{1}{k+2}}}\right) = \mathcal{O}\left(h^\infty\right) e^{-\Phi/h^{\frac{1}{k+2}}},$$

In addition, there exists $c_0 > 0$ *such that for all* $h \in (0, h_0)$

$$\mathcal{B}\Big(\lambda_{n,0}^{\mathsf{vf}} + \lambda_{n,1}^{\mathsf{vf}} h^{\frac{1}{k+2}}, c_0 h^{\frac{2}{k+2}}\Big) \cap \mathsf{sp}\left(\mathfrak{L}_h^{\mathsf{vf},[k]}\right) = \{\lambda_n^{\mathsf{vf}}(h)\},$$

and $\lambda_n^{\mathsf{vf}}(h)$ *is a simple eigenvalue.*

Remark 2.41. *In fact, if* $\gamma(s)^{-1}\gamma(0) - 1$ *is small enough (weak magnetic barrier), our construction of* Φ *can be made global, that is* $\mathcal{V} = \mathbb{R}$. *In this book, we will provide a proof of this theorem when* γ *is a polynomial.*

We will prove Theorem 2.40 in Chapter 11, Section 1.

3.3.3. *Along a varying edge.* Let us provide another example for which one can produce a WKB analysis. This one is motivated by the analysis of problems with singular boundaries (see Chapter 14 for a motivation). Here we are concerned with the case when the domain is a wedge with varying aperture, that is with the Neumann magnetic Laplacian $\mathfrak{L}_{h,\mathbf{A}}^{\mathsf{e}} = (-ih\nabla + \mathbf{A})^2$ on $\mathsf{L}^2(\mathcal{W}_{s\mapsto\alpha(s)}, \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z)$. Let us recall the definition of the magnetic wedge with constant aperture α. Many properties of this operator can be found in the thesis of Popoff [**145**]. We let

$$\mathcal{W}_\alpha = \mathbb{R} \times \mathcal{S}_\alpha,$$

where the 2D corner with fixed angle $\alpha \in (0, \pi)$ is defined by

$$\mathcal{S}_\alpha = \left\{(t,z) \in \mathbb{R}^2 : |z| < t \tan\left(\frac{\alpha}{2}\right)\right\}.$$

Definition 2.42. *Let $\mathfrak{L}^{\mathsf{e}}_\alpha$ be the Neumann realization on $\mathsf{L}^2(\mathcal{W}_\alpha, \mathrm{d}s\,\mathrm{d}t\,\mathrm{d}z)$ of*

$$D_t^2 + D_z^2 + (D_s - t)^2. \tag{2.3.8}$$

We denote by $\nu_1^{\mathsf{e}}(\alpha)$ the bottom of the spectrum of $\mathfrak{L}^{\mathsf{e}}_\alpha$.

Using the Fourier transform with respect to $\hat{s}$, we have the decomposition:

$$\mathfrak{L}^{\mathsf{e}}_\alpha = \int^{\oplus} \mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}\,\mathrm{d}\zeta, \tag{2.3.9}$$

where $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$ is the following Neumann realization on $\mathsf{L}^2(\mathcal{S}_\alpha, \mathrm{d}t\,\mathrm{d}z)$:

$$\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta} = D_t^2 + D_z^2 + (\zeta - t)^2, \tag{2.3.10}$$

where $\zeta \in \mathbb{R}$ is the Fourier parameter. As

$$\lim_{\substack{|(t,z)|\to+\infty \\ (t,z)\in\mathcal{S}_\alpha}} (\zeta - t)^2 = +\infty,$$

the Schrödinger operator $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$ has compact resolvent for all $(\alpha,\zeta) \in (0,\pi) \times \mathbb{R}$.

Notation 2.43. *For each $\alpha \in (0,\pi)$, we denote by $\nu_1^{\mathsf{e}}(\alpha,\eta)$ the lowest eigenvalue of $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$ and we denote by $u^{\mathsf{e}}_{\alpha,\zeta}$ a normalized corresponding eigenfunction.*

Using (2.3.9) we have:

$$\nu_1^{\mathsf{e}}(\alpha) = \inf_{\zeta\in\mathbb{R}} \nu_1^{\mathsf{e}}(\alpha,\zeta). \tag{2.3.11}$$

Let us gather a few elementary properties.

Lemma 2.44. *We have:*

(1) For all $(\alpha,\zeta) \in (0,\pi) \times \mathbb{R}$, $\nu_1^{\mathsf{e}}(\alpha,\zeta)$ is a simple eigenvalue of $\mathfrak{L}^{\mathsf{e}}_{\alpha,\zeta}$.
(2) The function $(0,\pi) \times \mathbb{R} \ni (\alpha,\zeta) \mapsto \nu_1^{\mathsf{e}}(\alpha,\zeta)$ is analytic.
(3) For all $\zeta \in \mathbb{R}$, the function $(0,\pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha,\zeta)$ is decreasing.
(4) The function $(0,\pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha)$ is non increasing.
(5) For all $\alpha \in (0,\pi)$, we have

$$\lim_{\eta\to-\infty} \nu_1^{\mathsf{e}}(\alpha,\zeta) = +\infty \quad \textit{and} \quad \lim_{\zeta\to+\infty} \nu_1^{\mathsf{e}}(\alpha,\zeta) = \mathfrak{s}\left(\tfrac{\pi-\alpha}{2}\right). \tag{2.3.12}$$

PROOF. We refer to [**145**, Section 3] for the first two statements. The monoticity comes from [**145**, Proposition 8.14] and the limits as ζ goes to $\pm\infty$ are computed in [**145**, Theorem 5.2]. □

Remark 2.45. *As* $\nu_1^{\mathsf{e}}(\pi) = \Theta_0$*, we have:*

$$\forall \alpha \in (0,\pi), \quad \nu_1^{\mathsf{e}}(\alpha) \geq \Theta_0. \tag{2.3.13}$$

Let us note that it is proved in [**145**, Proposition 8.13] *that* $\nu_1^{\mathsf{e}}(\alpha) > \Theta_0$ *for all* $\alpha \in (0,\pi)$*.*

Proposition 2.46. *There exists* $\tilde{\alpha} \in (0,\pi)$ *such that for* $\alpha \in (0,\tilde{\alpha})$*, the function* $\zeta \mapsto \nu_1^{\mathsf{e}}(\alpha,\zeta)$ *reaches its infimum and*

$$\nu_1^{\mathsf{e}}(\alpha) < \mathfrak{s}\left(\frac{\pi-\alpha}{2}\right), \tag{2.3.14}$$

where the spectral function $\mathfrak{s}$ *is defined in Chapter 1, Section 1.4.4.*

Remark 2.47. *By computing* C^{qm}*, we notice that* (2.3.14) *holds at least for* $\alpha \in (0, 1.2035)$*. Numerical computations show that in fact* (2.3.14) *seems to hold for all* $\alpha \in (0,\pi)$*.*

We will work under the following conjecture:

Conjecture 2.48. *For all* $\alpha \in (0,\pi)$*,* $\zeta \mapsto \nu_1^{\mathsf{e}}(\alpha,\zeta)$ *has a unique critical point denoted by* $\zeta_0^{\mathsf{e}}(\alpha)$ *and it is a non degenerate minimum.*

Remark 2.49. *A numerical analysis seems to indicate that Conjecture 2.48 is true (see* [**145**, Subsection 6.4.1]*).*

Under this conjecture and using the analytic implicit functions theorem, we deduce:

Lemma 2.50. *Under Conjecture 2.48, the function* $(0,\pi) \ni \alpha \mapsto \zeta_0^{\mathsf{e}}(\alpha)$ *is analytic and so is* $(0,\pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha)$*. Moreover the function* $(0,\pi) \ni \alpha \mapsto \nu_1^{\mathsf{e}}(\alpha)$ *is decreasing.*

We will assume that there is a unique point of maximal aperture (which is non-degenerate).

Assumption 2.51. *The function* $s \mapsto \alpha(s)$ *is analytic and admits a unique and non-degenerate maximum* α_0 *at* $s = 0$*.*

Notation 2.52. *We let* $\mathcal{T}(s) = \tan\left(\frac{\alpha(s)}{2}\right)$*.*

3.3.4. *WKB expansions near the maximal aperture.* We will need the Neumann realization of the operator defined on $\mathsf{L}^2(\mathcal{S}_{\alpha_0}, \mathrm{d}t\,\mathrm{d}z)$ by

$$\mathcal{M}_{s,\zeta}^{\mathsf{e}} = D_t^2 + \mathcal{T}(s)^{-2}\mathcal{T}(0)^2 D_z^2 + (\zeta - t)^2,$$

whose form domain is

$$\mathsf{Dom}\,(\mathcal{Q}_{s,\zeta}^{\mathsf{e}}) = \left\{\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}) : D_t\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}), D_z\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}), t\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0})\right\}$$

and with operator domain

$$\mathsf{Dom}\,(\mathcal{M}_{s,\zeta}^{\mathsf{e}}) = \left\{\psi \in \mathsf{Dom}\,(\mathcal{Q}_{s,\zeta}^{\mathsf{e}}) : \mathcal{M}_{s,\zeta}^{\mathsf{e}}\psi \in \mathsf{L}^2(\mathcal{S}_{\alpha_0}), \mathfrak{C}(s)\psi = 0\right\},$$

where

$$\mathfrak{C}(s) = -\mathsf{sgn}(z)D_t + \mathcal{T}(s)^{-2}\mathcal{T}(0)D_z.$$

The lowest eigenvalue of $\mathcal{M}^{\mathsf{e}}_{s,\zeta}$ is denoted by $\mu^{\mathsf{e}}(s,\zeta)$. Conjecture 2.48 can be reformulated as follows.

Conjecture 2.53. *For all $\alpha_0 \in (0,\pi)$, the function $\zeta \mapsto \mu^{\mathsf{e}}(0,\zeta)$ admits a unique critical point ζ_0^{e} (or simply ζ_0 if not ambiguous) which is a non-degenerate minimum.*

The following proposition shows that the operator symbol $\mathcal{M}^{\mathsf{e}}_{s,\zeta}$ satisfies generic properties.

Proposition 2.54. *Under Assumption 2.51 and if Conjecture 2.53 is true, the function μ^{e} admits a local non degenerate minimum at $(0,\zeta_0)$. Moreover the Hessian at $(0,\zeta_0)$ is given by*

$$4\kappa\mathcal{T}(0)^{-1}\|D_z u^{\mathsf{e}}_{0,\zeta_0}\|s^2 + \partial^2_\zeta \mu^{\mathsf{e}}(0,\zeta_0)\zeta^2, \tag{2.3.15}$$

where $\kappa = -\frac{\mathcal{T}''(0)}{2} > 0$.

Proof. The proof follows from the perturbation theory. We have the eigenvalue equation

$$\mathcal{M}^{\mathsf{e}}_{s,\zeta}u^{\mathsf{e}}_{s,\zeta} = \mu^{\mathsf{e}}(s,\zeta)u^{\mathsf{e}}_{s,\zeta}, \qquad \mathfrak{T}(s)u^{\mathsf{e}}_{s,\zeta} = 0.$$

We notice that $\mathfrak{T}'(0) = 0$ and $\mathfrak{T}''(0) = 4\kappa\mathcal{T}(0)^{-2}D_z$. Let us analyze the derivative with respect to s. We have

$$\left(\mathcal{M}^{\mathsf{e}}_{s,\zeta} - \mu^{\mathsf{e}}_1(s,\zeta)\right)\left(\partial_s u^{\mathsf{e}}\right)_{s,\zeta} = \partial_s\mu^{\mathsf{e}}(s,\zeta)u^{\mathsf{e}}_{s,\zeta} - \left(\partial_s\mathcal{M}^{\mathsf{e}}_{s,\zeta}\right)u^{\mathsf{e}}_{s,\zeta}.$$

We notice that $\partial_s\mathcal{M}^{\mathsf{e}}_{0,\zeta} = 0$ and $\mathfrak{T}(0)\left(\partial_s u^{\mathsf{e}}\right)_{0,\zeta} = 0$. This implies that $\partial_s\mu^{\mathsf{e}}(0,\zeta) = 0$ by the Fredholm condition. Therefore $(0,\zeta_0)$ is a critical point of μ^{e}_1. Let us now consider the derivative with respect to s and ζ. We have $\partial_s\partial_\zeta\mathcal{M}^{\mathsf{e}}_{s,\zeta} = 0$ and by the Feynman-Hellmann formula

$$\partial_s\mu^{\mathsf{e}}(0,\zeta) = \int_{\mathcal{S}_{\alpha_0}} \partial_s\mathcal{M}^{\mathsf{e}}_{0,\zeta}u^{\mathsf{e}}_{0,\zeta}u^{\mathsf{e}}_{0,\zeta}\,\mathrm{d}z\,\mathrm{d}t,$$

we get

$$\partial_s\partial_\zeta\mu^{\mathsf{e}}(0,\zeta_0) = 0.$$

We shall now analyze the second order derivative with respect to s:

$$\left(\mathcal{M}^{\mathsf{e}}_{0,\zeta_0} - \mu^{\mathsf{e}}_1(0,\zeta_0)\right)\left(\partial^2_s u^{\mathsf{e}}\right)_{0,\zeta_0} = \partial^2_s\mu^{\mathsf{e}}_1(0,\zeta_0)u^{\mathsf{e}}_{0,\zeta_0} - \partial^2_s\mathcal{M}^{\mathsf{e}}_{0,\zeta_0}u_{0,\zeta_0},$$

with boundary condition $\mathfrak{T}(0)\left(\partial^2_s u^{\mathsf{e}}\right)_{0,\zeta_0} = -\mathfrak{T}''(0)u^{\mathsf{e}}_{0,\zeta_0}$. We have $\partial^2_s\mathcal{M}^{\mathsf{e}}_{0,\zeta_0} = 4\kappa\mathcal{T}(0)^{-1}D_z^2$. With the Fredholm condition, we get $\partial^2_s\mu^{\mathsf{e}}(0,\zeta_0) = 4\kappa\mathcal{T}(0)^{-1}\|D_z u^{\mathsf{e}}_{0,\zeta_0}\|^2$. $\square$

In Chapter 11, Section 2, we will provide local (near the point of the edge giving the maximal aperture) WKB expansions of the lowest eigenfunctions.

3.3.5. *Curvature induced magnetic bound states.* As we have seen, in many situations the spectral splitting appears in the second term of the asymptotic expansion of the eigenvalues. It turns out that we can also deal with more degenerate situations. The next lines are motivated by the initial paper [**92**] whose main result is recalled in (1.1.2).

Their fundamental result establishes that a smooth Neumann boundary can trap the lowest eigenfunctions near the points of maximal curvature. These considerations are generalized in [**67**, Theorem 1.1] where the complete asymptotic expansion of the n-th eigenvalue of $\mathfrak{L}^{\mathsf{c}}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2$ is provided and satisfies in particular:

$$\Theta_0 h - C_1\kappa_{\max}h^{3/2} + (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}h^{7/4} + o(h^{7/4}), \tag{2.3.16}$$

where $k_2 = -\kappa''(0)$. In this book, as in [**67**], we will consider the magnetic Neumann Laplacian on a smooth domain Ω such that the algebraic curvature κ satisfies the following assumption.

Assumption 2.55. *The function κ is smooth and admits a unique and non-degenerate maximum.*

In Chapter 11, Section 3 we prove that the lowest eigenfunctions are approximated by local WKB expansions which can be made global when for instance $\partial\Omega$ is the graph of a smooth function. In particular we recover the term $C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}$ by a method different from the one of Fournais and Helffer and we explicitly provide a candidate to be the optimal distance of Agmon in the boundary. Since it is quite unusual to exhibit pure magnetic Agmon distance, let us provide a precise statement. For that purpose, let us consider the following Neumann realization on $\mathsf{L}^2(\mathbb{R}^2_+, m(s,t)\,\mathrm{d}s\,\mathrm{d}t)$, which is nothing but the expression of the magnetic Laplacian in curvilinear coordinates,

$$\begin{aligned}\mathcal{L}^{\mathsf{c}}_h &= m(s,t)^{-1}hD_t m(s,t)hD_t \\ &+ m(s,t)^{-1}\left(hD_s + \zeta_0 h^{\frac12} - t + \kappa(s)\frac{t^2}{2}\right)m(s,t)^{-1}\left(hD_s + \zeta_0 h^{\frac12} - t + \kappa(s)\frac{t^2}{2}\right),\end{aligned} \tag{2.3.17}$$

where $m(s,t) = 1 - t\kappa(s)$. Thanks to the rescaling

$$t = h^{1/2}\tau, \qquad s = \sigma,$$

and after division by h the operator $\mathcal{L}^{\mathsf{c}}_h$ becomes

$$\begin{aligned}\mathfrak{L}^{\mathsf{c}}_h &= m(\sigma, h^{1/2}\tau)^{-1}D_\tau m(\sigma, h^{1/2}\tau)D_\tau \\ &+ m(\sigma, h^{1/2}\tau)^{-1}\left(h^{1/2}D_\sigma + \zeta_0 - \tau + h^{1/2}\kappa(\sigma)\frac{\tau^2}{2}\right)m(\sigma, h^{1/2}\tau)^{-1}\left(h^{1/2}D_\sigma + \zeta_0 - \tau + h^{1/2}\kappa(\sigma)\frac{\tau^2}{2}\right),\end{aligned} \tag{2.3.18}$$

on the space $\mathsf{L}^2(m(\sigma, h^{1/2}\tau)\,\mathrm{d}\sigma\,\mathrm{d}\tau)$.

Theorem 2.56. *Under Assumption 2.51, there exist a function*

$$\Phi = \Phi(\sigma) = \left(\frac{2C_1}{\nu_1''(\zeta_0)}\right)^{1/2}\left|\int_0^\sigma (\kappa(0) - \kappa(s))^{1/2}\,\mathrm{d}s\right|$$

defined in a neighborhood $\mathcal{V}$ of $(0,0)$ such that $\operatorname{Re}\Phi''(0) > 0$*, and a sequence of real numbers* $(\lambda^{\mathsf{c}}_{n,j})$ *such that*

$$\lambda^{\mathsf{c}}_n(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda^{\mathsf{c}}_{n,j} h^{\frac{j}{4}}.$$

Besides there exists a formal series of smooth functions on $\mathcal{V}$*,*

$$a^{\mathsf{c}}_n \underset{h\to 0}{\sim} \sum_{j\geq 0} a^{\mathsf{c}}_{n,j} h^{\frac{j}{4}}$$

such that

$$\left(\mathfrak{L}^{\mathsf{c}}_h - \lambda^{\mathsf{c}}_n(h)\right)\left(a^{\mathsf{c}}_n e^{-\Phi/h^{\frac{1}{4}}}\right) = \mathcal{O}\left(h^{\infty}\right) e^{-\Phi/h^{\frac{1}{4}}}.$$

We also have that $\lambda^{\mathsf{c}}_{n,0} = \Theta_0$, $\lambda^{\mathsf{c}}_{n,1} = 0$, $\lambda^{\mathsf{c}}_{n,1} = -C_1\kappa_{\max}$ *and* $\lambda^{\mathsf{c}}_{n,3} = (2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}$. *The main term in the Ansatz is in the form*

$$a^{\mathsf{c}}_{n,0}(\sigma,\tau) = f^{\mathsf{c}}_{n,0}(\sigma) u_{\zeta_0}(\tau).$$

Moreover, for all $n \geq 1$*, there exist* $h_0 > 0$*,* $c > 0$ *such that for all* $h \in (0,h_0)$*, we have*

$$\mathcal{B}\Big(\lambda^{\mathsf{c}}_{n,0} + \lambda^{\mathsf{c}}_{n,2} h^{1/2} + \lambda^{\mathsf{c}}_{n,3} h^{\frac{3}{4}}, c h^{\frac{3}{4}}\Big) \cap \mathsf{sp}\left(\mathfrak{L}^{\mathsf{c}}_h\right) = \{\lambda^{\mathsf{c}}_n(h)\},$$

and $\lambda^{\mathsf{c}}_n(h)$ *is a simple eigenvalue.*

Remark 2.57. *In particular, Theorem 2.56 proves that there are no odd powers of* $h^{\frac{1}{8}}$ *in the expansion of the eigenvalues (see* [**67**, Theorem 1.1]*).*

CHAPTER 3

Semiclassical magnetic normal forms

> Now do you imagine he would have attempted to inquire or learn what he thought he knew, when he did not know it, until he had been reduced to the perplexity of realizing that he did not know, and had felt a craving to know?
>
> *Meno*, Plato

In this chapter we enlighten the normal form philosophy explained in Chapter 1, Section 3 by presenting four results of *magnetic harmonic approximation*. As we will see, each situation will present its specific features and difficulties:

- How can we deal with a vanishing magnetic field in dimension two?
- How can we treat a problem with smooth boundary in dimension three?
- Can we still display a precise semiclassical asymptotics in dimension three if the boundary is not smooth?
- In dimension two and without boundary, can we describe more than $\lambda_n(h)$ for fixed n?

1. Vanishing magnetic fields in dimension two

In this section we study the influence of the cancellation of the magnetic field along a smooth curve in dimension two.

1.1. Framework. We consider a vector potential $\mathbf{A} \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ and we consider the self-adjoint operator on $\mathsf{L}^2(\mathbb{R}^2)$ defined by:

$$\mathfrak{L}_{h,\mathbf{A}} = (-ih\nabla + \mathbf{A})^2.$$

Notation 3.1. *We will denote by $\lambda_n(h)$ the n-th eigenvalue of $\mathfrak{L}_{h,\boldsymbol{A}}$.*

1.1.1. *How does $\boldsymbol{B}$ vanish?* In order $\mathfrak{L}_{h,\mathbf{A}}$ to have compact resolvent, we will assume that:

$$\mathbf{B}(x) \underset{|x|\to+\infty}{\to} +\infty. \tag{3.1.1}$$

As in [**143, 85**], we will investigate the case when $\mathbf{B}$ cancels along a closed and smooth curve $\mathcal{C}$ in $\mathbb{R}^2$. We have already discussed the motivation in Chapter 2, Section 2. Let us notice that the assumption (3.1.1) could clearly be relaxed so that one could also consider a smooth, bounded and simply connected domain of $\mathbb{R}^2$ with Dirichlet or Neumann condition on the boundary as far as the magnetic field does not vanish near the boundary (in this case one should meet a model presented in Chapter 2, Section 2). We let:

$$\mathcal{C} = \{c(s), s \in \mathbb{R}\}.$$

We assume that $\mathbf{B}$ is non positive inside $\mathcal{C}$ and non negative outside. We introduce the standard tubular coordinates (s,t) near $\mathcal{C}$ defined by the map

$$(s,t) \mapsto c(s) + t\mathbf{n}(s),$$

where $\mathbf{n}(s)$ denotes the inward pointing normal to $\mathcal{C}$ at $c(s)$. The function $\tilde{\mathbf{B}}$ will denote $\mathbf{B}$ in the coordinates (s,t), so that $\tilde{\mathbf{B}}(s,0) = 0$.

1.1.2. *Heuristics and leading operator.* Let us adopt first a heuristic point of view to introduce the leading operator of the analysis presented in this section. We want to describe the operator $\mathfrak{L}_{h,\mathbf{A}}$ near the cancellation line of $\mathbf{B}$, that is near $\mathcal{C}$. In a rough approximation, near $(s_0, 0)$, we can imagine that the line is straight $(t = 0)$ and that the magnetic field cancels linearly so that we can consider $\tilde{\mathbf{B}}(s,t) = \gamma(s_0)t$ where $\gamma(s_0)$ is the derivative of $\tilde{\mathbf{B}}$ with respect to t. Therefore the operator to which we are reduced at the leading order near s_0 is:

$$h^2 D_t^2 + \left(hD_s - \gamma(s_0)\frac{t^2}{2}\right)^2.$$

This operator is a special case of the larger class introduced in Chapter 2, see also Chapter 10, Section 3.2.

1.2. Montgomery operator and rescaling. We will be led to use the Montgomery operator with parameters $\eta \in \mathbb{R}$ and $\gamma > 0$:

$$\mathfrak{L}^{[1]}_{\gamma,\zeta} = D_t^2 + \left(\zeta - \frac{\gamma}{2}t^2\right)^2. \tag{3.1.2}$$

The Montgomery operator has clearly compact resolvent and we can consider its lowest eigenvalue denoted by $\nu_1^{[1]}(\gamma,\zeta)$. In fact one can take $\gamma = 1$ up to the rescaling $t = \gamma^{-1/3}\tau$ and $\mathfrak{L}^{[1]}_{\gamma,\zeta}$ is unitarily equivalent to:

$$\gamma^{2/3}\left(D_\tau^2 + (-\eta\gamma^{-1/3} + \frac{1}{2}\tau^2)^2\right) = \gamma^{2/3}\mathfrak{L}^{[1]}_{1,\zeta\gamma^{-1/3}}.$$

Let us emphasize that this rescaling is related with the normal form analysis that we will use in the semiclassical spectral asymptotics. For all $\gamma > 0$, we have (see Chapter 2, Proposition 2.5):

(3.1.3) $\zeta \mapsto \nu_1^{[1]}(\gamma,\zeta)$ admits a unique and non-degenerate minimum at a point $\zeta_0^{[1]}(\gamma)$.

If $\gamma = 1$, we have $\zeta_0^{[1]}(1) = \zeta_0^{[1]}$. We may write:

$$\inf_{\zeta\in\mathbb{R}} \nu_1^{[1]}(\gamma,\zeta) = \gamma^{2/3}\nu_1^{[1]}(\zeta_0^{[1]}). \tag{3.1.4}$$

Let us recall some notation.

Notation 3.2. *We notice that* $\mathfrak{L}^{[1]}_{\zeta} = \mathfrak{L}^{[1]}_{1,\zeta}$ *and we denote by* $u^{[1]}_{\zeta}$ *the* L^2*-normalized and positive eigenfunction associated with* $\nu^{[1]}_1(\zeta)$*.*

For fixed $\gamma > 0$, the family $(\mathfrak{L}^{[1]}_{\gamma,\zeta})_{\eta\in\mathbb{R}}$ is an analytic family of type (B) so that the eigenpair $(\nu^{[1]}_1(\zeta), u^{[1]}_{\zeta})$ has an analytic dependence on ζ (see [**108**]).

1.3. Semiclassical asymptotics with vanishing magnetic fields. We consider the normal derivative of $\mathbf{B}$ on $\mathcal{C}$, i.e. the function $\gamma : s \mapsto \partial_t \tilde{\mathbf{B}}(s,0)$. We will assume that:

Assumption 3.3. γ *admits a unique, non-degenerate and positive minimum at* x_0*.*

We let $\gamma_0 = \gamma(0)$ and assume without loss of generality that $x_0 = (0,0)$. Let us state the main result of this section:

Theorem 3.4. *We assume Assumption 3.3. For all* $n \geq 1$*, there exists a sequence* $(\theta^n_j)_{j\geq 0}$ *such that we have:*

$$\lambda_n(h) \underset{h\to 0}{\sim} h^{4/3} \sum_{j\geq 0} \theta^n_j h^{j/6}$$

where:

$$\theta^n_0 = \gamma_0^{2/3} \nu^{[1]}_1(\zeta^{[1]}_0), \quad \theta^n_1 = 0, \quad \theta^n_2 = \gamma_0^{2/3} C_0 + \gamma_0^{2/3}(2n-1)\left(\frac{\alpha \nu^{[1]}_1(\eta_0)(\nu^{[1]}_1)''(\zeta^{[1]}_0)}{3}\right)^{1/2},$$

where we have let:

$$\alpha = \frac{1}{2}\gamma_0^{-1}\gamma''(0) > 0 \tag{3.1.5}$$

and:

$$C_0 = \langle L u^{[1]}_{\zeta^{[1]}_0}, u^{[1]}_{\zeta^{[1]}_0} \rangle_{\mathsf{L}^2(\mathbb{R}_{\hat{\tau}})}, \tag{3.1.6}$$

where:

$$L = 2k(0)\gamma_0^{-4/3}\left(\frac{\hat{\tau}^2}{2} - \zeta^{[1]}_0\right)\hat{\tau}^3 + 2\hat{\tau}\gamma_0^{-1/3}\kappa(0)\left(-\zeta^{[1]}_0 + \frac{\hat{\tau}^2}{2}\right)^2,$$

and:

$$k(0) = \frac{1}{6}\partial_t^2 \tilde{\boldsymbol{B}}(0,0) - \frac{\kappa(0)}{3}\gamma_0.$$

Remark 3.5. *This theorem is mainly motivated by the paper of Helffer and Kordyukov* [**85**] *(see also* [**83**, Section 5.2] *where the above result is presented as a conjecture and the paper* [**91**] *where the case of discrete wells is analyzed) where the authors prove a one term asymptotics for all the eigenvalues (see* [**85**, Corollary 1.1]*). Moreover, they also prove an accurate upper bound in* [**85**, Theorem 1.4] *thanks to a Grushin type method (see* [**80**]*). This result could be generalized to the case when the magnetic vanishes on hypersurfaces at a given order.*

2. Variable magnetic field and smooth boundary in dimension three

This section is devoted to the investigation of the relation between the boundary and the magnetic field in dimension three. We will see that the semiclassical structure is completely different from the one presented in the previous section.

2.1. A toy operator with variable magnetic field. Let us introduce the geometric domain

$$\Omega_0 = \{(x,y,z) \in \mathbb{R}^3 : |x| \leq x_0, |y| \leq y_0 \text{ and } 0 < z \leq z_0\},$$

where $x_0, y_0, z_0 > 0$. The part of the boundary which carries the Dirichlet condition is given by

$$\partial_{\mathsf{Dir}}\Omega_0 = \{(x,y,z) \in \Omega_0 : |x| = x_0 \text{ or } |y| = y_0 \text{ or } z = z_0\}.$$

2.1.1. *Definition of the operator.* For $h > 0$, $\alpha \geq 0$ and $\theta \in \left(0, \frac{\pi}{2}\right)$, we consider the self-adjoint operator:

$$\mathfrak{L}_{h,\alpha,\theta} = h^2 D_y^2 + h^2 D_z^2 + (hD_x + z\cos\theta - y\sin\theta + \alpha z(x^2+y^2))^2, \tag{3.2.1}$$

with domain:

$$\begin{aligned} \mathsf{Dom}\,(\mathfrak{L}_{h,\alpha,\theta}) = \{\psi \in L^2(\Omega_0) : \quad & \mathfrak{L}_{h,\alpha,\theta}\psi \in L^2(\Omega_0), \\ & \psi = 0 \quad \text{on} \quad \partial_{\mathsf{Dir}}\Omega_0 \quad \text{and} \quad \partial_z\psi = 0 \text{ on } z = 0\}. \end{aligned}$$

We denote by $(\lambda(h), u_h)$ an eigenpair and we let $\mathfrak{L}_h = \mathfrak{L}_{h,\alpha,\theta}$ (we omit the dependence on α and θ). The vector potential is expressed as:

$$\mathbf{A}(x,y,z) = (V_\theta(y,z) + \alpha z(x^2+y^2), 0, 0)$$

where

$$V_\theta(y,z) = z\cos\theta - y\sin\theta. \tag{3.2.2}$$

The associated magnetic field is given by:

$$\nabla \times \mathbf{A} = \mathbf{B} = (0, \cos\theta + \alpha(x^2+y^2), \sin\theta - 2\alpha yz). \tag{3.2.3}$$

2.1.2. *Constant magnetic field ($\alpha = 0$).* Let us examine the important case when $\alpha = 0$:

$$\mathfrak{L}_{h,0,\theta} = h^2 D_y^2 + h^2 D_z^2 + (hD_x + V_\theta(y,z))^2,$$

viewed as an operator on $\mathsf{L}^2(\mathbb{R}^3_+)$. We perform the rescaling:

$$x = h^{1/2}r, \quad y = h^{1/2}s, \quad z = h^{1/2}t \tag{3.2.4}$$

and the operator becomes (after division by h):

$$\mathfrak{L}_{1,0,\theta} = D_s^2 + D_t^2 + (D_r + V_\theta(s,t))^2.$$

Making a Fourier transform in the variable r denoted by $\mathcal{F}$, we get:

$$\mathcal{F}\mathfrak{L}_{1,0,\theta}\mathcal{F}^{-1} = D_s^2 + D_t^2 + (\eta + V_\theta(s,t))^2. \tag{3.2.5}$$

Then, we use a change of coordinates:

$$U_\theta(\eta, s, t) = (\rho, \sigma, \tau) = \left(\eta, s - \frac{\eta}{\sin\theta}, t\right) \tag{3.2.6}$$

and we obtain:

$$\mathfrak{H}_\theta^{\mathsf{Neu}} = U_\theta \mathcal{F}\mathfrak{L}_{1,0,\theta}\mathcal{F}^{-1}U_\theta^{-1} = D_\sigma^2 + D_\tau^2 + V_\theta(\sigma,\tau)^2.$$

Notation 3.6. *We denote by $\mathfrak{Q}_\theta^{\mathsf{Neu}}$ the quadratic form associated with $\mathfrak{H}_\theta^{\mathsf{Neu}}$.*

The operator $\mathfrak{H}_\theta^{\mathsf{Neu}}$ viewed as an operator acting on $\mathsf{L}^2(\mathbb{R}^2_+)$ is nothing but $\mathfrak{L}_\theta^{\mathsf{LP}}$ (see Chapter 1, Section 1.4.4). Let us also recall that the lower bound of the essential spectrum is related, through the Persson's theorem (see Chapter 5), to the following estimate:

$$\mathfrak{q}_\theta^{\mathsf{LP}}(\chi_R u) \geq (1 - \varepsilon(R))\|\chi_R u\|, \quad \forall u \in \mathsf{Dom}\,(\mathfrak{q}_\theta^{\mathsf{LP}}),$$

where $\mathfrak{q}_\theta^{\mathsf{LP}}$ is the quadratic form associated with $\mathfrak{L}_\theta^{\mathsf{LP}}$, where χ_R is a cutoff function away from the ball $B(0,R)$ and $\varepsilon(R)$ is tending to zero when R tends to infinity. Moreover, if we consider the Dirichlet realization $\mathfrak{L}_\theta^{\mathsf{LP,Dir}}$, we have:

$$\mathfrak{q}_\theta^{\mathsf{LP,Dir}}(u) \geq \|u\|^2, \quad \forall u \in \mathsf{Dom}\,(\mathfrak{q}_\theta^{\mathsf{LP,Dir}}). \tag{3.2.7}$$

2.1.3. *A "generic" model.* Let us explain why we are led to consider our model. Let us introduce the fundamental invariant in the case of variable magnetic field and our generic assumptions. We let:

$$\hat{\mathbf{B}}(x,y) = \mathfrak{s}(\theta(x,y))\|\mathbf{B}(x,y,0)\|,$$

where $\theta(x,y)$ is the angle of $\mathbf{B}(x,y,0)$ with the boundary $z=0$:

$$\|\mathbf{B}(x,y,0)\|\sin\theta(x,y) = \mathbf{B}(x,y,0)\cdot\mathbf{n}(x,y),$$

where $\mathbf{n}(x,y)$ is the inward normal at $(x,y,0)$. It is proved in [**124**] that the semiclassical asymptotics of the lowest eigenvalue is:

$$\lambda_1(h) = \min(\inf_{z=0}\hat{\mathbf{B}}, \inf_{\Omega_0}\|\mathbf{B}\|)h + o(h).$$

We are interested in the case when the following generic assumptions are satisfied:

$$\inf_{z=0}\hat{\mathbf{B}} < \inf_{\Omega_0}\|\mathbf{B}\| \tag{3.2.8}$$

(3.2.9) $\hat{\mathbf{B}}$ admits a unique and non degenerate minimum.

Under these assumptions, a three terms upper bound is proved for $\lambda_1(h)$ in [**150**] and the corresponding lower bound, for a general domain, is still an open problem.

For $\alpha > 0$, the toy operator (3.2.1) is the simplest example of a generic Schrödinger operator with variable magnetic field satisfying Assumptions (3.2.8) and (3.2.9). We have the Taylor expansion:

$$\hat{\mathbf{B}}(x,y)=\mathfrak{s}(\theta)+\alpha C(\theta)(x^2+y^2)+O(|x|^3+|y|^3). \tag{3.2.10}$$

with:

$$C(\theta)=\cos\theta\mathfrak{s}(\theta)-\sin\theta\mathfrak{s}'(\theta).$$

Moreover, it is proved in Chapter 6, Proposition 6.10 that $C(\theta)>0$, for $\theta\in\left(0,\frac{\pi}{2}\right)$. Thus, Assumption (3.2.9) is verified if x_0, y_0 and z_0 are fixed small enough. Using $\mathfrak{s}(\theta)<1$ when $\theta\in\left(0,\frac{\pi}{2}\right)$ and $\|\mathbf{B}(0,0,0)\|=1$, we get Assumption (3.2.8).

2.1.4. *Remark on the function $\hat{\boldsymbol{B}}$.* Using the explicit expression of the magnetic field, we have:

$$\hat{\mathbf{B}}(x,y)=\hat{\mathbf{B}}_{\mathsf{rad}}(R),\quad R=\alpha(x^2+y^2)$$

and an easy computation gives:

$$\hat{\mathbf{B}}_{\mathsf{rad}}(R)=\|\mathbf{B}_{\mathsf{rad}}(R)\|\mathfrak{s}\left(\arctan\left(\frac{\sin\theta}{\cos\theta+R}\right)\right),$$

with

$$\|\mathbf{B}_{\mathsf{rad}}(R)\|=\sqrt{(\cos\theta+R)^2+\sin^2\theta}.$$

The results of Chapter 6 imply that $\hat{\mathbf{B}}_{\mathsf{rad}}$ is strictly increasing and

$$\partial_R\hat{\mathbf{B}}_{\mathsf{rad}}(R=0)=C(\theta)>0.$$

Consequently, $\hat{\mathbf{B}}$ admits a unique and non degenerate minimum on $\mathbb{R}^3_+$ and tends to infinity far from 0. This is easy to see that:

$$\inf_{\mathbb{R}^3_+}\|\mathbf{B}\|=\cos\theta.$$

We deduce that, as long as $\mathfrak{s}(\theta)<\cos\theta$, the generic assumptions are satisfied with $\Omega_0=\mathbb{R}^3_+$.

2.1.5. *Three dimensional magnetic wells induced by the magnetic field and the (smooth) boundary.* Let us introduce the fundamental operator

$$\mathfrak{S}_\theta(D_\rho,\rho)=\left(2\int_{\mathbb{R}^2_+}\tau V_\theta(u_\theta^{\mathsf{LP}})^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\mathcal{H}_{\mathsf{harm}}+\left(\frac{2}{\sin\theta}\int_{\mathbb{R}^2_+}\tau V_\theta(u_\theta^{\mathsf{LP}})^2\,\mathrm{d}\sigma\,\mathrm{d}\tau\right)\rho+d(\theta),$$

where

$$\mathcal{H}_{\mathsf{harm}}=D_\rho^2+\frac{\rho^2}{\sin^2\theta}$$

and

$$d(\theta)=\sin^{-2}\theta\langle\tau(D_\sigma^2V_\theta+V_\theta D_\sigma^2)u_\theta^{\mathsf{LP}},u_\theta^{\mathsf{LP}}\rangle+2\int_{\mathbb{R}^2_+}\tau\sigma^2V_\theta(u_\theta^{\mathsf{LP}})^2\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We recall the important fact that (see [**150**, Formula (2.31)]):

$$2\int_{\mathbb{R}^2_+}tV_\theta(u_\theta^{\mathsf{LP}})^2\,\mathrm{d}s\,\mathrm{d}t=C(\theta)>0,$$

so that $\mathfrak{S}_\theta(D_\rho,\rho)$ can be viewed as the harmonic oscillator up to dilation and translations.

We can now state the main result of this section.

Theorem 3.7. *For all $\alpha > 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, there exist a sequence $(\mu_{j,n})_{j\geq 0}$ and $\varepsilon_0 > 0$ s. t. for $|x_0| + |y_0| + |z_0| \leq \varepsilon_0$,*

$$\lambda_n(h) \sim h \sum_{j\geq 0} \mu_{j,n} h^j$$

and we have $\mu_{0,n} = \mathfrak{s}(\theta)$ and $\mu_{1,n}$ is the n-th eigenvalue of $\mathfrak{S}_\theta(D_\rho, \rho)$.

3. When a magnetic field meets a curved edge

We analyze here the effect of an edge in the boundary and how its combines with the magnetic field to produce a spectral asymptotics.

3.1. Geometrical assumptions and local models.

3.1.1. *Description of the lens.* We first define the lens Ω.

Definition 3.8. *Let Σ be a smooth and connected surface in $\mathbb{R}^3$ and Π be the plane $x_3 = 0$. We assume that the intersection $\Sigma \cap \Pi$ is a smooth and closed curve and that Σ and Π intersect neither normally nor tangentially. Denoting by Σ^+ the set $\{\mathsf{x} \in \Sigma : x_3 > 0\}$ and by Σ^- its symmetric with respect to $x_3 = 0$, the lens Ω is the open set of the points lying between Σ^+ and Σ^- whereas the edge is*

$$E = \overline{\Sigma^+} \cap \overline{\Sigma^-}. \tag{3.3.1}$$

We define $\alpha(\mathsf{x})$ as the opening angle between Σ^- and Σ^+ at the point $\mathsf{x} \in E$. We assume that $\alpha(\mathsf{x}) \in (0, \pi)$ for all $\mathsf{x} \in E$.

In our situation the magnetic field $\mathbf{B} = (0, 0, 1)$ is normal to the plane where the edge lies. For $\mathsf{x} \in \partial\Omega \setminus E$ we introduce the angle $\theta(\mathsf{x})$ defined by:

$$\mathbf{B} \cdot \boldsymbol{n}(\mathsf{x}) = \sin\theta(\mathsf{x}). \tag{3.3.2}$$

A model lens with constant opening angle is given by two parts of a sphere glued together (see Figure 1). In this case we have

$$\forall \mathsf{x} \in \partial\Omega \setminus E, \quad \frac{\pi - \alpha}{2} < \theta(\mathsf{x}) \tag{3.3.3}$$

where $\alpha \in (0, \pi)$ is the opening angle of the lens and we notice that the magnetic field is nowhere tangent to the boundary. We will assume that the opening angle of the lens is variable. For a given point x of the boundary, we analyze the localized (in a neighborhood of x) magnetic Laplacian and we distinguish between x belonging to the edge and x belonging to the smooth part of the boundary.

3.1.2. *Leading Operator.* Let $\mathsf{x} \in E$ and V a small neighborhood of x in Ω. We suppose that the opening angle at x is α. There is a diffeomorphism, denoted by the local coordinates $(\hat{s}, \hat{t}, \hat{z})$, from V to an open subset of the infinite wedge $\mathcal{W}_\alpha$. This

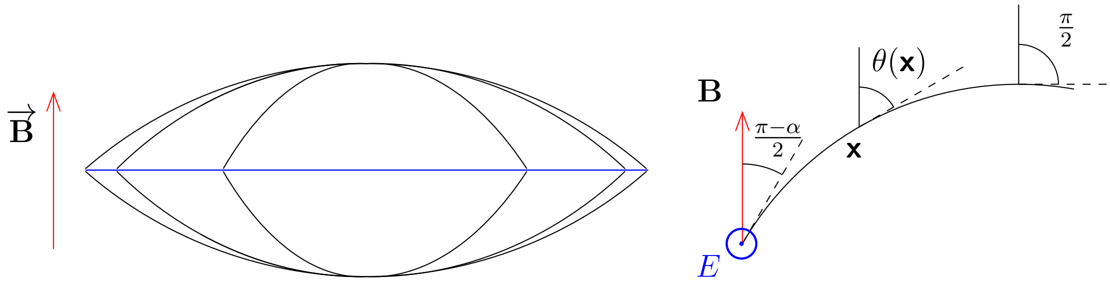


FIGURE 1. A lens Ω: the magnetic field is nowhere tangent to the boundary and it makes the angle $\theta(\mathbf{x})$ with the regular boundary.

diffeomorphism can be explicitly described. We refer to Chapter 2, Section 3.3.3 where some basic properties of the magnetic wedge were discussed.

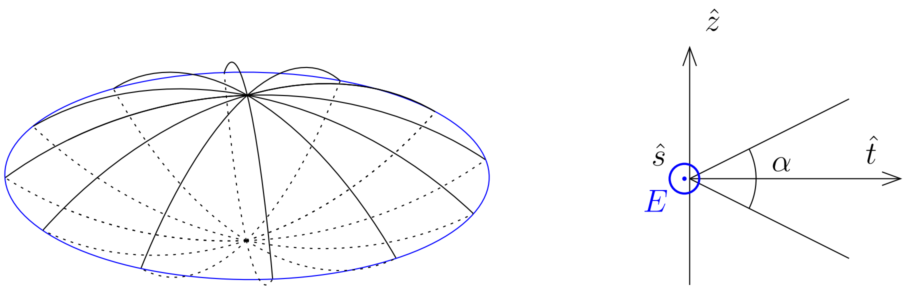


FIGURE 2. Using the local coordinates $(\hat{s}, \hat{t}, \hat{z})$, a neighborhood of a point of the edge can be described as a subset of the infinite wedge $\mathcal{W}_\alpha$.

The model situations (magnetic wedge and smooth boundary) lead to compare the following quantities:

$$\inf_{\mathbf{x}\in E} \nu_1^{\mathsf{e}}(\alpha(\mathbf{x})), \qquad \inf_{\mathbf{x}\in\partial\Omega\setminus E} \mathfrak{s}_1(\theta(\mathbf{x})),$$

where $\theta(\mathbf{x})$ is defined in (3.3.2), $\alpha(\mathbf{x})$ and E are defined in Definition 3.8. Let us state the different assumptions under which we work:

Assumption 3.9.

$$\inf_{\mathbf{x}\in E} \nu_1^{\mathsf{e}}(\alpha(\mathbf{x})) < \inf_{\mathbf{x}\in\partial\Omega\setminus E} \mathfrak{s}_1(\theta(\mathbf{x})). \tag{3.3.4}$$

Remark 3.10. *Using* (3.3.3)*, the fact that $\mathfrak{s}_1$ is increasing and Proposition 2.46, we check that, in the model case when Ω is made of two parts of a sphere glued together, Assumption 3.9 is satisfied for α small enough. By a continuity argument, Assumption 3.9 holds for not too large perturbations of this lens.*

From the properties of the leading operator we see that we will be led to work near the point of the edge of maximal opening. Therefore we will assume the following generic assumption:

Assumption 3.11. *We denote by $\alpha : E \mapsto (0,\pi)$ the opening angle of the lens. We assume that α admits a unique and non degenerate maximum at the point x_0 and we let*

$$\alpha_0 = \max_E \alpha.$$

We denote $\tau = \tan\frac{\alpha}{2}$ and $\tau_0 = \tan\frac{\alpha_0}{2}$.

In particular, under this assumption and Conjecture 2.48, the function $s \mapsto \nu_1^{\mathsf{e}}(\alpha(s))$ admits a unique and non-degenerate minimum.

3.2. Normal form. This is "classical" that Assumption 3.9 leads to localization properties of the eigenfunctions near the edge E and more precisely near the points of the edge where $E \ni \mathsf{x} \mapsto \nu(\alpha(\mathsf{x}))$ is minimal. Therefore, since ν is decreasing and thanks to Assumption 3.11, we expect that the first eigenfunctions concentrate near the point x_0 where the opening is maximal. This is possible to introduce, near each $\mathsf{x} \in E$, a local change of variables which transforms a neighborhood of x in Ω in a ε_0-neighborhood of $(0,0,0)$ of $\mathcal{W}_{\alpha(\mathsf{x})}$, denoted by $\mathcal{W}_{\alpha(\mathsf{x}),\varepsilon_0}$.

For the convenience of the reader, let us write below the expression of the magnetic Laplacian in the new local coordinates $(\check{s},\check{t},\check{z})$ where $\check{s}$ is a curvilinear abscissa of the edge. The magnetic Laplacian $\mathfrak{L}_h^{\mathsf{lens}}$ is given by the Laplace-Beltrami expression (on $\mathsf{L}^2(|\check{G}|^{1/2}\,\mathrm{d}\check{s}\,\mathrm{d}\check{t}\,\mathrm{d}\check{z})$):

$$\check{\mathfrak{L}}_h^{\mathsf{lens}} := |\check{G}|^{-1/2}\check{\nabla}_h|\check{G}|^{1/2}\check{G}^{-1}\check{\nabla}_h \tag{3.3.5}$$

where:

$$\check{\nabla}_h = \begin{pmatrix} hD_{\check{s}} \\ hD_{\check{t}} \\ h\tau(\check{s})^{-1}\tau(0)D_{\check{z}} \end{pmatrix} + \begin{pmatrix} -\check{t} + \zeta_0^{\mathsf{e}}h^{1/2} - h\frac{\tau'}{2\tau}(\check{z}D_{\check{z}} + D_{\check{z}}\check{z}) + \check{R}_1(\check{s},\check{t},\check{z}) \\ 0 \\ 0 \end{pmatrix}. \tag{3.3.6}$$

The precise forms of the Taylor expansions of the remainder $\check{R}_1$, the metric $\check{G}$ and the function $\check{s} \mapsto \tau(\check{s})$ are analyzed in [**147**]. The reader will not need them to understand the structure of the investigation.

Remark 3.12. *Such a normal form allows us to describe the leading structure of this magnetic Laplace-Beltrami operator. Indeed, if we just keep the main terms in* (3.3.5) *by neglecting formally the geometrical factors, our operator takes the simpler form:*

$$(hD_{\check{s}} - \check{t} + \zeta_0^{\mathsf{e}}h^{1/2})^2 + h^2D_{\check{t}}^2 + h^2\tau(0)^2\tau(\check{s})^{-2}D_{\check{z}}^2,$$

whose symbol with respect to s is discussed in Chapter 2, Section 3.3.4. Performing another formal Taylor expansion near $\check{s} = 0$, we are led to the following operator:

$$(hD_{\check{s}} - \check{t} + \zeta_0^{\mathsf{e}}h^{1/2})^2 + h^2D_{\check{t}}^2 + h^2D_{\check{z}}^2 + ch^2\check{s}^2D_{\check{z}}^2,$$

where $c > 0$. Using a scaling, we get a rescaled operator $\mathcal{L}_h$ whose first term is the leading operator $\mathfrak{L}_{\alpha_0}^{\mathsf{e}}$ and which allows to construct quasimodes. Moreover this form is suitable to establish microlocalization properties of the eigenfunctions with respect to $D_{\check{s}}$.

3.3. Magnetic wells induced by the variations of a singular geometry. The main result of this section is a complete asymptotic expansion of all the first eigenvalues of $\mathfrak{L}_h^{\mathsf{lens}}$:

Theorem 3.13. *We assume that Conjecture 2.48 is true. We also assume Assumptions 3.9 and 3.11. For all $n \geq 1$ there exists $(\mu_{j,n})_{j\geq 0}$ such that we have:*

$$\lambda_n(h) \underset{h\to 0}{\sim} h \sum_{j\geq 0} \mu_{j,n} h^{j/4}.$$

Moreover, we have:

$$\mu_{0,n} = \nu_1^{\mathsf{e}}(\alpha_0), \quad \mu_{1,n} = 0, \quad \mu_{2,n} = \omega_0 + (2n-1)\sqrt{\kappa\tau_0^{-1}\|D_{\hat{z}} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}\|^2 \partial_\zeta^2 \nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}})},$$

where the geometrical constants ω_0 and κ are respectively given in (14.1.13) *and* (14.1.6)*.*

Remark 3.14. *We observe that, for all $n \geq 1$, $\lambda_n(h)$ is simple for h small enough. This simplicity, jointly with a quasimodes construction, also provides an approximation of the corresponding normalized eigenfunction.*

4. Birkhoff normal form

Sections 1, 2 and 3 are mainly structured around the idea of normal forms. Indeed, in each case we have introduced an appropriate change of variable or equivalently a Fourier integral operator and we have *normalized* the magnetic Laplacian by transferring the magnetic geometry into the coefficients of the operator. We can interpret this normalization as a very explicit application of the Egorov theorem. Then, in the investigation, we are led to use the Feshbach projection to simplified again the situation. This projection method can also be heuristically interpreted as a normal form in the spirit of Egorov: taking the average of the operator in a certain quantum state is nothing but the quantum analog of averaging a full Hamiltonian with respect to a reduced Hamiltonian. In problems with boundaries or with vanishing magnetic fields it appears that the dynamics of the reduced Hamiltonian is less understood (due to the boundary conditions for instance) than the spectral theory of its quantization. Keeping this remark in mind it now naturally appears that we should implement a general normal form for instance in the simplest situation of dimension two, without boundary and with a non vanishing magnetic field.

4.1. Preliminary considerations. As we shall recall below, a particle in a magnetic field has a fast rotating motion, coupled to a slow drift. It is of course expected that the long-time behaviour of the particle is governed by this drift. From the quantum point of view we will see that this drift is governed by a reduced Hamiltonian which can be approximated by the magnetic field itself.

Let (e_1, e_2, e_3) be an orthonormal basis of $\mathbb{R}^3$ and let us consider the plane $\mathbb{R}^2 = \{q_1 e_1 + q_2 e_2;\ (q_1, q_2) \in \mathbb{R}^2\}$, and the magnetic field is $\mathbf{B} = B(q_1, q_2)e_3$. For the moment we only assume that $q = (q_1, q_2)$ belongs to an open set Ω where B does not vanish.

With appropriate constants, Newton's equation for the particle under the action of the Lorentz force writes

$$\ddot{q} = 2\dot{q} \times \mathbf{B}. \tag{3.4.1}$$

The kinetic energy $E = \frac{1}{4}\|\dot{q}\|^2$ is conserved. If the speed $\dot{q}$ is small, we may linearize the system, which amounts to have a constant magnetic field. Then, as is well known, the integration of Newton's equations gives a circular motion of angular velocity $\dot{\theta} = -2B$ and radius $\|\dot{q}\|/2B$. Thus, even if the norm of the speed is small, the angular velocity may be very important. Now, if B is in fact not constant, the particle may leave the region where the linearization is meaningful. This suggests a separation of scales (as in the semiclassical and quantum context of Sections 1 and 3), where the fast circular motion is superposed with a slow motion of the center.

It is known that the system (3.4.1) is Hamiltonian and that the usual kinetic energy has to be replaced by the so-called Peierls kinetic energy. Let $\mathbf{A} \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

As usual we may identify $\mathbf{A} = (A_1, A_2)$ with the 1-form $A = A_1\,\mathrm{d}q_1 + A_2\,\mathrm{d}q_2$. Then, as a differential 2-form, $\mathrm{d}A = (\frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2})\,\mathrm{d}q_1 \wedge \mathrm{d}q_2 = B\,\mathrm{d}q_1 \wedge \mathrm{d}q_2$. In terms of canonical variables $(q, p) \in T^*\mathbb{R}^2 = \mathbb{R}^4$ the Hamiltonian of our system is

$$H(q, p) = \|p - \mathbf{A}(q)\|^2. \tag{3.4.2}$$

We use here the Euclidean norm on $\mathbb{R}^2$, which allows the identification of $\mathbb{R}^2$ with $(\mathbb{R}^2)^*$ by

$$\forall (v, p) \in \mathbb{R}^2 \times (\mathbb{R}^2)^*, \qquad p(v) = \langle p, v\rangle. \tag{3.4.3}$$

Thus, the canonical symplectic structure ω on $T^*\mathbb{R}^2$ is given by

$$\omega((Q_1, P_1), (Q_2, P_2)) = \langle P_1, Q_2\rangle - \langle P_2, Q_1\rangle. \tag{3.4.4}$$

It is easy to check that Hamilton's equations for H imply Newton's equation (3.4.1). In particular, through the identification (3.4.3) we have $\dot{q} = 2(p - \mathbf{A})$.

4.2. Classical magnetic normal forms. Before considering the semiclassical magnetic Laplacian we shall briefly discuss some results concerning the classical dynamics for large time. We will not discuss the proofs in this book, but these considerations will give some insights to answer the semiclassical questions. As we have already mentioned in the introduction, the large time dynamics problem has to face the issue that the conservation of the energy H is not enough to confine the trajectories in a compact set .

The first result shows the existence of a smooth symplectic diffeomorphism that transforms the initial Hamiltonian into a normal form, up to any order in the distance to the zero energy surface.

Theorem 3.15. *Let*

$$H(q,p) := \|p - \boldsymbol{A}(q)\|^2 , \quad (q,p) \in T^*\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2,$$

where the magnetic potential $\boldsymbol{A} : \mathbb{R}^2 \to \mathbb{R}^2$ *is smooth. Let* $B := \frac{\partial A_2}{\partial q_1} - \frac{\partial A_1}{\partial q_2}$ *be the corresponding magnetic field. Let* $\Omega \subset \mathbb{R}^2$ *be a bounded open set where* B *does not vanish. Then there exists a symplectic diffeomorphism* Φ*, defined in an open set* $\tilde{\Omega} \subset \mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$*, with values in* $T^*\mathbb{R}^2$*, which sends the plane* $\{z_1 = 0\}$ *to the surface* $\{H = 0\}$*, and such that*

$$H \circ \Phi = |z_1|^2 f(z_2, |z_1|^2) + \mathcal{O}(|z_1|^\infty), \tag{3.4.5}$$

where $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ *is smooth. Moreover, the map*

$$\varphi : \Omega \ni q \mapsto \Phi^{-1}(q, \boldsymbol{A}(q)) \in (\{0\} \times \mathbb{R}^2_{z_2}) \cap \tilde{\Omega} \tag{3.4.6}$$

is a local diffeomorphism and

$$f \circ (\varphi(q), 0) = |B(q)| .$$

In the following theorem we denote by $K = |z_1|^2 f(z_2, |z_1|^2) \circ \Phi^{-1}$ the (completely integrable) normal form of H given be Theorem 3.15 above. Let φ_H^t be the Hamiltonian flow of H, and let φ_K^t be the Hamiltonian flow of K. Let us state, without proofs, the important dynamical consequences of Theorem 3.15 (see Figure 3).

Theorem 3.16. *Assume that the magnetic field* $B > 0$ *is confining: there exists* $C > 0$ *and* $M > 0$ *such that* $B(q) \geq C$ *if* $\|q\| \geq M$*. Let* $C_0 < C$*. Then*

(1) *The flow* φ_H^t *is uniformly bounded for all starting points* (q,p) *such that* $B(q) \leq C_0$ *and* $H(q,p) = \mathcal{O}(\epsilon)$ *and for times of order* $\mathcal{O}(1/\epsilon^N)$*, where* N *is arbitrary.*
(2) *Up to a time of order* $T_\epsilon = \mathcal{O}(|\ln \epsilon|)$*, we have*

$$\left\|\varphi_H^t(q,p) - \varphi_K^t(q,p)\right\| = \mathcal{O}(\epsilon^\infty) \tag{3.4.7}$$

for all starting points (q,p) *such that* $B(q) \leq C_0$ *and* $H(q,p) = \mathcal{O}(\epsilon)$*.*

It is interesting to notice that, if one restricts to regular values of B, one obtains the same control for a much longer time, as stated below.

Theorem 3.17. *Under the same confinement hypothesis as Theorem 3.16, let* $J \subset (0, C_0)$ *be a closed interval such that* dB *does not vanish on* $B^{-1}(J)$*. Then up to a time of order* $T = \mathcal{O}(1/\epsilon^N)$*, for an arbitrary* $N > 0$*, we have*

$$\left\|\varphi_H^t(q,p) - \varphi_K^t(q,p)\right\| = \mathcal{O}(\epsilon^\infty)$$

for all starting points (q,p) *such that* $B(q) \in J$ *and* $H(q,p) = \mathcal{O}(\epsilon)$*.*

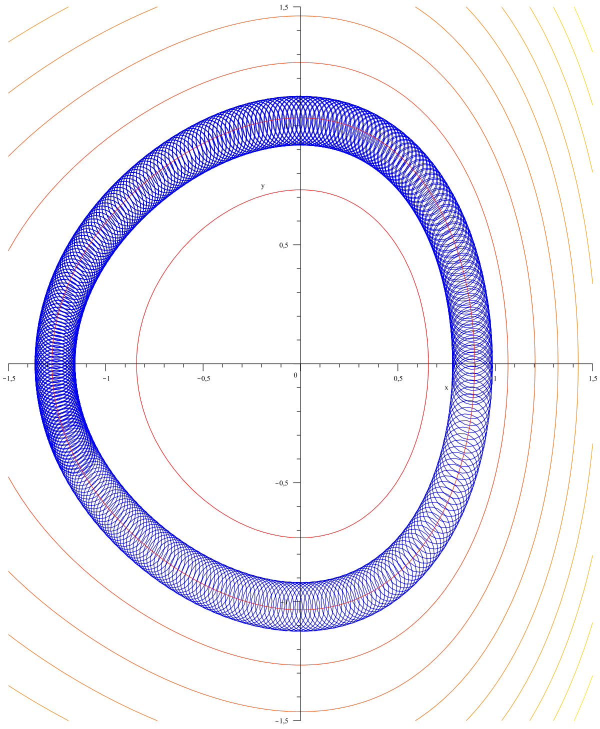


FIGURE 3. Numerical simulation of the flow of H when the magnetic field is given by $B(x,y) = 2 + x^2 + y^2 + \frac{x^3}{3} + \frac{x^4}{20}$, and $\epsilon = 0.05$, $t \in [0, 500]$. The picture also displays in red some level sets of B.

4.3. Semiclassical magnetic normal forms. We turn now to the quantum counterpart of these results. Let $\mathcal{L}_{h,\mathbf{A}} = (-ih\nabla - \mathbf{A})^2$ be the magnetic Laplacian on $\mathbb{R}^2$, where the potential $\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth, and such that $\mathcal{L}_{h,\mathbf{A}} \in S(m)$ for some order function m on $\mathbb{R}^4$ (see [**49**, Chapter 7]). We will work with the Weyl quantization; for a classical symbol $a = a(x,\xi) \in S(m)$, it is defined as:

$$\mathsf{Op}_h^w a\,\psi(x) = \frac{1}{(2\pi h)^2} \int\int e^{i(x-y)\cdot\xi/h} a\left(\frac{x+y}{2}, \xi\right) \psi(y)\,\mathrm{d}y\,\mathrm{d}\xi, \quad \forall\psi \in \mathcal{S}(\mathbb{R}^2).$$

The first result shows that the spectral theory of $\mathcal{L}_{h,\mathbf{A}}$ is governed at first order by the magnetic field itself, viewed as a symbol.

Theorem 3.18. *Assume that the magnetic field B is non vanishing on $\mathbb{R}^2$ and confining: there exist constants $\tilde{C}_1 > 0$, $M_0 > 0$ such that*

$$B(q) \geq \tilde{C}_1 \quad \textit{for} \quad |q| \geq M_0. \tag{3.4.8}$$

Let $\mathcal{H}_h^0 = \mathsf{Op}_h^w(H^0)$, where $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism. Then there exists a bounded classical pseudo-differential operator Q_h on $\mathbb{R}^2$, such that

- *Q_h commutes with $\mathsf{Op}_h^w(|z_1|^2)$;*
- *Q_h is relatively bounded with respect to $\mathcal{H}_h^0$ with an arbitrarily small relative bound;*
- *its Weyl symbol is $O_{z_2}(h^2 + h\,|z_1|^2 + |z_1|^4)$,*

so that the following holds. Let $0 < C_1 < \tilde{C}_1$. *Then the spectra of* $\mathcal{L}_{h,\boldsymbol{A}}$ *and* $\mathcal{L}_h^{\mathsf{No}} := \mathcal{H}_h^0 + Q_h$ *in* $(-\infty, C_1 h]$ *are discrete. We denote by* $0 < \lambda_1(h) \leq \lambda_2(h) \leq \cdots$ *the eigenvalues of* $\mathcal{L}_{h,\boldsymbol{A}}$ *and by* $0 < \mu_1(h) \leq \mu_2(h) \leq \cdots$ *the eigenvalues of* $\mathcal{L}_h^{\mathsf{No}}$. *Then for all* $j \in \mathbb{N}^*$ *such that* $\lambda_j(h) \leq C_1 h$ *and* $\mu_j(h) \leq C_1 h$, *we have*

$$|\lambda_j(h) - \mu_j(h)| = O(h^\infty).$$

The proof of Theorem 3.18 relies on the following theorem (see [**105**] where a close form of this theorem appears), which provides in particular an accurate description of Q_h. In the statement, we use the notation of Theorem 3.15. We recall that Σ is the zero set of the classical Hamiltonian H.

Theorem 3.19. *For* h *small enough there exists a Fourier Integral Operator* U_h *such that*

$$U_h^* U_h = I + Z_h, \qquad U_h U_h^* = I + Z_h',$$

where Z_h, Z_h' *are pseudo-differential operators that microlocally vanish in a neighborhood of* $\tilde{\Omega} \cap \Sigma$, *and*

$$U_h^* \mathcal{L}_{h,\boldsymbol{A}} U_h = \mathcal{L}_h^{\mathsf{No}} + R_h, \tag{3.4.9}$$

where

(1) $\mathcal{L}_h^{\mathsf{No}}$ *is a classical pseudo-differential operator in* $S(m)$ *that commutes with*

$$\mathcal{I}_h := -h^2 \frac{\partial^2}{\partial x_1^2} + x_1^2;$$

(2) *For any Hermite function* $h_n(x_1)$ *such that* $\mathcal{I}_h h_n = h(2n-1) h_n$, *the operator* $\mathcal{L}_h^{\mathsf{No},(n)}$ *acting on* $L^2(\mathbb{R}_{x_2})$ *by*

$$h_n \otimes \mathcal{L}_h^{\mathsf{No},(n)}(u) = \mathcal{L}_h^{\mathsf{No}}(h_n \otimes u)$$

is a classical pseudo-differential operator in $S_{\mathbb{R}^2}(m)$ *of* h*-order 1 with principal symbol*

$$F^{(n)}(x_2, \xi_2) = h(2n-1) B(q),$$

where $(0, x_2 + i\xi_2) = \varphi(q)$ *as in* (3.4.6);

(3) *Given any classical pseudo-differential operator* D_h *with principal symbol* d_0 *such that* $d_0(z_1, z_2) = c(z_2)|z_1|^2 + O(|z_1|^3)$, *and any* $N \geq 1$, *there exist classical pseudo-differential operators* $S_{h,N}$ *and* K_N *such that:*

$$R_h = S_{h,N}(D_h)^N + K_N + O(h^\infty), \tag{3.4.10}$$

with K_N *compactly supported away from a fixed neighborhood of* $|z_1| = 0$.

(4) $\mathcal{L}_h^{\mathsf{No}} = \mathcal{H}_h^0 + Q_h$, *where* $\mathcal{H}_h^0 = \mathsf{Op}_h^w(H^0)$, $H^0 = B(\varphi^{-1}(z_2))|z_1|^2$, *and the operator* Q_h *is relatively bounded with respect to* $\mathcal{H}_h^0$ *with an arbitrarily small relative bound.*

We recover the result of [**86**], adding the fact that no odd power of $h^{1/2}$ can show up in the asymptotic expansion (see the recent work [**89**] where a Grushin type method is used to obtain a close result).

Corollary 3.20 (Low lying eigenvalues)**.** *Assume that B has a unique non-degenerate minimum. Then there exists a constant c_0 such that for any j, the eigenvalue $\lambda_j(h)$ has a full asymptotic expansion in integral powers of h whose first terms have the following form:*

$$\lambda_j(h) \sim h \min B + h^2(c_1(2j-1) + c_0) + O(h^3),$$

with $c_1 = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2B \circ \varphi^{-1}(0)}$, where the minimum of B is reached at $\varphi^{-1}(0)$.

Proof. The first eigenvalues of $\mathcal{L}_{h,A}$ are equal to the eigenvalues of $\mathcal{L}_h^{\mathsf{No},(1)}$ (in point (2) of Theorem 3.19). Since B has a non-degenerate minimum, the symbol of $\mathcal{L}_h^{\mathsf{No},(1)}$ has a non-degenerate minimum, and the spectral asymptotics of the low-lying eigenvalues for such a 1D pseudo-differential operator are well known. We get

$$\lambda_j(h) \sim h \min B + h^2(c_1(2j-1) + c_0) + O(h^3),$$

with $c_1 = \sqrt{\det(B \circ \varphi^{-1})"(0)/2}$. One can easily compute

$$c_1 = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2\,|\det(D\varphi^{-1}(0))|} = \frac{\sqrt{\det(B" \circ \varphi^{-1}(0))}}{2B \circ \varphi^{-1}(0)}.$$

□

CHAPTER 4

Waveguides

Si on me presse de dire pourquoi je l'aimais, je sens que cela ne se peut exprimer qu'en répondant : Parce que c'était lui : parce que c'était moi.

Les Essais, Livre I, Chapitre XXVIII, Montaigne

This chapter presents recent progress in the spectral theory of waveguides. In Section 1 we describe magnetic waveguides in dimensions two and three and we analyze the spectral influence of the width ε of the waveguide and the intensity b if the magnetic field. In particular we investigate the limit $\varepsilon \to 0$. In Section 2 we describe the same problem in the case of layers. In Sections 3 and 4 the effect of a corner in dimension two is tackled.

1. Magnetic waveguides

This section is concerned with spectral properties of a curved quantum waveguide when a magnetic field is applied. We will give a precise definition of what a waveguide is in Sections 1.3 and 1.4. Without going into the details we can already mention that we will use the definition given in the famous (non magnetic) paper of Duclos and Exner [**53**] and its generalizations [**35, 113, 72**]. The waveguide is nothing but a tube Ω_ε about an unbounded curve γ in the Euclidean space $\mathbb{R}^d$, with $d \geq 2$, where ε is a positive shrinking parameter and the cross section is defined as $\varepsilon\omega = \{\varepsilon\tau : \tau \in \omega\}$.

More precisely this section is devoted to the spectral analysis of the magnetic operator with Dirichlet boundary conditions $\mathfrak{L}^{[d]}_{\varepsilon,b\mathbf{A}}$ defined as

$$(-i\nabla_x + b\,\mathbf{A}(x))^2 \qquad \text{on} \qquad \mathsf{L}^2(\Omega_\varepsilon,\, \mathrm{d}x). \tag{4.1.1}$$

where $b > 0$ is a positive parameter and $\mathbf{A}$ a smooth vector potential associated with a given magnetic field $\mathbf{B}$.

1.1. The result of Duclos and Exner.

One of the deep facts which is proved by Duclos and Exner is that the Dirichlet Laplacian on Ω_ε always has discrete spectrum below its essential spectrum when the waveguide is not straight and asymptotically straight. Let us sketch the proof of this result in the case of two dimensional waveguides.

Let us consider a smooth and injective curve $\gamma\colon \mathbb{R} \ni s \mapsto \gamma(s)$ which is parameterized by its arc length s. The normal to the curve at $\gamma(s)$ is defined as the unique unit vector $\boldsymbol{n}(s)$ such that $\gamma'(s) \cdot \nu(s) = 0$ and $\det(\gamma', \nu) = 1$. We have the relation $\gamma''(s) = \kappa(s)\boldsymbol{n}(s)$ where $\kappa(s)$ denotes the algebraic curvature at the point $\gamma(s)$. We can now define standard tubular coordinates. We consider:

$$\mathbb{R} \times (-\varepsilon, \varepsilon) \ni (s,t) \mapsto \Phi(s,t) = \gamma(s) + t\boldsymbol{n}(s).$$

We always assume

$$\Phi \text{ is injective} \qquad \text{and} \qquad \varepsilon \sup_{s\in\mathbb{R}} |\kappa(s)| < 1. \tag{4.1.2}$$

Then it is well known (see [**113**]) that Φ defines a smooth diffeomorphism from $\mathbb{R}\times(-\varepsilon,\varepsilon)$ onto the image $\Omega_\varepsilon = \Phi(\mathbb{R} \times (-\varepsilon, \varepsilon))$, which we identify with our waveguide. In these new coordinates, the operator becomes (exercise)

$$\mathfrak{L}^{[2]}_{\varepsilon,0} = -m^{-1}\partial_s m^{-1}\partial_s - m^{-1}\partial_t m \partial_t, \quad m(s,t) = 1 - t\kappa(s),$$

which is acting in the weighted space $\mathsf{L}^2(\mathbb{R} \times (-\varepsilon, \varepsilon), m(s,t)\,\mathrm{d}s\,\mathrm{d}t)$. We introduce the shifted quadratic form:

$$\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi) = \int_{\mathbb{R}\times(-\varepsilon,\varepsilon)} \left(m^{-2}|\partial_s(\phi)|^2 + |\partial_t\phi|^2 - \frac{\pi^2}{4\varepsilon^2}|\phi|^2 \right) m\,\mathrm{d}s\,\mathrm{d}t$$

and we let:

$$\phi_n(s,t) = \chi_0(n^{-1}s)\cos\left(\frac{\pi}{2\varepsilon}t\right),$$

where χ_0 is a smooth cutoff function which is 1 near 0. We can check that $\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n) \underset{n\to+\infty}{\to} 0$. Let us now consider a smooth cutoff function χ_1 which is 1 near a point where κ is not zero and define $\tilde{\phi}(s,t) = -\chi_1^2(s,t)\mathcal{L}^{[2],\mathsf{sh}}_{\varepsilon,0}\phi_n(s,t)$ which does not depend on n as soon as n is large enough. Then we have:

$$\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n + \eta\tilde{\phi}) = \mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n) - 2\eta\mathcal{B}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n, \chi_1(s)\mathcal{L}^{[2],\mathsf{sh}}_{\varepsilon,0}\phi_n) + \eta^2\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\tilde{\phi}).$$

For n large enough, the quantity $\mathcal{B}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n, \chi_1(s)\mathcal{L}^{[2],\mathsf{sh}}_{\varepsilon,0}\phi_n)$ does not depend on n and is positive. For such an n, we take η small enough and we find:

$$\mathcal{Q}^{[2],\mathsf{sh}}_{\varepsilon,0}(\phi_n + \eta\tilde{\phi}) < 0.$$

Therefore the bottom of the spectrum is an eigenvalue due to the min-max principle.

Duclos and Exner also investigate the limit $\varepsilon \to 0$ to show that the Dirichlet Laplacian on the tube Ω_ε converges in a suitable sense to the effective one dimensional operator

$$\mathcal{L}^{\mathsf{eff}} = -\partial_s^2 - \frac{\kappa(s)^2}{4} \qquad \text{on} \qquad \mathsf{L}^2(\gamma,\, \mathrm{d}s).$$

In addition it is proved in [**53**] that each eigenvalue of this effective operator generates an eigenvalue of the Dirichlet Laplacian on the tube.

As Duclos and Exner we are interested in approximations of $\mathfrak{L}^{[d]}_{\varepsilon,b\mathbf{A}}$ in the small cross section limit $\varepsilon \to 0$. Such an approximation might non trivially depends on the intensity of the magnetic field b especially if it is allowed to depend on ε.

1.2. Waveguides with more geometry. In dimension three it is also possible to twist the waveguide by allowing the cross section of the waveguide to non-trivially rotate by an angle function θ with respect to a relatively parallel frame of γ (then the velocity θ' can be interpreted as a "torsion"). It is proved in [**57**] that, whereas the curvature is favourable to discrete spectrum, the torsion plays against it. In particular, the spectrum of a straight twisted waveguide is stable under small perturbations (such as local electric field or bending). This repulsive effect of twisting is quantified in [**57**] (see also [**112, 116**]) by means of a Hardy type inequality. The limit $\varepsilon \to 0$ permits to compare the effects bending and twisting ([**26, 47, 115**]) and the effective operator is given by

$$\mathcal{L}^{\mathsf{eff}} = -\partial_s^2 - \frac{\kappa(s)^2}{4} + C(\omega)\theta'(s)^2 \qquad \text{on} \qquad \mathsf{L}^2(\gamma, \mathrm{d}s),$$

where $C(\omega)$ is a positive constant whenever ω is not a disk or annulus. Writing (4.1.1)

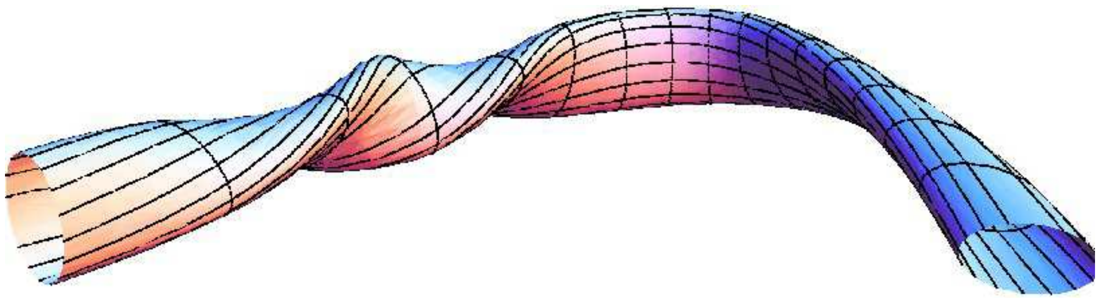


FIGURE 1. Torsion on the left and curvature on the right

in suitable curvilinear coordinates (see (4.1.9) below), one may notice similarities in the appearance of the torsion and the magnetic field in the coefficients of the operator and it therefore seems natural to ask the following question:

"Does the magnetic field act as the torsion ?"

In order to define our effective operators in the limit $\varepsilon \to 0$ we shall describe more accurately the geometry of our waveguides. This is the aim of the next two sections in which we will always assume that the geometry (curvature and twist) and the magnetic field are compactly supported.

1.3. Two-dimensional waveguides. Up to changing the gauge, the Laplace-Beltrami expression of $\mathfrak{L}^{[2]}_{\varepsilon,b\mathbf{A}}$ in these coordinates is given by

$$\mathfrak{L}^{[2]}_{\varepsilon,b\mathcal{A}} = (1-t\kappa(s))^{-1}(i\partial_s + b\mathcal{A}_1)(1-t\kappa(s))^{-1}(i\partial_s + b\mathcal{A}_1) - (1-t\kappa(s))^{-1}\partial_t(1-t\kappa(s))\partial_t,$$

with the gauge:

$$\mathcal{A}(s,t)=(\mathcal{A}_1(s,t),0),\quad \mathcal{A}_1(s,t)=\int_0^t (1-t'\kappa(s))\mathbf{B}(\Phi(s,t'))\,\mathrm{d}t'.$$

We let:

$$m(s,t)=1-t\kappa(s).$$

The self-adjoint operator $\mathfrak{L}^{[2]}_{\varepsilon,b\mathcal{A}}$ on $\mathsf{L}^2(\mathbb{R}\times(-\varepsilon,\varepsilon),m\,\mathrm{d}s\,\mathrm{d}t)$ is unitarily equivalent to the self-adjoint operator on $\mathsf{L}^2(\mathbb{R}\times(-\varepsilon,\varepsilon),\,\mathrm{d}s\,\mathrm{d}t)$:

$$\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}}=m^{1/2}\mathfrak{L}^{[2]}_{\varepsilon,b\mathbf{A}}m^{-1/2}.$$

Introducing the rescaling

$$t=\varepsilon\tau, \tag{4.1.3}$$

we let:

$$\mathcal{A}_\varepsilon(s,\tau)=(\mathcal{A}_{1,\varepsilon}(s,\tau),0)=(\mathcal{A}_1(s,\varepsilon\tau),0)$$

and denote by $\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon}$ the homogenized operator on $\mathsf{L}^2(\mathbb{R}\times(-1,1),\,\mathrm{d}s\,\mathrm{d}\tau)$:

$$\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon}=m_\varepsilon^{-1/2}(i\partial_s+b\mathcal{A}_{1,\varepsilon})m_\varepsilon^{-1}(i\partial_s+b\mathcal{A}_{1,\varepsilon})m_\varepsilon^{-1/2}-\varepsilon^{-2}\partial_\tau^2+V_\varepsilon(s,\tau), \tag{4.1.4}$$

with:

$$m_\varepsilon(s,\tau)=m(s,\varepsilon\tau),\quad V_\varepsilon(s,\tau)=-\frac{\kappa(s)^2}{4}(1-\varepsilon\kappa(s)\tau)^{-2}.$$

It is easy to verify that $\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}}$, defined as Friedrich extension of the operator initially defined on $\mathcal{C}_0^\infty(\mathbb{R}\times(-\varepsilon,\varepsilon))$, has form domain $\mathsf{H}^1_0(\mathbb{R}\times(-\varepsilon,\varepsilon))$. Similarly, the form domain of $\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon}$ is $\mathsf{H}^1_0(\mathbb{R}\times(-1,1))$.

1.4. Three-dimensional waveguides. The situation is geometrically more complicated in dimension 3. We consider a smooth curve γ which is parameterized by its arc length s and does not overlap itself. We use the so-called Tang frame (or the relatively parallel frame, see for instance [**115**]) to describe the geometry of the tubular neighbourhood of γ. Denoting the (unit) tangent vector by $T(s)=\gamma'(s)$, the Tang frame $(T(s),M_2(s),M_3(s))$ satisfies the relations:

$$\begin{aligned} T' &= \kappa_2 M_2+\kappa_3 M_3,\\ M_2' &= -\kappa_2 T,\\ M_3' &= -\kappa_3 T.\end{aligned}$$

The functions κ_2 and κ_3 are the curvatures related to the choice of the normal fields M_2 and M_3. We can notice that $\kappa^2=\kappa_2^2+\kappa_3^2=|\gamma''|^2$ is the square of the usual curvature of γ.

Let $\theta : \mathbb{R} \to \mathbb{R}$ a smooth function (twisting). We introduce the map $\Phi : \mathbb{R}\times(\varepsilon\omega) \to \Omega_\varepsilon$ defined by:

(4.1.5)
$$x = \Phi(s, t_2, t_3) = \gamma(s) + t_2(\cos\theta M_2(s) + \sin\theta M_3(s)) + t_3(-\sin\theta M_2(s) + \cos\theta M_3(s)).$$

Let us notice that s will often be denoted by t_1. As in dimension two, we always assume:

(4.1.6) $$\Phi \text{ is injective} \quad \text{and} \quad \varepsilon \sup_{(\tau_2,\tau_3)\in\omega} (|\tau_2| + |\tau_3|) \sup_{s\in\mathbb{R}} |\kappa(s)| < 1.$$

Sufficient conditions ensuring the injectivity hypothesis can be found in [**57**, App. A]. We define $\mathcal{A} = D\Phi\mathbf{A}(\Phi) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3)$,

$$\begin{aligned} h &= 1 - t_2(\kappa_2\cos\theta + \kappa_3\sin\theta) - t_3(-\kappa_2\sin\theta + \kappa_3\cos\theta),\\ h_2 &= -t_2\theta',\\ h_3 &= t_3\theta', \end{aligned}$$

and $\mathcal{R} = h_3 b\mathcal{A}_2 + h_2 b\mathcal{A}_3$. We also introduce the angular derivative $\partial_\alpha = t_3\partial_{t_2} - t_2\partial_{t_3}$. We will see in Section 2 that the magnetic operator $\mathfrak{L}^{[3]}_{\varepsilon,b\mathbf{A}}$ is unitarily equivalent to the operator on $\mathsf{L}^2(\Omega_\varepsilon, h\,\mathrm{d}t)$ given by

(4.1.7) $$\mathfrak{L}^{[3]}_{\varepsilon,b\mathcal{A}} = \sum_{j=2,3} h^{-1}(-i\partial_{t_j} + b\mathcal{A}_j)h(-i\partial_{t_j} + b\mathcal{A}_j) \\ + h^{-1}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R})h^{-1}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R}).$$

By considering the conjugate operator $h^{1/2}\mathfrak{L}^{[3]}_{\varepsilon,b\mathcal{A}}h^{-1/2}$, we find that $\mathfrak{L}^{[3]}_{\varepsilon,b\mathcal{A}}$ is unitarily equivalent to the operator defined on $\mathsf{L}^2(\mathbb{R}\times(\varepsilon\omega), \mathrm{d}s\,\mathrm{d}t_2\,\mathrm{d}t_3)$ given by:

(4.1.8) $$\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}} = \sum_{j=2,3} (-i\partial_{t_j} + b\mathcal{A}_j)^2 - \frac{\kappa^2}{4h^2} \\ + h^{-1/2}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R})h^{-1}(-i\partial_s + b\mathcal{A}_1 - i\theta'\partial_\alpha + \mathcal{R})h^{-1/2}.$$

Finally, introducing the rescaling

$$(t_2, t_3) = \varepsilon(\tau_2, \tau_3) = \varepsilon\tau,$$

we define the homogenized operator on $\mathsf{L}^2(\mathbb{R}\times\omega, \mathrm{d}s\,\mathrm{d}\tau)$:

(4.1.9) $$\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}_\varepsilon} = \sum_{j=2,3} (-i\varepsilon^{-1}\partial_{\tau_j} + b\mathcal{A}_{j,\varepsilon})^2 - \frac{\kappa^2}{4h_\varepsilon^2} \\ + h_\varepsilon^{-1/2}(-i\partial_s + b\mathcal{A}_{1,\varepsilon} - i\theta'\partial_\alpha + \mathcal{R}_\varepsilon)h_\varepsilon^{-1}(-i\partial_s + b\mathcal{A}_{1,\varepsilon} - i\theta'\partial_\alpha + \mathcal{R}_\varepsilon)h_\varepsilon^{-1/2},$$

where $\mathcal{A}_\varepsilon(s,\tau) = \mathcal{A}(s,\varepsilon\tau)$, $h_\varepsilon(s,\tau) = h(s,\varepsilon\tau)$ and $\mathcal{R}_\varepsilon = \mathcal{R}(s,\varepsilon\tau)$.

We leave as an exercise the verification that the form domains of $\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}}$ and $\mathcal{L}^{[3]}_{\varepsilon,b\mathcal{A}_\varepsilon}$ are $\mathsf{H}^1_0(\mathbb{R}\times(-\varepsilon,\varepsilon))$ and $\mathsf{H}^1_0(\mathbb{R}\times(-1,1))$, respectively.

1.5. Limiting models and asymptotic expansions. We can now state our main results concerning the effective models in the limit $\varepsilon \to 0$. We will denote by $\lambda_n^{\mathsf{Dir}}(\omega)$ the n-th eigenvalue of the Dirichlet Laplacian $-\Delta_\omega^{\mathsf{Dir}}$ on $\mathsf{L}^2(\omega)$. The first positive and L^2-normalized eigenfunction will be denoted by J_1.

Definition 4.1 (Case $d = 2$). *For $\delta \in (-\infty, 1)$, we define:*

$$\mathcal{L}_{\varepsilon,\delta}^{\mathsf{eff},[2]} = -\varepsilon^{-2}\Delta_\omega^{\mathsf{Dir}} - \partial_s^2 - \frac{\kappa(s)^2}{4}$$

and for $\delta = 1$, we let:

$$\mathcal{L}_{\varepsilon,1}^{\mathsf{eff},[2]} = -\varepsilon^{-2}\Delta_\omega^{\mathsf{Dir}} + \mathcal{T}^{[2]},$$

where

$$\mathcal{T}^{[2]} = -\partial_s^2 + \left(\frac{1}{3} + \frac{2}{\pi^2}\right)\boldsymbol{B}(\gamma(s))^2 - \frac{\kappa(s)^2}{4}.$$

Theorem 4.2 (Case $d = 2$). *There exists K such that, for all $\delta \in (-\infty, 1]$, there exist $\varepsilon_0 > 0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\left\|\left(\mathcal{L}_{\varepsilon,\varepsilon^{-\delta}\mathcal{A}_\varepsilon}^{[2]} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1} - \left(\mathcal{L}_{\varepsilon,\delta}^{\mathsf{eff},[2]} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1}\right\| \leq C\max(\varepsilon^{1-\delta}, \varepsilon), \text{ for } \delta < 1$$

and:

$$\left\|\left(\mathcal{L}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon}^{[2]} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1} - \left(\mathcal{L}_{\varepsilon,1}^{\mathsf{eff},[2]} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K\right)^{-1}\right\| \leq C\varepsilon.$$

In the critical regime $\delta = 1$, we deduce the following corollary providing the asymptotic expansions of the lowest eigenvalues $\lambda_n^{[2]}(\varepsilon)$ of $\mathcal{L}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon}^{[2]}$.

Corollary 4.3 (Case $d = 2$ and $\delta = 1$). *Let us assume that $\mathcal{T}^{[2]}$ admits N (simple) eigenvalues $\mu_0, \cdots, \mu_N$ below the threshold of the essential spectrum. Then, for all $n \in \{1, \cdots N\}$, there exist $(\gamma_{j,n})_{j\geq 0}$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\lambda_n^{[2]}(\varepsilon) \underset{\varepsilon\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n}\varepsilon^{-2+j},$$

with

$$\gamma_{0,n} = \frac{\pi^2}{4}, \quad \gamma_{1,n} = 0, \quad \gamma_{2,n} = \mu_n.$$

Thanks to the spectral theorem, we also get the approximation of the corresponding eigenfunctions at any order (see our quasimodes in (16.1.9)).

In order to present analogous results in dimension three, we introduce supplementary notation. The norm and the inner product in $\mathsf{L}^2(\omega)$ will be denoted by $\|\cdot\|_\omega$ and $\langle\cdot,\cdot\rangle_\omega$, respectively.

Definition 4.4 (Case $d = 3$). *For $\delta \in (-\infty, 1)$, we define:*

$$\mathcal{L}_{\varepsilon,\delta}^{\mathsf{eff},[3]} = -\varepsilon^{-2}\Delta_\omega^{\mathsf{Dir}} - \partial_s^2 - \frac{\kappa(s)^2}{4} + \|\partial_\alpha J_1\|_\omega^2\theta'^2$$

and for $\delta = 1$*, we let:*

$$\mathcal{L}^{\mathsf{eff},[3]}_{\varepsilon,1} = -\varepsilon^{-2}\Delta^{\mathsf{Dir}}_{\omega} + \mathcal{T}^{[3]},$$

where $\mathcal{T}^{[3]}$ *is defined by:*

$$\mathcal{T}^{[3]} = \langle(-i\partial_s - i\theta'\partial_\alpha - \mathcal{B}_{12}(s,0,0)\tau_2 - \mathcal{B}_{13}(s,0,0)\tau_3)^2 \mathsf{Id}(s)\otimes J_1, \mathsf{Id}(s)\otimes J_1\rangle_\omega$$
$$+ \mathcal{B}^2_{23}(s,0,0)\left(\frac{\|\tau J_1\|^2_\omega}{4} - \langle D_\alpha R_\omega, J_1\rangle_\omega\right) - \frac{\kappa^2(s)}{4},$$

with R_ω *being given in* (16.2.6) *and*

$$\begin{aligned}
\mathcal{B}_{23}(s,0,0) &= \boldsymbol{B}(\gamma(s))\cdot T(s),\\
\mathcal{B}_{13}(s,0,0) &= \boldsymbol{B}(\gamma(s))\cdot(\cos\theta\, M_2(s) - \sin\theta\, M_3(s)),\\
\mathcal{B}_{12}(s,0,0) &= \boldsymbol{B}(\gamma(s))\cdot(-\sin\theta\, M_2(s) + \cos\theta\, M_3(s)).
\end{aligned}$$

Theorem 4.5 (Case $d = 3$)**.** *There exists* K *such that for all* $\delta \in (-\infty, 1]$*, there exist* $\varepsilon_0 > 0, C > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*:*

$$\left\|\left(\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-\delta}\mathcal{A}_\varepsilon} - \varepsilon^{-2}\lambda^{\mathsf{Dir}}_1(\omega) + K\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[3]}_{\varepsilon,\delta} - \varepsilon^{-2}\lambda^{\mathsf{Dir}}_1(\omega) + K\right)^{-1}\right\| \leq C\max(\varepsilon^{1-\delta}, \varepsilon), \text{ for } \delta < 1$$

and:

$$\left\|\left(\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon} - \varepsilon^{-2}\lambda^{\mathsf{Dir}}_1(\omega) + K\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[3]}_{\varepsilon,1} - \varepsilon^{-2}\lambda^{\mathsf{Dir}}_1(\omega) + K\right)^{-1}\right\| \leq C\varepsilon.$$

In the same way, this theorem implies asymptotic expansions of eigenvalues $\lambda^{[3]}_n(\varepsilon)$ of $\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-1}\mathcal{A}_\varepsilon}$.

Corollary 4.6 (Case $d = 3$ and $\delta = 1$)**.** *Let us assume that* $\mathcal{T}^{[3]}$ *admits* N *(simple) eigenvalues* $\nu_0, \cdots, \nu_N$ *below the threshold of the essential spectrum. Then, for all* $n \in \{1, \cdots N\}$*, there exist* $(\gamma_{j,n})_{j\geq 0}$ *and* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*:*

$$\lambda^{[3]}_n(\varepsilon) \underset{\varepsilon\to 0}{\sim} \sum_{j\geq 0} \gamma_{j,n}\varepsilon^{-2+j},$$

with

$$\gamma_{0,n} = \lambda^{\mathsf{Dir}}_1(\omega), \quad \gamma_{1,n} = 0, \quad \gamma_{2,n} = \nu_n.$$

As in two dimensions, we also get the corresponding expansion for the eigenfunctions. Complete asymptotic expansions for eigenvalues in finite three-dimensional waveguides without magnetic field are also previously established in [**79, 22**]. Such expansions were also obtained in [**78**] in the case $\delta = 0$ in a periodic framework.

Remark 4.7. *As expected, when* $\delta = 0$ *that is when* b *is kept fixed, the magnetic field does not persists in the limit* $\varepsilon \to 0$ *as well in dimension two as in dimension three. Indeed, in this limit* Ω_ε *converges to the one dimensional curve* γ *and there is no magnetic field in dimension* 1.

1.6. Norm resolvent convergence. Let us state an auxiliary result, inspired by the approach of [**74**], which tells us that, in order to estimate the difference between two resolvents, it is sufficient to analyse the difference between the corresponding sesquilinear forms as soon as their domains are the same.

Lemma 4.8. *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be two positive self-adjoint operators on a Hilbert space* H. *Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be their associated sesquilinear forms. We assume that* $\mathsf{Dom}\,(\mathfrak{B}_1) = \mathsf{Dom}\,(\mathfrak{B}_2)$. *Assume that there exists $\eta > 0$ such that for all $\phi, \psi \in \mathsf{Dom}\,(\mathfrak{B}_1)$:*

$$|\mathfrak{B}_1(\phi,\psi) - \mathfrak{B}_2(\phi,\psi)| \leq \eta\sqrt{\mathfrak{Q}_1(\psi)}\sqrt{\mathfrak{Q}_2(\phi)},$$

where $\mathfrak{Q}_j(\varphi) = \mathfrak{B}_j(\varphi,\varphi)$ for $j = 1,2$ and $\varphi \in \mathsf{Dom}\,(\mathfrak{B}_1)$. Then, we have:

$$\|\mathfrak{L}_1^{-1} - \mathfrak{L}_2^{-1}\| \leq \eta\|\mathfrak{L}_1^{-1}\|^{1/2}\|\mathfrak{L}_2^{-1}\|^{1/2}.$$

Proof. The original proof can be found in [**115**, Prop. 5.3]. Let us consider $\tilde{\phi}, \tilde{\psi} \in \mathsf{H}$. We let $\phi = \mathfrak{L}_2^{-1}\tilde{\phi}$ and $\psi = \mathfrak{L}_1^{-1}\tilde{\psi}$. We have $\phi, \psi \in \mathsf{Dom}\,(\mathfrak{B}_1) = \mathsf{Dom}\,(\mathfrak{B}_2)$. We notice that:

$$\mathfrak{B}_1(\phi,\psi) = \langle \mathfrak{L}_2^{-1}\tilde{\phi}, \tilde{\psi}\rangle, \quad \mathfrak{B}_2(\phi,\psi) = \langle \mathfrak{L}_1^{-1}\tilde{\phi}, \tilde{\psi}\rangle$$

and:

$$\mathfrak{Q}_1(\psi) = \langle \tilde{\psi}, \mathfrak{L}_1^{-1}\tilde{\psi}\rangle, \quad \mathfrak{Q}_2(\phi) = \langle \tilde{\phi}, \mathfrak{L}_2^{-1}\tilde{\phi}\rangle.$$

We infer that:

$$\left|\langle(\mathfrak{L}_1^{-1} - \mathfrak{L}_2^{-1})\tilde{\phi}, \tilde{\psi}\rangle\right| \leq \eta\|\mathfrak{L}_1^{-1}\|^{1/2}\|\mathfrak{L}_2^{-1}\|^{1/2}\|\tilde{\phi}\|\|\tilde{\psi}\|$$

and the result elementarily follows. $\square$

1.7. A magnetic Hardy inequality. In dimension 2, the limiting model (with $\delta = 1$) enlightens the fact that the magnetic field plays against the curvature, whereas in dimension 3 this repulsive effect is not obvious (it can be seen that $\langle D_\alpha R_\omega, J_1\rangle_\omega \geq 0$). Nevertheless, if ω is a disk, we have $\langle D_\alpha R_\omega, J_1\rangle_\omega = 0$ and thus the component of the magnetic field parallel to γ plays against the curvature (in comparison, a pure torsion has no effect when the cross section is a disk). In the flat case ($\kappa = 0$), we can quantify this repulsive effect by means of a magnetic Hardy inequality (see [**56**] where this inequality is discussed in dimension two). We will not discuss the proof of this inequality in this book.

Theorem 4.9. *Let $d \geq 2$. Let us consider $\Omega = \mathbb{R} \times \omega$. For $R > 0$, we let:*

$$\Omega(R) = \{t \in \Omega : |t_1| < R\}.$$

Let $\boldsymbol{A}$ be a smooth vector potential such that $\sigma_{\boldsymbol{B}}$ is not zero on $\Omega(R_0)$ for some $R_0 > 0$. Then, there exists $C > 0$ such that, for all $R \geq R_0$, there exists $c_R(\boldsymbol{B}) > 0$ such that, we have:

$$\int_\Omega |(-i\nabla + \boldsymbol{A})\psi|^2 - \lambda_1^{\mathsf{Dir}}(\omega)|\psi|^2 \,\mathrm{d}t \geq \int_\Omega \frac{c_R(\boldsymbol{B})}{1+s^2}|\psi|^2\,\mathrm{d}t, \quad \forall\psi \in \mathcal{C}_0^\infty(\Omega). \tag{4.1.10}$$

Moreover we can take:

$$c_R(\boldsymbol{B}) = \left(1 + CR^{-2}\right)^{-1} \min\left(\frac{1}{4}, \lambda_1^{\mathsf{Dir},\mathsf{Neu}}(\boldsymbol{B}, \Omega(R)) - \lambda_1^{\mathsf{Dir}}(\omega)\right),$$

where $\lambda_1^{\mathsf{Dir},\mathsf{Neu}}(\boldsymbol{B}, \Omega(R))$ *denotes the first eigenvalue of the magnetic Laplacian on* $\Omega(R)$, *with Dirichlet condition on* $\mathbb{R} \times \partial\omega$ *and Neumann condition on* $\{|s| = R\} \times \omega$.

The inequality of Theorem 4.9 can be applied to prove certain stability of the spectrum of the magnetic Laplacian on Ω under local and small deformations of Ω. Let us fix $\varepsilon > 0$ and describe a generic deformation of the straight tube Ω. We consider the local diffeomorphism:

$$\Phi_\varepsilon(t) = \Phi_\varepsilon(s, t_2, t_3) = (s, 0, \cdots, 0) + \sum_{j=2}^{d} (t_j + \varepsilon_j(s)) M_j + \mathcal{E}_1(s),$$

where $(M_j)_{j=2}^{d}$ is the canonical basis of $\{0\} \times \mathbb{R}^{d-1}$. The functions ε_j and $\mathcal{E}_1$ are smooth and compactly supported in a compact set K. As previously we assume that Φ_ε is a global diffeomorphism and we consider the deformed tube $\Omega^{\mathsf{def},\varepsilon} = \Phi_\varepsilon(\mathbb{R} \times \omega)$.

Proposition 4.10. *Let* $d \geq 2$. *There exists* $\varepsilon_0 > 0$ *such that for* $\varepsilon \in (0, \varepsilon_0)$, *the spectrum of the Dirichlet realization of* $(-i\nabla + \boldsymbol{A})^2$ *on* $\Omega^{\mathsf{def},\varepsilon}$ *coincides with the spectrum of the Dirichlet realization of* $(-i\nabla + \boldsymbol{A})^2$ *on* Ω. *The spectrum is given by* $[\lambda_1^{\mathsf{Dir}}(\omega), +\infty)$.

By using a semiclassical argument, it is possible to prove a stability result which does not use the Hardy inequality.

Proposition 4.11. *Let* $R_0 > 0$ *and* $\Omega(R_0) = \{t \in \mathbb{R} \times \omega : |t_1| \leq R_0\}$. *Let us assume that* $\sigma_{\boldsymbol{B}} = \mathrm{d}\xi_{\boldsymbol{A}}$ *does not vanish on* $\Phi(\Omega(R_0))$ *and that on* $\Omega_1 \setminus \Phi(\Omega(R_0))$ *the curvature is zero. Then, there exists* $b_0 > 0$ *such that for* $b \geq b_0$, *the discrete spectrum of* $\mathfrak{L}_{1,b\boldsymbol{A}}^{[d]}$ *is empty.*

2. Magnetic layers

As we will sketch below, the philosophy of Duclos and Exner may also apply to thin quantum layers as we can see in the contributions [**54, 32, 120, 121, 122, 159**] and the related papers [**107, 41, 42, 164, 133, 75, 72, 171, 168, 117, 115**].

Let us consider Σ an hypersurface embedded in $\mathbb{R}^d$ with $d \geq 2$, and define a tubular neighbourhood about Σ,

$$\Omega_\varepsilon := \left\{ x + t\boldsymbol{n} \in \mathbb{R}^d \;\middle|\; (x, t) \in \Sigma \times (-\varepsilon, \varepsilon) \right\}, \tag{4.2.1}$$

where $\boldsymbol{n}$ denotes a unit normal vector field of Σ. We investigate:

$$\mathcal{L}_{\mathbf{A},\Omega_\varepsilon} = (-i\nabla + \mathbf{A})^2 \quad \text{on} \quad \mathsf{L}^2(\Omega_\varepsilon), \tag{4.2.2}$$

with Dirichlet boundary conditions on $\partial\Omega_\varepsilon$.

2.1. Normal form. As usual the game is to find an appropriate normal form for the magnetic Laplacian. Given $I := (-1,1)$ and $\varepsilon > 0$, we define a layer Ω_ε of width 2ε along Σ as the image of the mapping

$$\Phi : \Sigma \times I \to \mathbb{R}^d : \ \big\{(x,u) \mapsto x + \varepsilon u \boldsymbol{n}\big\} \tag{4.2.3}$$

Let us denote by $\tilde{\mathbf{A}}$ the components of the vector potential expressed in the curvilinear coordinates induced by the embedding (4.2.3). Moreover, assume

$$\tilde{A}_d = 0. \tag{4.2.4}$$

Thanks to the diffeomorphism $\Phi : \Sigma \times I \to \Omega_\varepsilon$, we may identify $\mathcal{L}_{\mathbf{A},\Omega_\varepsilon}$ with an operator $\hat{H}$ on $\mathsf{L}^2(\Sigma \times I,\, \mathrm{d}\Omega_\varepsilon)$ that acts, in the form sense, as

$$\hat{H} = |G|^{-1/2}(-i\partial_{x^\mu} + \tilde{A}_\mu)|G|^{1/2}G^{\mu\nu}(-i\partial_{x^\nu} + \tilde{A}_\nu) - \varepsilon^{-2}|G|^{-1/2}\partial_u|G|^{1/2}\partial_u\,.$$

Let us define

$$J := \frac{1}{4}\ln\frac{|G|}{|g|} = \frac{1}{2}\sum_{\mu=1}^{d-1}\ln(1-\varepsilon u\kappa_\mu) = \frac{1}{2}\ln\left[1 + \sum_{\mu=1}^{d-1}(-\varepsilon u)^\mu\binom{d-1}{\mu}K_\mu\right].$$

Using the unitary transform

$$U : \mathsf{L}^2(\Sigma \times I,\, \mathrm{d}\Omega_\varepsilon) \to \mathsf{L}^2(\Sigma \times I,\, \mathrm{d}\Sigma \wedge\, \mathrm{d}u) : \ \big\{\psi \mapsto e^J\psi\big\}\ ,$$

we arrive at the unitarily equivalent operator

$$H := U\hat{H}U^{-1} = |g|^{-1/2}(-i\partial_{x^\mu} + \tilde{A}_\mu)|g|^{1/2}G^{\mu\nu}(-i\partial_{x^\nu} + \tilde{A}_\nu) - \varepsilon^{-2}\partial_u^2 + V\,,$$

where

$$V := |g|^{-1/2}\,\partial_{x^i}\big(|g|^{1/2}G^{ij}(\partial_{x^j}J)\big) + (\partial_{x^i}J)G^{ij}(\partial_{x^j}J)\,.$$

We get

$$H = U\hat{U}(-\Delta_{D,A}^{\Omega_\varepsilon})\hat{U}^{-1}U^{-1}\,.$$

2.2. The effective operator. H is approximated in the norm resolvent sense (see [**114**] for the details) by

$$H_0 = h_{\mathrm{eff}} - \varepsilon^{-2}\partial_u^2 \ \simeq\ h_{\mathrm{eff}} \otimes 1 + 1 \otimes (-\varepsilon^{-2}\partial_u^2) \tag{4.2.5}$$

on $\mathsf{L}^2(\Sigma \times I,\, \mathrm{d}\Sigma \wedge\, \mathrm{d}u) \simeq \mathsf{L}^2(\Sigma,\, \mathrm{d}\Sigma) \otimes \mathsf{L}^2(I,\, \mathrm{d}u)$ with the effective Hamiltonian

$$h_{\mathrm{eff}} := |g|^{-1/2}\big(-i\partial_{x^\mu} + \tilde{A}_\mu(.,0)\big)|g|^{1/2}g^{\mu\nu}\big(-i\partial_{x^\nu} + \tilde{A}_\nu(.,0)\big) + V_{\mathrm{eff}}, \tag{4.2.6}$$

where

$$V_{\mathrm{eff}} := -\frac{1}{2}\sum_{\mu=1}^{d-1}\kappa_\mu^2 + \frac{1}{4}\left(\sum_{\mu=1}^{d-1}\kappa_\mu\right)^2. \tag{4.2.7}$$

3. Semiclassical triangles

As we would like to analyze the spectrum of broken waveguides (that is waveguides with an angle), this is natural to prepare the investigation by studying the Dirichlet eigenvalues of the Laplacian on some special shrinking triangles. This subject is already dealt with in [**71**, Theorem 1] where four-term asymptotics is proved for the lowest eigenvalue, whereas a three-term asymptotics for the second eigenvalue is provided in [**71**, Section 2]. We can mention the papers [**73, 74**] whose results provide two-term asymptotics for the thin rhombi and also [**23**] which deals with a regular case (thin ellipse for instance), see also [**24**]. We also invite the reader to take a look at [**102**]. For a complete description of the low lying spectrum of general shrinking triangles, one may consult the paper by Ourmières [**141**] (especially the existence of a boundary layer living near the shrinking height is proved, see also [**46**]) where tunnel effect estimates are also established. In dimension three the generalization to cones with small aperture is done in [**140**] and which is motivated by [**65**].

Let us define the isosceles triangle in which we are interested:

$$\mathsf{Tri}_\theta = \left\{(x_1,x_2)\in\mathbb{R}_-\times\mathbb{R} : x_1\tan\theta < |x_2| < \left(x_1+\frac{\pi}{\sin\theta}\right)\tan\theta\right\}. \tag{4.3.1}$$

We will use the coordinates

$$x = x_1\sqrt{2}\sin\theta,\quad y = x_2\sqrt{2}\cos\theta, \tag{4.3.2}$$

which transform Tri_θ into $\mathsf{Tri}_{\pi/4}$. The operator becomes:

$$\mathcal{D}_{\mathsf{Tri}}(h) = 2\sin^2\theta\,\partial_x^2 - 2\cos^2\theta\,\partial_y^2,$$

with Dirichlet condition on the boundary of Tri. We let $h=\tan\theta$; after a division by $2\cos^2\theta$, we get the new operator:

$$\mathcal{L}_{\mathsf{Tri}}(h) = -h^2\partial_x^2 - \partial_y^2. \tag{4.3.3}$$

This operator is thus in the "Born-Oppenheimer form" and we shall introduce its Born-Oppenheimer approximation which is the Dirichlet realization on $\mathsf{L}^2((-\pi\sqrt{2},0))$ of:

$$\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h) = -h^2\partial_x^2 + \frac{\pi^2}{4(x+\pi\sqrt{2})^2}. \tag{4.3.4}$$

Theorem 4.12. *The eigenvalues of $\mathcal{H}_{\mathsf{BO},\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{BO},\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{BO},\mathsf{Tri},n}(h) \underset{h\to0}{\sim} \sum_{j\geq0}\hat\beta_{j,n}h^{2j/3},\quad \textit{with}\quad \hat\beta_{0,n}=\frac{1}{8}\quad\textit{and}\quad \hat\beta_{1,n}=(4\pi\sqrt{2})^{-2/3}z_{\mathsf{Ai}^{\mathsf{rev}}}(n),$$

where $z_{\mathsf{Ai}^{\mathsf{rev}}}(n)$ is the n-th zero of the reversed Airy function $\mathsf{Ai}^{\mathsf{rev}}(x)=\mathsf{Ai}(-x)$.

We state the result for the scaled operator $\mathcal{L}_{\mathsf{Tri}}(h)$.

Theorem 4.13. *The eigenvalues of $\mathcal{L}_{\mathsf{Tri}}(h)$, denoted by $\lambda_{\mathsf{Tri},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Tri},n}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \beta_{j,n} h^{j/3} \quad \textit{with} \quad \beta_{0,n} = \frac{1}{8}, \quad \beta_{1,n} = 0, \quad \textit{and} \quad \beta_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{Ai}^{\mathsf{rev}}}(n),$$

the terms of odd rank being zero for $j \leq 8$. The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with both scales $x/h^{2/3}$ and x/h.

4. Broken waveguides

4.1. Physical motivation. As we have already recalled at the beginning of this chapter, it has been proved in [**53**] that a curved, smooth and asymptotically straight waveguide has discrete spectrum below its essential spectrum. Now we would like to explain the influence of a corner which is somehow an infinite curvature and extend the philosophy of the smooth case. This question is investigated with the L-shape waveguide in [**64**] where the existence of discrete spectrum is proved. For an arbitrary angle too, this existence is proved in [**6**] and an asymptotic study of the ground energy is done when θ goes to $\frac{\pi}{2}$ (where θ is the semi-opening of the waveguide). Another question which arises is the estimate of the lowest eigenvalues in the regime $\theta \to 0$. This problem is analyzed in [**31**] where a waveguide with corner is the model chosen to describe some electromagnetic experiments (see the experimental results in [**31**]). We also refer to our work [**45, 46**].

4.2. Geometric description. Let us denote by (x_1, x_2) the Cartesian coordinates of the plane and by $\mathbf{0} = (0,0)$ the origin. Let us define our so-called "broken waveguides". For any angle $\theta \in \left(0, \frac{\pi}{2}\right)$ we introduce

$$\Omega_\theta = \left\{(x_1,x_2) \in \mathbb{R}^2 : x_1 \tan\theta < |x_2| < \left(x_1 + \frac{\pi}{\sin\theta}\right)\tan\theta\right\}. \tag{4.4.1}$$

Note that its width is independent from θ, normalized to π, see Figure 2. The limit case where $\theta = \frac{\pi}{2}$ corresponds to the straight strip $(-\pi, 0) \times \mathbb{R}$.

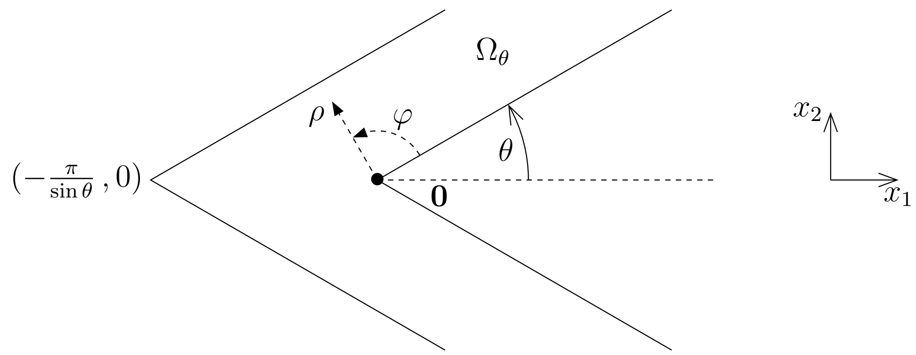


FIGURE 2. The broken guide Ω_θ (here $\theta = \frac{\pi}{6}$). Cartesian and polar coordinates.

The operator $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ is a positive unbounded self-adjoint operator with domain

$$\mathsf{Dom}\,(-\Delta^{\mathsf{Dir}}_{\Omega_\theta}) = \{\psi \in \mathsf{H}^1_0(\Omega_\theta) : \quad -\Delta\psi \in \mathsf{L}^2(\Omega_\theta)\}.$$

When $\theta \in \left(0, \frac{\pi}{2}\right)$, the boundary of Ω_θ is not smooth, it is polygonal. The presence of the non-convex corner with vertex $\mathbf{0}$ is the reason for the space $\mathsf{Dom}\,(-\Delta^{\mathsf{Dir}}_{\Omega_\theta})$ to be distinct from $\mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega_\theta)$. We have the following description of the domain (see the classical references [**111, 77**]):

$$\mathsf{Dom}\,(-\Delta^{\mathsf{Dir}}_{\Omega_\theta}) = \left(\mathsf{H}^2 \cap \mathsf{H}^1_0(\Omega_\theta)\right) \oplus [\psi^\theta_{\mathsf{sing}}] \tag{4.4.2}$$

where $[\psi^\theta_{\mathsf{sing}}]$ denotes the space generated by the singular function $\psi^\theta_{\mathsf{sing}}$ defined in the polar coordinates (ρ, φ) near the origin by

$$\psi^\theta_{\mathsf{sing}}(x_1, x_2) = \chi(\rho)\,\rho^{\pi/\omega} \sin\frac{\pi\varphi}{\omega} \quad \text{with} \quad \omega = 2(\pi - \theta) \tag{4.4.3}$$

where where χ is a radial cutoff function near the origin.

We gather in the following statement several important preliminary properties for the spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$. All these results are proved in the literature.

Proposition 4.14. (i) *If $\theta = \frac{\pi}{2}$, $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ has no discrete spectrum. Its essential spectrum is the closed interval $[1, +\infty)$.*

(ii) *For any θ in the open interval $(0, \frac{\pi}{2})$ the essential spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ coincides with $[1, +\infty)$.*

(iii) *For any $\theta \in (0, \frac{\pi}{2})$, the discrete spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ is* nonempty. (iv) *For any $\theta \in (0, \frac{\pi}{2})$ and any eigenvalue in the discrete spectrum of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$, the associated eigenvectors ψ are* even *with respect to the horizontal axis: $\psi(x_1, -x_2) = \psi(x_1, x_2)$.*

(v) *For any $\theta \in (0, \frac{\pi}{2})$, let $\mu_{\mathsf{Gui},n}(\theta)$, $n = 1, \ldots$, be the n-th Rayleigh quotient of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$. Then, for any $n \geq 1$, the function $\theta \mapsto \mu_{\mathsf{Gui},n}(\theta)$ is continuous and increasing.*

It is also possible to prove that the number of eigenvalues below the essential spectrum is exactly 1 as soon as θ is close enough to $\frac{\pi}{2}$ (see [**139**]). In this book we will provide an instructive proof of the following proposition which is inspired by [**136**, Theorem 2.1].

Proposition 4.15. *For any $\theta \in (0, \frac{\pi}{2})$, the number of eigenvalues of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ below 1, denoted by $\mathcal{N}(-\Delta^{\mathsf{Dir}}_{\Omega_\theta}, 1)$, is finite.*

As a consequence of the parity properties of the eigenvectors of $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$, cf. point (iv) of Proposition 4.14, we can reduce the spectral problem to the half-guide

$$\Omega^+_\theta = \{(x_1, x_2) \in \Omega_\theta : \ x_2 > 0\}\,. \tag{4.4.4}$$

We define the Dirichlet part of the boundary by $\partial_{\mathsf{Dir}}\Omega^+_\theta = \partial\Omega_\theta \cap \partial\Omega^+_\theta$, and the corresponding form domain

$$\mathsf{H}^1_{\mathsf{Mix}}(\Omega^+_\theta) = \left\{\psi \in \mathsf{H}^1(\Omega^+_\theta) : \quad \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\Omega^+_\theta\right\}.$$

Then the new operator of interest, denoted by $-\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$, is the Laplacian with mixed Dirichlet-Neumann conditions on Ω^+_θ. Its domain is:

$$\mathsf{Dom}\,(-\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}) = \left\{\psi \in \mathsf{H}^1_{\mathsf{Mix}}(\Omega^+_\theta) : \quad \Delta\psi \in \mathsf{L}^2(\Omega^+_\theta) \ \text{ and } \ \partial_2\psi = 0 \ \text{ on } \ x_2 = 0\right\}.$$

Then the operators $-\Delta^{\mathsf{Dir}}_{\Omega_\theta}$ and $-\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$ have the same eigenvalues below 1 and the eigenvectors of the latter are the restriction to Ω^+_θ of the former.

In order to analyze the asymptotics $\theta \to 0$, it is useful to rescale the integration domain and transfer the dependence on θ into the coefficients of the operator. For this reason, let us perform the following linear change of coordinates:

$$x = x_1\sqrt{2}\sin\theta, \quad y = x_2\sqrt{2}\cos\theta, \tag{4.4.5}$$

which maps Ω^+_θ onto the θ-independent domain $\Omega^+_{\pi/4}$, see Fig. 3. That is why we set for simplicity

$$\Omega := \Omega^+_{\pi/4}\,, \quad \partial_{\mathsf{Dir}}\Omega = \partial_{\mathsf{Dir}}\Omega^+_{\pi/4}\,, \quad \text{and} \quad \mathsf{H}^1_{\mathsf{Mix}}(\Omega) = \big\{\psi \in \mathsf{H}^1(\Omega) : \psi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\Omega\big\}. \tag{4.4.6}$$

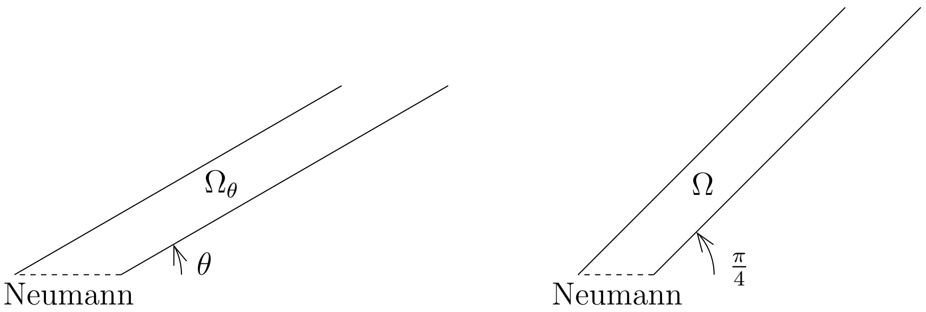


FIGURE 3. The half-guide Ω^+_θ for $\theta = \frac{\pi}{6}$ and the reference domain Ω.

Then, $\Delta^{\mathsf{Mix}}_{\Omega^+_\theta}$ is unitarily equivalent to the operator defined on Ω by:

$$\mathcal{D}_{\mathsf{Gui}}(\theta) := -2\sin^2\theta\,\partial_x^2 - 2\cos^2\theta\,\partial_y^2, \tag{4.4.7}$$

with Neumann condition on $y = 0$ and Dirichlet everywhere else on the boundary of Ω. We let $h = \tan\theta$; after a division by $2\cos^2\theta$, we get the new operator:

$$\mathcal{L}_{\mathsf{Gui}}(h) = -h^2\partial_x^2 - \partial_y^2, \tag{4.4.8}$$

with domain:

$$\mathsf{Dom}\,(\mathcal{L}_{\mathsf{Gui}}(h)) = \big\{\psi \in \mathsf{H}^1_{\mathsf{Mix}}(\Omega) : \ \ \mathcal{L}_{\mathsf{Gui}}(h)\psi \in L^2(\Omega) \ \text{ and } \ \partial_y\psi = 0 \ \text{ on } \ y = 0\big\}.$$

The Born-Oppenheimer approximation (see Chapter 9) is:

$$\mathcal{H}_{\mathsf{BO,Gui}}(h) = -h^2\partial_x^2 + V(x), \tag{4.4.9}$$

where

$$V(x) = \begin{cases} \dfrac{\pi^2}{4(x+\pi\sqrt{2})^2} & \text{when } x \in (-\pi\sqrt{2}, 0), \\ \dfrac{1}{2} & \text{when } x \geq 0. \end{cases}$$

4.3. Eigenvalues induced by a strongly broken waveguide. Let us now state the main result concerning the asymptotic expansion of the eigenvalues of the broken waveguide.

Theorem 4.16. *For all N_0, there exists $h_0 > 0$, such that for $h \in (0, h_0)$ the N_0 first eigenvalues of $\mathcal{L}_{\mathsf{Gui}}(h)$ exist. These eigenvalues, denoted by $\lambda_{\mathsf{Gui},n}(h)$, admit the expansions:*

$$\lambda_{\mathsf{Gui},n}(h) \underset{h \to 0}{\sim} \sum_{j \geq 0} \gamma_{j,n} h^{j/3} \quad \text{with} \quad \gamma_{0,n} = \frac{1}{8}, \quad \gamma_{1,n} = 0, \quad \text{and} \quad \gamma_{2,n} = (4\pi\sqrt{2})^{-2/3} z_{\mathsf{Ai}^{\mathsf{rev}}}(n)$$

and the term of order h is not zero. The corresponding eigenvectors have expansions in powers of $h^{1/3}$ with the scale x/h when $x > 0$, and both scales $x/h^{2/3}$ and x/h when $x < 0$.

4.4. A few numerical simulations. Let us provide some enlightening numerical simulations (using [**125**]) of the first eigenfunctions.

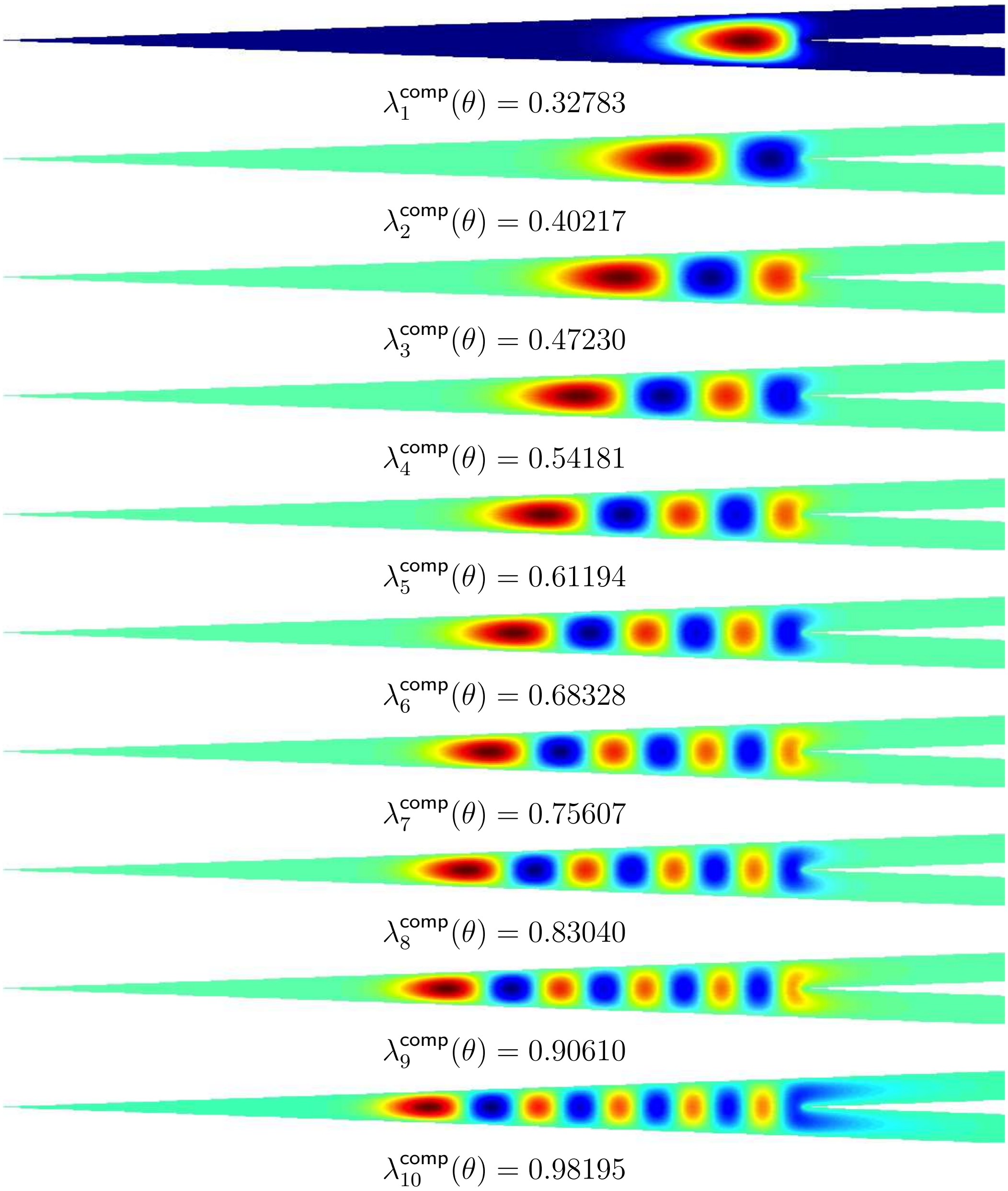


FIGURE 4. Computations for $\theta = 0.0226 * \pi/2 \sim 2^\circ$ with the mesh M64S. Numerical values of the 10 eigenvalues $\lambda_j(\theta) < 1$. Plots of the associated eigenfunctions in the physical domain.

Part 2

Models

CHAPTER 5

Spectrum and quasimodes

It will neither be necessary to deliberate nor to trouble ourselves, as if we shall do this thing, something definite will occur, but if we do not, it will not occur.

Organon, On Interpretation, Aristotle

This chapter is devoted to recall basic tools in spectral analysis.

1. Spectrum

1.1. Spectrum of an unbounded operator. Let $\mathcal{C}$ be an unbounded operator on an Hilbert space H with domain $\mathsf{Dom}\,(\mathcal{C})$. We recall the following characterizations of its spectrum $\mathsf{sp}(\mathcal{C})$, its essential spectrum $\mathsf{sp}_{\mathsf{ess}}(\mathcal{C})$ and its discrete spectrum $\mathsf{sp}_{\mathsf{dis}}(\mathcal{C})$:

- Spectrum: $\lambda \in \mathsf{sp}(\mathcal{C})$ if and only if $(\mathcal{C} - \lambda\,\mathsf{Id})$ is not invertible from $\mathsf{Dom}\,(A)$ onto H,
- Essential spectrum: $\lambda \in \mathsf{sp}_{\mathsf{ess}}(\mathcal{C})$ if and only if $(\mathcal{C} - \lambda\,\mathsf{Id})$ is not Fredholm[1] from $\mathsf{Dom}\,(\mathcal{C})$ into H (see [**155**, Chapter VI] and [**118**, Chapter 3]),
- Discrete spectrum: $\mathsf{sp}_{\mathsf{dis}}(\mathcal{C}) := \mathsf{sp}(\mathcal{C}) \setminus \mathsf{sp}_{\mathsf{ess}}(\mathcal{C})$.

Lemma 5.1 (Weyl criterion)**.** *We have $\lambda \in \mathsf{sp}_{\mathsf{ess}}(\mathcal{C})$ if and only if there exists a sequence $(u_n) \in \mathsf{Dom}\,(\mathcal{C})$ such that $\|u_n\|_{\mathsf{H}} = 1$, (u_n) has no subsequence converging in H and $(\mathcal{C} - \lambda\,\mathsf{Id})u_n \underset{n\to+\infty}{\to} 0$ in H.*

From this lemma, one can deduce (see [**118**, Proposition 2.21 and Proposition 3.11]):

Lemma 5.2. *The discrete spectrum is formed by isolated eigenvalues of finite multiplicity.*

1.2. The example of the magnetic Laplacian.

1.2.1. *The Dirichlet realization.* Let us consider the following quadratic form which is defined for $u \in \mathcal{C}_0^\infty(\Omega)$ by:

$$\mathcal{Q}_{h,\mathbf{A}}(u) = \int_\Omega |(-ih\nabla + \mathbf{A})u|^2 \,\mathrm{d}x \geq 0.$$

[1]We recall that an operator is said to be *Fredholm* if its kernel is finite dimensional, its range is closed and with finite codimension.

The standard Friedrichs procedure (see [**155**, p. 177]) allows to define a self-adjoint operator $\mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}}$ whose (closed) quadratic form is:

$$\mathcal{Q}_{h,\mathbf{A}}(u) = \int_{\Omega} |(-ih\nabla + \mathbf{A})u|^2 \,\mathrm{d}x \geq 0, \quad u \in \mathsf{H}^1_0(\Omega)$$

and such that:

$$\langle \mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}} u, v\rangle = Q_{h,\mathbf{A}}(u,v), \quad u,v \in \mathcal{C}^\infty_0(\Omega).$$

The domain of the Friedrichs extension is defined as:

$$\mathsf{Dom}\,(\mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}}) = \left\{u \in \mathsf{H}^1_0(\Omega) : \mathcal{L}_{h,\mathbf{A}} u \in \mathsf{L}^2(\Omega)\right\}.$$

When Ω is regular, we have the characterization:

$$\mathsf{Dom}\,(\mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}}) = \mathsf{H}^1_0(\Omega) \cap \mathsf{H}^2(\Omega).$$

1.2.2. *The Neumann realization.* We consider the other quadratic form defined by:

$$\mathcal{Q}_{h,\mathbf{A}}(u) = \int_{\Omega} |(-ih\nabla + \mathbf{A})u|^2 \,\mathrm{d}x, \quad u \in \mathcal{C}^\infty(\overline{\Omega}).$$

This form admits a Friedrichs extension (a closure) defined by:

$$\mathcal{Q}_{h,\mathbf{A}}(u) = \int_{\Omega} |(-ih\nabla + \mathbf{A})u|^2 \,\mathrm{d}x, \quad u \in \mathsf{H}^1(\Omega).$$

By the Friedrichs theorem, we can define a self-adjoint operator $\mathcal{L}^{\mathsf{Neu}}_{h,\mathbf{A}}$ whose domain is given by:

$$\mathsf{Dom}\,(\mathcal{L}^{\mathsf{Neu}}_{h,\mathbf{A}}) = \left\{u \in \mathsf{H}^1(\Omega) : \mathcal{L}_{h,\mathbf{A}} u \in \mathsf{L}^2(\Omega), (-ih\nabla + \mathbf{A})u \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\right\}.$$

When Ω is regular, this becomes:

$$\mathsf{Dom}\,(\mathcal{L}^{\mathsf{Neu}}_{h,\mathbf{A}}) = \left\{u \in \mathsf{H}^1(\Omega) : u \in \mathsf{H}^2(\Omega), (-ih\nabla + \mathbf{A})u \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\right\}.$$

1.2.3. *Discrete spectrum.* Since Ω is bounded and Lipschitzian, the form domains $\mathsf{H}^1_0(\Omega)$ and $\mathsf{H}^1(\Omega)$ are compactly included in $L^2(\Omega)$ (by the Riesz-Fréchet-Kolmogorov criterion, see [**28**]) so that the corresponding Friedrichs extensions $\mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}}$ and $\mathcal{L}^{\mathsf{Neu}}_{h,\mathbf{A}}$ have compact resolvents. Therefore these operators have discrete spectra and we can consider the non decreasing sequences of their eigenvalues.

Remark 5.3. *Let us give a basic example of Fredholm operator. We consider $P = \mathcal{L}^{\mathsf{Dir}}_{h,\boldsymbol{A}}$ when Ω is bounded and regular. Let us take λ an eigenvalue of P ($\lambda \in \mathbb{R}$ since P is self-adjoint). As we said* $\ker(P-\lambda)$ *has finite dimension. Since P is self-adjoint, we can write:*

$$\overline{\Im(P-\lambda)} = \ker(P-\lambda)^\perp.$$

This is easy to see that the image of $P-\lambda$ is closed by using that $K = (P-\lambda+i)^{-1}$ is compact. P is a Fredholm operator.

2. Min-max principle and spectral theorem

We give a standard method to estimate the discrete spectrum and the bottom of the essential spectrum of a self-adjoint operator $\mathcal{C}$ on an Hilbert space H. We recall first the definition of the Rayleigh quotients of a self-adjoint operator $\mathcal{C}$.

Definition 5.4. *The Rayleigh quotients associated with the self-adjoint operator $\mathcal{C}$ on H of domain $\mathsf{Dom}\,(\mathcal{C})$ are defined for all positive natural number j by*

$$\lambda_j(\mathcal{C}) = \inf_{\substack{u_1,\ldots,u_j \in \mathsf{Dom}\,(\mathcal{C}) \\ \textit{independent}}} \ \sup_{u \in [u_1,\ldots,u_j]} \frac{\langle \mathcal{C}u, u\rangle_{\mathsf{H}}}{\langle u, u\rangle_{\mathsf{H}}}.$$

Here $[u_1, \ldots, u_j]$ denotes the subspace generated by the j independent vectors $u_1, \ldots, u_j$.

The following statement gives the relation between Rayleigh quotients and eigenvalues.

Theorem 5.5. *Let $\mathcal{C}$ be a self-adjoint operator of domain $\mathsf{Dom}\,(\mathcal{C})$. We assume that $\mathcal{C}$ is semi-bounded from below. We set $\gamma = \min \mathsf{sp}_{\mathsf{ess}}(\mathcal{C})$. Then the Rayleigh quotients λ_j of $\mathcal{C}$ form a non-decreasing sequence and there holds*

(1) If $\lambda_j(\mathcal{C}) < \gamma$, it is an eigenvalue of $\mathcal{C}$,
(2) If $\lambda_j(\mathcal{C}) \geq \gamma$, then $\lambda_j = \gamma$,
(3) The j-th eigenvalue $< \gamma$ of $\mathcal{C}$ (if exists) coincides with $\lambda_j(\mathcal{C})$.

A consequence of this theorem which is often used is the following:

Proposition 5.6. *Suppose that there exists $a \in \mathbb{R}$ and an n-dimensional space $\mathsf{V} \subset \mathsf{Dom}\,\mathcal{C}$ such that:*

$$\langle \mathcal{C}\psi, \psi\rangle_{\mathsf{H}} \leq a\|\psi\|^2.$$

Then, we have:

$$\lambda_n(\mathcal{C}) \leq a.$$

Remark 5.7. *For the proof we refer to* [**118**, Proposition 6.17 and 13.1] *or to* [**156**, Chapter XIII].

Let us give a characterization of the bottom of the essential spectrum (see [**144**] and also [**68**]).

Theorem 5.8. *Let V be real-valued, semi-bounded potential and $\boldsymbol{A} \in \mathcal{C}^1(\mathbb{R}^n)$ a magnetic potential. Let $\mathcal{L}_{\boldsymbol{A},V}$ be the corresponding self-adjoint, semi-bounded Schrödinger operator. The, the bottom of the essential spectrum is given by:*

$$\inf \mathsf{sp}_{\mathsf{ess}}(\mathcal{L}_{\boldsymbol{A},V}) = \Sigma(\mathcal{L}_{\boldsymbol{A},V}),$$

where:

$$\Sigma(\mathcal{L}_{\boldsymbol{A},V}) = \sup_{K \subset \mathbb{R}^n} \left[\inf_{\|\phi\|=1} \langle \mathcal{L}_{\boldsymbol{A},V}\phi, \phi\rangle_{\mathsf{L}^2} \,|\, \phi \in \mathcal{C}_0^\infty(\mathbb{R}^n \setminus K) \right].$$

Let us notice that generalizations including the presence of a boundary are possible.

We state a theorem which will be one of the fundamental tools in this course.

Theorem 5.9. *Let us assume that $(\mathcal{C}, \mathsf{Dom}\,(\mathcal{C}))$ is a self-adjoint operator. Then, if $\lambda \notin \mathsf{sp}(\mathcal{C})$, we have:*

$$\|(\mathcal{C} - \lambda)^{-1}\| \leq \frac{1}{\mathrm{dist}(\lambda, \mathsf{sp}(\mathcal{C}))}.$$

Remark 5.10. *This theorem is known as the spectral theorem and a proof can be found in* [**156**] *and* [**108**, Section VI.5]. *An immediate consequence of this theorem is that, for all $\psi \in \mathsf{Dom}\,(\mathcal{C})$:*

$$\|\psi\|\mathrm{dist}(\lambda, \mathsf{sp}(\mathcal{C})) \leq \|(\mathcal{C} - \lambda)\psi\|.$$

In particular, if we find $\psi \in \mathsf{Dom}\,(\mathcal{C})$ such that $\|\psi\| = 1$ and $\|(\mathcal{C} - \lambda)\psi\| \leq \varepsilon$, we get: $\mathrm{dist}(\lambda, \mathsf{sp}(\mathcal{C})) \leq \varepsilon$.

3. Harmonic oscillator

Before going further we shall discuss the spectrum of the harmonic oscillator which we will encounter many times in this book. We are interested in the self-adjoint realization on $L^2(\mathbb{R})$ of:

$$\mathcal{H}_{\mathsf{harm}} = D_x^2 + x^2.$$

In terms of the philosophy of the last section, this operator is defined as the Friedrichs extension associated with the closed quadratic form defined by:

$$\mathcal{Q}_{\mathsf{harm}}(\psi) = \|\psi'\|^2 + \|x\psi\|^2, \quad \psi \in \mathsf{B}^1(\mathbb{R}),$$

where

$$\mathsf{B}^1(\mathbb{R}) = \{\psi \in L^2(\mathbb{R}) : \psi' \in \mathsf{L}^2(\mathbb{R}), x\psi \in \mathsf{L}^2(\mathbb{R})\}.$$

The domain of the operator can be characterized (thanks to the difference quotients method, see [**28**, Theorem IX. 25]) as:

$$\mathsf{Dom}\,(\mathcal{H}_{\mathsf{harm}}) = \{\psi \in \mathsf{L}^2(\mathbb{R}) : \psi'' \in \mathsf{L}^2(\mathbb{R}), x^2\psi \in \mathsf{L}^2(\mathbb{R})\}.$$

The self-adjoint operator $\mathcal{H}_{\mathsf{harm}}$ has compact resolvent since $\mathsf{B}^1(\mathbb{R})$ is compactly included in $\mathsf{L}^2(\mathbb{R})$. Its spectrum is a sequence of eigenvalues which tends to $+\infty$. Let us explain how we can get the spectrum of $\mathcal{H}_{\mathsf{harm}}$. We let:

$$a = \frac{1}{\sqrt{2}}\left(\frac{d}{dx} + x\right), \quad a^* = \frac{1}{\sqrt{2}}\left(-\frac{d}{dx} + x\right).$$

We have:

$$[a, a^*] = aa^* - a^*a = 1.$$

We let:

$$f_0(x) = e^{-x^2/2}.$$

We investigate the spectrum of a^*a. We have: $af_0 = 0$. We let $f_n = (a^*)^n f_0$. This is easy to prove that $a^*af_n = nf_n$ and that $af_n = nf_{n-1}$. The (f_n) form a Hilbertian basis of

$L^2(\mathbb{R})$. These functions are called Hermite's functions. The eigenvalues of $\mathcal{H}_{\mathsf{harm}}$ are the numbers $2n+1, n \in \mathbb{N}$. They are simple and associated with the normalized Hermite's functions.

Exercise. [2] We wish to study the $2D$ harmonic oscillator: $-\Delta + |x|^2$.

(1) Write the operator in terms of radial coordinates.
(2) Explain how the spectral analysis can be reduced to the study of:
$$-\partial_\rho^2 - \rho^{-1}\partial_\rho + \rho^{-2}m^2 + \rho^2,$$
on $\mathsf{L}^2(\rho d\rho)$ with $m \in \mathbb{Z}$.
(3) Perform the change of variable $t = \rho^2$.
(4) For which α is $t \mapsto t^\alpha e^{-t/2}$ an eigenfunction ?
(5) Conjugate the operator by $t^{-m/2}e^{t/2}$. On which space is the new operator $\mathcal{L}_m$ acting ? Describe the new scalar product.
(6) Find eigenvalues of $\mathcal{L}_m$ by noticing that $\mathbb{R}_N[X]$ is stable by $\mathcal{L}_m$.
(7) Conclude.

4. Harmonic approximation in dimension one

We illustrate the application of the spectral theorem in the case of the electric Laplacian $\mathcal{L}_{h,V} = -h^2\Delta + V(x)$. We assume that $V \in \mathcal{C}^\infty(\mathbb{R},\mathbb{R})$, that $V(x) \to +\infty$ when $|x| \to +\infty$ and that it admits a unique and non degenerate minimum at 0. This example is also the occasion to understand more in details how we construct quasi-eigenpairs in general. From a heuristic point of view, we guess that the lowest eigenvalues correspond to functions localized near the minimum of the potential (intuition coming from the classical mechanics). Therefore we can use a Taylor expansion of V near 0:
$$V(x) = \frac{V''(0)}{2}x^2 + O(|x|^3).$$
We can then try to compare $-h^2\Delta + V(x)$ with $-h^2\Delta + \dfrac{V''(0)}{2}x^2$. For an homogeneity reason, we try the rescaling $x = h^{1/2}y$. The electric operator becomes:
$$\tilde{\mathcal{L}}_{h,V} = -h\Delta_y + V(h^{1/2}y).$$
Let us use the Taylor formula:
$$V(h^{1/2}y) \sim \frac{V''(0)}{2}hy^2 + \sum_{j\geq 3} h^{j/2}\frac{V^{(j)}(0)}{j!}y^j.$$
This provides the formal expansion:
$$\tilde{\mathcal{L}}_{h,V} \sim h\left(L_0 + \sum_{j\geq 1} h^{j/2}L_j\right),$$

[2]This exercise is an example of exact WKB expansions. We will recognize Laguerre's polynomials.

where

$$L_0 = -\partial_y^2 + \frac{V''(0)}{2}y^2.$$

We look for a quasimode in the form:

$$u \sim \sum_{j\geq 0} u_j(y)h^{j/2}$$

and an eigenvalue:

$$\mu \sim h\sum_{j\geq 0} \mu_j h^{j/2}.$$

Let us investigate the system of PDE that we get when solving in the formal series:

$$\tilde{\mathcal{L}}_{h,V} u \sim \mu u.$$

We get the equation:

$$L_0 u_0 = \mu_0 u_0.$$

Therefore we can take for (μ_0, u_0) a L^2-normalized eigenpair of the harmonic oscillator. Then we solve:

$$(L_0 - \mu_0)u_1 = (\mu_1 - L_1)u_0.$$

We want to determine μ_1 and u_1. We can verify that $H_0 - \mu_0$ is a Fredholm operator so that a necessary and sufficient condition to solve this equation is given by:

$$\langle (\mu_1 - L_1)u_0, u_0\rangle_{\mathsf{L}^2} = 0.$$

Lemma 5.11. *Let us consider the equation:*

$$(L_0 - \mu_0)u = f, \tag{5.4.1}$$

with $f \in \mathcal{S}(\mathbb{R})$ such that $\langle f, u_0\rangle_{\mathsf{L}^2} = 0$. The (5.4.1) *admits a unique solution which is orthogonal to u_0 and this solution is in the Schwartz class.*

Proof. Let us just sketch the proof to enlighten the general idea. We know that we can find $u \in \mathsf{Dom}\,(H_0)$ and that u is determined modulo u_0 which is in the Schwartz class. Therefore, we have: $y^2u \in \mathsf{L}^2(\mathbb{R})$ and $u \in \mathsf{H}^2(\mathbb{R})$. Let us introduce a smooth cutoff function $\chi_R(y) = \chi(R^{-1}y)$. $\chi_R y^2 u$ is in the form domain of H_0 as well as in the domain of H_0 so that we can write:

$$\langle L_0(\chi_R y^2 u), \chi_R y^2 u\rangle_{\mathsf{L}^2} = \langle [L_0, \chi_R y^2]u, \chi_R y^2 u\rangle_{\mathsf{L}^2} + \langle \chi_R y^2 u(\mu_0 u + f), \chi_R y^2 u\rangle_{\mathsf{L}^2}.$$

The commutator can easily be estimated and, by dominate convergence, we find the existence of $C > 0$ such that for R large enough we have:

$$\|\chi_R y^3 u\|^2 \leq C.$$

The Fatou lemma involves:

$$y^3 u \in \mathsf{L}^2(\mathbb{R}).$$

This is then a standard iteration procedure which gives that $\partial_y^l(y^k u) \in L^2(\mathbb{R})$. The Sobolev injection ($\mathsf{H}^s(\mathbb{R}) \hookrightarrow \mathcal{C}^{s-\frac{1}{2}}(\mathbb{R})$ for $s > \frac{1}{2}$) gives the conclusion.

□

This determines a unique value of $\mu_1 = \langle L_1 u_0, u_0\rangle_{\mathsf{L}^2}$. For this value we can find a unique $u_1 \in \mathcal{S}(\mathbb{R})$ orthogonal to u_0.

This is easy to see that this procedure can be continued at any order.

Let us consider the (μ_j, u_j) that we have constructed and let us introduce:

$$U_{J,h} = \sum_{j=0}^{J} u_j(y) h^{j/2}, \quad \mu_{J,h} = h \sum_{j=0}^{J} \mu_j h^{j/2}.$$

We estimate:

$$\|(\tilde{\mathcal{L}}_{h,V} - \mu_{J,h}) U_{J,h}\|.$$

By using the Taylor formula and the definition of the μ_j and u_j, we have:

$$\|(\tilde{\mathcal{L}}_{h,V} - \mu_{J,h}) U_{J,h}\| \leq C_J h^{(J+1)/2},$$

since $h^{(J+1)/2}\|y^{(J+1)/2} U_{J,h}\| \leq C_J h^{(J+1)/2}$ due to the fact that $u_j \in \mathcal{S}(\mathbb{R})$. The spectral theorem implies:

$$\text{dist}\left(\mu_{J,h}, \mathsf{sp}_{\mathsf{dis}}(\tilde{\mathcal{L}}_{h,V})\right) \leq C_J h^{(J+1)/2}.$$

5. Helffer-Kordyukov's toy operator

Let us now give an explicit example of construction of quasimodes for the magnetic Laplacian in $\mathbb{R}^2$. We investigate the operator:

$$\mathcal{L}_{h,\mathbf{A}} = (hD_1 + A_1)^2 + (hD_2 + A_2)^2,$$

with domain:

$$\mathsf{Dom}\, \mathcal{L}_{h,\mathbf{A}} = \{\psi \in \mathsf{L}^2(\mathbb{R}^2) : \left((hD_1 + A_1)^2 + (hD_1 + A_2)^2\right)\psi \in \mathsf{L}^2(\mathbb{R}^2)\}.$$

5.1. Compact resolvent ? Let us state an easy lemma.

Lemma 5.12. *We have:*

$$\mathcal{Q}_{h,\boldsymbol{A}}(\psi) \geq \left|\int_{\mathbb{R}^2} h\boldsymbol{B}(x)|\psi|^2\, \mathrm{d}x\right|, \quad \forall \psi \in \mathcal{C}_0^\infty(\mathbb{R}^2).$$

Proof. We notice that:

$$[hD_1 + A_1, hD_2 + A_2] = -ih\mathbf{B}.$$

We find:

$$\langle [hD_1 + A_1, hD_2 + A_2]\psi, \psi\rangle_{\mathsf{L}^2} = -ih \int_{\mathbb{R}^2} \mathbf{B}|\psi|^2\, \mathrm{d}x.$$

By integration by parts, we deduce:

$$|\langle [hD_1 + A_1, hD_2 + A_2]\psi, \psi\rangle_{\mathsf{L}^2}| \leq 2\|(hD_1 + A_1)\psi\|\|(hD_2 + A_2)\psi\| \leq \mathcal{Q}_{h,\mathbf{A}}(\psi).$$

□

Proposition 5.13. *Suppose that $\boldsymbol{A} \in \mathcal{C}^\infty(\mathbb{R}^2)$ and that $\boldsymbol{B} = \nabla \times \boldsymbol{A} \geq 0$ and $\boldsymbol{B}(x) \underset{|x \to +\infty|}{\to} +\infty$. Then, $\mathcal{L}_{h,\boldsymbol{A}}$ has compact resolvent.*

Proof. This is an application of the Riesz-Fréchet-Kolmogorov theorem, see [**28**, Theorem IV.25] (the form domain has compact injection in $\mathsf{L}^2(\mathbb{R}^2)$). $\square$

5.2. Quasimodes. Let us give a simple example inspired by [**86**]. Let us choose $\mathbf{A}$ such that $\mathbf{B} = 1 + x^2 + y^2$. We take $A_1 = 0$ and $A_2 = x + \frac{x^3}{3} + y^2 x$. We study:

$$\mathcal{L}_{h,\mathbf{A}} = h^2 D_x^2 + \left(hD_y + x + \frac{x^3}{3} + y^2 x\right)^2.$$

Let us try the rescaling $x = h^{1/2}u$, $y = h^{1/2}v$. We get a new operator:

$$\tilde{\mathcal{L}}_{h,\mathbf{A}} = hD_u^2 + h\left(D_v + u + h\frac{u^3}{3} + hv^2 u\right)^2.$$

Let us conjugate by the partial Fourier transform with respect to v ; we get the unitarily equivalent operator:

$$\hat{\mathcal{L}}_{h,\mathbf{A}} = hD_u^2 + h\left(\xi + u + h\frac{u^3}{3} + huD_\xi^2\right)^2.$$

Let us now use the transvection: $u = \check{u} - \check{\xi}, \xi = \check{\xi}$. We have:

$$D_u = D_{\check{u}}, \quad D_\xi = D_{\check{u}} + D_{\check{\xi}}.$$

We are reduced to the study of:

$$\check{\mathcal{L}}_{h,\mathbf{A}} = hD_{\check{u}}^2 + h\left(\check{u} + h\frac{(\check{u} - \check{\xi})^3}{3} + h(\check{u} - \check{\xi})(D_\xi + D_{\check{u}})^2\right)^2$$

We can expand $\check{\mathcal{L}}_{h,\mathbf{A}}$ in formal power series:

$$\check{\mathcal{L}}_{h,\mathbf{A}} = hP_0 + h^2 P_1 + \cdots,$$

where $P_0 = D_{\check{u}}^2 + \check{u}^2$ and $P_1 = \frac{2}{3}\check{u}(\check{u} - \check{\xi})^3 + (\check{u} - \check{\xi})(D_{\check{\xi}} + D_{\check{u}})^2\check{u} + \check{u}(\check{u} - \check{\xi})(D_{\check{\xi}} + D_{\check{u}})^2$.

Let us look for quasi-eigenpairs in the form

$$\lambda \sim h\lambda_0 + h^2\lambda_1 + \cdots, \quad \psi \sim \psi_0 + h\psi_1 + \cdots$$

We solve the equation:

$$P_0\psi_0 = \lambda_0\psi_0.$$

We take $\lambda_0 = 1$ and $\psi_0(\check{u}, \check{\xi}) = g_0(\check{u})f_0(\check{\xi})$ where g_0 is the first normalized eigenfunction of the harmonic oscillator. f_0 is a function to be determined. The second equation of the formal system is:

$$(P_0 - \lambda_0)\psi_1 = (\lambda_1 - P_1)\psi_0.$$

The Fredholm condition gives, for all $\check{\xi}$:

$$\langle(\lambda_1 - P_1)\psi_0, g_0\rangle_{\mathsf{L}^2(\mathbb{R}_{\check{u}})} = 0.$$

Let us analyze the different terms which appear in this differential equation. There should be a term in $\check{\xi}^3$. Its coefficient is:

$$\int_{\mathbb{R}} \check{u} g_0(\check{u})^2 \, \mathrm{d}\check{u} = 0.$$

For the same parity reason, there is no term in $\check{\xi}$. Let us now analyze the term in $D_{\check{\xi}}$. Its coefficient is:

$$\langle (D_{\check{u}} \check{u} + \check{u} D_{\check{u}}) g_0, \check{u} g_0 \rangle_{\mathsf{L}^2(\mathbb{R}_{\check{u}})} = 0,$$

for a parity reason. In the same way, there is no term in $\check{\xi} D_{\check{\xi}}^2$. The coefficient of $\check{\xi} D_{\check{\xi}}$ is:

$$2 \int_{\mathbb{R}} (\check{u} D_{\check{u}} - D_{\check{u}} \check{u}) g_0 g_0 \, \mathrm{d}\check{u} = 0.$$

The compatibility equation is in the form:

$$(a D_{\check{\xi}}^2 + b \check{\xi}^2 + c) f_0 = \lambda_1 f_0.$$

It turns out that (exercise):

$$a = b = 2 \int_{\mathbb{R}} \check{u}^2 g_0^2 \, \mathrm{d}\check{u} = 1.$$

In the same way c can be explicitly found. This leads to a family of choices for (λ_1, f_0): We can take $\lambda_1 = c + (2m+1)$ and $f_0 = g_m$ the corresponding Hermite function.

This construction provides us a family of quasimodes (which are in the Schwartz class). By the spectral theorem, we infer that, for each $m \in \mathbb{N}$, there exists $C_m > 0$ such that:

$$\mathrm{dist}\left(h + (2m+1+c)h^2, \mathsf{sp}_{\mathsf{dis}}(P_{h,\mathbf{A}})\right) \leq C_m h^3.$$

Remark 5.14. *One could continue the expansion at any order and one could also consider the other possible values of λ_0 (next eigenvalues of the harmonic oscillator).*

Remark 5.15. *The fact that the construction can be continued as much as the appearance of the harmonic oscillator is a clue that our initial scaling is actually the good one. We can also guess that the lowest eigenfunctions are concentrated near zero at the scale $h^{1/2}$ if the quasimodes approximate the true eigenfunctions.*

CHAPTER 6

From local models to global estimates

> Zeno's reasoning, however, is fallacious, when he says that if everything when it occupies an equal space is at rest, and if that which is in locomotion is always occupying such a space at any moment, the flying arrow is therefore motionless. This is false, for time is not composed of indivisible moments any more than any other magnitude is composed of indivisibles.
>
> *Physics*, Aristotle

1. Some local models

As we mentioned in the introduction, the analysis of the magnetic Laplacian leads to the study of numerous model operators. We saw that the harmonic oscillator is such a model.

1.1. De Gennes Operator. The analysis of the $2D$ magnetic Laplacian with Neumann condition on $\mathbb{R}^2_+$ makes the so-called de Gennes operator to appear. We refer to [**44**] where this model is studied in details (see also [**68**]). This operator is defined as follows. For $\zeta \in \mathbb{R}$, we consider the Neumann realization $\mathfrak{L}^{[0]}_\zeta$ in $\mathsf{L}^2(\mathbb{R}_+)$ associated with the operator

$$D_t^2 + (\zeta - t)^2, \quad \mathsf{Dom}\,(\mathfrak{L}^{[0]}_\zeta) = \{u \in \mathsf{B}^2(\mathbb{R}_+) : u'(0) = 0\}. \tag{6.1.1}$$

The operator $\mathfrak{L}^{[0]}_\zeta$ has compact resolvent by standard arguments. By the Cauchy-Lipschitz theorem, all the eigenvalues are simple.

Notation 6.1. *The lowest eigenvalue of $\mathfrak{L}^{[0]}_\zeta$ is denoted $\nu^{[0]}_1(\zeta)$; the associated L^2-normalized and positive eigenfunction is denoted by $u^{[0]}_\zeta = u^{[0]}(\cdot, \zeta)$.*

We easily get that $u^{[0]}_\zeta$ is in the Schwartz class.

Lemma 6.2. *The function $\zeta \mapsto \nu^{[0]}_n(\zeta)$ is analytic and so is $\zeta \mapsto u^{[0]}(\cdot, \zeta)$.*

Proof. The family $(\mathfrak{L}^{[0]}_\zeta)_{\zeta\in\mathbb{R}}$ is analytic of type (A), see [**108**, p. 375]. Let us provide an elementary proof. Let us fix $\zeta_1 \in \mathbb{R}$ and prove that $\nu^{[0]}_n$ is continuous at ζ_1. We have, for all $\psi \in \mathsf{B}^1(\mathbb{R}_+)$,

$$\left|\mathfrak{Q}^{[0]}_{\zeta}(\psi) - \mathfrak{Q}^{[0]}_{\zeta_1}(\psi)\right| \leq |\zeta^2 - \zeta_1^2| \|\psi\|^2 + 2|\zeta - \zeta_1| \|t^{\frac{1}{2}}\psi\|^2$$

so that

$$\left|\mathfrak{Q}^{[0]}_{\zeta}(\psi) - \mathfrak{Q}^{[0]}_{\zeta_1}(\psi)\right| \leq |\zeta^2 - \zeta_1^2| \|\psi\|^2 + 4|\zeta - \zeta_1| \mathfrak{Q}^{[0]}_{\zeta_1}(\psi) + 4\zeta_1^2|\zeta - \zeta_1| \|\psi\|^2.$$

We deduce that

$$\mathfrak{Q}^{[0]}_{\zeta}(\psi) \leq (1 + 4|\zeta - \zeta_1|)\, \mathfrak{Q}^{[0]}_{\zeta_1}(\psi) + 4\zeta_1^2|\zeta - \zeta_1| \|\psi\|^2 + |\zeta^2 - \zeta_1^2| \|\psi\|^2$$

and

$$\mathfrak{Q}^{[0]}_{\zeta}(\psi) \geq (1 - 4|\zeta - \zeta_1|)\, \mathfrak{Q}^{[0]}_{\zeta_1}(\psi) - 4\zeta_1^2|\zeta - \zeta_1| \|\psi\|^2 - |\zeta^2 - \zeta_1^2| \|\psi\|^2.$$

It remains to apply the min-max principle and we get the comparisons between the eigenvalues. We shall now prove the analyticity. Let us fix $\zeta_1 \in \mathbb{R}$ and $z \in \mathbb{C} \setminus \mathsf{sp}(\mathfrak{L}^{[0]}_{\zeta_1})$. We observe that $t(\mathfrak{L}^{[0]}_{\zeta_1} - z)^{-1}$ is bounded with a uniform bound with respect to z so that for ζ close enough to ζ_1, $\mathfrak{L}^{[0]}_{\zeta} - z$ is invertible. We also infer that $\zeta \mapsto (\mathfrak{L}^{[0]}_{\zeta} - z)^{-1}$ is analytic near ζ_1. Since $\mathfrak{L}^{[0]}_{\zeta}$ has compact resolvent and is self-adjoint, the application

$$P_{\Gamma}(\zeta) = \frac{1}{2i\pi} \int_{\Gamma} (\mathfrak{L}^{[0]}_{\zeta} - z)^{-1}\, \mathrm{d}z$$

is the projection on the space generated by the eigenfunctions associated with eigenvalues enclosed by the smooth contour Γ. It is possible to consider a contour which encloses only $\nu^{[0]}_n(\zeta_1)$ and, thus, only $\nu^{[0]}_n(\zeta)$ as soon as ζ is close enough to ζ_1. The simplicity of the eigenvalues implies the analyticity. □

Lemma 6.3. $\zeta \mapsto \nu^{[0]}_1(\zeta)$ *admits a unique minimum and it is non degenerate.*

Proof. An easy application of the min-max principle gives:

$$\lim_{\zeta \to -\infty} \nu^{[0]}_1(\zeta) = +\infty.$$

Let us now show that:

$$\lim_{\zeta \to +\infty} \nu^{[0]}_1(\zeta) = 1.$$

The de Gennes operator is equivalent to the operator $-\partial_t^2 + t^2$ on $(-\zeta, +\infty)$ with Neumann condition at $-\zeta$. Let us begin with upper bound. An easy and explicit computation gives:

$$\nu^{[0]}_1(\zeta) \leq \langle(-\partial_t^2 + t^2)e^{-t^2/2}, e^{-t^2/2}\rangle_{\mathsf{L}^2((-\zeta,+\infty))} \underset{\zeta \to +\infty}{\to} 1.$$

Let us investigate the converse inequality. Let us prove some concentration of $u^{[0]}_{\zeta}$ near 0 when ζ increases (the reader can compare this with the estimates of Agmon of Section 3.4). We have:

$$\int_0^{+\infty} (t - \zeta)^2 |u^{[0]}_{\zeta}(t)|^2\, \mathrm{d}t \leq \nu^{[0]}_1(\zeta).$$

If $\lambda(\zeta)$ is the lowest Dirichlet eigenvalue, we have:

$$\nu_1^{[0]}(\zeta) \leq \lambda(\zeta).$$

By monotonicity of the Dirichlet eigenvalue with respect to the domain, we have, for $\zeta > 0$:

$$\lambda(\zeta) \leq \lambda(0) = 3.$$

It follows that:

$$\int_0^1 |u_\zeta^{[0]}(t)|^2 \, \mathrm{d}t \leq \frac{3}{(\zeta-1)^2}, \quad \zeta \geq 2.$$

Let us introduce the test function: $\chi(t)u_\zeta^{[0]}(t)$ with χ supported in $(0, +\infty)$ and being 1 for $t \geq 1$. We have:

$$\begin{aligned}\langle(-\partial_t^2 + (t-\zeta)^2)\chi(t)u_\zeta^{[0]}(t), \chi(t)u_\zeta^{[0]}(t)\rangle_{\mathsf{L}^2(\mathbb{R})} \geq \|\chi(\cdot+\zeta)u_\zeta^{[0]}(\cdot+\zeta)\|^2_{\mathsf{L}^2(\mathbb{R})} &= \|\chi u_\zeta^{[0]}\|^2_{\mathsf{L}^2(\mathbb{R})} \\ &= 1 + O(|\zeta|^{-2}).\end{aligned}$$

Moreover, we get:

$$\langle(-\partial_t^2+(t-\zeta)^2)\chi(t)u_\zeta^{[0]}(t), \chi(t)u_\zeta^{[0]}(t)\rangle_{\mathsf{L}^2(\mathbb{R})} = \langle(-\partial_t^2+(t-\zeta)^2)\chi(t)u_\zeta^{[0]}(t), \chi(t)u_\zeta^{[0]}(t)\rangle_{\mathsf{L}^2(\mathbb{R}_+)}.$$

We have:

$$\langle(-\partial_t^2 + (t-\zeta)^2)\chi(t)u_\zeta^{[0]}(t), \chi(t)u_\zeta^{[0]}(t)\rangle_{\mathsf{L}^2(\mathbb{R}_+)} = \nu_1^{[0]}(\zeta)\|\chi u_\zeta^{[0]}\|^2 + \|\chi' u_\zeta^{[0]}\|^2$$

which can be controlled by the concentration result. We infer that, for ζ large enough:

$$\nu_1^{[0]}(\zeta) \geq 1 - C|\zeta|^{-1}.$$

From these limits, we deduce the existence of a minimum strictly less than 1.

We now use the Feynman-Hellmann formula which will be established later. We have:

$$(\nu_1^{[0]})'(\zeta) = -2\int_{t>0} (t-\zeta)|u_\zeta^{[0]}(t)|^2 \, \mathrm{d}t.$$

For $\zeta < 0$, we get an increasing function. Moreover, we see that $\nu(0) = 1$. The minima are obtained for $\zeta > 0$.

We can write that:

$$(\nu_1^{[0]})'(\zeta) = 2\int_{t>0} (t-\zeta)^2 u_\zeta^{[0]}(u_\zeta^{[0]})' \, \mathrm{d}t + \zeta^2 u_\zeta^{[0]}(0)^2.$$

This implies:

$$(\nu_1^{[0]})'(\zeta) = (\zeta^2 - \nu_1^{[0]}(\zeta))u_\zeta^{[0]}(0)^2.$$

Let ζ_c a critical point for $\nu_1^{[0]}$. We get:

$$(\nu_1^{[0]})''(\zeta_c) = 2\zeta_c u_{\zeta_c}^{[0]}(0)^2.$$

The critical points are all non degenerate. They correspond to local minima.We conclude that there is only one critical point and that is the minimum. We denote it ζ_0 and we have $\nu_1^{[0]}(\zeta_0) = \zeta_0^2$. $\square$

We let:

$$\Theta_0 = \nu_1^{[0]}(\zeta_0), \tag{6.1.2}$$

$$C_1 = \frac{(u_{\zeta_0}^{[0]})^2(0)}{3}. \tag{6.1.3}$$

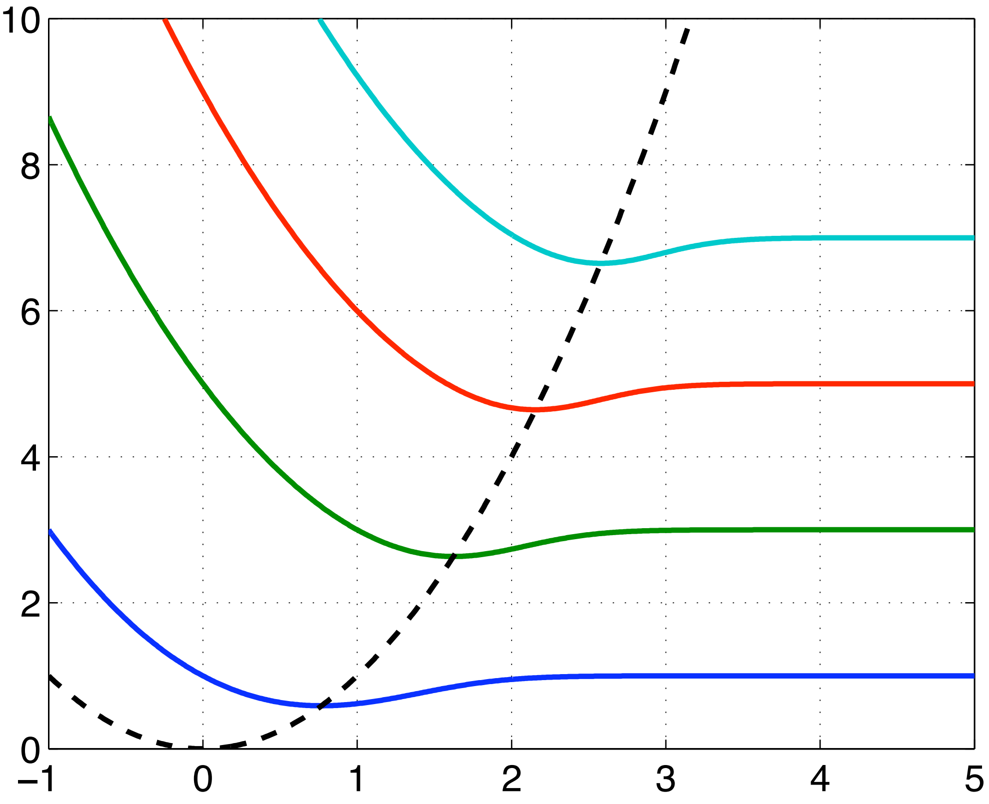


FIGURE 1. $\zeta \mapsto \nu_k^{[0]}(\zeta)$, for $k = 1, 2, 3, 4$

1.2. Lu-Pan Operator. Let us recall that $\mathfrak{L}_\theta^{\mathsf{LP}}$ is defined by:

$$\mathfrak{L}_\theta^{\mathsf{LP}} = -\Delta + V_\theta = D_s^2 + D_t^2 + V_\theta,$$

where V_θ is defined for any $\theta \in (0, \frac{\pi}{2})$ by

$$V_\theta\colon (s,t) \in \mathbb{R}_+^2 \longmapsto (t\cos\theta - s\sin\theta)^2.$$

We can notice that V_θ reaches its minimum 0 all along the line $t\cos\theta = s\sin\theta$, which makes the angle θ with $\partial\mathbb{R}_+^2$. We denote by $\mathsf{Dom}\,(\mathfrak{L}_\theta^{\mathsf{LP}})$ the domain of $\mathcal{L}_\theta$ and we consider the associated quadratic form $\mathfrak{Q}_\theta^{\mathsf{LP}}$ defined by:

$$\mathfrak{Q}_\theta^{\mathsf{LP}}(u) = \int_{\mathbb{R}_+^2} \left(|\nabla u|^2 + V_\theta |u|^2\right) \mathrm{d}s\,\mathrm{d}t,$$

whose domain $\mathsf{Dom}\,(\mathfrak{Q}_\theta^{\mathsf{LP}})$ is:

$$\mathsf{Dom}\,(\mathfrak{Q}_\theta^{\mathsf{LP}}) = \{u \in \mathsf{L}^2(\mathbb{R}^2_+),\ \nabla u \in \mathsf{L}^2(\mathbb{R}^2_+),\ \sqrt{V_\theta}\, u \in \mathsf{L}^2(\mathbb{R}^2_+)\}.$$

Let $\mathfrak{s}_n(\theta)$ denote the n-th Rayleigh quotient of $\mathfrak{L}_\theta^{\mathsf{LP}}$. Let us recall some fundamental spectral properties of $\mathfrak{L}_\theta^{\mathsf{LP}}$ when $\theta \in \left(0, \frac{\pi}{2}\right)$.

It is proved in [**93**] that $\mathsf{sp}_{\mathsf{ess}}(\mathfrak{L}_\theta^{\mathsf{LP}}) = [1, +\infty)$ and that $\theta \mapsto \mathfrak{s}_1(\theta)$ is non decreasing. Moreover, the function $(0, \frac{\pi}{2}) \ni \theta \mapsto \mathfrak{s}_1(\theta)$ is increasing, and corresponds to a simple eigenvalue < 1 associated with a positive eigenfunction (see [**124**, Lemma 3.6]). As a consequence $\theta \mapsto \mathfrak{s}_1(\theta)$ is analytic (see [**108**, Chapter 7]).

1.3. Kato Theory: Feynman-Hellmann Formulas. As we can notice, all the operators that we have introduced depend on parameters and are analytic of type (B) in terms of Kato's theory. Moreover, we also observe that the lowest eigenvalues of the previous model operators are simple, we systematically deduce that they analytically depend on the parameters.

In order to illustrate the Feynman-Hellmann formulas, let us examine a few examples.

1.3.1. *De Gennes operator.* Let us prove propositions which are often used in the study of the magnetic Laplacian.

For $\rho > 0$ and $\zeta \in \mathbb{R}$, let us introduce the Neumann realization on $\mathbb{R}_+$ of:

$$\mathfrak{L}_{\rho,\zeta}^{[0]} = -\rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \zeta)^2.$$

By scaling, we observe that $\mathfrak{L}_{\rho,\zeta}^{[0]}$ is unitarily equivalent to $\mathfrak{L}_\zeta^{[0]}$ and that $\mathfrak{L}_{1,\zeta}^{[0]} = \mathfrak{L}_\zeta^{[0]}$ (the corresponding eigenfunction is $u_{1,\zeta}^{[0]} = u_\zeta^{[0]}$).

Remark 6.4. *The introduction of the scaling parameter ρ is related to the Virial theorem (see* [**169**]*) which was used by physicists in the theory of superconductivity (see* [**52**] *and also* [**3, 33**]*). We also refer to the papers* [**150**] *and* [**153**] *where it is used many times.*

The form domain of $\mathfrak{L}_{\rho,\zeta}^{[0]}$ is $\mathsf{B}^1(\mathbb{R}_+)$ and is independent from ρ and ζ so that the family $\left(\mathfrak{L}_{\rho,\zeta}^{[0]}\right)_{\rho>0,\zeta\in\mathbb{R}}$ is an analytic family of type (B). The lowest eigenvalue of $H_{\rho,\zeta}$ is $\nu_1^{[0]}(\zeta)$ and we will denote by $u_{\rho,\zeta}$ the corresponding normalized eigenfunction:

$$u_{\rho,\zeta}^{[0]}(\tau) = \rho^{1/4} u_\zeta^{[0]}(\rho^{1/2}\tau).$$

Since $u_\zeta^{[0]}$ satisfies the Neumann condition, we observe that $\partial_\rho^m \partial_\zeta^n u_{\rho,\zeta}^{[0]}$ also satisfies it. In order to lighten the notation and when it is not ambiguous we will write $\mathfrak{L}$ for $\mathfrak{L}_{\rho,\zeta}^{[0]}$, u for $u_{\rho,\zeta}^{[0]}$ and ν for $\nu_1^{[0]}(\zeta)$.

The main idea is now to take derivatives of:

$$\mathfrak{L}u = \nu u \tag{6.1.4}$$

with respect to ρ and ζ. Taking the derivative with respect to ρ and ζ, we get the proposition:

Proposition 6.5. *We have:*

$$(\mathfrak{L}-\nu)\partial_\zeta u = 2(\rho^{1/2}\tau-\zeta)u+\nu'(\zeta)u \tag{6.1.5}$$

and

$$(\mathfrak{L}-\nu)\partial_\rho u = \left(-\rho^{-2}\partial_\tau^2-\zeta\rho^{-1}(\rho^{1/2}\tau-\zeta)-\rho^{-1}\tau(\rho^{1/2}\tau-\zeta)^2\right)u. \tag{6.1.6}$$

Moreover, we get:

$$(\mathfrak{L}-\nu)(Su) = Xu, \tag{6.1.7}$$

where

$$X = -\frac{\zeta}{2}\nu'(\zeta)+\rho^{-1}\partial_\tau^2+(\rho^{1/2}\tau-\zeta)^2$$

and

$$S = -\frac{\zeta}{2}\partial_\zeta-\rho\partial_\rho.$$

Proof. Taking the derivatives with respect to ζ and ρ of (6.1.4), we get:

$$(\mathfrak{L}-\nu)\partial_\zeta u = \nu'(\zeta)u-\partial_\zeta\mathfrak{L}u$$

and

$$(\mathfrak{L}-\nu)\partial_\rho u = -\partial_\rho\mathfrak{L}.$$

We have: $\partial_\zeta\mathfrak{L} = -2(\rho^{1/2}\tau-\zeta)$ and $\partial_\rho\mathfrak{L} = \rho^{-2}\partial_\tau^2+\rho^{-1/2}\tau(\rho^{1/2}\tau-\zeta)$. ☐

Taking $\rho=1$ and $\zeta=\zeta_0$ in (6.1.5), we deduce, with the Fredholm alternative:

Corollary 6.6. *We have:*

$$(\mathfrak{L}_{\zeta_0}^{[0]}-\nu(\zeta_0))v_{\zeta_0}^{[0]} = 2(t-\zeta_0)u_{\zeta_0}^{[0]},$$

with:

$$v_{\zeta_0}^{[0]} = \left(\partial_\zeta u_\zeta^{[0]}\right)_{|\zeta=\zeta_0}.$$

Moreover, we have:

$$\int_{\tau>0}(\tau-\zeta_0)(u_{\zeta_0}^{[0]})^2\,\mathrm{d}\tau = 0.$$

Corollary 6.7. *We have, for all $\rho>0$:*

$$\int_{\tau>0}(\rho^{1/2}\tau-\zeta_0)(u_{\rho,\zeta_0}^{[0]})^2\,\mathrm{d}\tau = 0$$

and:

$$\int_{\tau>0}(\tau-\zeta_0)\left(\partial_\rho u\right)_{\rho=1,\zeta=\zeta_0}u\,\mathrm{d}\tau = -\frac{\zeta_0}{4}.$$

Corollary 6.8. *We have:*

$$(\mathfrak{L}_{\zeta_0}^{[0]}-\nu(\zeta_0))S_0u = \left(\partial_\tau^2+(\tau-\zeta_0)^2\right)u_{\zeta_0}^{[0]},$$

where:

$$S_0u = -\left(\partial_\rho u_{\rho,\zeta}^{[0]}\right)_{|\rho=1,\zeta=\zeta_0}-\frac{\zeta_0}{2}v_{\zeta_0}^{[0]}.$$

Moreover, we have:

$$\|\partial_\tau u_{\zeta_0}^{[0]}\|^2 = \|(\tau-\zeta_0)u_{\zeta_0}^{[0]}\|^2 = \frac{\Theta_0}{2}.$$

The next proposition deals with the second derivative of (6.1.4) with respect to ζ.

Proposition 6.9. *We have:*

$$(\mathfrak{L}_\zeta^{[0]} - \nu_1^{[0]}(\zeta))w_{\zeta_0}^{[0]} = 4(\tau-\zeta_0)v_{\zeta_0}^{[0]} + ((\nu_1^{[0]})''(\zeta_0)-2)u_{\zeta_0}^{[0]},$$

with

$$w_{\zeta_0}^{[0]} = \left(\partial_\zeta^2 u_\zeta^{[0]}\right)_{|\zeta=\zeta_0}.$$

Moreover, we have:

$$\int_{\tau>0}(\tau-\zeta_0)v_{\zeta_0}^{[0]}u_{\zeta_0}^{[0]}\,\mathrm{d}\tau = \frac{2-(\nu_1^{[0]})''(\zeta_0)}{4}.$$

PROOF. Taking the derivative of (6.1.5) with respect to ζ (with $\rho=1$), we get:

$$(\mathfrak{L}_\zeta^{[0]} - \nu_1^{[0]}(\zeta))\partial_\zeta^2 u_\zeta^{[0]} = 2\nu'(\zeta)\partial_\zeta u_\zeta^{[0]} + 4(\tau-\zeta)\partial_\zeta u_\zeta^{[0]} + (\nu''(\zeta)-2)u_\zeta^{[0]}.$$

It remains to take $\zeta=\zeta_0$ and to write the Fredholm alternative. $\square$

1.3.2. *Lu-Pan operator.* The following result is obtained in [**16**].

Proposition 6.10. *For all $\theta\in\left(0,\frac{\pi}{2}\right)$, we have:*

$$\mathfrak{s}_1(\theta)\cos\theta - \mathfrak{s}_1'(\theta)\sin\theta > 0.$$

Moreover, we have:

$$\lim_{\substack{\theta\to\frac{\pi}{2}\\ \theta<\frac{\pi}{2}}} \mathfrak{s}_1'(\theta) = 0.$$

PROOF. For $\gamma\geq 0$, we introduce the operator (see [**152**]):

$$\mathfrak{L}_{\theta,\gamma}^{\mathsf{LP}} = D_s^2 + D_t^2 + (t(\cos\theta+\gamma) - s\sin\theta)^2$$

and we denote by $\mathfrak{s}_1(\theta,\gamma)$ the bottom of its spectrum. Let $\rho>0$ and $\alpha\in(0,\frac{\pi}{2})$ satisfy

$$\cos\theta+\gamma = \rho\cos\alpha \quad\text{and}\quad \sin\theta = \rho\sin\alpha.$$

We perform the rescaling $t=\rho^{-1/2}\hat{t}$, $s=\rho^{-1/2}\hat{s}$ and obtain that $\mathfrak{L}_{\theta,\gamma}^{\mathsf{LP}}$ is unitarily equivalent to:

$$\rho(D_{\hat{s}}^2 + D_{\hat{t}}^2 + (\hat{t}\cos\alpha - \hat{s}\sin\alpha)^2) = \rho\mathfrak{L}_\alpha^{\mathsf{LP}}.$$

In particular, we observe that $\mathfrak{s}_1(\theta,\gamma)=\rho\mathfrak{s}_1(\alpha)$ is a simple eigenvalue: there holds

$$\mathfrak{s}_1(\theta,\gamma) = \sqrt{(\cos\theta+\gamma)^2+\sin^2\theta}\;\;\mathfrak{s}_1\left(\arctan\left(\frac{\sin\theta}{\cos\theta+\gamma}\right)\right). \tag{6.1.8}$$

Performing the rescaling $\tilde{t}=(\cos\theta+\gamma)t$, we get the operator $\tilde{\mathfrak{L}}_{\theta,\gamma}^{\mathsf{LP}}$ which is unitarily equivalent to $\mathfrak{L}_{\theta,\gamma}^{\mathsf{LP}}$:

$$\tilde{\mathfrak{L}}_{\theta,\gamma}^{\mathsf{LP}} = D_s^2 + (\cos\theta+\gamma)^2 D_{\tilde{t}}^2 + (\tilde{t}-s\sin\theta)^2.$$

We observe that the domain of $\tilde{\mathfrak{L}}^{\mathsf{LP}}_{\theta,\gamma}$ does not depend on $\gamma \geq 0$. Denoting by $\tilde{u}_{\theta,\gamma}$ the L^2-normalized and positive eigenfunction of $\tilde{\mathfrak{L}}^{\mathsf{LP}}_{\theta,\gamma}$ associated with $\mathfrak{s}_1(\theta,\gamma)$, we write:

$$\tilde{\mathfrak{L}}^{\mathsf{LP}}_{\theta,\gamma}\tilde{u}^{\mathsf{LP}}_{\theta,\gamma} = \mathfrak{s}_1(\theta,\gamma)\tilde{u}^{\mathsf{LP}}_{\theta,\gamma}.$$

Taking the derivative with respect to γ, multiplying by $\tilde{u}^{\mathsf{LP}}_{\theta,\gamma}$ and integrating, we get the Feynman-Hellmann formula:

$$\partial_\gamma \mathfrak{s}_1(\theta,\gamma) = 2(\cos\theta+\gamma)\int_{\mathbb{R}^2_+} |D_t\tilde{u}^{\mathsf{LP}}_{\theta,\gamma}|^2 \,\mathrm{d}\mathbf{x} \geq 0.$$

We deduce that, if $\partial_\gamma\mathfrak{s}_1(\theta,\gamma) = 0$, then $D_t\tilde{u}^{\mathsf{LP}}_{\theta,\gamma} = 0$ and $\tilde{u}^{\mathsf{LP}}_{\theta,\gamma}$ only depends on s, which is a contradiction with $\tilde{u}^{\mathsf{LP}}_{\theta,\gamma} \in \mathsf{L}^2(\mathbb{R}^2_+)$. Consequently, we have $\partial_\gamma\mathfrak{s}_1(\theta,\gamma) > 0$ for any $\gamma \geq 0$. An easy computation using formula (6.1.8) provides:

$$\partial_\gamma\mathfrak{s}_1(\theta,0) = \mathfrak{s}_1(\theta)\cos\theta - \mathfrak{s}_1'(\theta)\sin\theta.$$

The function $\mathfrak{s}_1$ is analytic and increasing. Thus we deduce:

$$\forall\theta\in\left(0,\frac{\pi}{2}\right), \qquad 0 \leq \mathfrak{s}_1'(\theta) < \frac{\cos\theta}{\sin\theta}\mathfrak{s}_1(\theta).$$

We get:

$$0 \leq \liminf_{\substack{\theta\to\frac{\pi}{2}\\ \theta<\frac{\pi}{2}}} \mathfrak{s}_1'(\theta) \leq \limsup_{\substack{\theta\to\frac{\pi}{2}\\ \theta<\frac{\pi}{2}}} \mathfrak{s}_1'(\theta) \leq 0,$$

which ends the proof. $\square$

2. Estimating numbers of eigenvalues

2.1. Estimate of the number of eigenvalues thanks to local models. In this subsection we explain how we can estimate the number of eigenvalues of $h^2D_x^2 + V(x)$ by using the spirit of partitions of unity and reduction to local models. We propose to prove the following version of the Weyl's law in dimension one (see Remark 6.12).

Proposition 6.11. *Let us consider $V : \mathbb{R}\to\mathbb{R}$ a piecewise Lipschitzian with a finite number of discontinuities which satisfies:*

(1) V tends to $\ell_{\pm\infty}$ when $x\to\pm\infty$ with $\ell_{+\infty}\leq\ell_{-\infty}$,
(2) $\sqrt{(\ell_{+\infty}-V)_+}$ belongs to $L^1(\mathbb{R})$.

We consider the operator $\mathfrak{h}_h = h^2D_x^2+V(x)$ and we assume that the function $(0,1)\ni h\mapsto E(h)\in(-\infty,\ell_{+\infty})$ satisfies

(1) for any $h\in(0,1)$, $\{x\in\mathbb{R} : V(x)\leq E(h)\} = [x_{\min}(E(h)), x_{\max}(E(h))]$,
(2) $h^{1/3}(x_{\max}(E(h)) - x_{\min}(E(h))) \underset{h\to0}{\to} 0$,
(3) $E(h) \underset{h\to0}{\to} E_0 \leq \ell_{+\infty}$.

Then we have:

$$\mathsf{N}(\mathfrak{h}_h, E(h)) \underset{h\to0}{\sim} \frac{1}{\pi h}\int_{\mathbb{R}}\sqrt{(E_0-V)_+}\,\mathrm{d}x.$$

Proof. The strategy of the proof is well-known but we recall it since the usual result does not deal with a moving threshold $E(h)$. We consider a subdivision of the real axis $(s_j(h^\alpha))_{j\in\mathbb{Z}}$, which contains the discontinuities of V, such that there exists $c>0$, $C>0$ such that, for all $j\in\mathbb{Z}$ and $h>0$, $ch^\alpha \leq s_{j+1}(h^\alpha)-s_j(h^\alpha)\leq Ch^\alpha$, where $\alpha>0$ is to be determined. We introduce

$$J_{\min}(h^\alpha)=\min\{j\in\mathbb{Z} : s_j(h^\alpha)\geq x_{\min}(E(h))\},$$
$$J_{\max}(h^\alpha)=\max\{j\in\mathbb{Z} : s_j(h^\alpha)\leq x_{\max}(E(h))\}.$$

For $j\in\mathbb{Z}$ we may introduce the Dirichlet (resp. Neumann) realization on $(s_j(h^\alpha), s_{j+1}(h^\alpha))$ of $h^2D_x^2+V(x)$ denoted by $\mathfrak{h}_{h,j}^{\mathsf{Dir}}$ (resp. $\mathfrak{h}_{h,j}^{\mathsf{Neu}}$). The so-called Dirichlet-Neumann bracketing (see [**156**, Chapter XIII, Section 15]) implies:

$$\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} \mathsf{N}(\mathfrak{h}_{h,j}^{\mathsf{Dir}}, E(h)) \leq \mathsf{N}(\mathfrak{h}_h, E(h)) \leq \sum_{j=J_{\min}(h^\alpha)-1}^{J_{\max}(h^\alpha)+1} \mathsf{N}(\mathfrak{h}_{h,j}^{\mathsf{Neu}}, E(h)).$$

Let us estimate $\mathsf{N}(\mathfrak{h}_{h,j}^{\mathsf{Dir}}, E(h))$. If $\mathfrak{q}_{h,j}^{\mathsf{Dir}}$ denotes the quadratic form of $\mathfrak{h}_{h,j}^{\mathsf{Dir}}$, we have:

$$\mathfrak{q}_{h,j}^{\mathsf{Dir}}(\psi)\leq \int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)} h^2|\psi'(x)|^2+V_{j,\sup,h}|\psi(x)|^2\,\mathrm{d}x,\quad \forall\psi\in\mathcal{C}_0^\infty((s_j(h^\alpha),s_{j+1}(h^\alpha))),$$

where

$$V_{j,\sup,h}=\sup_{x\in(s_j(h^\alpha),s_{j+1}(h^\alpha))} V(x).$$

We infer that

$$\mathsf{N}(\mathfrak{h}_{h,j}^{\mathsf{Dir}}, E(h))\geq \#\left\{n\geq 1 : n\leq \frac{1}{\pi h}(s_{j+1}(h^\alpha)-s_j(h^\alpha))\sqrt{(E(h)-V_{j,\sup,h})_+}\right\}$$

so that:

$$\mathsf{N}(\mathfrak{h}_{h,j}^{\mathsf{Dir}}, E(h))\geq \frac{1}{\pi h}(s_{j+1}(h^\alpha)-s_j(h^\alpha))\sqrt{(E(h)-V_{j,\sup,h})_+}-1$$

and thus:

$$\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)} \mathsf{N}(\mathfrak{h}_{h,j}^{\mathsf{Dir}}, E(h))\geq$$
$$\frac{1}{\pi h}\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)}(s_{j+1}(h^\alpha)-s_j(h^\alpha))\sqrt{(E(h)-V_{j,\sup,h})_+}-(J_{\max}(h^\alpha)-J_{\min}(h^\alpha)+1).$$

Let us consider the function

$$f_h(x)=\sqrt{(E(h)-V(x))_+}$$

and analyze

$$\left|\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)}(s_{j+1}(h^\alpha)-s_j(h^\alpha))\sqrt{(E(h)-V_{j,\sup,h})_+}-\int_{\mathbb{R}}f_h(x)\,\mathrm{d}x\right|$$

$$\leq\left|\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)}\int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)}\sqrt{(E(h)-V_{j,\sup,h})_+}-f_h(x)\,\mathrm{d}x\right|$$

$$+\int_{s_{J_{\max}}(h^\alpha)}^{x_{\max}(E(h))}f_h(x)\,\mathrm{d}x+\int_{x_{\min}(E(h))}^{s_{J_{\min}(h^\alpha)}}f_h(x)\,\mathrm{d}x$$

$$\leq\left|\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)}\int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)}\sqrt{(E(h)-V_{j,\sup,h})_+}-f_h(x)\,\mathrm{d}x\right|+\tilde{C}h^\alpha.$$

Using the trivial inequality $|\sqrt{a_+}-\sqrt{b_+}|\leq\sqrt{|a-b|}$, we notice that

$$|f_h(x)-\sqrt{(E(h)-V_{j,\sup,h})_+}|\leq\sqrt{|V(x)-V_{j,\sup,h}|}.$$

Since V is Lipschitzian on $(s_j(h^\alpha),s_{j+1}(h^\alpha))$, we get:

$$\left|\sum_{j=J_{\min}(h^\alpha)}^{J_{\max}(h^\alpha)}\int_{s_j(h^\alpha)}^{s_{j+1}(h^\alpha)}\sqrt{(E(h)-V_{j,\sup,h})_+}-f_h(x)\,\mathrm{d}x\right|\leq(J_{\max}(h^\alpha)-J_{\min}(h^\alpha)+1)\tilde{C}h^\alpha h^{\alpha/2}.$$

This leads to the optimal choice $\alpha=\frac{2}{3}$ and we get the lower bound:

$$\sum_{j=J_{\min}(h^{2/3})}^{J_{\max}(h^{2/3})}\mathsf{N}(\mathfrak{h}_{h,j}^{\mathsf{Dir}},E(h))\geq\frac{1}{\pi h}\left(\int_{\mathbb{R}}f_h(x)\,\mathrm{d}x-\tilde{C}h(J_{\max}(h^{2/3})-J_{\min}(h^{2/3})+1)\right).$$

Therefore we infer

$$\mathsf{N}(\mathfrak{h}_h,E(h))\geq\frac{1}{\pi h}\left(\int_{\mathbb{R}}f_h(x)\,\mathrm{d}x-\tilde{C}h^{1/3}(x_{\max}(E(h))-x_{\min}(E(h))-\tilde{C}h\right).$$

We notice that: $f_h(x)\leq\sqrt{(\ell_{+\infty}-V(x))_+}$ so that we can apply the dominate convergence theorem. We can deal with the Neumann realizations in the same way. $\square$

Remark 6.12. *Classical results (see* [**156, 158, 49, 172**]*) impose a fixed security distance below the edge of the essential spectrum ($E(h)=E_0<l_{+\infty}$) or deal with non-negative potentials, V, with compact support. Both these cases are recovered by Proposition 6.11. In our result, the maximal threshold for which one can ensure that the semiclassical behavior of the counting function holds is dictated by the convergence rate of the potential towards its limit at infinity, through the assumption*

$$h^{1/3}(x_{\max}(E(h))-x_{\min}(E(h)))\underset{h\to0}{\to}0.$$

More precisely, assume that $l_{-\infty}>l_{+\infty}$ so that $x_{\min}(E(h))\geq x_{\min}(l_{+\infty})$ is uniformly bounded for $E(h)$ in a neighborhood of $l_{+\infty}$. Then

- *If $l_{+\infty} - V(x) \leq Cx^{-\gamma}$ for any $x \geq x_0$ and given $x_0, C > 0$ and $\gamma > 2$, then one can choose $E(h) = l_{+\infty} - Ch^{\rho}$ and $x_{\max}(E(h)) \leq h^{-\rho/\gamma}$, provided $\rho < \gamma/3$.*
- *If $l_{+\infty} - V(x) \leq C_1 \exp(-C_2 x)$ for any $x \geq x_0$ and given $x_0, C_1, C_2 > 0$, then one can choose $E(h) = l_{+\infty} - C_1 \exp(C_2 h^{-1/3} \times o(h))$ and the assumption is satisfied.*

2.2. An easy example of dimensional reduction coming from the theory of waveguides. This section is devoted to the proof of Proposition 4.15 stated in Chapter 4. We introduce the open set $\widetilde{\Omega}_\theta$ isometric to Ω_θ^+, see Figure 2,

$$\widetilde{\Omega}_\theta = \left\{(\tilde{x}, \tilde{y}) \in \left(-\frac{\pi}{\tan\theta}, +\infty\right) \times (0, \pi) : \quad \tilde{y} < \tilde{x}\tan\theta + \pi \ \text{ if } \ \tilde{x} \in \left(-\frac{\pi}{\tan\theta}, 0\right)\right\}.$$

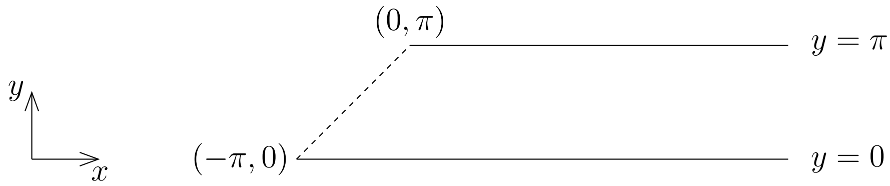


FIGURE 2. The reference half-guide $\widetilde{\Omega} := \widetilde{\Omega}_{\pi/4}$.

The part $\partial_{\mathsf{Dir}}\widetilde{\Omega}_\theta$ of the boundary carrying the Dirichlet condition is the union of its horizontal parts. Let us now perform the change of variable:

$$x = \tilde{x}\tan\theta, \quad y = \tilde{y},$$

so that the new integration domain $\widetilde{\Omega} := \widetilde{\Omega}_{\pi/4}$ is independent of θ. The bilinear form b on $\widetilde{\Omega}_\theta$ is transformed into the form b_θ on the fixed set $\widetilde{\Omega}$:

$$b_\theta(\psi, \psi') = \int_{\widetilde{\Omega}} \tan^2\theta\,(\partial_x\psi\,\partial_x\psi') + (\partial_y\psi\,\partial_y\psi')\,\mathrm{d}x\,\mathrm{d}y, \tag{6.2.1}$$

with associated form domain

$$V := \{\psi \in \mathsf{H}^1(\widetilde{\Omega}) : \psi = 0 \text{ on } \partial_{\mathsf{Dir}}\widetilde{\Omega}\} \tag{6.2.2}$$

independent of θ.

The opening θ being fixed, we drop the index θ in the notation of quadratic forms and write simply as Q the quadratic form associated with b_θ:

$$Q(\psi) = b_\theta(\psi, \psi) = \int_{\widetilde{\Omega}} \tan^2\theta\,|\partial_x\phi|^2 + |\partial_y\phi|^2\,\mathrm{d}x\,\mathrm{d}y.$$

We recall that the form domain V is the subspace of $\psi \in \mathsf{H}^1(\widetilde{\Omega})$ which satisfy the Dirichlet condition on $\partial_{\mathsf{Dir}}\widetilde{\Omega}$. We want to prove that

$$\mathsf{N}(Q, 1) \quad \text{is finite.}$$

We consider a partition of unity (χ_0, χ_1) such that

$$\chi_0(x)^2 + \chi_1(x)^2 = 1$$

with $\chi_0(x) = 1$ for $x < 1$ and $\chi_0(x) = 0$ for $x > 2$. For $R > 0$ and $\ell \in \{0, 1\}$, we introduce:

$$\chi_{\ell,R}(x) = \chi_\ell(R^{-1}x).$$

Thanks to the IMS formula, we can split the quadratic form as:

$$Q(\psi) = Q(\chi_{0,R}\psi) + Q(\chi_{1,R}\psi) - \|\chi'_{0,R}\psi\|^2_{\widetilde{\Omega}} - \|\chi'_{1,R}\psi\|^2_{\widetilde{\Omega}}\,. \tag{6.2.3}$$

We can write

$$|\chi'_{0,R}(x)|^2 + |\chi'_{1,R}(x)|^2 = R^{-2}W_R(x) \quad \text{with} \quad W_R(x) = |\chi'_0(R^{-1}x)|^2 + |\chi'_1(R^{-1}x)|^2\,.$$

Then

$$\begin{aligned}\|\chi'_{0,R}\psi\|^2_{\widetilde{\Omega}} + \|\chi'_{1,R}\psi\|^2_{\widetilde{\Omega}} &= \int_{\widetilde{\Omega}} R^{-2}W_R(x)|\psi|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\widetilde{\Omega}} R^{-2}W_R(x)\big(|\chi_{0,R}\psi|^2 + |\chi_{1,R}\psi|^2\big)\,\mathrm{d}x\,\mathrm{d}y. \end{aligned} \tag{6.2.4}$$

Let us introduce the subsets of $\widetilde{\Omega}$:

$$\mathcal{O}_{0,R} = \{(x,y) \in \widetilde{\Omega} : x < 2R\} \quad \text{and} \quad \mathcal{O}_{1,R} = \{(x,y) \in \widetilde{\Omega} : x > R\}$$

and the associated form domains

$$V_0 = \Big\{\phi \in H^1(\mathcal{O}_{0,R}) : \quad \phi = 0 \ \text{ on } \ \partial_{\mathsf{Dir}}\widetilde{\Omega} \cap \partial\mathcal{O}_{0,R} \ \text{ and on } \ \{2R\}\times(0,\pi)\Big\}$$

$$V_1 = \mathsf{H}^1_0(\mathcal{O}_{1,R}).$$

We define the two quadratic forms $Q_{0,R}$ and $Q_{1,R}$ by

$$Q_{\ell,R}(\phi) = \int_{\mathcal{O}_{\ell,R}} \tan^2\theta|\partial_x\phi|^2 + |\partial_y\phi|^2 - R^{-2}W_R(x)|\phi|^2 \,\mathrm{d}x\,\mathrm{d}y \quad \text{for} \quad \psi \in V_\ell, \quad \ell = 0, 1. \tag{6.2.5}$$

As a consequence of (6.2.3) and (6.2.4) we find

$$Q(\psi) = Q_{0,R}(\chi_{0,R}\psi) + Q_{1,R}(\chi_{1,R}\psi) \quad \forall \psi \in V. \tag{6.2.6}$$

Let us prove

Lemma 6.13. *We have:*

$$\mathsf{N}(Q, 1) \le \mathsf{N}(Q_{0,R}, 1) + \mathsf{N}(Q_{1,R}, 1).$$

Proof. We recall the formula for the j-th Rayleigh quotient of Q:

$$\lambda_j = \inf_{\substack{E\subset V\\ \dim E = j}} \sup_{\psi\in E} \frac{Q(\psi)}{\|\psi\|^2_{\widetilde{\Omega}}}\,.$$

The idea is now to give a lower bound for λ_j. Let us introduce:

$$\mathcal{J} : \left\{ \begin{array}{ccc} V & \to & V_0 \times V_1 \\ \psi & \mapsto & (\chi_{0,R}\psi\ ,\ \chi_{1,R}\psi)\,. \end{array} \right.$$

As $(\chi_{0,R},\chi_{1,R})$ is a partition of the unity, $\mathcal{J}$ is injective. In particular, we notice that $\mathcal{J}: V \to \mathcal{J}(V)$ is bijective so that we have:

$$\begin{aligned}\lambda_j &= \inf_{\substack{F\subset \mathcal{J}(V)\\ \dim F=j}} \sup_{\psi\in\mathcal{J}^{-1}(F)} \frac{Q(\psi)}{\|\psi\|^2_{\tilde\Omega}}\\ &= \inf_{\substack{F\subset \mathcal{J}(V)\\ \dim F=j}} \sup_{\psi\in\mathcal{J}^{-1}(F)} \frac{Q_{0,R}(\chi_{0,R}\psi)+Q_{1,R}(\chi_{1,R}\psi)}{\|\chi_{0,R}\psi\|^2_{\tilde\Omega}+\|\chi_{1,R}\psi\|^2_{\tilde\Omega}}\\ &= \inf_{\substack{F\subset \mathcal{J}(V)\\ \dim F=j}} \sup_{(\psi_0,\psi_1)\in F} \frac{Q_{0,R}(\psi_0)+Q_{1,R}(\psi_1)}{\|\psi_0\|^2_{\mathcal{O}_{0,R}}+\|\psi_1\|^2_{\mathcal{O}_{1,R}}}.\end{aligned}$$

As $\mathcal{J}(V)\subset V_0\times V_1$, we deduce:

$$\lambda_j \geq \inf_{\substack{F\subset V_0\times V_1\\ \dim F=j}} \sup_{(\psi_0,\psi_1)\in F} \frac{Q_{0,R}(\psi_0)+Q_{1,R}(\psi_1)}{\|\psi_0\|^2_{\mathcal{O}_{0,R}}+\|\psi_1\|^2_{\mathcal{O}_{1,R}}} =: \nu_j, \tag{6.2.7}$$

Let $A_{\ell,R}$ be the self-adjoint operator with domain $\mathsf{Dom}\,(A_{\ell,R})$ associated with the coercive bilinear form corresponding to the quadratic form $Q_{\ell,R}$ on V_ℓ. We see that ν_j in (6.2.7) is the j-th Rayleigh quotient of the diagonal self-adjoint operator A_R

$$\begin{pmatrix} A_{0,R} & 0\\ 0 & A_{1,R}\end{pmatrix} \quad \text{with domain} \quad \mathsf{Dom}\,(A_{0,R})\times \mathsf{Dom}\,(A_{1,R})\,.$$

The Rayleigh quotients of $A_{\ell,R}$ are associated with the quadratic form $Q_{\ell,R}$ for $\ell=0,1$. Thus ν_j is the j-th element of the ordered set

$$\{\lambda_k(Q_{0,R}),\ k\geq 1\}\cup\{\lambda_k(Q_{1,R}),\ k\geq 1\}.$$

Lemma 6.13 follows. □

The operator $A_{0,R}$ is elliptic on a bounded open set, hence has a compact resolvent. Therefore we get:

Lemma 6.14. *For all $R>0$, $\mathsf{N}(Q_{0,R},1)$ is finite.*

To achieve the proof of Proposition 4.15, it remains to establish the following lemma:

Lemma 6.15. *There exists $R_0>0$ such that, for $R\geq R_0$, $\mathsf{N}(Q_{1,R},1)$ is finite.*

Proof. For all $\phi\in V_1$, we write:

$$\phi = \Pi_0\phi + \Pi_1\phi,$$

where

$$\Pi_0\phi(x,y) = \Phi(x)\sin y \quad \text{with} \quad \Phi(x) = \int_0^\pi \phi(x,y)\sin y\,\mathrm{d}y \tag{6.2.8}$$

is the projection on the first eigenvector of $-\partial_y^2$ on $\mathsf{H}_0^1(0,\pi)$, and $\Pi_1 = \mathsf{Id} - \Pi_0$. We have, for all $\varepsilon > 0$:

$$Q_{1,R}(\phi) = Q_{1,R}(\Pi_0\phi) + Q_{1,R}(\Pi_1\phi) - 2\int_{\mathcal{O}_{1,R}} R^{-2}W_R(x)\Pi_0\phi\,\Pi_1\phi\,\mathrm{d}x\,\mathrm{d}y$$

$$\geq Q_{1,R}(\Pi_0\phi) + Q_{1,R}(\Pi_1\phi) - \varepsilon^{-1}\int_{\mathcal{O}_{1,R}} R^{-2}W_R(x)|\Pi_0\phi|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$- \varepsilon\int_{\mathcal{O}_{1,R}} R^{-2}W_R(x)|\Pi_1\phi|^2\,\mathrm{d}x\,\mathrm{d}y. \tag{6.2.9}$$

Since the second eigenvalue of $-\partial_y^2$ on $\mathsf{H}_0^1(0,\pi)$ is 4, we have:

$$\int_{\mathcal{O}_{1,R}} |\partial_y\Pi_1\phi|^2\,\mathrm{d}x\,\mathrm{d}y \geq 4\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}}\,.$$

Denoting by M the maximum of W_R (which is independent of R), and using (6.2.5) we deduce

$$Q_{1,R}(\Pi_1\phi) \geq (4 - MR^{-2})\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}}\,.$$

Combining this with (6.2.9) where we take $\varepsilon = 1$, and with the definition (6.2.8) of Π_0, we find

$$Q_{1,R}(\phi) \geq q_R(\Phi) + (4 - 2MR^{-2})\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}},$$

where

$$q_R(\Phi) = \int_R^\infty \tan^2\theta|\partial_x\Phi|^2 + |\Phi|^2 - R^{-2}W_R(x)|\Phi|^2\,\mathrm{d}x$$

$$\geq \int_R^\infty \tan^2\theta|\partial_x\Phi|^2 + |\Phi|^2 - R^{-2}M\mathbb{1}_{[R,2R]}|\Phi|^2\,\mathrm{d}x.$$

We choose $R = \sqrt{M}$ so that $(4 - 2MR^{-2}) = 2$, and then

$$Q_{1,R}(\phi) \geq \tilde{q}_R(\Phi) + 2\|\Pi_1\phi\|^2_{\mathcal{O}_{1,R}}, \tag{6.2.10}$$

where now

$$\tilde{q}_R(\Phi) = \int_R^\infty \tan^2\theta|\partial_x\Phi|^2 + (1 - \mathbb{1}_{[R,2R]})|\Phi|^2\,\mathrm{d}x. \tag{6.2.11}$$

Let $\tilde{a}_R$ denote the 1D operator associated with the quadratic form $\tilde{q}_R$. From (6.2.10)-(6.2.11), we deduce that the j-th Rayleigh quotient of $A_{1,R}$ admits as lower bound the j-th Rayleigh quotient of the diagonal operator:

$$\begin{pmatrix} \tilde{a}_R & 0 \\ 0 & 2\,\mathsf{Id} \end{pmatrix}$$

so that we find:

$$\mathsf{N}(Q_{1,R}, 1) \leq \mathsf{N}(\tilde{q}_R, 1).$$

Finally, the eigenvalues < 1 of $\tilde{a}_R$ can be computed explicitly and this is an elementary exercise to deduce that $\mathsf{N}(\tilde{q}_R, 1)$ is finite. $\square$

This concludes the proof of Proposition 4.15.

3. Low lying spectrum, local models and estimates of Agmon

We explain in this section how we can perform a reduction of the magnetic Laplacian to local models.

3.1. Partition of unity and localization Formula.

The presentation is inspired by [**40**]. We introduce the following partition of unity:

$$\sum_j \chi_{j,R}^2 = 1,$$

where the $\chi_{j,R}$ is a smooth cutoff function supported in a ball of center x_j and radius $R > 0$. Moreover, we can find such a partition of unity so that:

$$\sum_j \|\nabla \chi_{j,R}\|^2 \leq CR^{-2}.$$

The following formula is usually called "IMS formula" and allows to localize the electro-magnetic Laplacian.

Proposition 6.16. *Let $\psi \in \mathsf{Dom}\,(\mathcal{Q}_{h,\boldsymbol{A},V})$. We have:*

$$\mathcal{Q}_{h,\boldsymbol{A},V}(\psi) = \sum_j \mathcal{Q}_{h,\boldsymbol{A},V}(\chi_{j,R}\psi) - h^2 \sum_j \|\nabla \chi_{j,R}\psi\|^2.$$

Proof. The proof is easy and instructive. By a density argument, it is enough to prove this for $\psi \in \mathsf{Dom}\,(\mathcal{L}_{h,\mathbf{A},V})$. We can write:

$$\mathcal{Q}_{h,\mathbf{A},V}(\chi_{j,R}\psi) = \langle \mathcal{L}_{h,\mathbf{A},V}\chi_{j,R}\psi, \chi_{j,R}\psi\rangle_{\mathsf{L}^2}.$$

We let $P = hD_k + A_k$ and $\chi = \chi_{j,R}$. It is enough to estimate:

$$\begin{aligned}
\langle P\psi, P\chi^2\psi\rangle_{\mathsf{L}^2} &= \langle \chi P\psi, [P,\chi]\psi\rangle_{\mathsf{L}^2} + \langle \chi P\psi, P\chi\psi\rangle_{\mathsf{L}^2}\\
&= \langle \chi P\psi, [P,\chi]\psi\rangle_{\mathsf{L}^2} + \langle P\chi\psi, P\chi\psi\rangle_{\mathsf{L}^2} + \langle [\chi, P]\psi, P\chi\psi\rangle_{\mathsf{L}^2}\\
&= \langle P\chi\psi, P\chi\psi\rangle_{\mathsf{L}^2} - \|[P,\chi]\psi\|^2 + \langle \chi P\psi, [P,\chi]\psi\rangle_{\mathsf{L}^2} - \langle [P,\chi]\psi, \chi P\psi\rangle_{\mathsf{L}^2}.
\end{aligned}$$

Taking the real part, we find:

$$\langle P\psi, P\chi^2\psi\rangle_{\mathsf{L}^2} = \|P\chi\psi\|^2 - \|[P,\chi]\psi\|^2.$$

We have: $[P,\chi] = -ih\partial_k\chi$. It remains to take the sum and the conclusion follows. $\square$

3.2. A preliminary electric example.

Let us consider the operator

$$\hat{\mathcal{N}}_{\hat{\alpha},h}^{\mathsf{Neu}} = h^2 D_\tau^2 + \left(\frac{\tau^2}{2} - 1\right)^2,$$

on $\mathsf{L}^2((-\hat{\alpha},+\infty))$. We denote by $\hat{\nu}_1^{\mathsf{Neu}}(\hat{\alpha},h)$ the lowest eigenvalue of $\hat{\mathcal{N}}_{\hat{\alpha},h}^{\mathsf{Neu}}$. We aim at establishing a uniform lower bound with respect to $\hat{\alpha}$ of $\hat{\nu}_1^{\mathsf{Neu}}(\hat{\alpha},h)$ when $h\to 0$. Let us prove the following lemma which we will need in the next chapter.

Lemma 6.17. *There exist $C,h_0>0$ such that for all $h\in(0,h_0)$ and $\hat{\alpha}\geq -1$:*

$$\hat{\nu}_1^{\mathsf{Neu}}(\hat{\alpha},h)\geq Ch.$$

PROOF. We have to be careful with the dependence on $\hat{\alpha}$. We introduce a partition of unity on $\mathbb{R}$ with balls of size $r>0$ and centers τ_j and such that:

$$\sum_j \chi_{j,r}^2=1,\quad \sum_j \chi_{j,r}'^2\leq Cr^{-2}.$$

We can assume that there exist j_- and j_+ such that $\tau_{j_-}=-\sqrt{2}$ and $\tau_{j_+}=\sqrt{2}$. The "IMS" formula provides:

$$\hat{\mathcal{Q}}_{\hat{\alpha},h}^{\mathsf{Neu}}(\psi)\geq \sum_j \hat{\mathcal{Q}}_{\hat{\alpha},h}^{\mathsf{Neu}}(\chi_{j,r}\psi)-Ch^2r^{-2}\|\psi\|^2.$$

We let $V(\tau)=\left(\frac{\tau^2}{2}-1\right)^2$. Let us fix ε_0 such that

$$V(\tau)\geq \frac{V''(\tau_{j_\pm})}{4}(\tau-\tau_{j_\pm})^2\quad \text{if}\quad |\tau-\tau_{j_\pm}|\leq\varepsilon_0. \tag{6.3.1}$$

There exists $\delta_0>0$ such that

$$V(\tau)\geq\delta_0\quad\text{if}\quad |\tau-\tau_{j_\pm}|>\frac{\varepsilon_0}{4}. \tag{6.3.2}$$

Let us consider j such that $j=j_-$ or $j=j_+$. We can write the Taylor expansion:

$$V(\tau)=\frac{V''(\tau_{j_\pm})}{2}(\tau-\tau_{j_\pm})^2+\mathcal{O}(|\tau-\tau_{j_\pm}|^3)=2(\tau-\tau_{j_\pm})^2+\mathcal{O}(|\tau-\tau_{j_\pm}|^3). \tag{6.3.3}$$

We have:

$$\hat{\mathcal{Q}}_{\hat{\alpha},h}^{\mathsf{Neu}}(\chi_{j,r}\psi)\geq\sqrt{2}\Theta_0 h\|\chi_{j,r}\psi\|^2-Cr^3\|\chi_{j,r}\psi\|^2, \tag{6.3.4}$$

where $\Theta_0>0$ is the infimum of the bottom of the spectrum for the ζ-dependent family of de Gennes operators $D_\tau^2+(\tau-\zeta)^2$ on $\mathbb{R}_+$ with Neumann boundary condition. We are led to choose $r=h^{2/5}$. We consider now the other balls: $j\neq j_-$ and $j\neq j_+$. If the center τ_j satisfies $|\tau_j-\tau_{j_\pm}|\leq\varepsilon_0/2$, then, for all $\tau\in B(\tau_j,h^{2/5})$, we have for h small enough:

$$|\tau-\tau_{j_\pm}|\leq h^{2/5}+\frac{\varepsilon_0}{2}\leq\varepsilon_0.$$

If $|\tau_j-\tau_{j_\pm}|\leq 2h^{2/5}$, then for $\tau\in B(\tau_j,h^{2/5})$, we have $|\tau-\tau_{j_\pm}|\leq 3h^{2/5}$ and we can use the Taylor expansion (6.3.3). Thus (6.3.4) is still available.
We now assume that $|\tau_j-\tau_{j_\pm}|\geq 2h^{2/5}$ so that, on $B(\tau_j,h^{2/5})$, we have:

$$V(\tau)\geq\frac{V''(\tau_{j_\pm})}{4}h^{4/5}.$$

If the center τ_j satisfies $|\tau_j - \tau_{j\pm}| > \varepsilon_0/2$, then, for all $\tau \in B(\tau_j, h^{2/5})$, we have $|\tau - \tau_{j\pm}| \geq \varepsilon_0/4$ and thus:

$$V(\tau) \geq \delta_0.$$

Gathering all the contributions, we find:

$$\hat{\mathcal{Q}}^{\mathsf{Neu}}_{\hat{\alpha},h}(\psi) \geq (\sqrt{2}\Theta_0 h - Ch^{6/5})\|\psi\|^2.$$

The conclusion follows from the min-max principle. $\square$

3.3. Magnetic example. As we are going to see, this localization formula is very convenient to prove lower bounds for the spectrum. We consider an open bounded set $\Omega \subset \mathbb{R}^3$ and the Dirichlet realization of the magnetic Laplacian $\mathcal{L}^{\mathsf{Dir}}_{h,\mathbf{A}}$. Then we have the lower bound for the lowest eigenvalues.

Proposition 6.18. *For all $n \in \mathbb{N}^*$, there exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq \min_{\Omega} \|\boldsymbol{B}\|\, h - Ch^{5/4}.$$

PROOF. We introduce a partition of unity with radius $R > 0$ denoted by $(\chi_{j,R})_j$. Let us consider an eigenpair (λ, ψ). We have:

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) = \sum_j \mathcal{Q}_{h,\mathbf{A}}(\chi_{j,R}\psi) - h^2 \sum_j \|\nabla\chi_{j,R}\psi\|^2$$

so that:

$$\mathcal{Q}_{h,\mathbf{A}}(\psi) \geq \sum_j \mathcal{Q}_{h,\mathbf{A}}(\chi_{j,R}\psi) - CR^{-2}h^2\|\psi\|^2$$

and:

$$\lambda\|\psi\|^2 \geq \sum_j \mathcal{Q}_{h,\mathbf{A}}(\chi_{j,R}\psi) - CR^{-2}h^2\|\psi\|^2.$$

It remains to provide a lower bound for $\mathcal{Q}_{h,\mathbf{A}}(\chi_{j,R}\psi)$. We choose $R = h^{\rho}$ with $\rho > 0$, to be chosen. We approximate the magnetic field in each ball by the constant magnetic field $\mathbf{B}_j$:

$$|\mathbf{B} - \mathbf{B}_j| \leq C\|x - x_j\|.$$

In a suitable gauge, we have:

$$\|\mathbf{A} - \mathbf{A}^{\mathsf{lin}}_j\| \leq C\|x - x_j\|^2,$$

where $C > 0$ does not depend on j but only on the magnetic field. Then, we have, for all $\varepsilon \in (0, 1)$:

$$\mathcal{Q}_{h,\mathbf{A}}(\chi_{j,R}\psi) \geq (1-\varepsilon)\mathcal{Q}_{h,\mathbf{A}^{\mathsf{lin}}_j}(\chi_{j,R}\psi) - C^2\varepsilon^{-1}R^4\|\chi_{j,R}\psi\|^2.$$

From the min-max principle, we deduce:

$$\mathcal{Q}_{h,\mathbf{A}}(\chi_{j,R}\psi) \geq \left((1-\varepsilon)\|\mathbf{B}_j\|h - C^2\varepsilon^{-1}h^{4\rho}\right)\|\chi_{j,R}\psi\|^2.$$

Optimizing ε, we take: $\varepsilon = h^{2\rho-1/2}$ and it follows:

$$\mathcal{Q}_{h,\mathbf{A}}(\chi_{j,R}\psi) \geq \left(\|\mathbf{B}_j\|h - Ch^{2\rho+1/2}\right)\|\chi_{j,R}\psi\|^2.$$

We now choose ρ such that $2\rho + 1/2 = 2 - 2\rho$. We are led to take: $\rho = \frac{3}{8}$ and the conclusion follows. □

3.4. Agmon estimates. This section is devoted to the Agmon estimates in the semiclassical framework. We refer to the classical references [**1, 2, 82, 98, 99**].

Proposition 6.19. *Let Ω be a bounded open domain in $\mathbb{R}^m$ with Lipschitzian boundary. Let $V \in \mathcal{C}^0(\overline{\Omega}, \mathbb{R})$, $\boldsymbol{A} \in \mathcal{C}^0(\overline{\Omega}, \mathbb{R}^m)$ and Φ a real valued Lipschitzian function on $\overline{\Omega}$. Then, for $u \in \mathsf{Dom}\,(\mathcal{L}_{h,\boldsymbol{A},V})$ (with Dirichlet or magnetic Neumann condition), we have:*

$$\int_\Omega \|(-ih\nabla + \boldsymbol{A})e^\Phi u\|^2 \,\mathrm{d}x + \int_\Omega \left(V - h^2\|\nabla\Phi\|^2 e^{2\Phi}\right)|u|^2 \,\mathrm{d}x = \mathsf{Re}\,\langle \mathcal{L}_{h,\boldsymbol{A},V}u, e^{2\Phi}u\rangle_{\mathsf{L}^2(\Omega)}.$$

Proof. We give the proof when Φ is smooth. Let us use the Green-Riemann formula:

$$\sum_{k=1}^m \langle(-ih\partial_k + A_k)^2 u, e^{2\Phi}u\rangle_{\mathsf{L}^2} = \sum_{k=1}^m \langle(-ih\partial_k + A_k)u, (-ih\partial_k + A_k)e^{2\Phi}u\rangle_{\mathsf{L}^2},$$

where the boundary term has disappeared thanks to the boundary condition. In order to lighten the notation, we let $P = -ih\partial_k + A_k$.

$$\begin{aligned}\langle Pu, Pe^{2\Phi}u\rangle_{\mathsf{L}^2} &= \langle e^\Phi Pu, [P, e^\Phi]u\rangle_{\mathsf{L}^2} + \langle e^\Phi Pu, Pe^\Phi u\rangle_{\mathsf{L}^2}\\ &= \langle e^\Phi Pu, [P, e^\Phi]u\rangle_{\mathsf{L}^2} + \langle Pe^\Phi u, Pe^\Phi u\rangle_{\mathsf{L}^2} + \langle [e^\Phi, P]u, Pe^\Phi u\rangle_{\mathsf{L}^2}\\ &= \langle Pe^\Phi u, Pe^\Phi u\rangle_{\mathsf{L}^2} - \|[P, e^\Phi]u\|^2 + \langle e^\Phi Pu, [P, e^\Phi]u\rangle_{\mathsf{L}^2} - \langle [P, e^\Phi]u, e^\Phi Pu\rangle_{\mathsf{L}^2}.\end{aligned}$$

We deduce:

$$\mathsf{Re}\,\left(\langle Pu, Pe^{2\Phi}u\rangle_{\mathsf{L}^2}\right) = \langle Pe^\Phi u, Pe^\Phi u\rangle_{\mathsf{L}^2} - \|[P, e^\Phi u]\|^2.$$

This is then enough to conclude.

□

In fact we can prove a more general "IMS" formula (which generalizes Propositions 6.18 and 6.19).

Proposition 6.20 ("Localization" of P^2 with respect to A)**.** *Let $(\mathsf{H}, \langle\cdot,\cdot\rangle)$ be a Hilbert space and two unbounded operators P and $\mathfrak{A}$ defined on a domain $\mathsf{D} \subset \mathsf{H}$. We assume that P is symmetric and that $P(\mathsf{D}) \subset \mathsf{D}$, $\mathfrak{A}(\mathsf{D}) \subset \mathsf{D}$ and $\mathfrak{A}^*(\mathsf{D}) \subset \mathsf{D}$. Then, for $\psi \in \mathsf{D}$, we have:*

$$\begin{aligned}\mathsf{Re}\,\langle P^2\psi, \mathfrak{A}\mathfrak{A}^*\psi\rangle = \|P(\mathfrak{A}^*\psi)\|^2 - \|[\mathfrak{A}^*, P]\psi\|^2 + \mathsf{Re}\,\langle P\psi, [[P,\mathfrak{A}],\mathfrak{A}^*]\psi\rangle\\ + \mathsf{Re}\,\left(\langle P\psi, \mathfrak{A}^*[P,\mathfrak{A}]\psi\rangle - \overline{\langle P\psi, \mathfrak{A}[P,\mathfrak{A}^*]\psi\rangle}\right).\end{aligned} \tag{6.3.5}$$

Let us continue to study of the Helffer-Kordyukov operator.

Proposition 6.21. *There exist $\tilde{C} > 0, h_0 > 0$, $\varepsilon > 0$ such that, for $h \in (0, h_0)$ and (λ, ψ) an eigenpair of $\mathcal{L}^{\mathsf{ex}}_{h,\boldsymbol{A}}$ satisfying $\lambda \leq h + Ch^2$, we have:*

$$\int_{\mathbb{R}^2} e^{\varepsilon h^{-1/4}|x|}|\psi|^2 \, \mathrm{d}x \leq \tilde{C}\|\psi\|^2.$$

Proof. We consider an eigenpair (λ, ψ) as in the proposition and we use the Agmon identity, jointly with the "IMS" formula (with balls of size $h^{3/8}$):

$$\mathcal{Q}^{\mathsf{ex}}_{h,\mathbf{A}}(e^{\Phi/h^{\delta}}\psi) - h^{2-2\delta}\|\nabla\Phi e^{\Phi/h^{\delta}}\psi\|^2 = \lambda\|e^{\Phi/h^{\delta}}\psi\|,$$

where $\delta > 0$ and Φ are to be determined. For simplicity we choose $\Phi(\mathbf{x}) = \varepsilon\|\mathbf{x}\|$. We infer that:

$$\int_{\mathbb{R}^2} (h\mathbf{B}(x,y) - h - Ch^2 - 2\varepsilon^2 h^{2-2\delta})|e^{\Phi/h^{\delta}}\psi|^2 \, \mathrm{d}x \, \mathrm{d}y \leq 0.$$

We recall that $\mathbf{B}(x,y) = 1 + x^2 + y^2$. We choose δ so that $hh^{2\delta} = h^{2-2\delta}$ and we get $\delta = \frac{1}{4}$. We now split the integral into two parts: $\|\mathbf{x}\| \geq C_0 h^{1/4}$ and $\|\mathbf{x}\| \leq C_0 h^{1/4}$. If ε is small enough, we infer that:

$$\|e^{\Phi/h^{1/4}}\psi\| \leq \tilde{C}\|\psi\|.$$

$\square$

CHAPTER 7

Models for vanishing magnetic fields

> Sie begriffen, dass die Vernunft nur das einsieht, was sie selbst nach ihrem Entwurfe hervorbringt, dass sie mit Prinzipien ihrer Urteile nach beständigen Gesetzen vorangehen und die Natur nötigen müsse auf ihre Fragen zu antworten, nicht aber sich von ihr allein gleichsam am Leitbande gängeln lassen müsse; denn sonst hängen zufällige, nach keinem vorher entworfenen Plane gemachte Beobachtungen gar nicht in einem notwendigen Gesetze zusammen, welches doch die Vernunft sucht und bedarf.
>
> *Kritik der reinen Vernunft*, Vorrede, Kant

We will see that the properties of $\mathcal{M}_{x,\xi}^{\mathsf{Neu}}$ be can used to investigate those of $\mathcal{M}_{x,\xi}$. Therefore we begin by analyzing the family of operators $\mathcal{M}_{x,\xi}^{\mathsf{Neu}}$ and we prove Theorem 2.21 and apply it to prove Theorem 2.22.

1. Analysis of $\mathcal{M}_{x,\xi}^{\mathsf{Neu}}$

1.1. Changing the parameters. To analyze the family of operators $\mathcal{M}_{x,\xi}^{\mathsf{Neu}}$ depending on parameters (x,ξ), we introduce the new parameters (x,η) using a change of variables. Let us introduce the following change of parameters:

$$\mathcal{P}(x,\xi) = (x,\eta) = \left(x, \xi + \frac{x^2}{2}\right).$$

A straight forward computation provides that $\mathcal{P} : \mathbb{R}^2 \to \mathbb{R}^2$ is a $\mathcal{C}^\infty$-diffeomorphism such that:

$$|x| + |\xi| \to +\infty \Leftrightarrow |\mathcal{P}(x,\xi)| \to +\infty.$$

We have $\mathcal{M}_{x,\xi}^{\mathsf{Neu}} = \mathcal{N}_{x,\eta}^{\mathsf{Neu}}$, where:

$$\mathcal{N}_{x,\eta}^{\mathsf{Neu}} = D_t^2 + \left(\frac{(t-x)^2}{2} - \eta\right)^2,$$

with Neumann condition on $t = 0$. Let us denote by $\nu_1^{\mathsf{Neu}}(x,\eta)$ the lowest eigenvalue of $\mathcal{N}_{x,\eta}^{\mathsf{Neu}}$, so that:

$$\mu_1^{\mathsf{Neu}}(x,\xi) = \nu_1^{\mathsf{Neu}}(x,\eta) = \nu_1^{\mathsf{Neu}}\left(\mathcal{P}(x,\xi)\right).$$

We denote by $\mathsf{Dom}\,(\mathcal{Q}^{\mathsf{Neu}}_{x,\eta})$ the form domain of the operator and by $\mathcal{Q}^{\mathsf{Neu}}_{x,\eta}$ the associated quadratic form.

1.2. Existence of a minimum for $\mu^{\mathsf{Neu}}_1(x,\xi)$. To prove Theorem 2.21, we establish the following result:

Theorem 7.1. *The function $\mathbb{R}\times\mathbb{R}\ni(x,\eta)\mapsto\nu^{\mathsf{Neu}}_1(x,\eta)$ admits a minimum. Moreover we have:*

$$\liminf_{|x|+|\eta|\to+\infty}\nu^{\mathsf{Neu}}_1(x,\eta)\geq\nu_{\mathsf{Mo}}>\min_{(x,\eta)\in\mathbb{R}^2}\nu^{\mathsf{Neu}}_1(x,\eta).$$

To prove this result, we decompose the plane in subdomains and analyze in each part.

Lemma 7.2. *For all $(x,\eta)\in\mathbb{R}^2$ such that $\eta\geq\frac{x^2}{2}$, we have:*

$$-\partial_x\nu^{\mathsf{Neu}}_1(x,\eta)+\sqrt{2\eta}\,\partial_\eta\nu^{\mathsf{Neu}}_1(x,\eta)>0.$$

Thus there is no critical point in the area $\{\eta\geq\frac{x^2}{2}\}$.

Proof. The Feynman-Hellmann formulas provide:

$$\partial_x\nu^{\mathsf{Neu}}_1(x,\eta)=-2\int_0^{+\infty}\left(\frac{(t-x)^2}{2}-\eta\right)(t-x)u^2_{x,\eta}(t)\,\mathrm{d}t,$$
$$\partial_\eta\nu^{\mathsf{Neu}}_1(x,\eta)=-2\int_0^{+\infty}\left(\frac{(t-x)^2}{2}-\eta\right)u^2_{x,\eta}(t)\,\mathrm{d}t.$$

We infer:

$$-\partial_x\nu^{\mathsf{Neu}}_1(x,\eta)+\sqrt{2\eta}\,\partial_\eta\nu^{\mathsf{Neu}}_1(x,\eta)=\int_0^{+\infty}(t-x-\sqrt{2\eta})(t-x+\sqrt{2\eta})(t-x-\sqrt{2\eta})u^2_{x,\eta}(t)\,\mathrm{d}t.$$

We have:

$$\int_0^{+\infty}(t-x-\sqrt{2\eta})^2(t-x+\sqrt{2\eta})u^2_{x,\eta}(t)\,\mathrm{d}t>0.$$

$\square$

Lemma 7.3. *We have:*

$$\inf_{(x,\eta)\in\mathbb{R}^2}\nu^{\mathsf{Neu}}_1(x,\eta)<\nu_{\mathsf{Mo}}.$$

Proof. We apply Lemma 7.2 at $x=0$ and $\eta=\eta_{\mathsf{Mo}}$ to deduce that:

$$\partial_x\nu^{\mathsf{Neu}}_1(0,\eta_{\mathsf{Mo}})<0.$$

$\square$

The following lemma is obvious:

Lemma 7.4. *For all $\eta\leq0$, we have:*

$$\nu^{\mathsf{Neu}}_1(x,\eta)\geq\eta^2.$$

In particular, we have

$$\nu^{\mathsf{Neu}}_1(x,\eta)>\nu_{\mathsf{Mo}},\qquad\forall\eta<-\sqrt{\nu_{\mathsf{Mo}}}.$$

Lemma 7.5. *For $x \leq 0$ and $\eta \leq \frac{x^2}{2}$, we have:*

$$\nu_1^{\mathsf{Neu}}(x,\eta) \geq \nu_1^{[1]}(0) > \nu_{\mathsf{Mo}}.$$

Proof. We have, for all $\psi \in \mathsf{Dom}\,(\mathcal{Q}_{x,\eta}^{\mathsf{Neu}})$:

$$\mathcal{Q}_{x,\eta}^{\mathsf{Neu}}(\psi) = \int_{\mathbb{R}_+} |D_t\psi|^2 + \left(\frac{(t-x)^2}{2} - \eta\right)^2 |\psi|^2 \,\mathrm{d}t$$

and

$$\left(\frac{(t-x)^2}{2} - \eta\right)^2 = \left(\frac{t^2}{2} - xt + \frac{x^2}{2} - \eta\right)^2 \geq \frac{t^4}{4}.$$

The min-max principle provides:

$$\nu_1^{\mathsf{Neu}}(x,\eta) \geq \nu_1^{[1]}(0).$$

Moreover, thanks to the Feynman-Hellmann theorem, we get:

$$\left(\partial_\eta \nu_1^{[1]}(\eta)\right)_{\eta=0} = -\int_{\mathbb{R}_+} t^2 u_0(t)^2 \,\mathrm{d}t < 0.$$

$\square$

Lemma 7.6. *There exist $C, D > 0$ such that for all $x \in \mathbb{R}$ and $\eta \geq D$ such that $\frac{x}{\sqrt{\eta}} \geq -1$:*

$$\nu_1^{\mathsf{Neu}}(x,\eta) \geq C\eta^{1/2}.$$

Proof. For $x \in \mathbb{R}$ and $\eta > 0$, we can perform the change of variable:

$$\tau = \frac{t-x}{\sqrt{\eta}}.$$

The operator $\eta^{-2}\mathcal{N}_{x,\eta}^{\mathsf{Neu}}$ is unitarily equivalent to:

$$\hat{\mathcal{N}}_{\hat{x},h}^{\mathsf{Neu}} = h^2 D_\tau^2 + \left(\frac{\tau^2}{2} - 1\right)^2,$$

on $\mathsf{L}^2\left((-\hat{x}, +\infty)\right)$, with $\hat{x} = \frac{x}{\sqrt{\eta}}$ and $h = \eta^{-3/2}$. With Lemma 6.17, we infer, using the min-max principle:

$$\nu_1^{\mathsf{Neu}}(x,\eta) \geq c\eta^{-3/2},$$

for η large enough. $\square$

Lemma 7.7. *Let u_η be an eigenfunction associated with the first eigenvalue of $\mathfrak{L}_\eta^{\mathsf{Mo},+}$. Let $D > 0$. There exist $\varepsilon_0, C > 0$ such that, for all η such that $|\eta| \leq D$, we have:*

$$\int_0^{+\infty} e^{2\varepsilon_0 t^3} |u_\eta|^2 \,\mathrm{d}t \leq C\|u_\eta\|^2.$$

Proof. We let $\Phi_m = \varepsilon\chi_m(t)t^3$. The Agmon estimate provides:

$$\int_0^\infty \left(\frac{t^2}{2} - \eta\right)^2 |e^{\Phi_m} u_\eta|^2 \,\mathrm{d}t \leq \nu_1^{[1]}(\eta)\|e^{\Phi_m}u_\eta\|^2 + \|\nabla\Phi_m e^{\Phi_m} u_\eta\|^2.$$

It follows that:

$$\int_0^\infty \frac{t^4}{8}|e^{\Phi_m}u_\eta|^2\,\mathrm{d}t \le (\nu_1^{[1]}(\eta)+2\eta^2)\|e^{\Phi_m}u_\eta\|^2 + \|\nabla\Phi_m e^{\Phi_m}u_\eta\|^2.$$

We infer that:

$$\int_0^\infty t^4|e^{\Phi_m}u_\eta|^2\,\mathrm{d}t \le M(D)\|e^{\Phi_m}u_\eta\|^2 + 8\|\nabla\Phi_m e^{\Phi_m}u_\eta\|^2.$$

With our choice of Φ_m, we have

$$|\nabla\Phi_m|^2 \le 18\varepsilon^2\chi_m^2(t)t^4 + 2\varepsilon^2\chi_m'(t)^2t^6 \le 18\varepsilon^2t^4 + 2\varepsilon^2\chi_m'(t)^2t^6 \le C\varepsilon^2t^4,$$

since $\chi_m'(t)^2t^2$ is bounded. For ε fixed small enough, we deduce

$$\int_0^\infty t^4|e^{\Phi_m}u_\eta|^2\,\mathrm{d}t \le \frac{M(D)}{1-8C\varepsilon^2}\|e^{\Phi_m}u_\eta\|^2 \le \tilde{M}(D)\|e^{\Phi_m}u_\eta\|^2.$$

Let us choose $R>0$ such that: $R^4 - M(D) > 0$. We have:

$$(R^4-\tilde{M}(D))\int_R^{+\infty} e^{2\Phi_m}|u_\eta|^2\,\mathrm{d}t \le \tilde{M}(D)\int_0^R e^{2\Phi_m}|u_\eta|^2\,dy \le \tilde{M}(D)C(R)\|u_\eta\|^2,$$

and:

$$\int_R^{+\infty} e^{2\Phi_m}|u_\eta|^2\,\mathrm{d}t \le C(R,D)\|u_\eta\|^2.$$

We infer:

$$\int_0^{+\infty} e^{2\Phi_m}|u_\eta|^2\,\mathrm{d}t \le \tilde{C}(R,D)\|u_\eta\|^2.$$

It remains to take the limit $m\to+\infty$. $\square$

Lemma 7.8. *For all $D>0$, there exist $A>0$ and $C>0$ such that for all $|\eta|\le D$ and $x\ge A$, we have:*

$$\left|\nu_1(x,\eta)-\nu_1^{[1]}(\eta)\right| \le Cx^{-2}.$$

Proof. We perform the translation $\tau = t-x$, so that $\mathcal{N}_{x,\eta}^{\mathsf{Neu}}$ is unitarily equivalent to:

$$\tilde{\mathcal{N}}_{x,\eta}^{\mathsf{Neu}} = D_\tau^2 + \left(\frac{\tau^2}{2}-\eta\right)^2,$$

on $\mathsf{L}^2(-x,+\infty)$. The corresponding quadratic form writes:

$$\tilde{\mathcal{Q}}_{x,\eta}^{\mathsf{Neu}}(\psi) = \int_{-x}^{+\infty} |D_\tau\psi|^2 + \left(\frac{\tau^2}{2}-\eta\right)^2|\psi|^2\,\mathrm{d}\tau.$$

Let us first prove the upper bound. We take $\psi(\tau)=\chi_0(x^{-1}\tau)u_\eta(\tau)$. The "IMS" formula provides:

$$\tilde{\mathcal{Q}}_{x,\eta}^{\mathsf{Neu}}(\chi_0(x^{-1}\tau)u_\eta(\tau)) = \nu_1^{[1]}(\eta)\|\chi_0(x^{-1}\tau)u_\eta(\tau)\|^2 + \|(\chi_0(x^{-1}\tau))'u_\eta(\tau)\|^2.$$

Jointly min-max principle with Lemma 7.7, we infer that:

$$\begin{aligned}\nu_1(x,\eta) &\leq \nu_1^{[1]}(\eta)+\frac{\|(\chi_0(x^{-1}\tau))' u_\eta(\tau)\|^2}{\|\chi_0(x^{-1}\tau)u_\eta(\tau)\|^2}\\ &\leq \nu_1^{[1]}(\eta)+\frac{Cx^{-2}}{e^{2c\varepsilon_0 x^3}}.\end{aligned}$$

Let us now prove the lower bound. Let us now prove the converse inequality. We denote by $\tilde{u}_{x,\eta}$ the positive and L^2-normalized groundstate of $\tilde{\mathcal{N}}_{x,\eta}^{\mathsf{Neu}}$. On the one hand, with the "IMS" formula, we have:

$$\tilde{\mathcal{Q}}_{x,\eta}^{\mathsf{Neu}}(\chi_0(x^{-1}\tau)\tilde{u}_{x,\eta}) \leq \nu_1(x,\eta)\|\chi_0(x^{-1}\tau)\tilde{u}_{x,\eta}\|^2+Cx^{-2}.$$

On the other hand, we notice that:

$$\int_{-x}^{+\infty} t^4|\tilde{u}_{x,\eta}|^2 \, \mathrm{d}\tau \leq C, \qquad \int_{-x}^{-\frac{x}{2}} t^4|\tilde{u}_{x,\eta}|^2 \, \mathrm{d}\tau \leq C,$$

and thus:

$$\int_{-x}^{-\frac{x}{2}} |\tilde{u}_{x,\eta}|^2 \, \mathrm{d}\tau \leq \tilde{C}x^{-4}.$$

We infer that:

$$\tilde{\mathcal{Q}}_{x,\eta}^{\mathsf{Neu}}(\chi_0(x^{-1}\tau)\tilde{u}_{x,\eta}) \leq (\nu_1(x,\eta)+Cx^{-2})\|\chi_0(x^{-1}\tau)\tilde{u}_{x,\eta}\|^2.$$

We deduce that:

$$\nu_1^{[1]}(\eta) \leq \nu_1(x,\eta)+Cx^{-2}.$$

□

We have proved in Lemmas 7.4-7.6 and 7.8 that the limit inferior of $\nu_1(x,\eta)$ in these areas are not less than ν_{Mo}. Then, we deduce the existence of a minimum with Lemma 7.3.

2. Analysis of $\mathcal{M}_{x,\xi}$

Theorem 2.22 is a consequence of the following two lemmas.

Lemma 7.9. *We have:*

$$\underline{\mu}_1 < \nu_{\mathsf{Mo}}.$$

PROOF. We have

$$\underline{\mu}_1 = \inf_{(x,\xi)\in\mathbb{R}^2} \mu_1(x,\xi) \leq \inf_{x\in\mathbb{R}} \mu_1(x,0).$$

We use a finite element method, with the Finite Element Library MÉLINA (see [**125**]), on $[-10,10]$ with Dirichlet condition on the artificial boundary, with 1000 elements $\mathbb{P}_2$. The discretization space for the finite element method is included in the form domain of the operator and thus the computed eigenvalue provides a rigorous upper-bound (see [**14**, Section 2] and [**16**, Section 5.1]). For any x, these computations give a upper-bound of $\mu_1(x,0)$. Numerical computations and Proposition 2.6 give

$$\inf_{x\in\mathbb{R}} \mu_1(x,0) \leq 0.33227 < 0.5 < \nu_{\mathsf{Mo}},$$

In fact, numerical simulations suggest that $\inf_{x\in\mathbb{R}} \mu_1(x,0) \simeq 0.33227$ which is an approximation of the first eigenvalue for $x = 0.827$. $\square$

Lemma 7.10. *For all $(x,\xi) \in \mathbb{R}^2$, we have:*

$$\mu_1(x,\xi) \geq \min(\mu_1^{\mathsf{Neu}}(x,\xi), \mu_1^{\mathsf{Neu}}(x,-\xi)).$$

PROOF. Let u be a normalized eigenfunction associated with $\mu_1(x,\xi)$. We can split:

$$\begin{aligned}
\mu_1(x,\xi) &= \int_{-\infty}^{0} |D_t u|^2 + \left(\frac{t^2}{2} - xt - \xi\right)^2 |u|^2 \,\mathrm{d}t + \int_{0}^{+\infty} |D_t u|^2 + \left(\frac{t^2}{2} - xt - \xi\right)^2 |u|^2 \,\mathrm{d}t \\
&\geq \mu_1^{\mathsf{Neu}}(x,-\xi) \int_{-\infty}^{0} |u|^2 \,\mathrm{d}t + \mu_1^{\mathsf{Neu}}(x,\xi) \int_{0}^{\infty} |u|^2 \,\mathrm{d}t \\
&\geq \min(\mu_1^{\mathsf{Neu}}(x,-\xi), \mu_1^{\mathsf{Neu}}(x,\xi)).
\end{aligned}$$

$\square$

CHAPTER 8

Models for magnetic cones

Ignarus enim praeterquam quod a causis externis multis modis agitatur nec unquam vera animi acquiescentia potitur, vivit paeterea sui et Dei et rerum quasi inscius et simulac pati desinit, simul etiam esse desinit.

Ethica, Pars V, Spinoza

This chapter deals with the proof of Theorem 2.2.

1. Agmon estimates for $\beta \in \left[0, \frac{\pi}{2}\right]$

We start by proving the following fine estimate when $\beta \in \left[0, \frac{\pi}{2}\right)$.

Proposition 8.1. *Let $C_0 > 0$ and $\eta \in \left(0, \frac{1}{2}\right)$. For all $\beta \in \left[0, \frac{\pi}{2}\right)$, there exist $\alpha_0 > 0$, ε_0 and $C > 0$ such that for any $\alpha \in (0, \alpha_0)$ and for all eigenpair (λ, ψ) of $\mathcal{L}_{\alpha,\beta}$ satisfying $\lambda \leq C_0\alpha$:*

$$\int_{\mathcal{C}_\alpha} e^{2\varepsilon_0\alpha^{1/2}|z|}|\psi|^2 \,\mathrm{d}\mathbf{x} \leq C\|\psi\|^2. \tag{8.1.1}$$

Proof. Thanks to a change of gauge $\mathfrak{L}_{\mathbf{A}}$ is unitarily equivalent to the Neumann realization of:

$$\mathfrak{L}_{\hat{\mathbf{A}}} = D_z^2 + (D_x + z\sin\beta)^2 + (D_y + x\cos\beta)^2.$$

The associated quadratic form is:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(\psi) = \int |D_z\psi|^2 + |(D_x + z\sin\beta)\psi|^2 + |(D_y + x\cos\beta)\psi|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Let us introduce a smooth cut-off function χ such that $\chi = 1$ near 0 and let us also consider, for $R \geq 1$ and $\varepsilon_0 > 0$:

$$\Phi_R(z) = \varepsilon_0\alpha^{1/2}\chi\left(\frac{z}{R}\right)|z|.$$

The Agmon identity writes:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(e^{\Phi_R}\psi) = \lambda\|e^{\Phi_R}\psi\|^2 - \|\nabla\Phi_R e^{\Phi_R}\psi\|^2.$$

There exists $\alpha_0 > 0$ and $\tilde{C}_0$ such that for $\alpha \in (0, \alpha_0)$, $R \geq 1$ and $\varepsilon_0 \in (0, 1)$, we have:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(e^{\Phi_R}\psi) \leq \tilde{C}_0\alpha\|e^{\Phi_R}\psi\|^2.$$

We introduce a partition of unity with respect to z:

$$\chi_1^2(z) + \chi_2^2(z) = 1,$$

where $\chi_1(z) = 1$ for $0 \leq z \leq 1$ and $\chi_1(z) = 0$ for $z \geq 2$. For $j = 1, 2$ and $\gamma > 0$, we let:

$$\chi_{j,\gamma}(z) = \chi_j(\gamma^{-1} z),$$

so that:

$$\|\chi'_{j,\gamma}\| \leq C\gamma^{-1}.$$

The "IMS" formula provides:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(e^{\Phi_R}\chi_{1,\gamma}\psi) + \widehat{\mathfrak{Q}_{\mathbf{A}}}(e^{\Phi_R}\chi_{2,\gamma}\psi) - C^2\gamma^{-2}\|e^{\Phi_R}\psi\|^2 \leq \tilde{C}_0\alpha\|e^{\Phi_R}\psi\|^2. \tag{8.1.2}$$

We want to write a lower bound for $\widehat{\mathfrak{Q}_{\mathbf{A}}}(e^{\Phi_R}\chi_{2,\gamma}\psi)$. Integrating by slices we have for $\psi \in$:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(\psi) \geq \cos\beta \int \mu(\sqrt{\cos\beta}\, z \tan(\alpha/2))\|\psi\|^2 \,\mathrm{d}z$$

where $\mu(\rho)$ is the lowest eigenvalue of the magnetic Neumann Laplacian on the disk of center $(0,0)$ and radius ρ. There exists $c > 0$ such that for all $\rho \geq 0$:

$$\mu(\rho) \geq c\min(\rho^2, 1).$$

We infer:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(e^{\Phi_R}\chi_{2,\gamma}\psi) \geq \int c\cos\beta \min(z^2\alpha^2\cos\beta, 1)\|e^{\Phi_R}\chi_{2,\gamma}\psi\|^2 \,\mathrm{d}z.$$

We choose $\gamma = \varepsilon_0^{-1}\alpha^{-1/2}(\cos\beta)^{-1/2}$. On the support of $\chi_{2,\gamma}$ we have $z \geq \gamma$. It follows:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(e^{\Phi_R}\chi_{2,\gamma}\psi) \geq \int c\cos\beta \min(\varepsilon_0^{-2}\alpha, 1)\|e^{\Phi_R}\chi_{2,\gamma}\psi\|^2 \,\mathrm{d}z.$$

For α such that $\alpha \leq \varepsilon_0^2$, we have:

$$\mathfrak{Q}_{\hat{\mathbf{A}}}(e^{\Phi_R}\chi_{2,\gamma}\psi) \geq \int c\alpha\varepsilon_0^{-2}\cos\beta\|e^{\Phi_R}\chi_{2,\gamma}\psi\|^2 \,\mathrm{d}z.$$

We deduce that there exists $c > 0$, $C > 0$ and $\tilde{C}_0 > 0$ such that for all $\varepsilon_0 \in (0,1)$ there exists $\alpha_0 > 0$ such that for all $R \geq 1$ and $\alpha \in (0, \alpha_0)$:

$$(c\varepsilon_0^{-2}\cos\beta\alpha - C\alpha)\|\chi_{2,\gamma}e^{\Phi_R}\psi\|^2 \leq \tilde{C}_0\alpha\|\chi_{1,\gamma}e^{\Phi_R}\psi\|^2.$$

Since $\cos\beta > 0$ and $\eta > 0$, if we choose ε_0 small enough, this implies:

$$\|\chi_{2,\gamma}e^{\Phi_R}\psi\|^2 \leq \tilde{C}\|\chi_{1,\gamma}e^{\Phi_R}\psi\|^2 \leq \hat{C}\|\psi\|^2.$$

It remains to take the limit $R \to +\infty$. □

Remark 8.2. *It turns out that Proposition 8.1 is still true for $\beta = \frac{\pi}{2}$. In this case the argument must be changed as follows. Instead of decomposing the integration with respect to $z > 0$ one should integrate by slices along a fixed direction which is not parallel to the axis of the cone. Therefore we are reduced to analyze the bottom of the spectrum of the Neumann Laplacian on ellipses instead of circles. We leave the details to the reader.*

2. Construction of quasimodes when $\beta = 0$

This section deals with the proof of the following proposition.

Proposition 8.3. *For all $N \geq 1$ and $J \geq 1$, there exist $C_{N,J}$ and α_0 such that for all $1 \leq n \leq N$, and $0 < \alpha < \alpha_0$, we have:*

$$\operatorname{dist}\left(\mathsf{sp}_{\mathsf{dis}}(\mathfrak{L}_{\alpha,0}), \sum_{j=0}^{J} \gamma_{j,n}\alpha^{2j+1}\right) \leq C_{N,J}\ \alpha^{2J+3},$$

where $\gamma_{0,n} = \mathfrak{l}_N = 2^{-5/2}(4n-1)$.

Proof. We construct quasimodes which do not depend on θ. In other words, we look for quasimodes for:

$$\mathcal{L}_{\alpha,0} = -\frac{1}{\tau^2}\partial_\tau \tau^2 \partial_\tau + \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2 - \frac{1}{\alpha^2\ \tau^2 \sin(\alpha\varphi)}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi.$$

We write a formal Taylor expansion of $\mathcal{L}_{\alpha,0}$ in powers of α^2:

$$\mathcal{L}_{\alpha,0} \sim \alpha^{-2}\mathcal{M}_{-1} + \mathcal{M}_0 + \sum_{j\geq 1}\alpha^{2j}\mathcal{M}_j,$$

where:

$$\mathcal{M}_{-1} = -\frac{1}{\tau^2\varphi}\partial_\varphi \varphi \partial_\varphi, \qquad \mathcal{M}_0 = -\frac{1}{\tau^2}\partial_\tau \tau^2 \partial_\tau + \frac{\varphi^2\tau^2}{4} + \frac{1}{3\tau^2}\varphi\partial_\varphi.$$

We look for quasi-eigenpairs expressed as formal series:

$$\psi \sim \sum_{j\geq 0}\alpha^{2j}\psi_j, \qquad \lambda \sim \alpha^{-2}\lambda_{-1} + \lambda_0 + \sum_{j\geq 1}\alpha^{2j}\lambda_j,$$

so that, formally, we have:

$$\mathcal{L}_{\alpha,0}\psi \sim \lambda\psi.$$

We are led to solve the equation:

$$\mathcal{M}_{-1}\psi_0 = -\frac{1}{\tau^2\varphi}\partial_\varphi \varphi \partial_\varphi \psi_0 = \lambda_{-1}\psi_0.$$

We choose $\lambda_{-1} = 0$ and $\psi_0(\tau,\varphi) = f_0(\tau)$, with f_0 to be chosen in the next step. We shall now solve the equation:

$$\mathcal{M}_{-1}\psi_1 = (\lambda_0 - \mathcal{M}_0)\psi_0.$$

We look for ψ_1 in the form: $\psi_1(\tau,\varphi) = t^2\tilde{\psi}_1(\tau,\varphi) + f_1(\tau)$. The equation provides:

$$-\frac{1}{\varphi}\partial_\varphi \varphi \partial_\varphi \tilde{\psi}_1 = (\lambda_0 - \mathcal{M}_0)\psi_0. \tag{8.2.1}$$

For each $\tau > 0$, the Fredholm condition is $\langle(\lambda_0 - \mathcal{M}_0)\psi_0, 1\rangle_{\mathsf{L}^2((0,\frac{1}{2}),\varphi\,\mathrm{d}\varphi)} = 0$, that reads:

$$\int_0^{\frac{1}{2}} (\mathcal{M}_0\psi_0)(\tau,\varphi)\,\varphi\,\mathrm{d}\varphi = \frac{\lambda_0}{2^3}f_0(\tau).$$

Moreover we have:

$$\int_0^{\frac{1}{2}} (\mathcal{M}_0\psi_0)(\tau,\varphi)\,\varphi\,\mathrm{d}\varphi = -\frac{1}{2^3\tau^2}\partial_\tau\tau^2\partial_\tau f_0(\tau) + \frac{1}{2^8}\tau^2 f_0(\tau),$$

so that we get:

$$\left(-\frac{1}{\tau^2}\partial_\tau\tau^2\partial_\tau + \frac{1}{2^5}\tau^2\right) f_0 = \lambda_0 f_0.$$

We are led to take:

$$\lambda_0 = \mathfrak{l}_N \qquad \text{and} \qquad f_0(\tau) = \mathfrak{f}_n(\tau).$$

For this choice of f_0, we infer the existence of a unique function denoted by $\tilde{\psi}_1^\perp$ (in the Schwartz class with respect to t) orthogonal to 1 in $\mathsf{L}^2((0,\frac{1}{2}),\varphi\,\mathrm{d}\varphi)$ which satisfies (8.2.1). Using the decomposition of ψ_1, we have:

$$\psi_1(\tau,\varphi) = \tau^2\tilde{\psi}_1^\perp(\tau,\varphi) + f_1(\tau),$$

where f_1 has to be determined in the next step.

We leave the construction of the next terms to the reader.

We define:

$$\Psi_n^J(\alpha)(\tau,\theta,\varphi) = \sum_{j=0}^J \alpha^{2j}\psi_j(\tau,\varphi), \quad \forall(\tau,\theta,\varphi)\in\mathcal{P}, \tag{8.2.2}$$

$$\Lambda_n^J(\alpha) = \sum_{j=0}^J \alpha^{2j}\lambda_j. \tag{8.2.3}$$

Due to the exponential decay of the ψ_j and thanks to Taylor expansions, there exists $C_{n,J}$ such that:

$$\|\left(\mathcal{L}_\alpha - \Lambda_n^J(\alpha)\right)\Psi_n^J(\alpha)\|_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \le C_{n,J}\alpha^{2J+2}\|\Psi_n^J(\alpha)\|_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

Using the spectral theorem and going back to the operator $\mathfrak{L}_\alpha$ by change of variables, we conclude the proof of Proposition 8.3 with $\gamma_{j,n} = \lambda_j$. $\square$

Considering the main term of the asymptotic expansion, we deduce the three following corollaries.

Corollary 8.4. *For all $n\ge 1$, there exist $\alpha_0(n)>0$ and $C_n>0$ such that, for all $\alpha\in(0,\alpha_0(n))$, the n-th eigenvalue exists and satisfies:*

$$\lambda_n(\alpha) \le C_n\alpha,$$

or equivalently $\tilde{\lambda}_n(\alpha)\le C_n$.

Corollary 8.5. *For all $N\ge 1$, there exist C and α_0 and for all $1\le n\le N$ and $0\le\alpha\le\alpha_0$, there exists an eigenvalue $\tilde{\lambda}_{k(n,\alpha)}$ of $\mathcal{L}_\alpha$ such that*

$$|\tilde{\lambda}_{k(n,\alpha)} - \mathfrak{l}_N| \le C\alpha^2.$$

Corollary 8.6. *We observe that for $1 \leq n \leq N$ and $\alpha \in (0, \alpha_0)$:*

$$0 \leq \tilde{\lambda}_n(\alpha) \leq \tilde{\lambda}_{k(n,\alpha)} \leq \mathfrak{l}_N + C\alpha^2.$$

This last corollary proves Corollary 8.4.

3. Axisymmetry of the first eigenfunctions when $\beta = 0$

Notation 8.7. *From Propositions 2.1 and 8.3, we infer that, for all $n \geq 1$, there exists $\alpha_n > 0$ such that if $\alpha \in (0, \alpha_n)$, the n-th eigenvalue $\tilde{\lambda}_n(\alpha)$ of $\mathcal{L}_\alpha$ exists. Due to the fact that $-i\partial_\theta$ commutes with the operator, one deduces that for each $n \geq 1$, we can consider a basis $(\psi_{n,j}(\alpha))_{j=1,\cdots J(n,\alpha)}$ of the eigenspace of $\mathcal{L}_\alpha$ associated with $\tilde{\lambda}_n(\alpha)$ such that*

$$\psi_{n,j}(\alpha)(\tau, \theta, \varphi) = e^{im_{n,j}(\alpha)\theta}\Psi_{n,j}(\tau, \varphi).$$

As an application of the localization estimates of Section 1, we prove the following proposition.

Proposition 8.8. *For all $n \geq 1$, there exists $\alpha_n > 0$ such that if $\alpha \in (0, \alpha_n)$, we have:*

$$m_{n,j}(\alpha) = 0, \quad \forall j = 1, \ldots, J(n, \alpha).$$

In other words, the functions of the n-th eigenspace are independent from θ as soon as α is small enough.

In order to succeed, we use a contradiction argument: We consider an L^2-normalized eigenfunction of $\mathcal{L}_\alpha$ associated to $\lambda_n(\alpha)$ in the form $e^{im(\alpha)\theta}\Psi_\alpha(\tau, \varphi)$ and we assume that there exists $\alpha > 0$ as small as we want such that $m(\alpha) \neq 0$ or equivalently $|m(\alpha)| \geq 1$.

3.1. Dirichlet condition on the axis $\varphi = 0$.

Let us prove the following lemma.

Lemma 8.9. *For all $t > 0$, we have $\Psi_\alpha(t, 0) = 0$.*

Proof. We recall the eigenvalue equation:

$$\mathcal{L}_{\alpha,0,m(\alpha)}\Psi_\alpha = \tilde{\lambda}_n(\alpha)\Psi_\alpha.$$

We deduce:

$$\mathcal{Q}_{\alpha,0,m(\alpha)}(\Psi_\alpha) \leq C\|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R}, \mathrm{d}\mu)}.$$

This implies:

$$\int_{\mathcal{R}} \frac{1}{\tau^2 \sin^2(\alpha\varphi)} \left(m(\alpha) + \frac{\sin^2(\alpha\varphi)}{2\alpha}\tau^2\right)^2 |\Psi_\alpha(\tau, \varphi)|^2 \,\mathrm{d}\mu \leq C\|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R}, \mathrm{d}\mu)} < +\infty.$$

Using the inequality $(a + b)^2 \geq \frac{1}{2}a^2 - 2b^2$, it follows:

$$\frac{m(\alpha)^2}{2} \int_{\mathcal{R}} \frac{1}{\tau^2 \sin^2(\alpha\varphi)} |\Psi_\alpha(\tau, \varphi)|^2 \,\mathrm{d}\mu - 2\int_{\mathcal{R}} \frac{\tau^2 \sin^2(\alpha\varphi)}{4\alpha^2} |\Psi_\alpha(\tau, \varphi)|^2 \,\mathrm{d}\mu < +\infty,$$

so that:

$$m(\alpha)^2 \int_{\mathcal{R}} \frac{1}{\tau^2 \sin^2(\alpha\varphi)} |\Psi_\alpha(\tau,\varphi)|^2 \,\mathrm{d}\mu < +\infty,$$

and:

$$\int_{\mathcal{R}} \frac{1}{\tau^2 \sin^2(\alpha\varphi)} |\Psi_\alpha(\tau,\varphi)|^2 \,\mathrm{d}\mu < +\infty. \tag{8.3.1}$$

Therefore, for almost all $\tau > 0$, we have:

$$\int_0^{\frac{1}{2}} \frac{1}{\sin^2(\alpha\varphi)} |\Psi_\alpha(\tau,\varphi)|^2 \sin(\alpha\varphi) \,\mathrm{d}\varphi < +\infty. \tag{8.3.2}$$

The function $\mathcal{R} \ni (\tau,\varphi) \mapsto \Psi_\alpha(\tau,\varphi)$ is smooth by elliptic regularity inside $\mathcal{C}_\alpha$ (thus $\mathcal{R}$). In particular, it is continuous at $\varphi = 0$. By the integrability property (8.3.2), this imposes that, for all $\tau > 0$, we have $\Psi_\alpha(\tau, 0) = 0$. $\square$

3.2. The operator $-(\sin(\alpha\varphi))^{-1}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi$.

Notation 8.10. *For $\alpha \in (0,\pi)$, let us consider the operator on $\mathsf{L}^2\left((0,\frac{1}{2}), \sin(\alpha\varphi)\,\mathrm{d}\varphi\right)$ defined by:*

$$\mathfrak{P}_\alpha = -\frac{1}{\sin(\alpha\varphi)}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi,$$

with domain:

$$\mathsf{Dom}\,(\mathfrak{P}_\alpha) = \Big\{\psi \in \mathsf{L}^2\left((0,\tfrac{1}{2}), \sin(\alpha\varphi)\,\mathrm{d}\varphi\right), \\ \frac{1}{\sin(\alpha\varphi)}\partial_\varphi \sin(\alpha\varphi)\partial_\varphi\psi \in \mathsf{L}^2\left((0,\tfrac{1}{2}), \sin(\alpha\varphi)\,\mathrm{d}\varphi\right), \partial_\varphi\psi\left(\tfrac{1}{2}\right) = 0,\ \psi(0) = 0\Big\}.$$

We denote by $\nu_1(\alpha)$ its first eigenvalue.

The aim of this subsection is to establish the following lemma:

Lemma 8.11. *There exists $c_0 > 0$ such that for all $\alpha \in (0,\pi)$:*

$$\nu_1(\alpha) \geq c_0.$$

Proof. We consider the associated quadratic form $\mathfrak{p}_\alpha$:

$$\mathfrak{p}_\alpha(\psi) = \int_0^{\frac{1}{2}} \sin(\alpha\varphi)|\partial_\varphi\psi|^2 \,\mathrm{d}\varphi.$$

We have the elementary lower bound:

$$\mathfrak{p}_\alpha(\psi) \geq \int_0^{\frac{1}{2}} \alpha\varphi\left(1 - \frac{(\alpha\varphi)^2}{6}\right)|\partial_\varphi\psi|^2 \,\mathrm{d}\varphi \geq \frac{1}{2}\int_0^{\frac{1}{2}} \alpha\varphi|\partial_\varphi\psi|^2 \,\mathrm{d}\varphi,$$

since $0 \leq \alpha\varphi \leq \frac{\pi}{2}$. We are led to analyze the lowest eigenvalue $\gamma \geq 0$ of the operator on $\mathsf{L}^2\left((0,\frac{1}{2}), \varphi\,\mathrm{d}\varphi\right)$ defined by $-\frac{1}{\varphi}\partial_\varphi\varphi\partial_\varphi$ with Dirichlet condition at $\varphi = 0$ and Neumann condition at $\varphi = \frac{1}{2}$. Let us prove that $\gamma > 0$. If it were not the case, the corresponding eigenvector ψ would satisfy:

$$-\frac{1}{\varphi}\partial_\varphi \varphi \partial_\varphi \psi = 0,$$

so that:

$$\psi(\varphi) = c \ln \varphi + d, \qquad \text{with } c, d \in \mathbb{R}.$$

The boundary conditions provide $c = d = 0$ and thus $\psi = 0$. By contradiction, we infer that $\gamma > 0$.

We deduce that:

$$\mathfrak{p}_\alpha(\psi) \geq \frac{\gamma}{2} \int_0^{\frac{1}{2}} \alpha\varphi |\psi|^2 \,\mathrm{d}\varphi \geq \frac{\gamma}{2} \int_0^{\frac{1}{2}} \sin(\alpha\varphi) |\psi|^2 \,\mathrm{d}\varphi.$$

By the min-max principle, we conclude that, for all $\alpha \in (0, \pi)$:

$$\nu_1(\alpha) \geq \frac{\gamma}{2} =: c_0 > 0.$$

□

3.3. End of the proof of Proposition 8.8**.** We have:

$$\mathcal{L}_{\alpha,0,m(\alpha)}(\tau \Psi_\alpha) = \tilde{\lambda}_n(\alpha) \tau \Psi_\alpha + [\mathcal{L}_{\alpha,0,m(\alpha)}, \tau] \Psi_\alpha. \tag{8.3.3}$$

We have:

$$[\mathcal{L}_{\alpha,0,m(\alpha)}, \tau] = [-\tau^{-2}\partial_\tau \tau^2 \partial_\tau, \tau] = -2\partial_\tau - \frac{2}{\tau}.$$

We take the scalar product of the equation (8.3.3) with $t\Psi_\alpha$. We notice that:

$$\langle [\mathcal{L}_{\alpha,0,m(\alpha)}, \tau] \Psi_\alpha, \tau \Psi_\alpha \rangle_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} = -2 \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} + 3 \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} = \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)}.$$

The Agmon estimates provide:

$$|\langle \tau [\mathcal{L}_{\alpha,0,m(\alpha)}, \chi_{\alpha,\eta}] \Psi_\alpha, \tau \Psi_\alpha \rangle_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)}| = \mathcal{O}(\alpha^\infty) \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)}.$$

We infer:

$$\mathcal{Q}_{\alpha,0,m(\alpha)}(\tau \Psi_\alpha) \leq C(\|\tau \Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} + \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)}),$$

and especially:

$$\alpha^{-2} \int_{\mathcal{R}} |\partial_\varphi \Psi_\alpha|^2 \,\mathrm{d}\mu \leq C \left(\|t \Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} + \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} \right).$$

Lemmas 8.9 and 8.11 imply that:

$$c_0 \alpha^{-2} \int_{\mathcal{R}} |\Psi_\alpha|^2 \,\mathrm{d}\mu \leq C \left(\|\tau \Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} + \|\Psi^{\mathsf{cut}}\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} \right).$$

With the estimates of Agmon, we have:

$$c_0 \alpha^{-2} \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)} \leq \tilde{C} \|\Psi_\alpha\|^2_{\mathsf{L}^2(\mathcal{R},\, \mathrm{d}\mu)}.$$

We infer that, for α small enough, $\Psi_\alpha = 0$ and this is a contradiction. This ends the proof of Proposition 8.8.

4. Accurate estimate of the spectral gap when $\beta = 0$

This section is devoted to the proof of the following proposition.

Proposition 8.12. *For all $n \geq 1$, there exists $\alpha_0(n) > 0$ such that, for all $\alpha \in (0, \alpha_0(n))$, the n-th eigenvalue exists and satisfies:*

$$\lambda_n(\alpha, 0) \geq \gamma_{0,n}\alpha + o(\alpha),$$

or equivalently $\tilde{\lambda}_n(\alpha, 0) \geq \gamma_{0,n} + o(1)$.

We first establish approximation results satisfied by the eigenfunctions in order to catch their behavior with respect to the t-variable. Then, we can apply a reduction of dimension and we are reduced to a family of 1D model operators.

4.1. Approximation of the eigenfunctions . Let us consider $N \geq 1$ and let us introduce:

$$\mathfrak{E}_N(\alpha) = \mathsf{span}\{\psi_{n,1}(\alpha), 1 \leq n \leq N\},$$

where $\psi_{n,1}(\alpha)(t, \theta, \psi) = \Psi_{n,1}(t, \varphi)$ are considered as functions defined in $\mathcal{P}$.

Proposition 8.13. *For all $N \geq 1$, there exist $\alpha_0(N) > 0$ and $C_N > 0$ such that, for all $\psi \in \mathfrak{E}_N(\alpha)$:*

$$\|\tau^{-1}(\psi - \underline{\psi})\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \leq C_N\alpha^2\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}, \tag{8.4.1}$$

$$\|\psi - \underline{\psi}\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \leq C_N\alpha^2\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}, \tag{8.4.2}$$

$$\|\tau(\psi - \underline{\psi})\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \leq C_N\alpha^2\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}, \tag{8.4.3}$$

where:

$$\underline{\psi}(\tau) = \frac{1}{\int_0^{\frac{1}{2}} \varphi\,\mathrm{d}\varphi} \int_0^{\frac{1}{2}} \psi(\tau, \varphi)\,\varphi\,\mathrm{d}\varphi. \tag{8.4.4}$$

Proof. It is sufficient to prove the proposition for $\psi = \psi_{n,1}(\alpha)$ and $1 \leq n \leq N$. We have:

$$\mathcal{L}_\alpha\Psi_{n,1}(\alpha) = \tilde{\lambda}_n(\alpha)\Psi_{n,1}(\alpha). \tag{8.4.5}$$

We have:

$$\mathcal{Q}_\alpha(\psi) \leq C\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})},$$

and thus, seeing ψ as a function on $\mathcal{P}$:

$$\frac{1}{\alpha^2}\int_{\mathcal{P}} \tau^{-2}|\partial_\varphi\psi|^2\,\mathrm{d}\tilde{\mu} \leq C\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

We get:

$$\int_{\mathcal{P}} |\partial_\varphi\psi|^2\,\sin\alpha\varphi\,\mathrm{d}\tau\,\mathrm{d}\theta\,\mathrm{d}\varphi \leq C\alpha^2\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})},$$

so that (using the inequality $\sin(\alpha\varphi) \geq \frac{\alpha\varphi}{2}$):

$$\int_{\mathcal{P}} \frac{\alpha\varphi}{2} |\partial_\varphi \psi|^2 \,\mathrm{d}\tau\,\mathrm{d}\theta\,\mathrm{d}\varphi \leq C\alpha^2 \|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

We infer:

$$\int_{\mathcal{P}} \alpha\varphi |\partial_\varphi (\psi - \underline{\psi})|^2 \,\mathrm{d}\tau\,\mathrm{d}\theta\,\mathrm{d}\varphi \leq C\alpha^2 \|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

Let us consider the Neumann realization of the operator $-\frac{1}{\varphi}\partial_\varphi \varphi \partial_\varphi$ on $\mathsf{L}^2((0,\frac{1}{2}), \varphi\,\mathrm{d}\varphi)$. The first eigenvalue is simple, equal to 0 and associated to constant functions. Let $\delta > 0$ be the second eigenvalue. The function $\psi - \underline{\psi}$ is orthogonal to constant functions in $\mathsf{L}^2((0,\frac{1}{2})\varphi\,\mathrm{d}\varphi)$ by definition (8.4.4). Then, we apply the min-max principle to $\psi - \underline{\psi}$ and deduce:

$$\int_{\mathcal{P}} \delta\alpha\varphi |\psi - \underline{\psi}|^2 \,\mathrm{d}\tau\,\mathrm{d}\theta\,\mathrm{d}\varphi \leq C\alpha^2 \|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})},$$

and:

$$\int_{\mathcal{P}} \tau^{-2} |\psi - \underline{\psi}|^2 \,\mathrm{d}\tilde{\mu} \leq \tilde{C}\alpha^2 \|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})},$$

which ends the proof of (8.4.1). We multiply (8.4.5) by t and we take the scalar product with $\tau\psi$ to get:

$$\mathcal{Q}_\alpha(\tau\psi) \leq \tilde{\lambda}_n(\alpha) \|\tau\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} + \left| \langle [-\tau^{-2}\partial_\tau \tau^2 \partial_\tau, \tau]\psi, \tau\psi \rangle_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \right|.$$

We recall that:

$$[-\tau^{-2}\partial_\tau \tau^2 \partial_\tau, \tau] = -2\partial_\tau - \frac{2}{\tau}.$$

We get:

$$\mathcal{Q}_{\alpha,0}(t\psi) \leq C\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

We deduce (8.4.2) in the same way as (8.4.1).
Finally, we easily get:

$$\mathcal{Q}_{\alpha,0}(\tau^2\psi) \leq \tilde{\lambda}_n(\alpha) \|\tau^2\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} + \left| \langle [-\tau^{-2}\partial_\tau \tau^2 \partial_\tau, \tau^2]\psi, \tau^2\psi \rangle_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} \right|.$$

The commutator is:

$$[-\tau^{-2}\partial_\tau \tau^2 \partial_\tau, \tau^2] = -6 - 4\tau\partial_\tau.$$

This implies:

$$\mathcal{Q}_{\alpha,0}(\tau^2\psi) \leq C\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

The approximation (8.4.3) follows. □

4.2. Proof of Proposition 8.12. We have now the elements to prove Proposition 8.12. The main idea is to apply the min-max principle to the quadratic form $\mathcal{Q}_{\alpha,0}$ and to the space $\mathfrak{E}_N(\alpha)$.

Lemma 8.14. *For all $N \geq 1$, there exist $\alpha_N > 0$ and $C_N > 0$ such that, for all $\alpha \in (0, \alpha_N)$ and for all $\psi \in \mathfrak{E}_N(\alpha)$:*

$$\int_{\mathcal{P}} \left(|\partial_\tau \psi|^2 + 2^{-5}|\tau\psi|^2 + \frac{1}{\alpha^2\tau^2}|\partial_\varphi \psi|^2 \right) \mathrm{d}\tilde{\mu} \leq \tilde{\lambda}_n(\alpha)\|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})} + C_N\alpha \|\psi\|^2_{\mathsf{L}^2(\mathcal{P},\,\mathrm{d}\tilde{\mu})}.$$

PROOF. We recall that, for all $\psi \in \mathfrak{E}_n(\alpha)$, we have:

$$\mathcal{Q}_{\alpha,0}(\psi) \leq \tilde{\lambda}_n(\alpha)\|\psi\|^2_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

We infer that:

$$\int_{\mathcal{P}} \left(|\partial_\tau \psi|^2 + \frac{\sin^2(\alpha\varphi)}{4\alpha^2}|\tau\psi|^2 + \frac{1}{\alpha^2\tau^2}|\partial_\varphi \psi|^2 \right) \mathrm{d}\tilde{\mu} \leq \tilde{\lambda}_n(\alpha)\|\psi\|^2_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

We shall analyze the term $\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}|\tau\psi|^2 \,\mathrm{d}\tilde{\mu}$. We get:

$$\left| \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\psi|^2 \,\mathrm{d}\tilde{\mu} - \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\underline{\psi}|^2 \,\mathrm{d}\tilde{\mu} \right| \leq C\|\tau\psi - \tau\underline{\psi}\|_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}\|\psi\|_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})},$$

and thus:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\psi|^2 \,\mathrm{d}\tilde{\mu} \geq \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\underline{\psi}|^2 \,\mathrm{d}\tilde{\mu} - C\|\tau\psi - \tau\underline{\psi}\|_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}\|\psi\|_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

Proposition 8.13 provides:

$$\|\tau\psi - \tau\underline{\psi}\|_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})} \leq C\alpha\|\psi\|_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}, \tag{8.4.6}$$

so that:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\psi|^2 \,\mathrm{d}\tilde{\mu} \geq \int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\underline{\psi}|^2 \,\mathrm{d}\tilde{\mu} - C\alpha^{1/2-\eta}\|\psi\|^2_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

We deduce:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\psi|^2 \,\mathrm{d}\tilde{\mu} \geq (2^{-5} - C\alpha^2)\int_{\mathcal{P}} |\tau\underline{\psi}|^2 \,\mathrm{d}\tilde{\mu} - C\alpha\|\psi\|^2_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}. \tag{8.4.7}$$

Proposition 8.1 and (8.4.7) provide:

$$\int_{\mathcal{P}} \frac{\sin^2(\alpha\varphi)}{4\alpha^2}\tau^2|\psi|^2 \,\mathrm{d}\tilde{\mu} \geq 2^{-5}\int_{\mathcal{P}} |\tau\psi|^2 \,\mathrm{d}\tilde{\mu} - C\alpha\|\psi\|^2_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\tilde{\mu})}.$$

□

An straightforward consequence of Lemma 8.14 is:

Lemma 8.15. *For all $N \geq 1$, there exist $\alpha_N > 0$ and $C_N > 0$ such that, for all $\alpha \in (0, \alpha_N)$ and for all $\psi \in \mathfrak{E}_N(\alpha)$:*

$$\int_{\mathcal{P}} \left(|\partial_\tau \psi|^2 + 2^{-5}|\tau\psi|^2 + \frac{1}{\alpha^2\tau^2}|\partial_\varphi \psi|^2 \right) \mathrm{d}\breve{\mu} \leq \left(\tilde{\lambda}_n(\alpha) + C_N\alpha \right) \|\psi\|^2_{\mathsf{L}^2(\mathcal{P}, \mathrm{d}\breve{\mu})},$$

with $\mathrm{d}\breve{\mu} = t^2\varphi \,\mathrm{d}\tau \,\mathrm{d}\varphi \,\mathrm{d}\theta$.

PROOF. It is sufficient to write for any $\varphi \in (0, \frac{1}{2})$:

$$\varphi = \frac{1}{\alpha}\sin(\alpha\varphi)\frac{\alpha\varphi}{\sin(\alpha\varphi)} = \frac{1}{\alpha}\sin(\alpha\varphi)(1 + \mathcal{O}(\alpha^2)) \qquad \text{as } \alpha \to 0.$$

□

With Lemma 8.15, we deduce (from the min-max principle) that there exists α_N such that

$$\forall \alpha \in (0, \alpha_N), \qquad \tilde{\lambda}_n(\alpha) \geq \mathfrak{l}_N - C\alpha.$$

This achieves the proof of Proposition 8.12.

5. Case when $\beta \in \left[0, \frac{\pi}{2}\right]$

By using commutator formulas in the spirit of Proposition 6.20 jointly with the estimates of Agmon, one can prove that:

Lemma 8.16. *Let $k \geq 0$ and $C_0 > 0$. There exist $\alpha_0 > 0$ and $C > 0$ such that for all $\alpha \in (0, \alpha_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}_{\alpha,\beta}$ such that $\lambda \leq C_0$:*

$$\|\tau^k \psi - \tau^k \underline{\psi}_\theta\| \leq C\alpha^{1/2}\|\psi\|,$$

with

$$\underline{\psi}_\theta(\tau, \varphi) = \frac{1}{2\pi}\int_0^{2\pi} \psi(\tau, \theta, \varphi)\,\mathrm{d}\theta.$$

We also get an approximation of $D_t\psi$.

Lemma 8.17. *Let $C_0 > 0$. There exist $\alpha_0 > 0$ and $C > 0$ such that for all $\alpha \in (0, \alpha_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}_{\alpha,\beta}$ such that $\lambda \leq C_0$, we have:*

$$\|D_\tau \psi - D_\tau \underline{\psi}_\theta\| \leq C\alpha^{1/2}\|\psi\|.$$

The last two lemmas imply the following proposition:

Proposition 8.18. *There exist $C > 0$ and $\alpha_0 > 0$ such that for any $\alpha \in (0, \alpha_0)$ and all $\psi \in \mathfrak{E}_N(\alpha)$, we have*

$$\mathcal{Q}_{\alpha,\beta}(\psi) \geq (1-\alpha)\mathcal{Q}^{\mathsf{model}}_{\alpha,\beta}(\psi) - C\alpha^{1/2}\|\psi\|^2, \tag{8.5.1}$$

where:

$$\mathcal{Q}^{\mathsf{model}}_{\alpha,\beta}(\psi) =$$
$$\int_{\mathcal{P}} |D_\tau\psi|^2\,\mathrm{d}\tilde{\mu} + \frac{1}{2^4}\int_{\mathcal{P}} \cos^2(\alpha\varphi)\tau^2\sin^2\beta|\psi|^2\,\mathrm{d}\tilde{\mu} + \int_{\mathcal{P}} \frac{1}{\tau^2\sin^2(\alpha\varphi)}|(D_\theta + A_{\theta,1})\psi|^2\,\mathrm{d}\tilde{\mu} + \|P_3\psi\|^2.$$

The spectral analysis is then reduced to an axisymmetric case.

Part 3

Spectral reductions

CHAPTER 9

Born-Oppenheimer approximation

Le *cogito* d'un rêveur crée son propre cosmos, un cosmos singulier, un cosmos bien à lui. Sa rêverie est dérangée, son cosmos est troublé si le rêveur a la certitude que la rêverie d'un autre oppose un monde à son propre monde.

La flamme d'une chandelle, Bachelard

This chapter presents the main idea behind the electric Born-Oppenheimer approximation (see [**37, 126**]). We prove Theorem 2.27.

1. Basic estimates

Let us informally explain the main steps in the construction of quasimodes behind Theorem 2.27. We have:

$$\mathcal{V}(s)u_s = \nu(s)u_s.$$

This is easy to prove that (the details are left as an exercise):

$$\langle \mathcal{V}'(s_0)u_{s_0}, u_{s_0}\rangle = 0,$$

$$(\mathcal{V}(s_0) - \nu(s_0))\left(\frac{d}{ds}u_s\right)_{|s=s_0} = -\mathcal{V}'(s_0)u_{s_0}$$

and:

$$\left\langle \mathcal{V}'(s_0)\left(\frac{d}{ds}u_s\right)_{|s=s_0} + \frac{\mathcal{V}''(s_0)}{2}u_{s_0}, u_{s_0}\right\rangle = \frac{\nu''(s_0)}{2}.$$

Notation 9.1. *We let:*

$$v_{s_0}(\tau) = \left(\frac{d}{ds}u_s\right)_{|s=s_0}, \quad w_{s_0}(\tau) = \left(\frac{d^2}{ds^2}u_s\right)_{|s=s_0}.$$

As usual we begin with the construction of suitable quasimodes. We let

$$s = s_0 + h^{1/2}\sigma, \quad t = \tau$$

and, instead of $\mathfrak{H}_h$, we study:

$$\mathcal{H}_h = hD_\sigma^2 + \mathcal{V}(s_0 + h^{1/2}\sigma).$$

In terms of formal power series, we have:

$$\mathcal{H}_h = \mathcal{V}(s_0) + h^{1/2}\sigma\mathcal{V}'(s_0) + h\left(\sigma^2\frac{\mathcal{V}''(s_0)}{2} + D_\sigma^2\right) + \cdots$$

We look for quasi-eigenpairs in the form:

$$\lambda \sim \lambda_0 + h^{1/2}\lambda_1 + h\lambda_2 + \cdots, \quad \psi \sim \psi_0 + h^{1/2}\psi_1 + h\psi_2 + \cdots$$

We must solve:

$$\mathcal{V}(s_0)\psi_0 = \lambda_0\psi_0.$$

Therefore, we choose $\lambda_0 = \nu(s_0)$ and $\psi_0(\sigma,\tau) = u_{s_0}(\tau)f_0(\sigma)$.

We now meet the following equation:

$$(\mathcal{V}(s_0) - \lambda_0)\psi_1 = (\lambda_1 - \sigma\mathcal{V}'(s_0))\psi_0.$$

The Feynman-Hellmann formula jointly with the Fredholm alternative implies that: $\lambda_1 = 0$ and that we can take:

$$\psi_1(\sigma,\tau) = \sigma f_0(\sigma)v_{s_0} + \sigma f_1(\sigma)u_{s_0}.$$

The crucial equation is given by:

$$(\mathcal{V}(s_0) - \nu(s_0))\psi_2 = \lambda_2\psi_0 - \sigma\mathcal{V}'(s_0)\psi_1 - \left(\sigma^2\frac{\mathcal{V}''(s_0)}{2} + D_\sigma^2\right)\psi_0.$$

The Fredholm alternative jointly with the Feynman-Hellmann formula provides:

$$\left(D_\sigma^2 + \frac{\nu''(s_0)}{2}\sigma^2\right)f_0 = \lambda_2 f_0.$$

This is an easy exercise to prove that this construction can be continued at any order.

2. Essential spectrum and Agmon estimates

Let us briefly discuss the properties related to the essential spectrum. From Assumption 2.26, we infer (exercise), as a consequence of the theorem of Persson (see Theorem 5.8):

Proposition 9.2. *Under Assumption 2.26, we have:*

$$\inf_{h>0} \inf \mathsf{sp}_{\mathsf{ess}}(\mathcal{H}_h) > \nu(s_0).$$

We also get:

Proposition 9.3. *Under Assumption 2.26, there exists $h_0 > 0, C > 0, \varepsilon_0 > 0$ such that, for $h \in (0,h_0)$, for all eigenpair (λ,ψ) such that $\lambda \leq \nu(s_0) + C_0 h$, we have:*

$$\int_{\mathbb{R}\times\Omega} e^{2\varepsilon_0(|s|+|\tau|)}|\psi|^2 \,\mathrm{d}s\,\mathrm{d}\tau \leq C\|\psi\|^2.$$

We are now led to prove some localization behavior of the eigenfunctions associated with eigenvalues λ such that: $|\lambda - \nu(s_0)| \leq C_0 h$.

Proposition 9.4. *There exist $\varepsilon_0, h_0, C > 0$ such that for all eigenpair (λ, ψ) such that $|\lambda - \nu(s_0)| \leq C_0 h$, we have:*

$$\int_{\mathbb{R}\times\Omega} e^{2\varepsilon_0 h^{-1/2}|s|}|\psi|^2 \,\mathrm{d}x \leq C\|\psi\|^2.$$

and:

$$\int_{\mathbb{R}\times\Omega} \left|h\partial_s\left(e^{\varepsilon_0 h^{-1/2}|s|}\psi\right)\right|^2 \,\mathrm{d}x \leq Ch\|\psi\|^2.$$

Proof. Let us write an estimate of Agmon:

$$\mathcal{Q}_h(e^{h^{-1/2}\varepsilon_0|s|}\psi) - h\varepsilon_0^2\|e^{h^{-1/2}\varepsilon_0|s|}\psi\|^2 = \lambda\|e^{h^{-1/2}\varepsilon_0|s|}\psi\|^2 \leq (\nu(s_0) + C_0 h)\|e^{h^{-1/2}\varepsilon_0|s|}\psi\|^2.$$

But we notice that:

$$\mathcal{Q}_h(e^{h^{-1/2}\varepsilon_0|s|}\psi) \geq \int_{\mathbb{R}\times\Omega} h^2\left|\partial_s\left(e^{h^{-1/2}\varepsilon_0|s|}\psi\right)\right|^2 + \nu(s)\left|\left(e^{h^{-1/2}\varepsilon_0|s|}\psi\right)\right|^2 \,\mathrm{d}x$$

This implies the inequality:

$$\int_{\mathbb{R}\times\Omega}(\nu(s) - \nu(s_0) - C_0 h - \varepsilon_0^2 h)\left|\left(e^{h^{-1/2}\varepsilon_0|s|}\psi\right)\right|^2 \,\mathrm{d}x \leq 0.$$

We leave the conclusion as an exercise. □

3. Projection method

As we have observed, it can be more convenient to study $\tilde{\mathcal{H}}_h$ instead of $\mathcal{H}_h$. Let us introduce the Feshbach-Grushin projection (see [**80**]) on u_{s_0}:

$$\Pi_0\psi = \langle\psi, u_{s_0}\rangle_{\mathsf{L}^2(\Omega)} u_{s_0}(\tau).$$

We want to estimate the projection of the eigenfunctions associated with eigenvalues λ such that: $|\lambda - \nu(s_0)| \leq C_0 h$. For that purpose, let us introduce the quadratic form:

$$q_0(\psi) = \int_{\mathbb{R}\times\Omega} |\partial_\tau\psi|^2 + P(\tau, s_0)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

This quadratic form is associated with the operator: $\mathsf{Id}_\sigma \otimes \mathcal{V}(s_0)$ whereas Π_0 is the projection on its first eigenspace.

Proposition 9.5. *There exist $C, h_0 > 0$ such that, for $h \in (0, h_0)$, for all eigenpair (λ, ψ) of $\tilde{\mathcal{H}}(h)$ such that $\lambda \leq \nu(s_0) + C_0 h$:*

$$0 \leq q_0(\psi) - \nu(s_0)\|\psi\|^2 \leq Ch^{1/2}\|\psi\|^2.$$

Moreover, we have:

$$\|\psi - \Pi_0\psi\| + \|\partial_\tau(\psi - \Pi_0\psi)\| \leq Ch^{1/4}\|\psi\|.$$

Proof. The proof is rather easy. We write:

$$h\|\partial_\sigma\psi\|^2 + \|\partial_\tau\psi\|^2 + \int_{\mathbb{R}\times\Omega} P(\tau, s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}s\,\mathrm{d}\tau \leq (\lambda + C_0 h)\|\psi\|^2. \tag{9.3.1}$$

Using the fact that P is a polynomial and the fact that, for $k, n \in \mathbb{N}$:

$$\int |\tau|^n |\sigma|^k |\psi|^2 \, \mathrm{d}\sigma \, \mathrm{d}\tau \leq C\|\psi\|^2,$$

we get the first estimate. For the second one, we notice that:

$$q_0(\psi) - \nu(s_0)\|\psi\|^2 = q_0(\psi - \Pi_0\psi) - \nu(s_0)\|\psi - \Pi_0\psi\|^2,$$

due to the fact that $\Pi_0\psi$ belongs to the kernel of $\mathsf{Id}_u \otimes \mathcal{V}(s_0) - \nu(s_0)\mathsf{Id}$. We observe then that:

$$q_0(\psi - \Pi_0\psi) - \nu(s_0)\|\psi - \Pi_0\psi\|^2 \geq \int_{\mathbb{R}} \int_{\Omega} |\partial_\tau(\psi - \Pi_0\psi)|^2 + P(\tau, s_0)|(\psi - \Pi_0\psi)|^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma.$$

Since for each u, we have: $\langle \psi - \Pi_0\psi, u_{s_0} \rangle_{\mathsf{L}^2(\Omega)} = 0$, we have the lower bound (min-max principle):

$$q_0(\psi - \Pi_0\psi) - \nu(s_0)\|\psi - \Pi_0\psi\|^2 \geq \int_{\mathbb{R}} (\nu_2(s_0) - \nu(s_0)) \int_{\Omega} |\psi - \Pi_0\psi|^2 \, \mathrm{d}\tau \, \mathrm{d}\sigma.$$

$\square$

Proposition 9.6. *There exist $C, h_0 > 0$ such that, for $h \in (0, h_0)$, for all eigenpair (λ, ψ) of $\tilde{\mathcal{H}}_h$ such that $\lambda \leq \nu(s_0) + C_0 h$:*

$$0 \leq q_0(\sigma\psi) - \nu(s_0)\|\sigma\psi\|^2 \leq Ch^{1/2}\|\psi\|^2$$

and

$$0 \leq q_0(\partial_\sigma\psi) - \nu(s_0)\|\partial_\sigma\psi\|^2 \leq Ch^{1/4}\|\psi\|^2$$

Moreover, we have:

$$\|\sigma\psi - \sigma\Pi_0\psi\| + \|u\partial_t(\psi - \sigma\Pi_0\psi)\| \leq Ch^{1/4}\|\psi\|$$

and

$$\|\partial_\sigma(\psi - \Pi_0\psi)\| + \|\partial_\sigma(\partial_t(\psi - \Pi_0\psi))\| \leq Ch^{1/8}\|\psi\|.$$

Proof. Using the "IMS" formula, we get:

$$q_h(\sigma\psi) = \lambda\|\sigma\psi\|^2 + h\|\psi\|^2 \leq (\nu(s_0) + C_0 h)\|\sigma\psi\|^2 + h\|\psi\|^2.$$

Using the estimates of Agmon, we find:

$$q_0(\sigma\psi) - \nu(s_0)\|\sigma\psi\|^2 \leq Ch^{1/2}\|\psi\|^2.$$

Let us analyze the estimate with ∂_σ. We take the derivative with respect to u in the eigenvalue equation:

$$\left(hD_\sigma^2 + D_t^2 + P(\tau, s_0 + h^{1/2}\sigma)\right)\partial_\sigma\psi = \lambda\partial_\sigma\psi + [P(\tau, s_0 + h^{1/2}\sigma), \partial_\sigma]\psi. \tag{9.3.2}$$

Taking the scalar product with $\partial_\sigma\psi$, we find (exercise):

$$q_h(\partial_\sigma\psi) \leq (\nu(s_0) + C_0 h)\|\partial_\sigma\psi\|^2 + Ch^{1/2}\|\psi\|^2 \tag{9.3.3}$$

and:

$$q_0(\partial_\sigma\psi) - \nu(s_0)\|\partial_\sigma\psi\|^2 \leq Ch^{1/4}\|\psi\|^2,$$

where we have used: $\|\partial_\sigma^2\psi\| \leq Ch^{-1/4}\|\psi\| + C\|\partial_\sigma\psi\|$ which is a consequence of (9.3.3) and $\|\partial_\sigma\psi\| \leq C\|\psi\|$ which comes from (9.3.1). $\square$

We can now use our approximation results to reduce the investigation to a model operator in dimension one.

4. Accurate lower bound

For all $N \geq 1$, let us consider the L^2-normalized eigenpairs $(\lambda_n(h), \psi_{n,h})_{1\leq n\leq N}$ such that $\langle\psi_{n,h}, \psi_{m,h}\rangle = 0$ when $n \neq m$. We consider the N dimensional space defined by:

$$\mathfrak{E}_N(h) = \underset{1\leq n\leq N}{\mathsf{span}}\ \psi_{n,h}.$$

It is rather easy to observe that, for $\psi \in \mathfrak{E}_N(h)$:

$$\mathcal{Q}_h(\psi) \leq \lambda_N(h)\|\psi\|^2.$$

We are going to prove a lower bound of q_h on $\mathfrak{E}_N(h)$. We notice that:

$$\mathcal{Q}_h(\psi) \geq \int h|\partial_\sigma\psi|^2 + \nu(s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We have:

$$\int h|\partial_\sigma\psi|^2 + \nu(s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t = \int_{|uh^{1/2}|\leq\varepsilon_0} h|\partial_\sigma\psi|^2 + \nu(s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$+ \int_{|\sigma h^{1/2}|\geq\varepsilon_0} h|\partial_\sigma\psi|^2 + \nu(s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

With the Taylor formula, we can write:

$$\int_{|\sigma h^{1/2}|\leq\varepsilon_0} h|\partial_\sigma\psi|^2 + \nu(s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau \geq$$
$$\int_{|\sigma h^{1/2}|\leq\varepsilon_0} h|\partial_\sigma\psi|^2 + \nu(s_0) + h\frac{\nu''(s_0)}{2}\sigma^2|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau - Ch^{3/2}\int_{|\sigma h^{1/2}|\leq\varepsilon_0} |\sigma|^3|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

The estimates of Agmon give:

$$\int_{|\sigma h^{1/2}|\leq\varepsilon_0} h|\partial_\sigma\psi|^2 + \nu(s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}t$$
$$\geq \int_{|\sigma h^{1/2}|\leq\varepsilon_0} h|\partial_\sigma\psi|^2 + \nu(s_0)|\psi|^2 + h\frac{\nu''(s_0)}{2}\sigma^2|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau - Ch^{3/2}\|\psi\|^2.$$

Moreover, we have:

$$\int_{|\sigma h^{1/2}|\geq\varepsilon_0} h|\partial_\sigma\psi|^2 + \nu(s_0 + h^{1/2}\sigma)|\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau \geq (\nu(s_0) + \eta_0)\int_{|\sigma h^{1/2}|\geq\varepsilon_0} |\psi|^2 \,\mathrm{d}\sigma\,\mathrm{d}\tau$$
$$= O(h^\infty)\|\psi\|^2.$$

We observe that:

$$\int_{|\sigma h^{1/2}|\geq\varepsilon_0} h|\partial_\sigma\psi|^2+\nu(s_0)|\psi|^2+h\frac{\nu''(s_0)}{2}\sigma^2|\psi|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau=O(h^\infty)\|\psi\|^2.$$

It follows that:

$$\mathcal{Q}_h(\psi)\geq\int_{\mathbb{R}\times\Omega} h|\partial_\sigma\psi|^2+\nu(s_0)|\psi|^2+h\frac{\nu''(s_0)}{2}\sigma^2|\psi|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau-Ch^{3/2}\|\psi\|^2.$$

We can now use the approximation result and we infer (exercise):

$$\lambda_N(h)\|\psi\|^2\geq\mathcal{Q}_h(\psi)\geq\nu(s_0)\|\psi\|^2+\int_{\mathbb{R}\times\Omega} h|\partial_\sigma\Pi_0\psi|^2+h\frac{\nu''(s_0)}{2}\sigma^2|\Pi_0\psi|^2\,\mathrm{d}\sigma\,\mathrm{d}\tau+o(h)\|\psi\|^2.$$

This becomes:

$$\int_{\mathbb{R}} h|\partial_\sigma\langle\psi,v_{z_0}\rangle|^2+h\frac{\nu''(s_0)}{2}\sigma^2|\langle\psi,v_{z_0}\rangle|^2\,\mathrm{d}\sigma\leq(\lambda_N(h)-\nu(s_0)+o(h))\|\langle\psi,u_{s_0}\rangle\|^2_{\mathsf{L}^2(\mathbb{R}_\sigma)}.$$

By the min-max principle, we deduce:

$$\lambda_N(h)\geq\nu(s_0)+(2N-1)h\left(\frac{\nu''(s_0)}{2}\right)^{1/2}+o(h).$$

4.1. Examples. Let us now give examples which can be treated as exercises.

4.1.1. *Lu-Pan/de Gennes operator.* Our first example (which comes from [**16**] and [**153**]) is the Neumann realization of the operator acting on $\mathsf{L}^2(\mathbb{R}^2_+,\,\mathrm{d}\zeta\,\mathrm{d}\tau)$:

$$h^2D_\zeta^2+D_\tau^2+(\zeta-\tau)^2,$$

where $\mathbb{R}^2_+=\{(\zeta,\tau)\in\mathbb{R}^2:\tau>0\}$.

4.1.2. *Montgomery operator.* The second example (which is the core of [**51**]) is the self-adjoint realization on $\mathsf{L}^2(\,\mathrm{d}\zeta\,\mathrm{d}\tau)$ of:

$$h^2D_\zeta^2+D_\tau^2+\left(\zeta-\frac{\tau^2}{2}\right)^2.$$

4.1.3. *Popoff operator.* Our last example (which comes from [**147**]) corresponds to the Neumann realization on $\mathsf{L}^2(\mathcal{E}_\alpha,\,\mathrm{d}\zeta\,\mathrm{d}z\,\mathrm{d}\tau)$ of:

$$h^2D_\zeta^2+D_\tau^2+D_z^2+(\zeta-\tau)^2.$$

5. A non example

Let us now simultaneously prove Theorems 2.30 and 2.32 related to the δ-interactions.

5.1. Double δ-well. For $x\geq 0$, we introduce the quadratic form $\mathfrak{q}_x$ defined for $\psi\in H^1(\mathbb{R})$ by

$$\mathfrak{q}_x(\psi)=\int_{\mathbb{R}}|\psi'(y)|^2\,\mathrm{d}y-|\psi(-x)|^2-|\psi(x)|^2. \tag{9.5.1}$$

This is standard (see [**4**, Chapter II.2] and also [**27**]) that $\mathfrak{q}_x$ is a semi-bounded and closed quadratic form on $H^1(\mathbb{R})$. Therefore we may introduce the associated self-adjoint operator denoted by $\mathfrak{D}_x$ whose domain is

$$\mathsf{Dom}\,(\mathfrak{D}_x) = \left\{\psi \in H^1(\mathbb{R}) \cap H^2(\mathbb{R} \setminus \{\pm x\}) : \psi(\pm x^+) - \psi(\pm x^-) = -\psi(\pm x)\right\}$$

and defined as $\mathfrak{D}_x \psi(y) = -\psi''(y)$. We can write formally

$$\mathfrak{D}_x = D_y^2 - \delta_{-x} - \delta_x.$$

Let us describe the spectrum of $\mathfrak{D}_x$. The following lemma is obvious.

Lemma 9.7. *For all $x \geq 0$, the essential spectrum of $\mathfrak{D}_x$ is given by*

$$\mathsf{sp}_{\mathsf{ess}}(\mathfrak{D}_x) = [0, +\infty).$$

Notation 9.8. *For $x \geq 0$, we denote by $\mu_1(x)$ the lowest eigenvalue of $\mathfrak{D}_x$ and by u_x the corresponding positive and L^2-normalized eigenfunction.*

In fact we can give an explicit expression of the pair $(\mu_1(x), u_x)$. The following proposition is left as an exercise for the reader.

Proposition 9.9. *For $x \geq 0$, we have*

$$\mu_1(x) = -\left(\frac{1}{2} + \frac{1}{2x} W(xe^{-x})\right)^2.$$

The second eigenvalue $\mu_2(x)$ only exists for $x > 1$ and is given by

$$\mu_2(x) = -\left(\frac{1}{2} + \frac{1}{2x} W(-xe^{-x})\right)^2.$$

By convention we set $\mu_2(x) = 0$ when $x \leq 1$. In particular we have the following properties:

(1) $\mu_1(x) \underset{x \to 0}{=} -1 + 2x + O(x^2)$,
(2) $\mu_1(x) \underset{x \to +\infty}{=} -\frac{1}{4} - \frac{e^{-x}}{2} + O(xe^{-2x})$, $\mu_2(x) \underset{x \to +\infty}{=} -\frac{1}{4} + \frac{e^{-x}}{2} + O(xe^{-2x})$,
(3) For all $x \geq 0$, $-1 \leq \mu_1(x) < -\frac{1}{4}$ and for all $x > 1$, $\mu_2(x) > -\frac{1}{4}$,
(4) μ_1 admits a unique minimum at 0,
(5) For all $x \geq 0$ and all $\psi \in H^1(\mathbb{R})$, we have $\mathfrak{q}_x(\psi) \geq -\|\psi\|^2$,
(6) $R(x) := \|\partial_x u_x\|^2_{L^2(\mathbb{R}_y)}$ defines a bounded function for $x > 0$.
(7) $\|\partial_y u_x\|^2_{L^2(\mathbb{R}_y)}$ defines a bounded function for $x \geq 0$.

5.2. Dimensional reduction. Let us introduce the following extension of u_x.

Notation 9.10. *Let us define*

$$\tilde{u}_x(y) = \begin{cases} u_x(y) \text{ if } x \geq 0 \\ u_0(y) \text{ if } x < 0 \end{cases}.$$

We also introduce the projections defined for $\psi \in L^2(\mathbb{R}^2)$ by

$$\Pi_x\psi(x,y) = \langle \psi, \tilde{u}_x\rangle_{L^2(\mathbb{R}_y)}\tilde{u}_x(y), \quad \Pi_x^\perp\psi(x,y) = \psi(x,y) - \Pi_x\psi(x,y).$$

The following proposition establishes the spectral reduction to dimension one.

Proposition 9.11. *For all $f \in H^1(\mathbb{R})$, we let*

$$\mathfrak{Q}_h^{\mathsf{mod1}}(f) = \int_{\mathbb{R}} h^2|f'(x)|^2 + \hat{\mu}_1(x)|f(x)|^2\,\mathrm{d}x,$$

$$\mathfrak{Q}_h^{\mathsf{mod2}}(f) = \int_{\mathbb{R}} h^2|f'(x)|^2 + \tilde{\mu}_1(x)|f(x)|^2\,\mathrm{d}x,$$

and we denote by $\mathfrak{H}_h^{\mathsf{mod}j}$ the corresponding Friedrichs extensions. Set $M' > M$, where we denote

$$M = \sup_{x>0} R(x) = \sup_{x>0} \|\partial_x u_x\|^2_{L^2(\mathbb{R}_y)},$$

bounded by Proposition 9.9. Then there exists $M_0, h_0 > 0$ such that for all $h \in (0,h_0)$ and all $C_h \geq M_0 h$:

$$\mathcal{N}\left(\mathfrak{H}_h^{\mathsf{mod1}}, -\frac{1}{4} - C_h - h^2M\right) \leq \mathcal{N}\left(\mathfrak{H}_h, -\frac{1}{4} - C_h\right) \leq \mathcal{N}\left(\mathfrak{H}_h^{\mathsf{mod2}}, \frac{-\frac{1}{4} - C_h}{1-h} + (4M'+1)h\right)$$

and

$$(1-h)\left\{\lambda_n^{\mathsf{mod2}}(h) - (4M'+1)h\right\} \leq \lambda_n(h) \leq \lambda_n^{\mathsf{mod1}}(h) + h^2M.$$

In the next lines we only sketch the main steps of the proof. The lower bound is essentially a consequence of the following lemma.

Lemma 9.12. *For all $\psi \in \mathsf{Dom}\,(\mathfrak{Q}_h)$, the function $\Pi_x\psi$ belongs to $\mathsf{Dom}\,(\mathfrak{Q}_h)$ and we have*

$$\mathfrak{Q}_h(\Pi_x\psi) = \int_{\mathbb{R}_x} h^2|f'(x)|^2 + (\hat{\mu}_1(x) + h^2\tilde{R}(x))|f(x)|^2\,\mathrm{d}x, \qquad \text{with } f(x) = \langle\psi, \tilde{u}_x\rangle_{L^2(\mathbb{R}_y)},$$

where $\hat{\mu}_1(x) = \mu_1(x)$ for $x \geq 0$ and $\hat{\mu}_1(x) = 1$ for $x < 0$ and $\tilde{R}(x) = R(x)$ for $x > 0$ and $\tilde{R}(x) = 0$ for $x \leq 0$.

The proof of the upper bound is slightly more difficult and is a consequence of the following two propositions based on the orthogonal decomposition with respect to $\tilde{u}_x$.

Proposition 9.13. *For all $\psi \in \mathsf{Dom}\,(\mathfrak{Q}_h)$ and all $\varepsilon \in (0,1)$, we have*

$$\mathfrak{Q}_h(\psi) \geq \int_{\mathbb{R}_x} (1-\varepsilon)h^2|f'(x)|^2 + \big(\tilde{\mu}_1(x) - 4\varepsilon^{-1}h^2\tilde{R}(x)\big)|f(x)|^2\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}_x} (1-\varepsilon)h^2\|\partial_x\Pi_x^\perp\psi\|^2 + \big(\tilde{\mu}_2(x) - 4\varepsilon^{-1}h^2\tilde{R}(x)\big)\|\Pi_x^\perp\psi\|^2_{L^2(\mathbb{R}_y)}\,\mathrm{d}x,$$

where $\tilde{\mu}_i(x) = \mu_i(x)$ for $x \geq 0$ and $\tilde{\mu}_i(x) = 0$ for $x < 0$ ($i \in \{1,2\}$); $\tilde{R}(x) = R(x)$ for $x > 0$ and $\tilde{R}(x) = 0$ for $x \leq 0$.

Proposition 9.14. *Let us consider the following quadratic form, defined on the product $H^1(\mathbb{R})\times H^1(\mathbb{R}^2)$, by*

$$\mathfrak{Q}_h^{\mathsf{tens}}(f,\varphi)=$$
$$\int_{\mathbb{R}_x}(1-h)h^2|f'(x)|^2+\big(\tilde\mu_1(x)-4Mh\big)|f(x)|^2\,\mathrm{d}x+\int_{\mathbb{R}^2}(1-h)h^2|\partial_x\varphi|^2+\big(\tilde\mu_2(x)-4Mh\big)|\varphi|^2\,\mathrm{d}x\,\mathrm{d}y,$$
$$\forall (f,\varphi)\in H^1(\mathbb{R})\times H^1(\mathbb{R}^2).$$

If $\mathfrak{H}_h^{\mathsf{tens}}$ denotes the associated operator, then we have, for all $n\geq 1$

$$\lambda_n(h)\geq \lambda_n^{\mathsf{tens}}(h).$$

Proof. We use Proposition 9.13 with $\varepsilon=h$ and we get, for all $\psi\in\mathsf{Dom}\,(\mathfrak{Q}_h)$,

$$\mathfrak{Q}_h(\psi)\geq\int_{\mathbb{R}_x}(1-h)h^2|f'|^2+\big(\tilde\mu_1(x)-4Mh\big)|f|^2\,\mathrm{d}x$$
$$+\int_{\mathbb{R}^2}(1-h)h^2|\partial_x\Pi_x^\perp\psi|^2+\big(\tilde\mu_2(x)-4Mh\big)|\Pi_x^\perp\psi|^2\,\mathrm{d}x\,\mathrm{d}y.$$

Thus we have

$$\mathfrak{Q}_h(\psi)\geq\mathfrak{Q}_h^{\mathsf{tens}}(\langle\psi,\tilde u_x\rangle,\Pi_x^\perp\psi),\quad \|\psi\|^2=\|f\|^2+\|\Pi_x^\perp\psi\|^2. \tag{9.5.2}$$

With (9.5.2) we infer

$$\lambda_n(h)\geq\inf_{\substack{G\subset H^1(\mathbb{R}^2)\\ \dim G=n}}\sup_{\psi\in G}\frac{\mathfrak{Q}_h^{\mathsf{tens}}(\langle\psi,\tilde u_x\rangle,\Pi_x^\perp\psi)}{\|\Pi_x\psi\|^2+\|\Pi_x^\perp\psi\|^2}\,.$$

Now, we define the linear injection

$$\mathcal{J}:\left\{\begin{array}{ccc} H^1(\mathbb{R}^2) & \to & H^1(\mathbb{R})\times H^1(\mathbb{R}^2)\\ \psi & \mapsto & (\langle\psi,\tilde u_x\rangle\ ,\ \Pi_x^\perp\psi)\end{array}\right. .$$

so that we have

$$\inf_{\substack{G\subset H^1(\mathbb{R}^2)\\ \dim G=n}}\sup_{\psi\in G}\frac{\mathfrak{Q}_h^{\mathsf{tens}}(\Pi_x\psi,\Pi_x^\perp\psi)}{\|\Pi_x\psi\|^2+\|\Pi_x^\perp\psi\|^2}=\inf_{\substack{\tilde G\subset\mathcal{J}(H^1(\mathbb{R}^2))\\ \dim\tilde G=n}}\sup_{(f,\varphi)\in\tilde G}\frac{\mathfrak{Q}_h^{\mathsf{tens}}(f,\varphi)}{\|f\|^2+\|\varphi\|^2}$$

and

$$\inf_{\substack{\tilde G\subset\mathcal{J}(H^1(\mathbb{R}^2))\\ \dim\tilde G=n}}\sup_{(f,\varphi)\in\tilde G}\frac{\mathfrak{Q}_h^{\mathsf{tens}}(f,\varphi)}{\|f\|^2+\|\varphi\|^2}\geq\inf_{\substack{\tilde G\subset H^1(\mathbb{R})\times H^1(\mathbb{R}^2)\\ \dim\tilde G=n}}\sup_{(f,\varphi)\in\tilde G}\frac{\mathfrak{Q}_h^{\mathsf{tens}}(f,\varphi)}{\|f\|^2+\|\varphi\|^2}.$$

We recognize the n-th Rayleigh quotient of $\mathfrak{H}_h^{\mathsf{tens}}$ and the conclusion follows. $\square$

It remains to use Theorem 6.11 and Proposition 9.11 implies Theorem 2.30. Theorem 2.32 is a consequence of the analysis related to (4.4.9).

CHAPTER 10

Magnetic Born-Oppenheimer approximation

Pour l'achèvement de la science, il faut passer en revue une à une toutes les choses qui se rattachent à notre but par un mouvement de pensée continu et sans nulle interruption, et il faut les embrasser dans une énumération suffisante et méthodique.

Règles pour la direction de l'esprit,
Descartes

We explain in this chapter the main steps to the proof of Theorem 2.38. In particular the reader is supposed to be familiar with the basics of pseudo-differential calculus. We establish general Feynman-Hellmann formulas and we also recall the fundamental properties of coherent states.

1. Formal series

This section is devoted to the proof of the following proposition.

Proposition 10.1. *Let us assume Assumption 2.33. For all $n \geq 1$, there exist a sequence $(\gamma_{j,n})_{j\geq 0}$ such that for all $J \geq 0$ there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$:*

$$\mathrm{dist}\left(\sum_{j=0}^{J} \gamma_{j,n} h^{j/2}, \mathsf{sp}(\mathfrak{L}_h)\right) \leq C h^{(J+1)/2},$$

where:

$$\gamma_{0,n} = \mu_0, \quad \gamma_{1,n} = 0, \quad \gamma_{2,n} = \nu_n\left(\frac{1}{2}\mathsf{Hess}_{x_0,\xi_0}\,\mu(\sigma, D_\sigma)\right).$$

In order to perform the investigation we use the following rescaling:

$$s = h^{1/2}\sigma$$

so that $\mathfrak{L}_h$ becomes:

$$\mathcal{L}_h = (-i\nabla_\tau + A_2(x_0 + h^{1/2}\sigma, \tau))^2 + (\xi_0 - ih^{1/2}\nabla_\sigma + A_1(x_0 + h^{1/2}\sigma, \tau))^2. \tag{10.1.1}$$

We will also need generalizations of the Feynman-Hellmann formulas which are obtained by taking the derivative of the eigenvalue equation

$$\mathcal{M}_{x,\xi} u_{x,\xi} = \mu(x, \xi) u_{x,\xi}$$

with respect to x_j and ξ_k.

Proposition 10.2. *We have:*

$$(\mathcal{M}_{x,\xi}-\mu(x,\xi))(\partial_\eta u)_{x,\xi}=(\partial_\eta\mu(x,\xi)-\partial_\eta\mathcal{M}_{x,\xi})u_{x,\xi} \tag{10.1.2}$$

and:

$$\begin{aligned}(10.1.3)\quad &(\mathcal{M}_{x_0,\xi_0}-\mu_0)(\partial_\eta\partial_\theta u)_{x_0,\xi_0}\\ &=\partial_\eta\partial_\theta\mu(x_0,\xi_0)u_{x_0,\xi_0}-\partial_\eta\mathcal{M}_{x_0,\xi_0}(\partial_\theta u)_{x_0,\xi_0}-\partial_\theta\mathcal{M}_{x_0,\xi_0}(\partial_\eta u)_{x_0,\xi_0}-\partial_\eta\partial_\theta\mathcal{M}_{x_0,\xi_0}u_{x_0,\xi_0},\end{aligned}$$

where η and θ denote one of the x_j or ξ_k. Moreover we have

$$\partial_\eta\mu(x,\xi)=\int_{\mathbb{R}^n}\partial_\eta\mathcal{M}_{x,\xi}\,u_{x,\xi}(\tau)u_{x,\xi}(\tau)\,\mathrm{d}\tau \tag{10.1.4}$$

We can now prove Proposition 10.1. Since A_1 and A_2 are polynomials, we can write, for some $M\in\mathbb{N}$:

$$\mathcal{L}_h=\sum_{j=0}^{M}h^{j/2}\mathcal{L}_j$$

with:

$$\mathcal{L}_0=\mathcal{M}_{x_0,\xi_0},\quad \mathcal{L}_1=\sum_{j=1}^{m}(\partial_{x_j}\mathcal{M})_{x_0,\xi_0}\sigma_j+\sum_{j=1}^{m}(\partial_{\xi_j}\mathcal{M})_{x_0,\xi_0}D_{\sigma_j},$$

$$\begin{aligned}\mathcal{L}_2=\frac{1}{2}\sum_{k,j=1}^{m}\Big(&(\partial_{x_j}\partial_{x_k}\mathcal{M})_{x_0,\xi_0}\sigma_j\sigma_k+(\partial_{\xi_j}\partial_{\xi_k}\mathcal{M})_{x_0,\xi_0}D_{\sigma_j}D_{\sigma_k}+(\partial_{\xi_j}\partial_{x_k}\mathcal{M})_{x_0,\xi_0}D_{\sigma_j}\sigma_k\\ &+(\partial_{x_k}\partial_{\xi_j}\mathcal{M})_{x_0,\xi_0}\sigma_kD_{\sigma_j}\Big).\end{aligned}$$

We look for quasimodes in the form:

$$\psi\sim\sum_{j\geq0}h^{j/2}\psi_j$$

and quasi-eigenvalues in the form:

$$\gamma\sim\sum_{j\geq0}h^{j/2}\gamma_j$$

so that they solve in the sense of formal series:

$$\mathcal{L}_h\psi\sim\gamma\psi.$$

By collecting the terms of order h^0, we get the equation:

$$\mathcal{M}_{x_0,\xi_0}\psi_0=\gamma_0\psi_0.$$

This leads to take $\gamma_0=\mu_0$ and :

$$\psi_0(\sigma,\tau)=f_0(\sigma)u_0(\tau),$$

where $u_0 = u_{x_0,\xi_0}$ and f_0 is a function to be determined in the Schwartz class. By collecting the terms of order $h^{1/2}$, we find:

$$(\mathcal{M}_{x_0,\xi_0} - \mu(x_0,\xi_0))\psi_1 = (\gamma_1 - \mathcal{L}_1)\psi_0.$$

By using (10.1.2) and the Fredholm alternative (applied for σ fixed) we get $\gamma_1 = 0$ and the solution:

$$\psi_1(\sigma,\tau) = \sum_{j=1}^{m} (\partial_{x_j} u)_{x_0,\xi_0}\, \sigma_j f_0 + \sum_{j=1}^{m} (\partial_{\xi_j} u)_{x_0,\xi_0}\, D_{\sigma_j} f_0 + f_1(\sigma) u_0(\tau), \tag{10.1.5}$$

where f_1 is a function to be determined in the Schwartz class. The next equation reads:

$$(\mathcal{M}_{x_0,\xi_0} - \mu(x_0,\xi_0))\psi_2 = (\gamma_2 - \mathcal{L}_2)\psi_0 - \mathcal{L}_1\psi_1.$$

The Fredholm condition is:

$$\langle \mathcal{L}_2\psi_0 + \mathcal{L}_1\psi_1, u_0\rangle_{L^2(\mathbb{R}^n,\,\mathrm{d}\tau)} = \gamma_2 f_0. \tag{10.1.6}$$

We obtain (exercise):

$$\frac{1}{2}\mathsf{Hess}\,\mu(x_0,\xi_0)(\sigma, D_\sigma) f_0 = \gamma_2 f_0.$$

We take γ_2 in the spectrum of $\frac{1}{2}\mathsf{Hess}\,\mu(x_0,\xi_0)(\sigma,D_\sigma)$ and we choose f_0 a corresponding normalized eigenfunction. The construction can be continued at any order.

We deduce from Propositions 2.37 and 10.1:

Corollary 10.3. *For all $n \geq 1$ there exist $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$ the n-th eigenvalue of $\mathfrak{L}_h$ exists and satisfies:*

$$\lambda_n(h) \leq \mu_0 + Ch.$$

2. Rough estimates of the eigenfunctions

This section is devoted to recall the basic and rough localization and microlocalization estimates satisfied by the eigenfunctions resulting from Assumptions 2.33 and 2.35 and Corollary 10.3.

Proposition 10.4. *Let $C_0 > 0$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ,ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$ we have:*

$$\left\| e^{\varepsilon_0|\tau|}\psi\right\|^2 \leq C\|\psi\|^2, \quad \mathfrak{Q}_h\left(e^{\varepsilon_0|\tau|}\psi\right) \leq C\|\psi\|^2.$$

Proposition 10.5. *Let $C_0 > 0$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ,ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\left\| e^{\varepsilon_0|s|}\psi\right\|^2 \leq C\|\psi\|^2, \quad \mathfrak{Q}_h\left(e^{\varepsilon_0|s|}\psi\right) \leq C\|\psi\|^2.$$

We deduce from Propositions 10.4 and 10.5 the following corollary.

Corollary 10.6. *Let $C_0 > 0$ and $k, l \in \mathbb{N}$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\|\tau^k s^l \psi\| \leq C\|\psi\|, \quad \mathfrak{Q}_h(\tau^k s^l \psi) \leq C\|\psi\|^2,$$
$$\| - i\nabla_\tau s^l \tau^k \psi\| \leq C\|\psi\|^2, \quad \| - ih\nabla_s s^l \tau^k \psi\| \leq C\|\psi\|^2.$$

Taking successive derivatives of the eigenvalue equation we deduce by induction:

Corollary 10.7. *Let $C_0 > 0$ and $k, l, p \in \mathbb{N}$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$ and all $h \in (0, h_0)$, we have:*

$$\|(-i\nabla_\tau)^p s^l \tau^k \psi\| \leq C\|\psi\|^2, \quad \|(-ih\nabla_s)^p s^l \tau^k \psi\| \leq C\|\psi\|^2.$$

Using again Propositions 10.4 and 10.5 and an induction argument we get:

Proposition 10.8. *Let $k \in \mathbb{N}$. Let $\eta > 0$ and χ a smooth cutoff function being zero in a neighborhood of 0. There exists $h_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$ and all $h \in (0, h_0)$, we have:*

$$\|\chi(h^\eta s)\psi\|_{\mathsf{B}^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|, \quad \|\chi(h^\eta \tau)\psi\|_{\mathsf{B}^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|,$$

where $\| \cdot \|_{\mathsf{B}^k(\mathbb{R}^{n+m})}$ is the standard norm on:

$$\mathsf{B}^k(\mathbb{R}^{m+n}) = \{\psi \in L^2(\mathbb{R}^{m+n}) : y_j^q \partial_{y_l}^p \psi \in \mathsf{L}^2(\mathbb{R}^{n+m}), \forall j, k \in \{1, \cdots, m+n\},\ p + q \leq k\}.$$

By using a rough pseudo-differential calculus jointly with the space localization of Proposition 10.8 and standard elliptic estimates, we get:

Proposition 10.9. *Let $k \in \mathbb{N}$. Let $\eta > 0$ and χ a smooth cutoff function being zero in a neighborhood of 0. There exists $h_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\|\chi(h^\eta h D_s)\psi\|_{\mathsf{B}^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|, \quad \|\chi(h^\eta D_\tau)\psi\|_{\mathsf{B}^k(\mathbb{R}^{m+n})} \leq O(h^\infty)\|\psi\|.$$

3. Coherent states and microlocalization

3.1. A first lower bound. By using the formalism introduced in Chapter 2, Section 3.2.2, we get the following proposition.

Proposition 10.10. *There exist $h_0, C > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$ and all $h \in (0, h_0)$, we have*

$$\mathcal{Q}_h(\psi) \geq \int_{\mathbb{R}^{2m}} Q_{h,u,p}(\psi_{u,p})\,\mathrm{d}u\,\mathrm{d}p - Ch\|\psi\|^2 \geq (\mu(x_0, \xi_0) - Ch)\|\psi\|^2, \tag{10.3.1}$$

where $Q_{h,u,p}$ is the quadratic form associated with the operator $\mathcal{M}_{x_0 + h^{1/2}u, \xi_0 + h^{1/2}p}$.

Proof. We use (2.3.6). Then the terms of $\mathcal{R}_h$ (see (2.3.7)) are in the form $hh^{p/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta$ with $l + q \leq p$ and $\beta = 0, 1$. With Corollary 10.7 and the rescaling (4.1.3), we have:

$$\|h^{p/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta \psi\| \leq C\|\psi\|$$

and the conclusion follows. $\square$

3.2. Localization in the phase space. This section is devoted to elliptic regularity properties (both in space and frequency) satisfied by the eigenfunctions. We will use the generalization of the "IMS" formula given in Chapter 6, Formula (6.3.5). The following lemma is a straightforward consequence of Assumption 2.33.

Lemma 10.11. *Under Assumption 2.33, there exist $\varepsilon_0 > 0$ and $c > 0$ such that*

$$\mu(x_0 + x, \xi_0 + \xi) - \mu(x_0, \xi_0) \geq c(|x|^2 + |\xi|^2), \qquad \forall (x, \xi) \in \mathcal{B}(\varepsilon_0),$$

and

$$\mu(x_0 + x, \xi_0 + \xi) - \mu(x_0, \xi_0) \geq c, \qquad \forall (x, \xi) \in \complement\mathcal{B}(\varepsilon_0),$$

where $\mathcal{B}(\varepsilon_0) = \{(x, \xi), |x| + |\xi| \leq \varepsilon_0\}$ and $\complement\mathcal{B}(\varepsilon_0)$ is its complement.

Notation 10.12. *In what follows we will denote by $\tilde{\eta} > 0$ all the quantities which are multiples of $\eta > 0$,i.e. in the form $p\eta$ for $p \in \mathbb{N} \setminus \{0\}$. We recall that $\eta > 0$ can be chosen arbitrarily small.*

Proposition 10.13. *There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\|\sigma\psi\|^2 + \|\nabla_\sigma \psi\|^2 \leq C\|\psi\|^2.$$

Proof. Let (λ, ψ) be an eigenpair such that $\lambda \leq \mu_0 + C_0 h$. We recall that (10.3.1) holds. We have

$$\mathcal{Q}_h(\psi) = \lambda\|\psi\|^2 \leq (\mu_0 + C_0 h)\|\psi\|^2.$$

We deduce that

$$\int_{\mathbb{R}^{2m}} Q_{h,u,p}(\psi_{u,p}) - \mu_0 |\psi_{u,p}|^2 \,\mathrm{d}u \,\mathrm{d}p \leq Ch\|\psi\|^2$$

and thus by the min-max principle

$$\int_{\mathbb{R}^{2m}} \left(\mu(x_0 + h^{1/2}u, \xi_0 + h^{1/2}p) - \mu_0\right) |\psi_{u,p}|^2 \,\mathrm{d}u \,\mathrm{d}p \leq Ch\|\psi\|^2.$$

We use the $\varepsilon_0 > 0$ given in Lemma 10.11 and we split the integral into two parts. Therefore, we find:

$$\int_{\mathcal{B}(h^{-1/2}\varepsilon_0)} (|u|^2 + |p|^2)|\psi_{u,p}|^2 \,\mathrm{d}u \,\mathrm{d}p \leq C\|\psi\|^2, \tag{10.3.2}$$

$$\int_{\complement\mathcal{B}(h^{-1/2}\varepsilon_0)} |\psi_{u,p}|^2 \,\mathrm{d}u \,\mathrm{d}p \leq Ch\|\psi\|^2. \tag{10.3.3}$$

The first inequality is not enough to get the conclusion. We also need a control of momenta in the region $\complement\mathcal{B}(h^{-1/2}\varepsilon_0)$. For that purpose, we write:

$$\mathcal{Q}_h(\mathfrak{a}_j^*\psi) = \int_{\mathbb{R}^{2m}} Q_{h,u,p}\left(\frac{u_j - ip_j}{\sqrt{2}}\psi_{u,p}\right) \mathrm{d}u \,\mathrm{d}p + \langle \mathcal{R}_h \mathfrak{a}_j^*\psi, \mathfrak{a}_j^*\psi\rangle. \tag{10.3.4}$$

Up to lower order terms we must estimate terms in the form:

$$h\langle h^{p/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta \mathfrak{a}_j^*\psi, \mathfrak{a}_j^*\psi\rangle,$$

with $l+q=p$, $\alpha\in\mathbb{N}$ and $\beta=0,1$. By using the *a priori* estimates of Propositions 10.8 and 10.9, we have:

$$\|h^{p/2}\sigma^l D_\sigma^q \tau^\alpha D_\tau^\beta \mathfrak{a}_j^*\psi\| \leq Ch^{-\tilde{\eta}}\|\mathfrak{a}_j^*\psi\|.$$

The remainder is controlled by:

$$|\langle \mathcal{R}_h \mathfrak{a}_j^*\psi, \mathfrak{a}_j^*\psi\rangle| \leq Ch^{1-\tilde{\eta}}(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2).$$

Then we analyze $\mathcal{Q}_h(\mathfrak{a}_j^*\psi)$ by using (6.3.5) (Chapter 6) with $\mathfrak{A}=\mathfrak{a}_j$. We need to estimate the different remainder terms. We notice that:

$$\begin{aligned}
\|[\mathfrak{a}_j^*, P_{k,r,h}]\psi\| &\leq Ch^{1/2}\|\psi\|,\\
|\langle P_{k,r,h}\psi, \mathfrak{a}_j^*[P_{k,r,h},\mathfrak{a}_j]\psi\rangle| &\leq \|P_{k,r,h}\psi\|\ \|\mathfrak{a}_j^*[P_{k,r,h},\mathfrak{a}_j]\psi\|,\\
|\langle P_{k,r,h}\psi, \mathfrak{a}_j[P_{k,r,h},\mathfrak{a}_j^*]\psi\rangle| &\leq \|P_{k,r,h}\psi\|\ \|\mathfrak{a}_j[P_{k,r,h},\mathfrak{a}_j^*]\psi\|,\\
|\langle P_{k,r,h}\psi, [[P_{k,r,h},\mathfrak{a}_j],\mathfrak{a}_j^*]\psi\rangle| &\leq \|P_{k,r,h}\psi\|\ \|[[P_{k,r,h},\mathfrak{a}_j],\mathfrak{a}_j^*]\psi\|,
\end{aligned}$$

where $P_{1,r,h}$ denotes the magnetic momentum $h^{1/2}D_{\sigma_r}+A_{1,r}(x_0+h^{1/2}\sigma,\tau)$ and $P_{2,r,h}$ denotes $D_{\tau_r}+A_{2,r}(x_0+h^{1/2}\sigma,\tau)$. We have:

$$\|P_{k,r,h}\psi\|\leq C\|\psi\|$$

and:

$$\|\mathfrak{a}_j^*[P_{k,r,h},\mathfrak{a}_j]\psi\| \leq Ch^{1/2}\|\mathfrak{a}_j^* Q(h^{1/2}\sigma,\tau)\psi\|,$$

where Q is polynomial. The other terms can be bounded in the same way. We apply the estimates of Propositions 10.8 and 10.9 to get:

$$\|\mathfrak{a}_j^* Q(h^{1/2}\sigma,\tau)\psi\| \leq Ch^{-\tilde{\eta}}\|\mathfrak{a}_j^*\psi\|.$$

We have:

$$\mathcal{Q}_h(\mathfrak{a}_j^*\psi)=\lambda\|\mathfrak{a}_j^*\psi\|^2+O(h)\|\psi\|^2+O(h^{\frac{1}{2}-\tilde{\eta}})(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2).$$

so that:

$$\mathcal{Q}_h(\mathfrak{a}_j^*\psi)\leq \mu(x_0,\xi_0)\|\mathfrak{a}_j^*\psi\|^2+Ch\|\psi\|^2+O(h^{\frac{1}{2}-\tilde{\eta}})(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2).$$

By using (10.3.4) and splitting again the integral into two parts, it follows:

$$\begin{aligned}
\int_{\mathcal{B}(h^{-1/2}\varepsilon_0)}(|u|^2+|p|^2)|(u_j-ip_j)\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p &\leq C\|\psi\|^2+Ch^{-\frac{1}{2}-\tilde{\eta}}(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2),\\
\int_{\complement\mathcal{B}(h^{-1/2}\varepsilon_0)}|(u_j-ip_j)\psi_{u,p}|^2\,\mathrm{d}u\,\mathrm{d}p &\leq Ch\|\psi\|^2+Ch^{\frac{1}{2}-\tilde{\eta}}(\|\nabla_\sigma\psi\|^2+\|\sigma\psi\|^2).
\end{aligned}$$

Combining the last inequality with the first one of (10.3.2) and the Parseval formula we get the conclusion.

$\square$

By using the same ideas, we can establish the following proposition.

Proposition 10.14. *Let $P \in \mathbb{C}_2[X_1, \ldots, X_{2n}]$. There exist $h_0, C, \varepsilon_0 > 0$ such that for all eigenpairs (λ, ψ) of $\mathfrak{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\|P(\sigma, D_\sigma)\psi\|^2 \leq Ch^{-\frac{1}{2}-\tilde{\eta}}\|\psi\|^2.$$

3.3. Approximation lemmas. We can prove a first approximation.

Proposition 10.15. *There exist $h_0, C > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\|\psi - \Pi_0\psi\| \leq Ch^{1/2-\tilde{\eta}}\|\psi\|$$

Proof. We can write:

$$(\mathcal{L}_0 - \mu_0)\psi = (\lambda - \mu_0)\psi - h^{1/2}\mathcal{L}_1\psi - h\mathcal{L}_2\psi + \ldots - h^{M/2}\mathcal{L}_M\psi.$$

By using the rough microlocalization given in Propositions 10.8 and 10.9 and Proposition 10.14, we infer that for $p \geq 2$:

$$h^{p/2}\|\tau^\alpha D_\tau^\beta \sigma^l D_\sigma^q \psi\| \leq Ch^{\frac{p}{2}-\frac{p-2}{2}-\frac{1}{4}-\tilde{\eta}}\|\psi\| = Ch^{\frac{3}{4}-\tilde{\eta}}\|\psi\|, \tag{10.3.5}$$

and thanks to Proposition 10.13:

$$\|\mathcal{L}_1\psi\| \leq Ch^{-\tilde{\eta}}\|\psi\|,$$

so that:

$$\|(\mathcal{L}_0 - \mu_0)\psi\| \leq Ch^{\frac{1}{2}-\tilde{\eta}}\|\psi\|,$$

and the conclusion follows. □

Corollary 10.16. *There exist $h_0, C > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\|\sigma(\psi - \Pi_0\psi)\| \leq Ch^{1/4-\tilde{\eta}}\|\psi\|, \quad \|D_\sigma(\psi - \Pi_0\psi)\| \leq Ch^{1/4-\tilde{\eta}}\|\psi\|$$

We can now estimate $\psi - \Pi_h\psi$.

Proposition 10.17. *There exist $h_0, C > 0$ such that for all eigenpairs (λ, ψ) of $\mathcal{L}_h$ such that $\lambda \leq \mu_0 + C_0 h$, we have:*

$$\|\psi - \Pi_h\psi\| \leq Ch^{3/4-\tilde{\eta}}\|\psi\|.$$

Proof. Let us write:

$$\mathcal{L}_h\psi = \lambda\psi.$$

We have:

$$(\mathcal{L}_0 + h^{1/2}\mathcal{L}_1)\psi = (\mu_0 + O(h))\psi - h\mathcal{L}_2\psi - \ldots - h^{M/2}\mathcal{L}_M\psi.$$

Let us notice that, as in (10.3.5), for $p \geq 2$:

$$h^{p/2}\|\mathcal{L}_p\psi\| \leq Ch^{\frac{3}{4}-\tilde{\eta}}\|\psi\|.$$

We get:

$$(\mathcal{L}_0 - \mu_0)R_h = -h^{1/2}\mathcal{L}_1(\psi - \Psi_0) + O(h)\psi - h\mathcal{L}_2\psi - \ldots - h^{M/2}\mathcal{L}_M\psi$$

It remains to apply Corollary 10.16 to get:

$$h^{1/2}\|\mathcal{L}_1(\psi - \Psi_0)\| \leq \tilde{C}h^{\frac{3}{4}-\tilde{\eta}}\|\psi\|.$$

□

Let us introduce a subspace of dimension $P \geq 1$. For $j \in \{1, \cdots, P\}$ we can consider a L^2-normalized eigenfunction of $\mathcal{L}_h$ denoted by $\psi_{j,h}$ and so that the family $(\psi_{j,h})_{j\in\{1,\cdots,P\}}$ is orthogonal. We let:

$$\mathcal{E}_P(h) = \underset{j\in\{1,\cdots,P\}}{\mathsf{span}} \psi_{j,h}.$$

Remark 10.18. *We can extend all the local and microlocal estimates as well as our approximations to $\psi \in \mathcal{E}_P(h)$.*

Then we can prove a lower bound for the quadratic form on $\mathcal{E}_P(h)$ by replacing $\psi \in \mathcal{E}_P(h)$ by $\Pi_h\psi$, in the spirit of Chapter 9.

CHAPTER 11

Examples of magnetic WKB constructions

A single soul dwelling in two bodies.
Aristotle

1. Vanishing magnetic fields

This section in devoted to the proof of Theorem 2.40. The fundamental ingredients to succeed are a normal form procedure, an operator valued WKB construction and a complex extension of the standard model operators.

Lemma 11.1. *For $r > 0$, let us consider a holomorphic function $\nu : D(0, r) \to \mathbb{C}$ such that $\nu(0) = \nu'(0) = 0$ and $\nu''(0) \in \mathbb{R}_+$. Let us also introduce a smooth F defined in a real neighborhood of $\sigma = 0$ such that $\sigma = 0$ is a non degenerate maximum. Then, there exists a neighborhood of $\sigma = 0$ such that the equation*

$$\nu(i\varphi(\sigma)) = F(\sigma) \tag{11.1.1}$$

admits a smooth solution φ solution such that $\varphi(0) = 0$ and $\varphi'(0) > 0$.

Proof. We can apply the Morse lemma to deduce that (11.1.1) is equivalent to

$$\tilde{\nu}(i\varphi(\sigma))^2 = -f(\sigma)^2,$$

where f is a non negative function such that $f'(0) = \sqrt{-\frac{F''(0)}{2}}$ and $F(\sigma) = -f(\sigma)^2$ and $\tilde{\nu}$ is a holomorphic function in a neighborhood of 0 such that $\tilde{\nu}^2 = \nu$ and $\tilde{\nu}'(0) = \sqrt{\frac{\nu''(0)}{2}}$. This provides the equations

$$\tilde{\nu}(i\varphi(\sigma)) = if(\sigma), \quad \tilde{\nu}(i\varphi(\sigma)) = -if(\sigma).$$

Since $\tilde{\nu}$ is a local biholomorphism and $f(0) = 0$, we can write the equivalent equations

$$\varphi(\sigma) = -i\tilde{\nu}^{-1}(if(\sigma)), \quad \varphi(\sigma) = -i\tilde{\nu}^{-1}(-if(\sigma)).$$

The function $\varphi(s) = -i\tilde{\nu}^{-1}(if(s))$ satisfies our requirements since $\varphi'(0) = \sqrt{-\frac{F''(0)}{\nu''(0)}}$. □

1.1. Renormalization. We use the canonical transformation associated with the change of variables:

$$t = (\gamma(\sigma))^{-\frac{1}{k+2}}\tau, \quad s = \sigma, \tag{11.1.2}$$

we deduce that $\mathfrak{L}_h^{[k]}$ is unitarily equivalent to the operator on $L^2(\,\mathrm{d}\sigma\,\mathrm{d}\tau)$:

$$\mathfrak{L}_h^{[k],\mathsf{new}} = \gamma(\sigma)^{\frac{2}{k+2}} D_\tau^2 + \left(hD_\sigma - \gamma(\sigma)^{\frac{1}{k+2}} \frac{\tau^{k+1}}{k+1} + \frac{h}{2(k+2)} \frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau + D_\tau \tau)\right)^2 .$$

We may change the gauge

$$\begin{aligned} & e^{-ig(\sigma)/h}\mathfrak{L}_h^{[k],\mathsf{new}} e^{ig(\sigma)/h} \\ &= \gamma(\sigma)^{\frac{2}{k+2}} D_\tau^2 + \left(hD_\sigma + \zeta_0^{[k]}\gamma(\sigma)^{\frac{1}{k+2}} - \gamma(\sigma)^{\frac{1}{k+2}} \frac{\tau^{k+1}}{k+1} + \frac{h}{2(k+2)} \frac{\gamma'(\sigma)}{\gamma(\sigma)}(\tau D_\tau + D_\tau \tau)\right)^2 . \end{aligned}$$

with

$$g(\sigma) = \zeta_0^{[k]} \int_0^\sigma \gamma(\tilde\sigma)^{\frac{1}{k+2}}\,\mathrm{d}\tilde\sigma.$$

For some function $\Phi = \Phi(\sigma)$ to be determined, we consider

$$\mathfrak{L}_h^{[k],\mathsf{wgt}} = e^{\Phi/h} e^{-ig(\sigma)/h}\mathfrak{L}_h^{[k],\mathsf{new}} e^{ig(\sigma)/h} e^{-\Phi/h} = \mathfrak{L}^{[k],\mathsf{wgt},0} + h\mathfrak{L}^{[k],\mathsf{wgt},1} + h^2\mathfrak{L}^{[k],\mathsf{wgt},2},$$

with

$$\begin{aligned} \mathfrak{L}^{[k],\mathsf{wgt},0} &= \gamma(\sigma)^{\frac{2}{k+2}} \mathfrak{L}_{w(\sigma)}^{[k]}, \\ \mathfrak{L}^{[k],\mathsf{wgt},1} &= \tfrac{1}{2}\left(\gamma(\sigma)^{\frac{1}{k+2}}\partial_\zeta \mathfrak{L}_\zeta^{[k]} D_\sigma + D_\sigma \gamma(\sigma)^{\frac{1}{k+2}}\partial_\zeta \mathfrak{L}_\zeta^{[k]}\right) + \mathfrak{R}_1(\sigma,\tau;D_\tau), \\ \mathfrak{L}^{[k],\mathsf{wgt},2} &= D_\sigma^2 + \mathfrak{R}_2(\sigma,\tau;D_\sigma,D_\tau), \end{aligned}$$

where

$$w(\sigma) = \zeta_0^{[k]} + i\gamma(\sigma)^{-\frac{1}{k+2}}\Phi'.$$

and where the $\mathfrak{R}_1(\sigma,\tau;D_\tau)$ is of order zero in D_σ and cancels for $\sigma = 0$ whereas $\mathfrak{R}_2(\sigma,\tau;D_\sigma,D_\tau)$ is of order one with respect to D_σ.

Now, let us try to solve, as usual, the eigenvalue equation

$$\mathfrak{L}_h^{[k],\mathsf{wgt}} a = \lambda a$$

in the sense of formal series in h:

$$a \sim \sum_{j\geq 0} h^j a_j, \quad \lambda \sim \sum_{j\geq 0} h^j \lambda_j.$$

1.2. Solving the operator valued eikonal equation. The first equation is

$$\mathfrak{L}^{[k],\mathsf{wgt},0} a_0 = \lambda_0 a_0.$$

We must choose

$$\lambda_0 = \gamma_0^{\frac{2}{k+2}} \nu_1(\zeta_0^{[k]})$$

and we are led to take

$$a_0(\sigma,\tau) = f_0(\sigma) u_{w(\sigma)}^{[k]}(\tau) \tag{11.1.3}$$

so that the equation becomes

$$\nu_1^{[k]}(w(\sigma)) - \nu_1(\zeta_0^{[k]}) = \left(\gamma_0^{\frac{2}{k+2}}\gamma(\sigma)^{-\frac{2}{k+2}} - 1\right)\nu_1^{[k]}(\zeta_0^{[k]}).$$

Therefore we are in the framework of Lemma 11.1. We use the lemma with $F(\sigma) = \left(\gamma_0^{\frac{2}{k+2}}\gamma(\sigma)^{-\frac{2}{k+2}} - 1\right)\nu_1^{[k]}(\zeta_0^{[k]})$ and, for the function φ given by the lemma, we have

$$\Phi'(\sigma) = \gamma(\sigma)^{\frac{1}{k+2}}\varphi(\sigma)$$

and we take

$$\Phi(\sigma) = \int_0^\sigma \gamma(\tilde{\sigma})^{\frac{1}{k+2}}\varphi(\tilde{\sigma})\,\mathrm{d}\tilde{\sigma},$$

which is defined in a fixed neighborhood of 0 and satisfies $\Phi(0) = \Phi'(0) = 0$ and

$$\Phi''(0) = \gamma_0^{\frac{1}{k+2}}\sqrt{\frac{2}{k+2}\frac{\gamma''(0)\nu_1^{[k]}(\zeta_0^{[k]})}{\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\gamma(0)}} > 0. \tag{11.1.4}$$

Therefore (11.1.3) is well defined in a neighborhood of $\sigma = 0$.

1.3. Solving the transport equation. We can now deal with the operator valued transport equation

$$(\mathfrak{L}^{[k],\mathsf{wgt},0} - \lambda_0)a_1 = (\lambda_1 - \mathfrak{L}^{[k],\mathsf{wgt},1})a_0.$$

For each σ the Fredholm condition is

$$\left\langle(\lambda_1 - \mathfrak{L}^{[k],\mathsf{wgt},1})a_0, \overline{u_{w(\sigma)}^{[k]}}\right\rangle_{\mathsf{L}^2(\mathbb{R}_\tau)} = 0,$$

where the complex conjugation is needed since $\mathfrak{L}^{[k],\mathsf{wgt},1}$ is not necessarily self-adjoint. Let us examine

$$\left\langle\mathfrak{L}^{[k],\mathsf{wgt},1}a_0, \overline{u_{w(\sigma)}^{[k]}}\right\rangle_{\mathsf{L}^2(\mathbb{R}_\tau)}.$$

We recall the Feynman-Hellmann formula

$$\frac{1}{2}\left(\nu_1^{[k]}\right)'(\zeta) = \int_{\mathbb{R}}\left(\zeta - \frac{\tau^{k+1}}{k+1}\right)u_\zeta^{[k]}u_\zeta^{[k]}\,\mathrm{d}\tau,$$

and the formula

$$\int_{\mathbb{R}} u_\zeta^{[k]}u_\zeta^{[k]}\,\mathrm{d}\tau = 1$$

which are valid for $\zeta \in \mathbb{C}$ close to $\zeta_0^{[k]}$ by holomorphic extension of the formulas valid for $\zeta \in \mathbb{R}$. We get an equation in the form

$$\begin{aligned}&\left\langle\mathfrak{L}^{[k],\mathsf{wgt},1}a_0, \overline{u_{w(\sigma)}^{[k]}}\right\rangle_{\mathsf{L}^2(\mathbb{R}_\tau)}\\ &\qquad= \tfrac{1}{2}\left\{\gamma(\sigma)^{\frac{1}{k+2}}\left(\nu_1^{[k]}\right)'(w(\sigma))D_\sigma + D_\sigma\gamma(\sigma)^{\frac{1}{k+2}}\left(\nu_1^{[k]}\right)'(w(\sigma))\right\}a_0 + R^{[k]}(\sigma)a_0,\end{aligned}$$

where $R^{[k]}$ is smooth an vanishes at $\sigma = 0$. Thus we are reduced to solve the transport equation

$$\frac{1}{2}\left\{\gamma(\sigma)^{\frac{1}{k+2}}\left(\nu_1^{[k]}\right)'(w(\sigma))D_\sigma + D_\sigma\gamma(\sigma)^{\frac{1}{k+2}}\left(\nu_1^{[k]}\right)'(w(\sigma))\right\}a_0 \;+\; R^{[k]}(\sigma)a_0 \;=\; \lambda_1 a_0.$$

The only point that we should verify is that the linearized transport equation near $\sigma = 0$ is indeed a transport equation in the sense of [**49**, Chapter 3] so that we have just to consider the linearization of the first part of the equation. The linearized operator is

$$\frac{\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\Phi''(0)}{2}(\sigma\partial_\sigma + \partial_\sigma\sigma).$$

The eigenvalues of this operator are

$$\left\{\frac{\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\Phi''(0)}{2}(2j+1), \quad j \in \mathbb{N}\right\}. \tag{11.1.5}$$

Let us notice that

$$\frac{\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})\Phi''(0)}{2} = \frac{\gamma_0^{\frac{1}{k+2}}}{2}\sqrt{\frac{2}{k+2}\frac{\gamma''(0)\nu_1^{[k]}(\zeta_0^{[k]})\left(\nu_1^{[k]}\right)''(\zeta_0^{[k]})}{\gamma(0)}}.$$

This is exactly the expected expression for the second term in the asymptotic expansion of the eigenvalues (see Theorem 2.38). Therefore λ_1 has to be chosen in the set (11.1.5), the transport equation can be solved in a neighborhood of $\sigma = 0$ and the construction can be continued at any order (see [**49**, Chapter 3]). Since the first eigenvalues are simple, the spectral theorem implies that the constructed functions $f_0(\sigma)u^{[k]}_{\zeta_0^{[k]}+i\gamma(\sigma)^{-\frac{1}{k+2}}\Phi'}(\tau)e^{-\frac{\Phi(\sigma)}{h}}$ are approximations of the true eigenfunctions of $e^{-ig(\sigma)}\mathfrak{L}_h^{[k],\mathsf{new}}e^{ig(\sigma)}$. This is the content of Theorem 2.40.

2. Along a varying egde

2.1. Normal form. We introduce the change of variables

$$\check{s} = s, \quad \check{t} = t, \quad \check{z} = \mathcal{T}(s)^{-1}\mathcal{T}(0)z$$

and we let:

$$\check{\nabla}_h = \begin{pmatrix} hD_{\check{s}} \\ hD_{\check{t}} \\ h\mathcal{T}(\check{s})^{-1}\mathcal{T}(0)D_{\check{z}} \end{pmatrix} + \begin{pmatrix} -\check{t} - h\frac{\mathcal{T}'}{2\mathcal{T}}(\check{z}D_{\check{z}} + D_{\check{z}}\check{z}) \\ 0 \\ 0 \end{pmatrix}.$$

The operator $\mathfrak{L}^{\mathsf{e}}_{h,\mathbf{A}}$ is unitarily equivalent to the operator, on $L^2(\mathcal{W}_{\alpha_0}, \mathrm{d}\check{s}\,\mathrm{d}\check{t}\,\mathrm{d}\check{z})$, $\check{\mathfrak{L}}^{\mathsf{e}}_h$ defined by

$$\check{\mathfrak{L}}^{\mathsf{e}}_h = \left(hD_{\check{s}} - \check{t} - h\frac{\mathcal{T}'}{2\mathcal{T}}(\check{z}D_{\check{z}} + D_{\check{z}}\check{z})\right)^2 + h^2D_{\check{t}}^2 + h^2\mathcal{T}(\check{s})^{-2}\mathcal{T}(0)^2D_{\check{z}}^2.$$

The boundary condition becomes, on $\partial\mathcal{W}_{\alpha_0}$,

$$\check{\nabla}_h \check{a} \cdot \check{\mathbf{n}} = 0$$

where

$$\check{\mathbf{n}} = \begin{pmatrix} -\mathcal{T}'(\check{s})\check{t} \\ -\mathcal{T}(\check{s}) \\ \pm 1 \end{pmatrix}. \tag{11.2.1}$$

We now perform the scaling which preserves $\mathcal{W}_{\alpha_0}$:

$$\check{s} = \sigma, \quad \check{t} = h^{1/2}\tau, \quad \check{z} = h^{1/2}\mathcal{z}.$$

The operator $h^{-1}\check{\mathfrak{L}}_h^{\mathsf{e}}$ becomes $\mathfrak{L}_h^{\mathsf{e}}$:

$$\mathfrak{L}_h^{\mathsf{e}} = \left(h^{1/2}D_\sigma - \tau - h^{1/2}\frac{\mathcal{T}'}{2\mathcal{T}}(\mathcal{z}D_{\mathcal{z}} + D_{\mathcal{z}}\mathcal{z})\right)^2 + D_\tau^2 + \mathcal{T}(\sigma)^{-2}\mathcal{T}(0)^2 D_{\mathcal{z}}^2.$$

Now, the boundary condition is, on $\partial\mathcal{W}_{\alpha_0}$,

$$\hat{\nabla}_h \hat{a} \cdot \hat{\mathbf{n}} = 0$$

where

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_0 + h^{1/2}\mathbf{n}_1 \tag{11.2.2}$$

with

$$\hat{\mathbf{n}}_0 = \begin{pmatrix} 0 \\ -\mathcal{T}(\sigma) \\ \pm 1 \end{pmatrix}, \quad \hat{\mathbf{n}}_1 = \begin{pmatrix} -\mathcal{T}'(\sigma)\tau \\ 0 \\ 0 \end{pmatrix}$$

and

$$\hat{\nabla}_h = \begin{pmatrix} h^{1/2}D_\sigma \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_{\mathcal{z}} \end{pmatrix} + \begin{pmatrix} -\tau - h^{1/2}\frac{\mathcal{T}'}{2\mathcal{T}}(\mathcal{z}D_{\mathcal{z}} + D_{\mathcal{z}}\mathcal{z}) \\ 0 \\ 0 \end{pmatrix}.$$

Theorem 11.2. *Under Assumption 2.51 and Conjecture 2.53, there exist a function $\Phi = \Phi(\sigma)$ defined in a neighborhood $\mathcal{V}$ of $(0,0,0)$ and a real positive number C such that* $\mathsf{Re}\,\mathsf{Hess}\Phi(0) > 0$ *on $\mathcal{V}$ and sequence of real numbers $(\lambda_{n,j}^{\mathsf{e}})$ such that the n-th eigenvalue of $\mathfrak{L}_h^{\mathsf{e}}$ satisfies*

$$\lambda_n^{\mathsf{e}}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \lambda_{n,j}^{\mathsf{e}} h^{\frac{j}{2}}.$$

in the sense of formal series. Besides there exists a formal series of smooth functions $(a_{n,j}^{\mathsf{e}}(\sigma,\tau,\mathcal{z}))$,

$$a_n^{\mathsf{e}} \underset{h\to 0}{\sim} \sum_{j\geq 0} a_{n,j}^{\mathsf{e}} h^{\frac{j}{2}}$$

with $a_{n,j}^{\mathsf{e}}(0,0)$ such that

$$(\mathcal{L}_h^{\mathsf{e}} - \lambda_n^{\mathsf{e}}(h))\left(a_n^{\mathsf{e}} e^{-\Phi/h^{1/2}}\right) = \mathcal{O}(h^\infty)\, e^{-\Phi/h^{1/2}}.$$

We also have that $\lambda^{\mathsf{e}}_{n,0} = \mu^{\mathsf{e}}(0,\zeta^{\mathsf{e}}_0)$ *and that* $\lambda^{\mathsf{e}}_{n,1}$ *is the* n*-th eigenvalue of the operator*

$$\frac{1}{2}\mathsf{Hess}\,\mu^{\mathsf{e}}(0,\zeta^{\mathsf{e}}_0)(\sigma, D_\sigma). \tag{11.2.3}$$

The main term in the Ansatz is in the form $a^{\mathsf{e}}_{n,0} = f^{\mathsf{e}}_{n,0}(\sigma)u^{\mathsf{e}}_{\sigma,i\Phi'(\sigma)}(\tau,\mathfrak{z})$*. Moreover, for all* $n \geq 1$*, there exist* $h_0 > 0$*,* $c > 0$ *such that for all* $h \in (0,h_0)$*, we have*

$$\mathcal{B}(\lambda^{\mathsf{e}}_{n,0}(h) + \lambda^{\mathsf{e}}_{n,1}(h), ch) \cap \mathsf{sp}\,(\mathfrak{L}^{\mathsf{e}}_h) = \{\lambda^{\mathsf{e}}_n(h)\},$$

and $\lambda^{\mathsf{e}}_n(h)$ *is a simple eigenvalue.*

2.2. WKB expansion for the normal form. Let us consider the conjugate operator

$$e^{\Phi(\sigma)/h^{1/2}}\mathfrak{L}^{\mathsf{e}}_h e^{-\Phi(\sigma)/h^{1/2}} = \left(h^{1/2}D_\sigma + i\Phi'(\sigma) - \tau - h^{1/2}\frac{\mathcal{T}'}{2\mathcal{T}}(\mathfrak{z}D_\mathfrak{z} + D_\mathfrak{z}\mathfrak{z})\right)^2 + D^2_\tau + \mathcal{T}(\sigma)^{-2}\mathcal{T}(0)^2D^2_\mathfrak{z},$$

with the corresponding boundary conditions. We can write the formal power series expansion:

$$e^{\Phi(\sigma)/h^{1/2}}\mathfrak{L}^{\mathsf{e}}_h e^{-\Phi(\sigma)/h^{1/2}} \sim \sum_{j\geq 0} h^{j/2}\mathfrak{L}_j$$

with

$$\mathfrak{L}_0 = \mathcal{M}^{\mathsf{e}}_{s,i\Phi'(s)},$$

$$\mathfrak{L}_1 = \frac{1}{2}\left(\partial_\eta\mathcal{M}^{\mathsf{e}}_{s,\eta}D_\sigma + D_\sigma\partial_\eta\mathcal{M}^{\mathsf{e}}_{s,\eta}\right) - (i\Phi'(\sigma) - \tau)\frac{\mathcal{T}'}{\mathcal{T}}(\mathfrak{z}D_\mathfrak{z} + D_\mathfrak{z}\mathfrak{z}).$$

Our Ansatze are in the form:

$$a \sim \sum_{j\geq 0} h^{j/2}a_j, \quad \mu \sim \sum_{j\geq 0} h^{j/2}\mu_j.$$

The first equation is given by

$$\mathfrak{L}_0 a_0 = \mu_0 a_0,$$

with boundary condition (which is in fact a Neumann condition):

$$\begin{pmatrix} -\tau \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_\mathfrak{z} \end{pmatrix} a_0 \cdot \mathbf{n}_0 = 0.$$

We take

$$a_0(\sigma,\tau,\mathfrak{z}) = f_0(\sigma)u^{\mathsf{e}}_{\sigma,i\Phi'(\sigma)}(\tau,\mathfrak{z})$$

and $\mu_0 = \mu^{\mathsf{e}}(0,\eta_0)$. The equation becomes

$$\mu^{\mathsf{e}}(\sigma, i\Phi'(\sigma)) = \mu_0.$$

This equation cannot be solved thanks to Lemma 11.1. This can be done by using the more general theory of [**18**]. Let us admit that we have found a suitable Φ with a positive real part at $s = 0$.

The second equation is

$$(\mathfrak{L}_0 - \mu_0)a_1 = (\mu_1 - \mathfrak{L}_1)a_0$$

with boundary condition:

$$\begin{pmatrix} -\tau \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_\mathcal{z} \end{pmatrix} a_0 \cdot \mathbf{n}_1 + \begin{pmatrix} -\tau \\ D_\tau \\ \mathcal{T}(\sigma)^{-1}\mathcal{T}(0)D_\mathcal{z} \end{pmatrix} a_1 \cdot \mathbf{n}_0 = 0.$$

The Fredholm condition can be rewritten in the form (thanks to the same calculations as in Section 1):

$$\left\{\frac{1}{2}\left(\partial_\eta \mu^{\mathsf{e}}(\sigma, i\Phi'(\sigma))D_\sigma + D_\sigma \partial_\eta \mu^{\mathsf{e}}(\sigma, i\Phi'(\sigma))\right) + R^{\mathsf{e}}(\sigma)\right\} f_0 = \mu_1 f_0,$$

where the smooth function $\sigma \mapsto R^{\mathsf{e}}(\sigma)$ vanishes at $\sigma = 0$ since $\mathcal{T}'(0) = 0$.

3. Curvature induced magnetic bound states

3.1. A higher order degeneracy. Let us prove Theorem 2.56.

3.2. WKB expansion. Let us introduce a phase function $\Phi = \Phi(\sigma)$ defined in a neighborhood of $\sigma = 0$ the unique and non-degenerate maximum of the curvature κ. We consider the conjugate operator

$$\mathfrak{L}_h^{\mathsf{c,wgt}} = e^{\Phi(\sigma)/h^{\frac{1}{4}}} \mathfrak{L}_h^{\mathsf{c}} e^{-\Phi(\sigma)/h^{\frac{1}{4}}}.$$

As usual, we look for

$$a \sim \sum_{j \geq 0} h^{\frac{j}{4}} a_j, \quad \lambda \sim \sum_{j \geq 0} \lambda_j h^{\frac{j}{4}}$$

such that, in the sense of formal series we have

$$\mathfrak{L}_h^{\mathsf{c,wgt}} a \sim \lambda a.$$

We may write

$$\mathfrak{L}_h^{\mathsf{c,wgt}} \sim \mathfrak{L}_0 + h^{\frac{1}{4}}\mathfrak{L}_1 + h^{\frac{1}{2}}\mathfrak{L}_2 + h^{\frac{3}{4}}\mathfrak{L}_3 + \dots,$$

where

$$\mathfrak{L}_0 = D_\tau^2 + (\zeta_0 - \tau)^2,$$
$$\mathfrak{L}_1 = 2(\zeta_0 - \tau)i\Phi'(\sigma),$$
$$\mathfrak{L}_2 = \kappa(\sigma)\partial_\tau + 2\left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right)(\zeta_0 - \tau) - \Phi'(\sigma)^2 + 2\kappa(\sigma)(\zeta_0 - \tau)^2\tau,$$
$$\mathfrak{L}_3 = \left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right)(i\Phi'(\sigma)) + (i\Phi'(\sigma))\left(D_\sigma + \kappa(\sigma)\frac{\tau^2}{2}\right) + 4i\Phi'(\sigma)\tau\kappa(\sigma)(\zeta_0 - \tau).$$

Let us now solve the formal system. The first equation is

$$\mathfrak{L}_0 a_0 = \lambda_0 a_0$$

and leads to take

$$\lambda_0 = \Theta_0, \quad a_0(\sigma,\tau) = f_0(\sigma)u_{\zeta_0}(\tau),$$

where f_0 has to be determined. The second equation is

$$(\mathfrak{L}_0 - \lambda_0)a_1 = (\lambda_1 - \mathfrak{L}_1)a_0 = (\lambda_1 - 2(\zeta_0 - \tau))u_{\zeta_0}(\tau)i\Phi'(\sigma)f_0$$

and, due to the Fredholm alternative, we must take $\lambda_1 = 0$ and we take

$$a_1(\sigma,\tau) = i\Phi'(\sigma)f_0(\sigma)\left(\partial_\zeta u\right)_{\zeta_0}(\tau) + f_1(\sigma)u_{\zeta_0}(\tau),$$

where f_1 is to be determined in a next step. Then the third equation is

$$(\mathfrak{L}_0 - \lambda_0)a_2 = (\lambda_2 - \mathfrak{L}_2)a_0 - \mathfrak{L}_1 a_1.$$

Let us explicitly write the r.h.s. It equals

$$\lambda_2 u_{\zeta_0} f_0 + \Phi'^2(u_{\zeta_0} + 2(\zeta_0 - \tau)(\partial_\zeta u)_{\zeta_0})f_0 - 2(\zeta_0 - \tau)u_{\zeta_0}(i\Phi' f_1 - i\partial_\sigma f_0) \\ + \kappa(\sigma)f_0(\partial_\tau u_{\zeta_0} - 2(\zeta_0 - \tau)^2\tau u_{\zeta_0} - \tau^2(\zeta_0 - \tau)u_{\zeta_0}).$$

Therefore the equation becomes

$$(\mathfrak{L}_0 - \lambda_0)\tilde{a}_2 = \lambda_2 u_{\zeta_0} f_0 + \frac{\nu''(\zeta_0)}{2}\Phi'^2 u_{\zeta_0} f_0 + \kappa(\sigma)f_0(-\partial_\tau u_{\zeta_0} - 2(\zeta_0 - \tau)^2\tau u_{\zeta_0} - \tau^2(\zeta_0 - \tau)u_{\zeta_0}),$$

where

$$\tilde{a}_2 = a_2 - v_{\zeta_0}(i\Phi' f_1 - i\partial_\sigma f_0) + \tfrac{1}{2}(\partial_\zeta^2 u)_{\zeta_0}\Phi'^2 f_0.$$

Let us now use the Fredholm alternative (with respect to τ). We will need the following lemma the proof of which relies on Feynman-Hellmann formulas (like in Proposition 10.2) and on [**67**, p. 19] (for the last one).

Lemma 11.3. *We have:*

$$\int_{\mathbb{R}_+}(\zeta_0 - \tau)u_{\zeta_0}^2(\tau)\,\mathrm{d}\tau = 0, \qquad \int_{\mathbb{R}_+}(\partial_\zeta u)_{\zeta_0}(\tau)u_{\zeta_0}(\tau)\,\mathrm{d}\tau = 0,$$

$$2\int_{\mathbb{R}_+}(\zeta_0 - \tau)(\partial_\zeta u)_{\zeta_0}(\tau)u_{\zeta_0}(\tau)\,\mathrm{d}\tau = \frac{\nu''(\zeta_0)}{2} - 1,$$

$$\int_{\mathbb{R}_+} 2\tau(\zeta_0 - \tau)^2 + \tau^2(\zeta_0 - \tau)u_{\zeta_0}^2 + u_{\zeta_0}\partial_\tau u_{\zeta_0}\,\mathrm{d}\tau = -C_1.$$

We get the equation

$$\lambda_2 + \frac{\nu''(\zeta_0)}{2}\Phi'^2(\sigma) + C_1\kappa(\sigma) = 0, \qquad C_1 = \frac{u_{\zeta_0}^2(0)}{3}.$$

This eikonal equation is the eikonal equation of a pure electric problem in dimension one whose potential is given by the curvature. Thus we take

$$\lambda_2 = -C_1\kappa(0),$$

and

$$\Phi(\sigma) = \left(\frac{2C_1}{\nu''(\zeta_0)}\right)^{1/2}\left|\int_0^\sigma(\kappa(0) - \kappa(s))^{1/2}\,\mathrm{d}s\right|.$$

In particular we have:

$$\Phi''(0) = \left(\frac{k_2 C_1}{\nu''(\zeta_0)}\right)^{1/2},$$

where $k_2 = -\kappa''(0) > 0$.
This leads to take

$$a_2 = f_0\hat{a}_2 + (\partial_\zeta u)_{\zeta_0}(i\Phi' f_1 - i\partial_\sigma f_0) - \tfrac{1}{2}(\partial_\eta^2 u)_{\zeta_0}\Phi'^2 f_0 + f_2 u_{\zeta_0},$$

where $\hat{a}_2$ is the unique solution, orthogonal to u_{ζ_0} for all σ, of

$$(\mathfrak{L}_0 - \nu_0)\hat{a}_2 = \nu_2 u_{\zeta_0} + \frac{\nu''(\zeta_0)}{2}\Phi'^2 u_{\zeta_0} + \kappa(\sigma)\left(-\partial_\tau u_{\zeta_0} - 2(\zeta_0-\tau)^2\tau u_{\zeta_0} - \tau^2(\zeta_0-\tau)u_{\zeta_0}\right),$$

and f_2 has to be determined.
Finally we must solve the fourth equation given by

$$(\mathfrak{L}_0 - \lambda_0)a_3 = (\lambda_3 - \mathfrak{L}_3)a_0 + (\lambda_2 - \mathfrak{L}_2)a_1 - \mathfrak{L}_1 a_2.$$

The Fredholm condition provides the following equation in the variable σ:

$$\langle \mathfrak{L}_3 a_0 + (\mathfrak{L}_2 - \lambda_2)a_1 + \mathfrak{L}_1 a_2, u_{\zeta_0}\rangle_{\mathsf{L}^2(\mathbb{R}_+,\,\mathrm{d}\tau)} = \lambda_3 f_0.$$

Using the previous steps of the construction, it is not very difficult to see that this equation does not involve f_1 and f_2 (due to the choice of Φ and λ_2 and Feynman-Hellmann formulas). Using the same formulas, we may write it in the form

$$\frac{\nu''(\zeta_0)}{2}\left(\Phi'(\sigma)\partial_\sigma + \partial_\sigma\Phi'(\sigma)\right)f_0 + F(\sigma)f_0 = \lambda_3 f_0, \tag{11.3.1}$$

where F is a smooth function which vanishes at $\sigma = 0$. Therefore the linearized equation at $\sigma = 0$ is given by

$$\Phi''(0)\frac{\nu''(\zeta_0)}{2}\left(\sigma\partial_\sigma + \partial_\sigma\sigma\right)f_0 = \lambda_3 f_0.$$

We recall that

$$\frac{\nu''(\zeta_0)}{2} = 3C_1\Theta_0^{1/2}$$

so that the linearized equation becomes

$$C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}}\left(\sigma\partial_\sigma + \partial_\sigma\sigma\right)f_0 = \lambda_3 f_0.$$

We have to choose λ_3 in the spectrum of this transport equation, which is given by the set

$$\left\{(2n-1)C_1\Theta_0^{1/4}\sqrt{\frac{3k_2}{2}},\quad n \geq 1\right\}.$$

If λ_3 belongs to this set, we may solve locally the transport equation (11.3.1) and thus find f_0. This procedure can be continued at any order.

Part 4

Semiclassical Magnetic Normal Forms

CHAPTER 12

Vanishing magnetic fields in dimension two

> For it is not from any sureness in myself that I cause others to doubt: it is from being in more doubt than anyone else that I cause doubt in others.
>
> *Meno*, Plato

This chapter presents the main elements of the proof of Theorem 3.4. We provide a flexible and "elementary" proof which can be adapted to other situations, especially less regular situations as in Chapter 14. A more conceptual proof, using a WKB method, is possible by using the material introduced in Chapter 10, Section 3.2. Nevertheless, the approach chosen for this chapter has the interest to reduce explicitly the spectral analysis to an electric Laplacian in the electric Born-Oppenheimer form. In particular, we do not need the notions of coherent states and of microlocalization in full generality.

1. Normal form

1.1. Toward a normal form. We can write (exercise !) the operator near the cancellation line in the coordinates (s,t):

$$\widetilde{\mathfrak{L}}_{h,\mathbf{A}} = h^2(1-t\kappa(s))^{-1}D_t(1-t\kappa(s))D_t + (1-t\kappa(s))^{-1}\tilde{P}(1-t\kappa(s))^{-1}\tilde{P},$$

where

$$\tilde{P} = ih\partial_s + \tilde{A}(s,t)$$

with:

$$\tilde{A}(s,t) = \int_0^t (1-k(s)t')\tilde{\mathbf{B}}(s,t')\,\mathrm{d}t'.$$

In terms of the quadratic form, we can write:

$$\tilde{\mathfrak{Q}}_{h,\mathbf{A}}(\psi) = \int \left(|hD_t\psi|^2 + (1-t\kappa(s))^{-2}|\tilde{P}\psi|^2\right)\, m(s,t)\,\mathrm{d}s\,\mathrm{d}t,$$

with:

$$m(s,t) = 1-t\kappa(s).$$

We consider the following operator on $\mathsf{L}^2(\mathbb{R}^2)$ which is unitarily equivalent to $\widetilde{\mathfrak{L}}_{h,\mathbf{A}}$ (see [**104**, Theorem 18.5.9 and below]):

$$\mathfrak{L}^{\mathsf{new}}_{h,\mathbf{A}} = m^{1/2}\widetilde{\mathfrak{L}}_{h,\mathbf{A}}m^{-1/2} = P_1^2 + P_2^2 - \frac{h^2\kappa(s)^2}{4m^2},$$

with $P_1 = m^{-1/2}(-hD_s + \tilde{A}(s,t))m^{-1/2}$ and $P_2 = hD_t$.

We wish to use a system of coordinates more adapted to the magnetic situation. Let us perform a Taylor expansion near $t = 0$. We have:

$$\tilde{\mathbf{B}}(s,t) = \gamma(s)t + \partial_t^2\tilde{\mathbf{B}}(s,0)\frac{t^2}{2} + O(t^3).$$

This provides:

$$\tilde{A}(s,t) = \frac{\gamma(s)}{2}t^2 + k(s)t^3 + O(t^4),$$

with:

$$k(s) = \frac{1}{6}\partial_t^2\tilde{\mathbf{B}}(s,0) - \frac{\kappa(s)}{3}\gamma(s)$$

This suggests, as for the model operator, to introduce the new magnetic coordinates in a fixed neighborhood of $(0,0)$:

$$\check{t} = \gamma(s)^{1/3}t, \quad \check{s} = s.$$

This change of variable is fundamental in the analysis of the models introduced in Chapter 10, Section 3.2. The change of coordinates for the derivatives is given by:

$$D_t = \gamma(\check{s})^{1/3}D_{\check{t}}, \quad D_s = D_{\check{s}} + \frac{1}{3}\gamma'\gamma^{-1}\check{t}D_{\check{t}}.$$

The space $\mathsf{L}^2(\,\mathrm{d}s\,\mathrm{d}t)$ becomes $\mathsf{L}^2(\gamma(\check{s})^{-1/3}\,\mathrm{d}\check{s}\,\mathrm{d}\check{t})$. In the same way as previously, we shall conjugate $\mathfrak{L}_{h,\mathbf{A}}^{\mathsf{new}}$. We introduce the self-adjoint operator on $\mathsf{L}^2(\mathbb{R}^2)$:

$$\check{\mathfrak{L}}_{h,\mathbf{A}} = \gamma^{-1/6}\mathcal{L}_{h,\mathbf{A}}^{\mathsf{new}}\gamma^{1/6}.$$

We deduce:

$$\check{\mathfrak{L}}_{h,\mathbf{A}} = h^2\gamma(\check{s})^{2/3}D_{\check{t}}^2 + \check{P}^2,$$

where:

$$\check{P} = \gamma^{-1/6}\check{m}^{-1/2}\left(-hD_{\check{s}} + \check{A}(\check{s},\check{t}) - h\frac{1}{3}\gamma'\gamma^{-1}\check{t}D_{\check{t}}\right)\check{m}^{-1/2}\gamma^{1/6},$$

with:

$$\check{A}(\check{s},\check{t}) = \tilde{A}(\check{s},\gamma(\check{s})^{-1/3}\check{t}).$$

A straight forward computation provides:

$$\check{P} = \check{m}^{-1/2}\left(-hD_{\check{s}} + \check{A}(\check{s},\check{t}) - h\frac{1}{6}\gamma'\gamma^{-1}(\check{t}D_{\check{t}} + D_{\check{t}}\check{t})\right)\check{m}^{-1/2},$$

where we make the generator of dilations $\check{t}D_{\check{t}} + D_{\check{t}}\check{t}$ to appear (and which is related to the virial theorem, see [**150, 153**] where this theorem is often used). Up to a change of gauge, we can replace $\check{P}$ by:

$$\check{m}^{-1/2}\left(-hD_{\check{s}} - \zeta_0^{[1]}(\gamma(\check{s}))^{1/3}h^{2/3} + \check{A}(\check{s},\check{t}) - h\frac{1}{6}\gamma'\gamma^{-1}(\check{t}D_{\check{t}} + D_{\check{t}}\check{t})\right)\check{m}^{-1/2}.$$

1.2. Normal form $\check{\mathfrak{L}}_{h,\mathbf{A}}$. Therefore, the operator takes the form “à la Hörmander”:

$$\check{\mathfrak{L}}_{h,\mathbf{A}} = P_1(h)^2 + P_2(h)^2 - \frac{h^2\kappa(\check{s})^2}{4m(\check{s},\gamma(\check{s})^{1/3}\check{t})^2}, \tag{12.1.1}$$

where:

$$P_1(h) = \check{m}^{-1/2}\left(-hD_{\check{s}} - \zeta_0^{[1]}(\gamma(\check{s}))^{1/3}h^{2/3} + \check{A}(\check{s},\check{t}) - h\frac{1}{6}\gamma'\gamma^{-1}(\check{t}D_{\check{t}} + D_{\check{t}}\check{t})\right)\check{m}^{-1/2},$$

$$P_2(h) = h\gamma(\check{s})^{1/3}D_{\check{t}}.$$

Computing a commutator, we can rewrite $P_1(h)$:

(12.1.2)

$$P_1(h) = \check{m}^{-1}\left(-hD_{\check{s}} - \zeta_0^{[1]}(\gamma(\check{s}))^{1/3}h^{2/3} + \check{A}(\check{s},\check{t}) - h\frac{1}{6}\gamma'\gamma^{-1}(\check{t}D_{\check{t}} + D_{\check{t}}\check{t})\right) + C_h,$$

where:

$$C_h = -h\check{m}^{-1/2}(D_{\check{s}}\check{m}^{-1/2}) - \frac{h\gamma'\gamma^{-1}}{3}\check{t}\check{m}^{-1/2}(D_{\check{t}}\check{m}^{-1/2}).$$

Notation 12.1. *The quadratic form corresponding to $\check{\mathcal{L}}_h$ will be denoted by $\check{\mathcal{Q}}_h$.*

1.3. Quasimodes. We can construct quasimodes using the classical recipe (see Chapter 10) and the scaling:

$$\check{t} = h^{1/3}\tau, \quad \check{s} = h^{1/6}\sigma. \tag{12.1.3}$$

Notation 12.2. *The operator $h^{-4/3}\check{\mathfrak{L}}_{h,\boldsymbol{A}}$ will be denoted by $\mathcal{L}_h$ in these new coordinates.*

This provides the following proposition.

Proposition 12.3. *We assume* (3.3)*. For all $n \geq 1$, there exist a sequence $(\theta_j^n)_{j\geq 0}$ such that, for all $J \geq 0$, there exists $h_0 > 0$ such that for $h \in (0, h_0)$, we have:*

$$\operatorname{dist}\left(h^{4/3}\sum_{j=0}^{J}\theta_j^n h^{j/6}, \mathsf{sp}(\mathfrak{L}_{h,\boldsymbol{A}})\right) \leq Ch^{4/3}h^{(J+1)/6}.$$

Moreover, we have:

$$\theta_0^n = \gamma_0^{2/3}\nu_1^{[1]}(\zeta_0^{[1]}), \quad \theta_1^n = 0, \quad \theta_2^n = \gamma_0^{2/3}C_0 + \gamma_0^{2/3}(2n-1)\left(\frac{\alpha\nu_1^{[1]}(\zeta_0^{[1]})(\nu_1^{[1]})''(\zeta_0^{[1]})}{3}\right)^{1/2}.$$

Thanks to the "IMS" formula and a partition of unity, we may prove the following proposition (exercise: use Lemma 5.12).

Proposition 12.4. *For all $n \geq 1$, there exist $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq \gamma_0^{2/3}\nu_1^{[1]}(\zeta_0^{[1]})h^{4/3} - Ch^{4/3+2/15}.$$

2. Agmon estimates

Two kinds of Agmon's estimates can be proved using the stand partition of unity arguments. We leave their proofs to the reader.

Proposition 12.5. *Let (λ,ψ) be an eigenpair of $\mathfrak{L}_{h,\boldsymbol{A}}$. There exist $h_0>0$, $C>0$ and $\varepsilon_0>0$ such that, for $h\in(0,h_0)$:*

$$\int e^{2\varepsilon_0|t(\mathbf{x})|h^{-1/3}}|\psi|^2\,\mathrm{d}\mathbf{x}\leq C\|\psi\|^2 \tag{12.2.1}$$

and:

$$\mathfrak{Q}_{h,\boldsymbol{A}}(e^{\varepsilon_0|t(\mathbf{x})|h^{-1/3}}\psi)\leq Ch^{4/3}\|\psi\|^2. \tag{12.2.2}$$

Proposition 12.6. *Let (λ,ψ) be an eigenpair of $\mathfrak{L}_{h,\boldsymbol{A}}$. There exist $h_0>0$, $C>0$ and $\varepsilon_0>0$ such that, for $h\in(0,h_0)$:*

$$\int e^{2\chi(t(\mathbf{x}))|s(\mathbf{x})|h^{-1/15}}|\psi|^2\,\mathrm{d}\mathbf{x}\leq C\|\psi\|^2 \tag{12.2.3}$$

and:

$$\mathfrak{Q}_{h,\boldsymbol{A}}(e^{\chi(t(\mathbf{x}))|s(\mathbf{x})|h^{-1/15}}\psi)\leq Ch^{4/3}\|\psi\|^2, \tag{12.2.4}$$

where χ is a fixed smooth cutoff function being 1 near 0.

From Propositions 12.5 and 12.6, we are led to introduce a cutoff function living near x_0. We take $\varepsilon>0$ and we let:

$$\chi_{h,\varepsilon}(\mathbf{x})=\chi\left(h^{-1/3+\varepsilon}t(\mathbf{x})\right)\chi\left(h^{-1/15+\varepsilon}s(\mathbf{x})\right).$$

where χ is a fixed smooth cutoff function supported near 0.

Notation 12.7. *We will denote by $\check{\psi}$ the function $\chi_{h,\varepsilon}(\mathbf{x})\psi(\mathbf{x})$ in the coordinates $(\check{s},\check{t})$.*

From the normal estimates of Agmon, we deduce the proposition:

Proposition 12.8. *For all $n\geq 1$, there exist $h_0>0$ and $C>0$ s. t., for $h\in(0,h_0)$:*

$$\lambda_n(h)\geq\gamma_0^{2/3}\nu_1^{[1]}(\zeta_0^{[1]})h^{4/3}-Ch^{5/3}.$$

We provide the proof of this proposition to understand the main idea of the lower bound.

Proof. We consider an eigenpair $(\lambda_n(h),\psi_{n,h})$ and we use the IMS formula:

$$\check{\mathfrak{Q}}_h(\check{\psi}_{n,h})=\lambda_n(h)\|\check{\psi}_{n,h}\|^2+O(h^\infty)\|\check{\psi}_{n,h}\|^2.$$

We have (cf. (12.1.1)):

$$\begin{aligned}&\check{\mathfrak{Q}}_h(\check{\psi}_{n,h})\geq\\&\int\check{m}^{-2}\left|\left(-hD_{\check{s}}-\zeta_0^{[1]}\gamma^{1/3}h^{2/3}+\check{A}-\frac{h}{6}\gamma'\gamma^{-1}(\check{t}D_{\check{t}}+D_{\check{t}}\check{t})+C_h\right)\check{\psi}_{n,h}\right|^2\mathrm{d}\check{s}\,\mathrm{d}\check{t}\\&+h^2\gamma_0^{2/3}\|D_{\check{t}}\check{\psi}_{n,h}\|^2-Ch^2\|\check{\psi}_{n,h}\|^2.\end{aligned}$$

Let us deal with the terms involving C_h in the double product produced by the expansion of the square. We have to estimate:

$$h\left|\mathsf{Re}\,\langle \check{m}^{-2}\gamma'\gamma^{-1}(\check{t}D_{\check{t}}+D_{\check{t}}\check{t})\check{\psi}_{n,h}, C_h\check{\psi}_{n,h}\rangle\right|$$

We have :

$$\|C_h\check{\psi}_{n,h}\| = o(h)\|\check{\psi}_{n,h}\|$$

and, with the estimates of Agmon (and the fact that 0 is a critical point of γ):

$$\|\gamma'\gamma^{-1}(\check{t}D_{\check{t}}+D_{\check{t}}\check{t})\check{\psi}_{n,h}\| = o(1)\|\check{\psi}_{n,h}\|.$$

Moreover, we have in the same way:

$$h\left|\mathsf{Re}\,\langle \check{A}\check{\psi}_{n,h}, C_h\check{\psi}_{n,h}\rangle\right| = o(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

Then, we have the control:

$$h\left|\mathsf{Re}\,\langle hD_{\check{s}}\check{\psi}_{n,h}, C_h\check{\psi}_{n,h}\rangle\right| = o(h^{5/3})\|\check{\psi}_{n,h}\|^2,$$

where we have used the rough estimate:

$$\|hD_{\check{s}}\check{\psi}_{n,h}\| \leq Ch^{2/3}\|\check{\psi}_{n,h}\|.$$

We have:

$$\begin{aligned}(12.2.5)\qquad &\check{\mathfrak{Q}}_h(\check{\psi}_{n,h}) \geq \\ &\int \check{m}^{-2}\left|\left(-hD_{\check{s}}-\zeta_0^{[1]}\gamma^{1/3}h^{2/3}+\check{A}-\frac{h}{6}\gamma'\gamma^{-1}(\check{t}D_{\check{t}}+D_{\check{t}}\check{t})\right)\check{\psi}_{n,h}\right|^2 \mathrm{d}\check{s}\,\mathrm{d}\check{t}\\ &+h^2\gamma_0^{2/3}\|D_{\check{t}}\check{\psi}_{n,h}\|^2+o(h^{5/3})\|\check{\psi}_{n,h}\|^2.\end{aligned}$$

We now deal with the term involving $\check{t}D_{\check{t}}+D_{\check{t}}\check{t}$. With the estimates of Agmon, we have:

$$h\left|\mathsf{Re}\,\langle \check{m}^{-2}\gamma'\gamma^{-1}(\check{t}D_{\check{t}}+D_{\check{t}}\check{t})\check{\psi}_{n,h}, (-hD_{\check{s}}-\zeta_0^{[1]}\gamma^{1/3}h^{2/3}+\check{A})\check{\psi}_{n,h}\rangle\right| = o(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

This implies:

$$\begin{aligned}\check{\mathfrak{Q}}_h(\check{\psi}_{n,h}) \geq &\gamma_0^{2/3}h^2\|D_{\check{t}}\check{\psi}_{n,h}\|^2+\int \check{m}^{-2}\left|\left(-hD_{\check{s}}-\zeta_0^{[1]}\gamma^{1/3}h^{2/3}+\check{A}\right)\check{\psi}_{n,h}\right|^2 \mathrm{d}\check{s}\,\mathrm{d}\check{t}\\ &+o(h^{5/3})\|\check{\psi}_{n,h}\|^2.\end{aligned}$$

With the same kind of arguments, it follows:

(12.2.6)

$$\begin{aligned}\check{\mathfrak{Q}}_h(\check{\psi}_{n,h}) \geq &h^2\gamma_0^{2/3}\|D_{\check{t}}\check{\psi}_{n,h}\|^2+\int \check{m}^{-2}\left|\left(-hD_{\check{s}}-\zeta_0^{[1]}\gamma^{1/3}h^{2/3}+\gamma^{1/3}\frac{\check{t}^2}{2}\right)\check{\psi}_{n,h}\right|^2 \mathrm{d}\check{s}\,\mathrm{d}\check{t}\\ &+O(h^{5/3})\|\check{\psi}_{n,h}\|^2\end{aligned}$$

and

$$\check{\mathfrak{Q}}_h(\check{\psi}_{n,h}) \geq h^2\gamma_0^{2/3}\|D_{\check{t}}\check{\psi}_{n,h}\|^2 + \int \left|\left(-hD_{\check{s}} - \zeta_0^{[1]}\gamma^{1/3}h^{2/3} + \gamma^{1/3}\frac{\check{t}^2}{2}\right)\check{\psi}_{n,h}\right|^2 \mathrm{d}\check{s}\,\mathrm{d}\check{t} + O(h^{5/3})\|\check{\psi}_{n,h}\|^2. \tag{12.2.7}$$

We get:

$$\check{\mathfrak{Q}}_h(\check{\psi}_{n,h}) \geq h^2\gamma_0^{2/3}\|D_{\check{t}}\check{\psi}_{n,h}\|^2 + \int \gamma_0^{2/3}\left|\left(-h\gamma^{-1/3}D_{\check{s}} - \zeta_0^{[1]}h^{2/3} + \frac{\check{t}^2}{2}\right)\check{\psi}_{n,h}\right|^2 \mathrm{d}\check{s}\,\mathrm{d}\check{t} + O(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

Then, we write:

$$\gamma^{-1/3}D_{\check{s}} = \gamma^{-1/6}D_{\check{s}}\gamma^{-1/6} + i\gamma^{-1/6}(\gamma^{-1/6})'$$

and deduce (by estimating the double product involved by $i\gamma^{-1/6}(\gamma^{-1/6})'$):

$$\check{\mathfrak{Q}}_h(\check{\psi}_{n,h}) \geq h^2\gamma_0^{2/3}\|D_{\check{t}}\check{\psi}_{n,h}\|^2 + \int \gamma_0^{2/3}\left|\left(-h\gamma^{-1/6}D_{\check{s}}\gamma^{-1/6} - \zeta_0^{[1]}h^{2/3} + \frac{\check{t}^2}{2}\right)\check{\psi}_{n,h}\right|^2 \mathrm{d}\check{s}\,\mathrm{d}\check{t} + o(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

We can apply the functional calculus to the self-adjoint operator $\gamma^{-1/6}D_{\check{s}}\gamma^{-1/6}$ and the following lower bound follows:

$$\check{\mathfrak{Q}}_h(\check{\psi}_{n,h}) \geq h^{4/3}\gamma_0^{2/3}\nu_1^{[1]}(\zeta_0^{[1]}) + O(h^{5/3})\|\check{\psi}_{n,h}\|^2.$$

$\square$

Exercise. Let γ be a smooth and bounded (so as its derivatives) and positive function on $\mathbb{R}$. Find a unitary transform which diagonalizes the self-adjoint realization of $\gamma D_\sigma \gamma$ on $\mathsf{L}^2(\mathbb{R}, \mathrm{d}\sigma)$. Notice that such a transform exists by the spectral theorem.

For all $N \geq 1$, let us consider L^2-normalized eigenpairs $(\lambda_n(h), \psi_{n,h})_{1\leq n\leq N}$ such that $\langle \psi_{n,h}, \psi_{m,h}\rangle = 0$ if $n \neq m$. We consider the N dimensional space defined by:

$$\mathfrak{E}_N(h) = \underset{1\leq n\leq N}{\mathsf{span}}\, \check{\psi}_{n,h}.$$

The next two propositions provide control with respect to σ and D_σ. We leave the proof to the reader and refer to [**51**] and also to the spirit of the proof of Proposition 12.8.

Proposition 12.9. *There exist $h_0 > 0$, $C > 0$ such that, for $h \in (0, h_0)$ and for all $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\|\check{s}\check{\psi}\| \leq Ch^{1/6}\|\check{\psi}\|.$$

Proposition 12.10. *There exist $h_0 > 0$, $C > 0$ such that, for $h \in (0, h_0)$ and for all $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\|D_{\check{s}}\check{\psi}\| \leq Ch^{-1/6}\|\check{\psi}\|.$$

With Proposition 12.9, we have a better lower bound for the quadratic form.

Proposition 12.11. *There exists $h_0 > 0$ such that for $h \in (0, h_0)$ and $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\check{\mathcal{Q}}_h(\check{\psi}) \geq \gamma_0^{2/3} \int (1 + 2\kappa_0 \check{t} \gamma_0^{-1/3}) |(\gamma^{-1/6} i h \partial_{\check{s}} \gamma^{-1/6} + \zeta_0^{[1]} h^{2/3} + \frac{\check{t}^2}{2} + \gamma_0^{-4/3} k(0) \check{t}^3) \check{\psi}|^2 \, \mathrm{d}\check{s} \, \mathrm{d}\check{t}$$
$$+ \int \gamma_0^{2/3} |h D_{\check{t}} \check{\psi}|^2 \, \mathrm{d}\check{s} \, \mathrm{d}\check{t} + \frac{2}{3} \gamma_0^{2/3} \alpha \nu_1^{[1]} (\zeta_0^{[1]}) h^{4/3} \|\check{s} \check{\psi}\|^2 + o(h^{5/3}) \|\check{\psi}\|^2.$$

3. Projection method

We can now prove an approximation result for the eigenfunctions. Let us recall the rescaled coordinates (see (12.1.3)):

$$\check{s} = h^{1/6} \sigma, \quad \check{t} = h^{1/3} \tau. \tag{12.3.1}$$

Notation 12.12. *$\mathcal{L}_h$ denotes $h^{-4/3} \check{\mathfrak{L}}_h$ in the coordinates (σ, τ). The corresponding quadratic form will be denoted by $\mathcal{Q}_h$. We will use the notation $\mathcal{E}_N(h)$ to denote $\mathfrak{E}_N(h)$ after rescaling.*

We introduce the Feshbach-Grushin projection:

$$\Pi_0 \phi = \langle \phi, u^{[1]}_{\zeta_0^{[1]}} \rangle_{\mathsf{L}(\mathbb{R}_\tau)} u^{[1]}_{\zeta_0^{[1]}}(\tau).$$

We will need to consider the quadratic form:

$$\hat{\mathcal{Q}}_0(\phi) = \gamma_0^{2/3} \int |D_\tau \phi|^2 + \left| \left(-\zeta_0^{[1]} + \frac{\tau^2}{2} \right) \phi \right|^2 \mathrm{d}\sigma \, \mathrm{d}\tau.$$

The fundamental approximation result is given in the following proposition.

Proposition 12.13. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and $\hat{\psi} \in \mathcal{E}_N(h)$:*

$$0 \leq \mathcal{Q}_0(\hat{\psi}) - \gamma_0^{2/3} \nu_1^{[1]} (\zeta_0^{[1]}) \|\hat{\psi}\|^2 \leq C h^{1/6} \|\hat{\psi}\|^2 \tag{12.3.2}$$

and:

$$\begin{aligned} &\|\Pi_0 \hat{\psi} - \hat{\psi}\| \leq C h^{1/12} \|\hat{\psi}\| \\ &\|D_\tau (\Pi_0 \hat{\psi} - \hat{\psi})\| \leq C h^{1/12} \|\hat{\psi}\|, \\ &\|\tau^2 (\Pi_0 \hat{\psi} - \hat{\psi})\| \leq C h^{1/12} \|\hat{\psi}\|. \end{aligned} \tag{12.3.3}$$

This permits to simplify the lower bound (see (3.1.6)).

Proposition 12.14. *There exist $h_0 > 0$, $C > 0$ such that, for $h \in (0, h_0)$ and $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\check{\mathfrak{Q}}_h(\check{\psi}) \geq \int \gamma_0^{2/3} \left(|h D_{\check{t}} \check{\psi}|^2 + |(\gamma^{-1/6} i h \partial_{\check{s}} \gamma^{-1/6} - \zeta_0^{[1]} h^{2/3} + \frac{\check{t}^2}{2}) \check{\psi}|^2 \right) \mathrm{d}\check{s} \, \mathrm{d}\check{t}$$
$$+ \frac{2}{3} \gamma_0^{2/3} \alpha \nu_1^{[1]} (\zeta_0^{[1]}) h^{4/3} \|\check{s} \check{\psi}\|^2 + C_0 h^{5/3} \|\check{\psi}\|^2 + o(h^{5/3}) \|\check{\psi}\|^2.$$

It remains to diagonalize $\gamma^{-1/6} i \partial_{\check{s}} \gamma^{-1/6}$:

Corollary 12.15. *There exist $h_0 > 0$, $C > 0$ such that, for $h \in (0, h_0)$ and $\check{\psi} \in \mathfrak{E}_N(h)$:*

$$\check{\mathcal{Q}}_h(\check{\psi}) \geq \int \gamma_0^{2/3} \left(|hD_{\check{t}}\check{\phi}|^2 + |(-h\mu - \zeta_0^{[1]} h^{2/3} + \frac{\check{t}^2}{2})\check{\phi}|^2 \right) \mathrm{d}\mu \, \mathrm{d}\check{t}$$
$$+ \frac{2}{3}\gamma_0^{2/3} \alpha \nu_1(\zeta_0^{[1]}) h^{4/3} \|D_\mu \check{\phi}\|^2 + C_0 h^{5/3} \|\check{\phi}\|^2 + o(h^{5/3})\|\check{\phi}\|^2,$$

with $\check{\phi} = \mathcal{F}_\gamma \check{\psi}$.

Let us introduce the operator on $\mathsf{L}^2(\mathbb{R}^2, \mathrm{d}\mu \, \mathrm{d}\check{t})$:

$$\frac{2}{3}\gamma_0^{2/3} \alpha \nu_1^{[1]}(\zeta_0^{[1]}) h^{4/3} D_\mu^2 + \gamma_0^{2/3} \left(h^2 D_{\check{t}}^2 + \left(-h\mu - \zeta_0^{[1]} h^{2/3} + \frac{\check{t}^2}{2} \right)^2 \right) + C_0 h^{5/3}. \tag{12.3.4}$$

Exercise. Determine the asymptotic expansion of the lowest eigenvalues of this operator thanks to the Born-Oppenheimer theory and prove:

Theorem 12.16. *We assume (3.3). For all $n \geq 1$, there exists $h_0 > 0$ such that for $h \in (0, h_0)$, we have:*

$$\lambda_n(h) \geq \theta_0^n h^{4/3} + \theta_2^n h^{5/3} + o(h^{5/3}).$$

This implies Theorem 3.4.

CHAPTER 13

A regular boundary in dimension three

Sedulo curavi, humanas actiones non ridere,
non lugere, neque detestari, sed intelligere.

Tractatus politicus, Spinoza

This chapter is devoted to the proof of Theorem 3.7. We keep the notation of Chapter 3, Section 2. We analyze here how a smooth boundary combines with the magnetic field to generate a magnetic harmonic approximation.

1. Quasimodes

Theorem 13.1. *For all $\alpha > 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, there exists a sequence $(\mu_{j,n})_{j\geq 0}$ and there exist positive constants C, h_0 such that for $h \in (0, h_0)$:*

$$\operatorname{dist}\left(\mathsf{sp}(\mathfrak{L}_h, h\sum_{j=0}^{J}\mu_{j,n}h^j\right) \leq Ch^{J+2}$$

and we have $\mu_{0,n} = \mathfrak{s}(\theta)$ and $\mu_{1,n}$ is the n-th eigenvalue of $\mathfrak{S}_\theta(D_\rho, \rho)$.

Proof. We perform the scaling (3.2.4) and, after division by h, $\mathfrak{L}_{h,\alpha,\theta}$ becomes:

$$\mathcal{L}_h = D_s^2 + D_t^2 + (D_r + t\cos\theta - s\sin\theta + h\alpha t(r^2 + s^2)).$$

Using the Fourier transform $\mathcal{F}$ (see (3.2.5)) and the translation U_θ (see (3.2.6)), we have:

$$U_\theta\mathcal{F}\mathcal{L}_h\mathcal{F}^{-1}U_\theta^{-1} = D_\sigma^2 + D_\tau^2 + \left(V_\theta(\sigma,\tau) + h\alpha\tau\left(D_\rho - \frac{D_\sigma}{\sin\theta}\right)^2 + \left(\sigma + \frac{\rho}{\sin\theta}\right)^2\right)^2.$$

This normal form will be denoted by $\mathcal{L}_h^{\mathsf{No}}$ and the corresponding quadratic form by $\mathcal{Q}_h^{\mathsf{No}}$. We write:

$$\mathcal{L}_h^{\mathsf{No}} = \mathfrak{L}_\theta^{\mathsf{LP}} + hL_1 + h^2L_2,$$

where:

$$L_1 = \alpha\tau\left\{\left(D_\rho - \frac{D_\sigma}{\sin\theta}\right)^2 V_\theta + V_\theta\left(D_\rho - \frac{D_\sigma}{\sin\theta}\right)^2 + 2V_\theta\left(\sigma + \frac{\rho}{\sin\theta}\right)^2\right\},$$

$$L_2 = \alpha^2\tau^2\left\{\left(D_\rho - \frac{D_\sigma}{\sin\theta}\right)^2 + \left(\sigma + \frac{\rho}{\sin\theta}\right)^2\right\}^2 \geq 0.$$

We look for quasi-eigenpairs in the form:

$$\mu \sim \sum_{j\geq 0} \mu_j h^j, \quad \psi \sim \sum_{j\geq 0} \psi_j h^j.$$

We solve the following problem in the sense of formal series:

$$\mathcal{L}_h^{\mathsf{No}} \psi \sim \mu\psi.$$

The term in h^0 leads to solve:

$$\mathfrak{H}_\theta^{\mathsf{Neu}} \psi_0 = \mu_0 \psi_0.$$

We take: $\mu_0 = \mathfrak{s}(\theta)$ and:

$$\psi_0(\rho, \sigma, \tau) = u_\theta^{\mathsf{LP}}(\sigma, \tau) f_0(\rho),$$

f_0 being to be determined. Then, we must solve:

$$(\mathfrak{H}_\theta^{\mathsf{Neu}} - \mathfrak{s}(\theta))\psi_1 = (\mu_1 - L_1)\psi_0.$$

We apply the Fredholm alternative and we write:

$$\langle (\mu_1 - L_1)\psi_0, u_\theta^{\mathsf{LP}} \rangle_{\mathsf{L}^2(\mathbb{R}^2_{+,\hat{s},\hat{t}})} = 0.$$

The compatibility equation rewrites:

$$\mathfrak{S}_\theta(D_\rho, \rho) f_0 = \mu_1 f_0$$

and we take μ_1 in the spectrum of $\mathfrak{S}_\theta(D_\rho, \rho)$ and for f_0 the corresponding L^2-normalized eigenfunction. Then, we can write the solution ψ_1 in the form:

$$\psi_1 = \psi_1^\perp + f_1(\rho) u_\theta(\sigma, \tau)$$

where $\psi_1^\perp$ is the unique solution orthogonal to u_θ^{LP}. We notice that it is the the Schwarz class. This construction can be continued at any order.

$\square$

2. Localization estimates

Let us first recall standard Agmon's estimates with respect to (x, y) satisfied by an eigenfunction u_h associated with $\lambda_n(h)$. It possible to establish the following lower bound by using the techniques of Chapter 6, Section 3 (see [**124**] and [**68**, Theorem 9.1.1]).

Proposition 13.2. *There exist $C > 0$ and $h_0 > 0$ such that, for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq \mathfrak{s}(\theta)h - Ch^{5/4}.$$

2.1. Agmon estimates of first order.

Using the techniques of Chapter 6, Section 3, we can obtain:

Proposition 13.3. *For all $\delta > 0$, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$:*

$$\int_{\Omega_0} e^{\delta(x^2+y^2)/h^{1/4}} |u_h|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \leq C \|u_h\|^2,$$

$$\int_{\Omega_0} e^{\delta(x^2+y^2)/h^{1/4}} |\nabla u_h|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq Ch^{-1}\|u_h\|^2.$$

Combining Proposition 13.2 and Theorem 13.1, this is standard to deduce the following normal Agmon estimates:

Proposition 13.4. *There exist $\delta > 0$, $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$, we have :*

$$\int_{\Omega_0} e^{\delta h^{-1/2} z}(|u_h|^2 + h^{-1}|(ih\nabla + \boldsymbol{A})u_h|^2) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq C\|u_h\|^2.$$

Corollary 13.5. *For all $\eta > 0$, we have:*

$$\int_{|x|+|y|\geq h^{1/8-\gamma}} |x|^k|y|^l|z|^m(|u_h|^2 + |D_x u_h|^2 + |D_y u_h|^2 + |D_z u_h|^2) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = O(h^\infty)\|u_h\|^2.$$

$$\int_{z\geq h^{1/2-\gamma}} |x|^k|y|^l|z|^m(|u_h|^2 + |D_x u_h|^2 + |D_y u_h|^2 + |D_z u_h|^2) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = O(h^\infty)\|u_h\|^2.$$

Let us consider $\eta > 0$ small enough and introduce the cutoff function defined by:

$$\chi_h(x, y) = \chi_0\left(h^{-1/8+\gamma}x, h^{-1/8+\gamma}y, h^{-1/2+\gamma}z\right),$$

where χ_0 is a smooth cutoff function being 1 near $(0, 0, 0)$. We can notice, by elliptic regularity, that $\chi_h u_h$ is smooth (as it is supported away from the vertices).

Let us consider $N \geq 1$. For $n = 1, \cdots, N$, let us consider $u_{n,h}$ a L^2-normalized associated with $\lambda_n(h)$ so that $\langle u_{n,h}, u_{m,h}\rangle = 0$ for $n \neq m$. We let:

$$\mathfrak{E}_N(h) = \operatorname*{span}_{n=1,\cdots,N} u_{n,h}.$$

We notice that Propositions 13.4 and 13.3 hold for the elements of $\mathfrak{E}_N(h)$. As a consequence of Propositions 13.4 and 13.3, we have:

Corollary 13.6. *We have:*

$$\mathfrak{Q}_h(\tilde{u}_h) \leq \lambda_N(h) + O(h^\infty), \qquad \textit{with } \tilde{u}_h = \chi_h u_h,$$

where $u_h \in \mathfrak{E}_N(h)$ and where $\mathfrak{Q}_h$ denotes the quadratic form associated with $\mathfrak{L}_h$.

2.2. Agmon estimates of higher order. In the last section we stated estimates of Agmon for u_h and its first derivatives. We will also need estimates for the higher order derivatives. The main idea to obtain such estimates can be found for instance in [**82**].

Proposition 13.7. *For all $\nu \in \mathbb{N}^3$, there exist $\delta > 0$, $\gamma \geq 0$, $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$:*

$$\int e^{\delta h^{-1/2} z}|D^\nu \tilde{u}_h|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq Ch^{-\gamma}\|\tilde{u}_h\|^2.$$

$$\int e^{\delta h^{-1/4}(x^2+y^2)}|D^\nu \tilde{u}_h|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \leq Ch^{-\gamma}\|\tilde{u}_h\|^2,$$

where $u_h \in \mathfrak{E}_N(h)$.

We infer:

Corollary 13.8. *For all $\eta > 0$, we have, for all $\nu \in \mathbb{N}^3$:*

$$\int_{|x|+|y|\geq h^{1/8-\gamma}} |x|^k |y|^l |z|^m |D^\nu \tilde{u}_h|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = O(h^\infty)\|\tilde{u}_h\|^2,$$

$$\int_{z\geq h^{1/2-\gamma}} |x|^k |y|^l |z|^m |D^\nu \tilde{u}_h|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = O(h^\infty)\|\tilde{u}_h\|^2,$$

where $u_h \in \mathfrak{E}_N(h)$.

2.3. Normal form. For $u_h \in \mathfrak{E}_N(h)$, we let:

(13.2.1)

$$w_h(r,s,t) = \chi_h^{\mathsf{resc}}(r,s,t) u_h^{\mathsf{resc}}(r,s,t) = \chi_0(h^{3/8+\gamma} r, h^{3/8+\gamma} s, h^\gamma t) u_h(h^{1/2} r, h^{1/2} s, h^{1/2} t)$$

and

$$v_h(\rho,\sigma,\tau) = U_\theta \mathcal{F}_{r\to\eta} w_h.$$

We consider $\mathcal{F}_N(h)$ the image of $\mathfrak{E}_N(h)$ by these transformations. We can reformulate Corollary 13.6.

Corollary 13.9. *With the previous notation, we have the lower bound, for $v_h \in \mathcal{F}_N(h)$:*

$$\mathcal{Q}_h^{\mathsf{No}}(v_h) \leq \lambda_N^{\mathsf{resc}}(h) + O(h^\infty),$$

where $\lambda_N^{\mathsf{resc}}(h) = h^{-1}\lambda_N(h)$.

We can also notice that, when u_h is an eigenfunction associated with $\lambda_p(h)$:

$$\mathcal{L}_h^{\mathsf{No}} v_h = \lambda_p^{\mathsf{resc}} v_h + r_h, \tag{13.2.2}$$

where the remainder r_h is $O(h^\infty)$ in the sense of Corollary 13.8.

In the following, we aim at proving localization and approximation estimates for v_h rather than u_h. Moreover, these approximations will allow us to estimate the energy $\mathcal{Q}_h^{\mathsf{No}}(v_h)$.

3. Relative polynomial localizations in the phase space

This section aims at estimating momenta of v_h with respect to polynoms in the phase space. Before starting the analysis, let us recall the link (cf. (3.2.6)) between the variables (η, s, t) and (ρ, σ, τ):

$$D_\rho = D_\eta + \frac{1}{\sin\theta} D_s, \quad D_\sigma = D_s, \quad D_\tau = D_t. \tag{13.3.1}$$

We will use the following obvious remark:

Remark 13.10. *We can notice that if ϕ is supported in $\mathsf{supp}(\chi_h)$, we have:*

$$\mathcal{Q}_h(\phi) \geq (1-\varepsilon)\mathfrak{Q}_{1,0,\theta}(\phi) - Ch^{1/2-6\gamma}\varepsilon^{-1}\|\phi\|^2.$$

Optimizing in ε, we have:

$$\mathcal{Q}_h(\phi) \geq (1 - h^{1/4-3\gamma})\mathfrak{Q}_{1,0,\theta}(\phi) - Ch^{1/4-3\gamma}\|\phi\|^2.$$

Moreover, when the support of ϕ avoids the boundary, we have:

$$\mathfrak{Q}_{1,0,\theta}(\phi) \geq \|\phi\|^2.$$

3.1. Localizations in σ and τ. This section is concerned with many localizations lemmas with respect to σ and τ.

3.1.1. *Estimates with respect to σ and τ.* We begin to prove estimates depending only on the variables σ and τ.

Lemma 13.11. *Let $N \geq 1$. For all k, n, there exist $h_0 > 0$ and $C(k,n) > 0$ such that, for all $h \in (0, h_0)$:*

$$\|\tau^k \sigma^{n+1} v_h\| \leq C(k,n)\|v_h\|, \tag{13.3.2}$$

$$\|\tau^k D_\sigma(\sigma^n v_h)\| \leq C(k,n)\|v_h\| \tag{13.3.3}$$

$$\|\tau^k D_\tau(\sigma^n v_h)\| \leq C(k,n)\|v_h\|., \tag{13.3.4}$$

for $v_h \in \mathcal{F}_N(h)$.

Proof. We prove the estimates when v_h is the image of an eigenfunction associated to $\lambda_p(h)$ with $p = 1, \cdots, N$.

Let us analyze the case $n = 0$. (13.3.4) follows from the normal Agmon estimates. We have:

$$\mathcal{Q}_h^{\mathsf{No}}(\tau^k v_h) \leq \lambda_p^{\mathsf{resc}}\|\tau^k v_h\|^2 + |\langle [D_\tau^2, \tau^k] w_h, \tau^k v_h\rangle| + O(h^\infty)\|v_h\|^2.$$

The normal Agmon estimates provide:

$$|\langle [D_\tau^2, \tau^k] v_h, \tau^k v_h\rangle| \leq C\|v_h\|^2$$

and thus:

$$\mathcal{Q}_h^{\mathsf{No}}(\tau^k v_h) \leq C\|v_h\|^2.$$

We deduce (13.3.3). We also have:

$$\|\tau^k(-\sigma\sin\theta + \tau\cos\theta + R_h)v_h\|^2 \leq C\|v_h\|^2,$$

where

$$R_h = h\alpha\tau\left\{\left(D_\rho - (\sin\theta)^{-1}D_\sigma\right)^2 + \left(\sigma + (\sin\theta)^{-1}\rho\right)^2\right\}. \tag{13.3.5}$$

We use the basic lower bound:

$$\|\tau^k(-\sigma\sin\theta + \tau\cos\theta + R_h)v_h\|^2 \geq \frac{1}{2}\|\tau^k\sigma\sin\theta v_h\|^2 - 2\|(\tau^{k+1}\cos\theta + \tau^k R_h)v_h\|^2.$$

Moreover, we have (using the support of χ_h^{resc}):

$$\|\tau^k R_h v_h\| \leq Ch(h^{-3/8-\gamma})^2\|\tau^{k+1}v_h\| \leq Ch(h^{-3/8-\gamma})^2\|v_h\|,$$

the last inequality coming from the normal Agmon estimates. Thus, we get:

$$\|\tau^k \sigma v_h\|^2 \leq C\|v_h\|^2.$$

We now proceed by induction. We apply $\tau^k\sigma^{n+1}$ to (13.2.2), take the scalar product with $\tau^k\sigma^{n+1}v_h$ and it follows:

$$\begin{aligned}
\mathcal{Q}_h^{\mathsf{No}}(\tau^k\sigma^{n+1}v_h) \leq{}& \lambda_p^{\mathsf{resc}}(h)\|\tau^k\sigma^{n+1}v_h\|^2 + C\|\tau^{k-2}\sigma^{n+1}v_h\|\|\tau^k\sigma^{n+1}v_h\| \\
&+ C\|\tau^{k-1}D_\tau\sigma^n v_h\|\|\tau^k\sigma^{n+1}v_h\| + C\|\tau^k D_\sigma\sigma^n w_h\|\|\tau^k\sigma^{n+1}v_h\| \\
&+ C\|\tau^k\sigma^{n-1}v_h\|\|\tau^k\sigma^{n+1}v_h\| + |\langle\tau^k[\sigma^{n+1},(-\sigma\sin\theta+\tau\cos\theta+R_h)^2]v_h,\tau^k\sigma^{n+1}\rangle|.
\end{aligned}$$

We have:

$$\begin{aligned}
&[\sigma^{n+1},(-\sigma\sin\theta+\tau\cos\theta+R_h)^2] \\
&= [\sigma^{n+1},R_h](-\sigma\sin\theta+\tau\cos\theta+R_h) + (-\sigma\sin\theta+\tau\cos\theta+R_h)[\sigma^{n+1},R_h].
\end{aligned}$$

Let us analyze the commutator $[\sigma^{n+1},R_h]$. We can write:

$$[\sigma^{n+1},R_h] = \alpha h\tau[\sigma^{n+1},\left(D_\rho-(\sin\theta)^{-1}D_\sigma\right)^2]$$

and:

$$\begin{aligned}
[\left(D_\rho-(\sin\theta)^{-1}D_\sigma\right)^2,\sigma^{n+1}] ={}& (\sin\theta)^{-2}n(n+1)\sigma^{n-1} \\
&+ 2i(\sin\theta)^{-1}(n+1)(D_\rho-(\sin\theta)^{-1}D_\sigma)\sigma^n
\end{aligned}$$

we infer:

$$\begin{aligned}
&[\sigma^{n+1},(-\sigma\sin\theta+\tau\cos\theta+R_h)^2] \\
&= \left(\alpha h\tau(\sin\theta)^{-2}n(n+1)\sigma^{n-1} + 2i\alpha h\tau(\sin\theta)^{-1}(n+1)(D_\rho-(\sin\theta)^{-1}D_\sigma)\sigma^n\right)(V_\theta+R_h) \\
&+ (V_\theta+R_h)\left(\alpha h\tau(\sin\theta)^{-2}n(n+1)\sigma^{n-1} + 2i\alpha h\tau(\sin\theta)^{-1}(n+1)(D_\rho-(\sin\theta)^{-1}D_\sigma)\sigma^n\right)
\end{aligned}$$

After having computed a few more commutators, the terms of $[\sigma^{n+1},(-\sigma\sin\theta+\tau\cos\theta+R_h)^2]$ are in the form:

$$\begin{gathered}
\tau^l\sigma^m, h\tau^l(D_\rho-(\sin\theta)^{-1}D_\sigma)\sigma^m, h^2\tau^l(D_\rho-(\sin\theta)^{-1}D_\sigma)^3\sigma^m, \\
h^2\tau^l(\sigma+(\sin\theta)^{-1}\rho)^2(D_\rho+(\sin\theta)^{-1}D_\sigma)\sigma^m
\end{gathered}$$

with $m \leq n+1$ and $l = 0,1,2$.

Let us examine for instance the term $h^2\tau^l(\sigma+(\sin\theta)^{-1}\rho)^2(D_\rho+(\sin\theta)^{-1}D_\sigma)\sigma^m$. We have, after the inverse Fourier transform and translation:

$$h^2\|\tau^l(\sigma+(\sin\theta)^{-1}\rho)^2(D_\rho+(\sin\theta)^{-1}D_\sigma)\sigma^m v_h\| \leq Ch^2(h^{-3/8-\gamma})^3\|\tau^l\sigma^m v_h\|$$

where we have used the support of χ_h^{resc} (see (13.2.1)). We get:

$$|\langle\tau^k[\sigma n+1,(-\sigma\sin\theta+\tau\cos\theta+R_h)^2]v_h,\tau^k\sigma^{n+1}v_h\rangle| \leq C\|\tau^k\sigma^{n+1}v_h\|\sum_{j=0}^{n+1}\sum_{l=0}^{k+2}\|\tau^l\sigma^j v_h\|.$$

We deduce by the induction assumption:

$$\mathcal{Q}_h^{\mathsf{No}}(\tau^k\sigma^{n+1}v_h) \leq C\|v_h\|^2.$$

We infer that, for all k:

$$\|D_\tau(\tau^k\sigma^{n+1})v_h\| \leq C\|v_h\| \text{ and } \|D_\sigma(\tau^k\sigma^{n+1})v_h\| \leq C\|v_h\|.$$

Moreover, we also deduce:

$$\|(V_\theta + R_h)\tau^k\sigma^{n+1}v_h\| \leq C\|v_h\|,$$

from which we find:

$$\|\tau^k\sigma^{n+2}v_h\| \leq C\|v_h\|.$$

□

We also need a control of the derivatives with respect to σ. The next lemma is left to the reader as an exercise.

Lemma 13.12. *For all m, n, k, there exist $h_0 > 0$ and $C(m,n,k) > 0$ such that for all $h \in (0, h_0)$:*

$$\|\tau^k D_\sigma^{m+1}\sigma^n v_h\| \leq C(k,m,n)\|v_h\| \tag{13.3.6}$$

$$\|\tau^k D_\sigma^m D_\tau\sigma^n v_h\| \leq C(k,m,n)\|v_h\|, \tag{13.3.7}$$

for $v_h \in \mathcal{F}_N(h)$.

We now establish partial Agmon estimates with respect to σ and τ. Roughly speaking, we can write the previous lemmas with ρv_h and $D_\rho v_h$ instead of v_h.

3.1.2. *Partial estimates involving ρ.* Let us begin to prove that:

Lemma 13.13. *For all $k \geq 0$, there exist $h_0 > 0$ and $C(k) > 0$ such that, for all $h \in (0, h_0)$:*

$$\begin{aligned}
\|\tau^k\rho v_h\| \quad &\leq C(\|\rho v_h\| + \|v_h\|),\\
\|\tau^k D_\tau\rho v_h\| \quad &\leq C(\|\rho v_h\| + \|v_h\|),\\
\|\tau^k D_\sigma\rho v_h\| \quad &\leq C(\|\rho v_h\| + \|v_h\|),
\end{aligned}$$

for $v_h \in \mathcal{F}_N(h)$.

Proof. For $k = 0$, we multiply (13.2.2) by ρ and take the scalar product with ρv_h. There is only one commutator to analyze:

$$[(V_\theta + R_h)^2, \rho] = [(V_\theta + R_h), \rho](V_\theta + R_h) + (V_\theta + R_h)[(V_\theta + R_h), \rho]$$

so that:

$$[(V_\theta + R_h)^2, \rho] = [R_h, \rho](V_\theta + R_h) + (V_\theta + R_h)[R_h, \rho].$$

We deduce, thanks to the support of w_h:

$$|\langle[(V_\theta + R_h)^2, \rho]v_h, \rho v_h\rangle| \leq C\|v_h\|\|\rho v_h\| \leq C(\|\rho v_h\|^2 + \|v_h\|^2)$$

and we infer:

$$\mathcal{Q}_h^{\mathsf{No}}(\rho v_h) \leq C(\|\rho v_h\|^2 + \|v_h\|^2).$$

We get:

$$\|D_\tau \rho v_h\| \leq C(\|\rho v_h\| + \|v_h\|) \text{ and } \|D_\sigma \rho v_h\| \leq C(\|\rho v_h\| + \|v_h\|).$$

Then it remains to prove the case $k \geq 1$ by induction (use Remark 13.10 and that $\mathfrak{s}(\theta) < 1$). □

As an easy consequence of the proof of Lemma 13.13, we have:

Lemma 13.14. *For all $k \geq 0$, there exist $h_0 > 0$ and $C(k) > 0$ such that, for all $h \in (0, h_0)$:*

$$\|\tau^k \sigma \rho v_h\| \leq C(k)(\|\rho v_h\| + \|v_h\|),$$

for $v_h \in \mathcal{F}_N(h)$.

We can now deduce the following lemma (exercise):

Lemma 13.15. *For all k, n, there exist $h_0 > 0$ and $C(k, n) > 0$ such that, for all $h \in (0, h_0)$:*

$$\|\rho\tau^k\sigma^{n+1}v_h\| \leq C(k, n)(\|\rho v_h\| + \|v_h\|), \tag{13.3.8}$$

$$\|\rho\tau^k D_\sigma(\sigma^n v_h)\| \leq C(k, n)(\|\rho v_h\| + \|v_h\|) \tag{13.3.9}$$

$$\|\rho\tau^k D_\tau(\sigma^n v_h)\| \leq C(k, n)(\|\rho v_h\| + \|v_h\|), \tag{13.3.10}$$

for $v_h \in \mathcal{F}_N(h)$.

From this lemma, we deduce a stronger control with respect to the derivative with respect to σ:

Lemma 13.16. *For all m, n, k, there exist $h_0 > 0$ and $C(m, n, k) > 0$ such that for all $h \in (0, h_0)$:*

$$\|\rho\tau^k D_\sigma^{m+1}\sigma^n v_h\| \leq C(k, m, n)(\|\rho v_h\| + \|v_h\|), \tag{13.3.11}$$

$$\|\rho\tau^k D_\sigma^m D_\tau\sigma^n v_h\| \leq C(k, m, n)(\|\rho v_h\| + \|v_h\|), \tag{13.3.12}$$

for $v_h \in \mathcal{F}_N(h)$.

Proof. The proof can be done by induction. The case $m = 0$ comes from the previous lemma. Then, the recursion is the same as for the proof of Lemma 13.12 and uses Lemma 13.12 to control the additional commutators. □

By using the symmetry between ρ and D_ρ, we have:

Lemma 13.17. *For all m, n, k, there exist $h_0 > 0$ and $C(m,n,k) > 0$ such that for all $h \in (0, h_0)$:*

$$\|D_\rho \tau^k D_\sigma^{m+1} \sigma^n v_h\| \leq C(k,m,n)(\|D_\rho v_h\| + \|v_h\|), \tag{13.3.13}$$

$$\|D_\rho \tau^k D_\sigma^{m} D_\tau \sigma^n v_h\| \leq C(k,m,n)(\|D_\rho v_h\| + \|v_h\|), \tag{13.3.14}$$

for $v_h \in \mathcal{F}_N(h)$.

In the next section, we prove that v_h behaves like $u_\theta^{\mathsf{LP}}(\sigma, \tau)$ with respect to σ and τ.

3.2. Approximation of v_h. Let us state the approximation result of this section:

Proposition 13.18. *There exists $C > 0$ and $h_0 > 0$ such that, for $h \in (0, h_0)$:*

$$\|v_h - \Pi v_h\| + \|V_\theta v_h - V_\theta \Pi v_h\| + \|\nabla_{\sigma,t}(v_h - \Pi v_h)\| \leq C h^{1/4 - 2\gamma} \|v_h\|,$$

where Π is the projection on u_θ^{LP} and $v_h \in \mathcal{F}_N(h)$.

Proof. As usual, we start to prove the inequality when v_h is the image of an eigenfunction associated with $\lambda_p(h)$, the extension to $v_h \in \mathcal{F}_N(h)$ being standard. We want to estimate

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \mathfrak{s}(\theta)) v_h\|.$$

We have:

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \mathfrak{s}(\theta)) v_h\| \leq \|(\mathfrak{H}^{\mathsf{Neu}}(\theta) - \lambda_p(h)) v_h\| + C h^{1/4} \|v_h\|.$$

With the definition of v_h and with Corollary 13.8, we have:

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \lambda_p(h)) v_h\| \leq h \|L_1 v_h\| + h^2 \|L_2 v_h\| + O(h^\infty) \|v_h\|.$$

Then, we can write:

$$\|L_1 v_h\| \leq C \left\| \tau V_\theta \left(D_\rho - \frac{D_\sigma}{\sin\theta} \right)^2 v_h \right\| + C \left\| \tau \left(D_\rho - \frac{D_\sigma}{\sin\theta} \right)^2 V_\theta v_h \right\| + C \left\| \tau V_\theta \left(\sigma + \frac{\rho}{\sin\theta} \right)^2 v_h \right\|$$

With Lemma 13.11 and the support of u_h, we infer:

$$h \|L_1 v_h\| \leq C h^{1/4 - 2\gamma} \|v_h\|.$$

In the same way, we get:

$$h^2 \|L_2 v_h\| \leq C h^{1/2 - 4\gamma} \|v_h\|.$$

We deduce:

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \mathfrak{s}(\theta)) v_h\| \leq C h^{1/4 - 2\gamma} \|v_h\|.$$

Let us write

$$v_h = v_h^\perp + \Pi v_h$$

We have:

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \mathfrak{s}(\theta)) v_h^\perp\| \leq C h^{1/4 - 2\gamma} \|v_h\|.$$

The resolvent, valued in the form domain, being bounded, the result follows. □

4. Localization induced by the effective harmonic oscillator

In this section, we prove Theorem 3.7. In order to do that, we first prove a localization with respect to ρ and then use it to improve the approximation of Proposition 13.18.

4.1. Control of v_h with respect to ρ. Let us prove an optimal localization estimate of the eigenfunctions with respect to $\hat{\eta}$. Thanks to our relative boundedness lemmas we can compare the original quadratic form with the model quadratic form.

Proposition 13.19. *There exist $h_0 > 0$ and $C > 0$ such that for all $C_0 > 0$ and $h \in (0, h_0)$:*

$$\begin{aligned} \mathcal{Q}_h^{\mathsf{No}}(v_h) \geq \quad & (1 - C_0 h)\left(\|D_\tau v_h\|^2 + \|D_\sigma v_h\|^2 + \|(V_\theta(\sigma,\tau) + \alpha h \tau \mathcal{H}_{\mathsf{harm}})\, v_h\|^2\right) \\ & - \frac{C}{C_0} h \langle \mathcal{H}_{\mathsf{harm}} v_h, v_h\rangle - Ch\|v_h\|^2, \end{aligned}$$

for $v_h \in \mathcal{F}_N(h)$.

Proof. Let us consider

$$\mathcal{Q}_h^{\mathsf{No}}(v_h) = \|D_\tau v_h\|^2 + \|D_\sigma v_h\|^2 + \left\|\left(V_\theta(\sigma,\tau) + \alpha h \tau \left\{\mathcal{H}_{\mathsf{harm}} + L(\rho, D_\rho, \sigma, D_\sigma)\right\}\right) v_h\right\|^2.$$

where

$$L(\rho, D_\rho, \sigma, D_\sigma) = (\sin\theta)^{-2}(-2\sin\theta D_\sigma D_\rho + 2\sin\theta \sigma\rho + D_\sigma^2 + \sigma^2).$$

For all $\varepsilon > 0$, we have:

$$\begin{aligned} \mathcal{Q}_h^{\mathsf{No}}(v_h) \geq \quad & (1 - \varepsilon)\left(\|D_\tau v_h\|^2 + \|D_\sigma v_h\|^2 + \|(V_\theta(\sigma,\tau) + \alpha h \tau \mathcal{H}_{\mathsf{harm}})\, v_h\|^2\right) \\ & - \varepsilon^{-1}\alpha^2 h^2 \|\tau L(\rho, D_\rho, \sigma, D_\sigma) v_h\|^2 \end{aligned}$$

We take $\varepsilon = C_0 h$. We apply Lemmas 13.12, 13.16 and 13.17 to get:

$$\|\tau L(\rho, D_\rho, \sigma, D_\sigma) v_h\|^2 \leq C(\|D_\rho v_h\|^2 + \|\rho v_h\|^2 + \|v_h\|^2).$$

$\square$

From the last proposition, we are led to study the model operator:

$$\mathcal{H}_h = D_\sigma^2 + D_\tau^2 + (V_\theta(\sigma,\tau) + \alpha h \tau \mathcal{H}_{\mathsf{harm}})^2.$$

We can write $\mathcal{H}_h$ as a direct sum:

$$\mathcal{H}_h = \bigoplus_{n \geq 1} \mathcal{H}_h^n,$$

with

$$\mathcal{H}_h^n = D_\sigma^2 + D_\tau^2 + (V_\theta(\sigma,\tau) + \alpha h \tau \mu_n)^2,$$

where μ_n is the n-the eigenvalue of $\mathcal{H}_{\mathsf{harm}}$. Therefore we shall analyze (see Chapter 6, Section 1.3.2):

$$\mathfrak{L}_{\theta,g}^{\mathsf{LP}} = D_\sigma^2 + D_\tau^2 + (V_\theta(\sigma,\tau) + g\tau)^2.$$

We deduce the existence of $c>0$ such that, for all $g \geq 0$:

$$\mathfrak{s}(\theta, g) \geq \mathfrak{s}(\theta) + cg.$$

Taking C_0 large enough in Proposition 13.19, we deduce the proposition:

Proposition 13.20. *There exist $C>0$ and $h_0>0$ such that:*

$$\langle \mathcal{H}_{\mathsf{harm}} v_h, v_h \rangle \leq C\|v_h\|^2, \text{ for } v_h \in \mathcal{F}_N(h)$$

and:

$$\lambda_N^{\mathsf{resc}}(h) \geq \sigma(\theta) - Ch.$$

4.2. Refined approximation of v_h. The control of v_h with respect to ρ provided by Proposition 13.20 permits to improve the approximation of v_h.

Proposition 13.21. *There exist $C>0$, $h_0>0$ and $\gamma>0$ such that, if $h \in (0, h_0)$:*

$$\|V_\theta D_\rho v_h - V_\theta D_\rho \Pi v_h\| + \|D_\rho v_h - D_\rho \Pi v_h\| + \|\nabla_{\sigma,\tau}(D_\rho v_h - D_\rho \Pi v_h)\| \leq Ch^\gamma \|v_h\|,$$
$$\|V_\theta \rho v_h - V_\theta \rho \Pi v_h\| + \|\rho v_h - \rho \Pi v_h\| + \|\nabla_{\sigma,\tau}(\rho v_h - \rho \Pi v_h)\| \leq Ch^\gamma \|v_h\|,$$

for $v_h \in \mathcal{F}_N(h)$.

PROOF. Let us apply D_ρ to (13.2.2). We have the existence of $\gamma > 0$ such that:

$$\|[\mathcal{L}_h^{\mathsf{No}}, D_\rho] v_h\| \leq Ch^\gamma \|v_h\|.$$

We can write:

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \sigma(\theta)) D_\rho v_h\| \leq \|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \lambda_p^{\mathsf{resc}}(h)) D_\rho v_h\| + Ch^{1/4}\|D_\rho v_h\|.$$

Proposition 13.20 provides:

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \sigma(\theta)) D_\rho v_h\| \leq \|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \lambda_p^{\mathsf{resc}}(h)) D_\rho v_h\| + Ch^{1/4}\|v_h\|.$$

Then, we get:

$$\|hL_1 D_\rho v_h\| \leq Ch^{1/4-2\gamma}\|v_h\|$$

and:

$$\|h^2 L_2 D_\rho v_h\| \leq Ch^{1/2-4\gamma}\|v_h\|.$$

We deduce:

$$\|(\mathfrak{H}_\theta^{\mathsf{Neu}} - \mathfrak{s}(\theta)) D_\rho v_h\| \leq Ch^{1/4-\gamma}\|v_h\|.$$

The conclusion is the same as for the proof of Proposition 13.18. The analysis for ρ can be done exactly in the same way. ☐

4.3. Conclusion: proof of Theorem 3.7. We recall that:

$$\mathcal{Q}_h^{\mathsf{No}}(v_h) = \|D_\tau v_h\|^2 + \|D_\sigma v_h\|^2 + \|(V_\theta(\sigma,\tau) + \alpha h\tau \{\mathcal{H}_{\mathsf{harm}} + L(\rho, D_\rho, \sigma, D_\sigma)\}) v_h\|^2$$

so that we get:

$$
\begin{aligned}
&\mathcal{Q}_h^{\mathsf{No}}(v_h) \geq \mathfrak{s}(\theta)\|v_h\|^2 \\
&+\alpha h\langle 2\tau V_\theta(\sigma,\tau)\mathcal{H}_{\mathsf{harm}} + \tau V_\theta L(\rho, D_\rho, \sigma, D_\sigma) + \tau L(\rho, D_\rho, \sigma, D_\sigma)V_\theta(\sigma,\tau)v_h, v_h\rangle
\end{aligned}
$$

It remains to approximate v_h by Πv_h modulo lower order remainders (exercise!). This implies:

$$
\mathcal{Q}_h^{\mathsf{No}}(v_h) \geq \mathfrak{s}(\theta)\|v_h\|^2 + \alpha h\langle \mathfrak{S}_\theta(D_\rho,\rho)\phi_h, \phi_h\rangle_{\mathsf{L}^2(\mathbb{R}_\rho)} + o(h)\|v_h\|^2,
$$

where $\phi_h = \langle v_h, u_\theta\rangle_{\mathsf{L}^2(\mathbb{R}_{\sigma,\tau})}$ and $v_h \in \mathcal{F}_N(h)$. With the min-max principle, we deduce the spectral gap between the lowest eigenvalues and it remains to use Proposition 13.18.

CHAPTER 14

When a magnetic field meets a curved edge

On oublie vite du reste ce qu'on n'a pas pensé avec profondeur, ce qui vous a été dicté par l'imitation, par les passions environnantes.

À la recherche du temps perdu,
La Prisonnière, Proust

This chapter is devoted to the proof of Theorem 3.13 announced in Chapter 3, Section 3. We focus on the specific features induced by the presence of a non smooth boundary.

1. Quasimodes

Before starting the analysis, we use the following scaling:

$$\check{s} = h^{1/4}\sigma, \quad \check{t} = h^{1/2}\tau, \quad \check{z} = h^{1/2}\mathscr{z} \tag{14.1.1}$$

so that we denote by $\mathcal{L}_h$ and $\mathcal{C}_h$ the operators $h^{-1}\check{\mathcal{L}}_h$ and $h^{-1/2}\check{\mathcal{C}}_h$ in the coordinates $(\sigma, \tau, \mathscr{z})$.

Using Taylor expansions, we can write in the sense of formal power series the magnetic Laplacian near the edge and the associated magnetic Neumann boundary condition:

$$\mathcal{L}_h \underset{h\to 0}{\sim} \sum_{j\geq 0} \mathcal{L}_j h^{j/4}$$

and

$$\mathcal{C}_h \underset{h\to 0}{\sim} \sum_{j\geq 0} \mathcal{C}_j h^{j/4},$$

where the first $\mathcal{L}_j$ and $\mathcal{T}_j$ are given by (see Conjecture 2.48):

$$\mathcal{L}_0 = D_\tau^2 + D_{\mathscr{z}}^2 + (\tau - \zeta_0^{\mathsf{e}})^2, \tag{14.1.2}$$

$$\mathcal{L}_1 = -2(\tau - \zeta_0^{\mathsf{e}})D_\sigma, \tag{14.1.3}$$

$$\mathcal{L}_2 = D_\sigma^2 + 2\kappa\mathcal{T}_0^{-1}\sigma^2 D_{\mathscr{z}}^2 + L_2, \tag{14.1.4}$$

where

$$L_2 = 2(\zeta_0^{\mathsf{e}} - \tau)\hat{r}_1 - \frac{\hat{l}}{2}\hat{P}\hat{P} + \hat{P}\frac{\hat{l}}{2}\hat{P} + \hat{P}\hat{L}\hat{P}, \quad \hat{P} = \begin{pmatrix} \zeta_0^{\mathsf{e}} - \tau \\ D_\tau \\ D_{\mathscr{z}} \end{pmatrix}, \tag{14.1.5}$$

and:

$$\mathcal{C}_0 = (-\tau + \zeta_0^{\mathsf{e}}, D_\tau, D_{z}),$$
$$\mathcal{C}_1 = (D_\sigma, 0, 0),$$
$$\mathcal{C}_2 = (0, 0, \kappa \mathcal{T}_0^{-1}\sigma^2 D_{z}) + \frac{\hat{l}}{2}\hat{P} + \hat{L}\hat{P},$$

with

(14.1.6) $$\kappa = -\frac{\mathcal{T}''(0)}{2} > 0.$$

We recall that $\mathcal{T}_0 = \mathcal{T}(0)$. We have used the notation

(14.1.7) $$\hat{r}_1(\tau, z) = h^{-1}\check{r}_1(h^{1/2}\tau, h^{1/2}z),$$

(14.1.8) $$\hat{l}(\tau, z) = h^{-1/2}\check{l}(h^{1/2}\tau, h^{1/2}\hat{z}),$$

(14.1.9) $$\hat{L}(\tau, z) = h^{-1/2}\check{L}(h^{1/2}\tau, h^{1/2}\hat{z}),$$

where $\check{r}_1$ is an homogeneous polynomial of degree 2 and where $\check{L}$ and $\check{l}$ depend linearly on $(\check{t}, \check{z})$. We will also use an asymptotic expansion of the normal $\hat{\boldsymbol{n}}(h)$. We recall that we have $\check{\boldsymbol{n}} = (-\mathcal{T}'(\check{s})\check{t}, -\mathcal{T}(\check{s}), \pm 1)$ so that we get:

$$\hat{\boldsymbol{n}}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \boldsymbol{n}_j h^{j/4},$$

with:

(14.1.10) $$\boldsymbol{n}_0 = (0, -\mathcal{T}_0, \pm 1), \quad \boldsymbol{n}_1 = (0,0,0), \quad \boldsymbol{n}_2 = (0, \kappa\sigma^2, 0).$$

We look for $(\hat{\lambda}(h), \hat{\psi}(h))$ in the form:

$$\hat{\lambda}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \mu_j h^{j/4},$$

$$\hat{\psi}(h) \underset{h\to 0}{\sim} \sum_{j\geq 0} \psi_j h^{j/4},$$

which satisfies, in the sense of formal series, the following boundary value problem:

(14.1.11) $$\begin{cases} \mathcal{L}_h\hat{\psi}(h) \underset{h\to 0}{\sim} \hat{\lambda}(h)\hat{\psi}(h), \\ \hat{\boldsymbol{n}} \cdot \mathcal{C}_h\hat{\psi}(h) \underset{h\to 0}{\sim} 0 \quad \text{on} \quad \partial_{\mathsf{Neu}}\mathcal{W}_{\alpha_0}. \end{cases}$$

This provides an infinite system of PDE's. We will use Notation 9.1 introduced in Chapter 9.

1.1. Terms in h^0. We solve the equation:

$$\mathcal{L}_0\psi_0 = \mu_0\psi_0, \text{ in } \mathcal{W}_{\alpha_0}, \quad \boldsymbol{n}_0 \cdot \mathcal{C}_0\psi_0 = 0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{W}_{\alpha_0}.$$

We notice that the boundary condition is exactly the Neumann condition. We are led to choose $\mu_0 = \nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}})$ and $\psi_0(\sigma, \tau, z) = u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}(\tau, z) f_0(\sigma)$ where f_0 will be chosen (in the Schwartz class) in a next step.

1.2. Terms in $h^{1/4}$. Collecting the terms of size $h^{1/4}$, we find the equation:

$$(\mathcal{L}_0 - \mu_0)\psi_1 = (\mu_1 - \mathcal{L}_1)\psi_0, \quad \boldsymbol{n}_0 \cdot \mathcal{C}_0\psi_1 = 0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{W}_{\alpha_0}.$$

As in the previous step, the boundary condition is just the Neumann condition. We use the Feynman-Hellmann formulas to deduce:

$$(\mathcal{L}_0 - \mu_0)(\psi_1 + v^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}(\tau, \mathcal{z})D_\sigma f_0(\sigma)) = \mu_1\psi_0, \quad \boldsymbol{n}_0 \cdot \mathcal{C}_0\psi_1 = 0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{W}_{\alpha_0}.$$

Taking the scalar product of the r.h.s. of the first equation with $u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}$ with respect to $(\tau, \mathcal{z})$ and using the Neumann boundary condition for $v^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}$ and ψ_1 when integrating by parts, we find $\mu_1 = 0$. This leads to choose:

$$\psi_1(\sigma, \tau, \mathcal{z}) = v^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}(\tau, \mathcal{z})D_\sigma f_0(\sigma) + f_1(\sigma)u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}(\tau, \mathcal{z}),$$

where f_1 will be determined in a next step.

1.3. Terms in $h^{1/2}$. Let us now deal with the terms of order $h^{1/2}$:

$$(\mathcal{L}_0 - \mu_0)\psi_2 = (\mu_2 - \mathcal{L}_2)\psi_0 - \mathcal{L}_1\psi_1, \quad \boldsymbol{n}_0 \cdot \mathcal{C}_0\psi_2 = -\boldsymbol{n}_0 \cdot \mathcal{C}_2\psi_0 - \boldsymbol{n}_2 \cdot \mathcal{C}_0\psi_0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{W}_{\alpha_0}.$$

We analyze the boundary condition:

$$\begin{aligned}
\boldsymbol{n}_0 \cdot \mathcal{C}_2\psi_0 + \boldsymbol{n}_2 \cdot \mathcal{C}_0\psi_0 &= \pm\kappa\mathcal{T}_0^{-1}\sigma^2 D_{\mathcal{z}}\psi_0 + \kappa\sigma^2 D_\tau\psi_0 + \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 + \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0 \\
&= \kappa\mathcal{T}_0^{-1}\sigma^2(\pm D_{\mathcal{z}} + \mathcal{T}_0 D_\tau)\psi_0 + \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 + \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0 \\
&= \pm 2\kappa\mathcal{T}_0^{-1}\sigma^2 D_{\mathcal{z}}\psi_0 + \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 + \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0.
\end{aligned}$$

where we have used the Neumann boundary condition of ψ_0. Then, we use the Feynman-Hellmann formulas together with (14.1.3) and (14.1.4) to get:
(14.1.12)

$$(\mathcal{L}_0 - \mu_0)(\psi_2 - v^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}D_\sigma f_1 - \frac{w^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}}{2}D_\sigma^2 f_0) = \mu_2\psi_0 - \frac{\partial_\zeta^2\nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}})}{2}D_\sigma^2\psi_0 - 2\kappa\mathcal{T}_0^{-1}\sigma^2 D_{\mathcal{z}}^2\psi_0 - L_2\psi_0,$$

with boundary condition:

$$\boldsymbol{n}_0 \cdot \mathcal{C}_0\psi_2 = \mp 2\kappa\sigma^2\mathcal{T}_0^{-1}D_{\mathcal{z}}\psi_0 - \boldsymbol{n}_0 \cdot \frac{\hat{l}}{2}\hat{P}\psi_0 - \boldsymbol{n}_0 \cdot \hat{L}\hat{P}\psi_0, \text{ on } \partial_{\mathsf{Neu}}\mathcal{W}_{\alpha_0}.$$

We use the Fredholm condition by taking the scalar product of the r.h.s. of (14.1.12) with $u^{\mathsf{e}}_{\alpha_0, \zeta_0^{\mathsf{e}}}$ with respect to $(\tau, \mathcal{z})$. Integrating by parts and using the Green-Riemann formula (the boundary terms cancel), this provides the equation:

$$\mathcal{H}^{\mathsf{e}}_{\mathsf{harm}} f_0 = (\mu_2 - \omega_0)f_0,$$

with:

$$\mathcal{H}^{\mathsf{e}}_{\mathsf{harm}} = \frac{\partial_\eta^2\nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}})}{2}D_\sigma^2 + 2\kappa\mathcal{T}_0^{-1}\|D_{\mathcal{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|^2_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}\sigma^2$$

and:

(14.1.13)

$$\omega_0 = \langle 2(\zeta_0^{\mathsf{e}} - \tau)\hat{r}_1 u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}, u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}} \rangle_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})} - \frac{\nu_1^{\mathsf{e}}(\alpha_0,\zeta_0^{\mathsf{e}})}{2} \langle \hat{l} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}, u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}} \rangle_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})} + \frac{1}{2} \langle \hat{l}\hat{P} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}, \hat{P} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}} \rangle_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})} + \langle \hat{L}\hat{P} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}, \hat{P} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}} \rangle_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}.$$

Up to a scaling, the 1D-operator $\mathcal{H}^{\mathsf{e}}_{\mathsf{harm}}$ is the harmonic oscillator on the line (we have used that Conjecture 2.48 is true). Its spectrum is given by:

$$\left\{ (2n-1)\sqrt{\kappa \mathcal{T}_0^{-1} \|D_{\mathcal{Z}} u_{\zeta_0^{\mathsf{e}}}\|^2 \partial_\zeta^2 \nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}})}, \quad n \geq 1 \right\}.$$

Therefore for μ_2 we take:

$$\mu_2 = \omega_0 + (2n-1)\sqrt{\kappa \mathcal{T}_0^{-1} \|D_{\mathcal{Z}} u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}\|^2_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})} \partial_\zeta^2 \nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}})} \tag{14.1.14}$$

with $n \in \mathbb{N}^*$ and for f_0 the corresponding normalized eigenfunction. With this choice we deduce the existence of $\psi_2^\perp$ such that:

$$(\mathcal{L}_0 - \mu_0)\psi_2^\perp = \mu_2 \psi_0 - \frac{\partial_\zeta^2 \nu_1^{\mathsf{e}}(\alpha_0,\zeta_0^{\mathsf{e}})}{2} D_\sigma^2 \psi_0 - 2\kappa \mathcal{T}_0^{-1} \sigma^2 D_{\mathcal{Z}}^2 \psi_0, \text{ and } \langle \psi_2^\perp, u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}} \rangle_{\tau,\mathcal{Z}} = 0. \tag{14.1.15}$$

We can write ψ_2 in the form:

$$\psi_2 = \psi_2^\perp + v_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}} D_\sigma f_1 + D_\sigma^2 f_0 \frac{w_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}}}{2} + f_2(\sigma) u_{\zeta_0^{\mathsf{e}}}^{\mathsf{e}},$$

where f_2 has to be determined in a next step.

The construction can be continued (exercise).

2. Agmon estimates

Thanks to a standard partition of unity, we can establish the following estimate for the eigenvalues.

Proposition 14.1. *There exist C and $h_0 > 0$ such that, for $h \in (0, h_0)$:*

$$\lambda_n(h) \geq \nu(\alpha_0)h - Ch^{5/4}.$$

From Proposition 14.1, we infer a localization near E.

Proposition 14.2. *There exist $\varepsilon_0 > 0, h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$:*

$$\int_\Omega e^{2\varepsilon_0 h^{-1/2} d(\mathbf{x}, E)} |\psi|^2 \, \mathrm{d}\mathbf{x} \leq C\|\psi\|^2,$$

$$\mathfrak{Q}_h(e^{\varepsilon_0 h^{-1/2} d(\mathbf{x}, E)} \psi) \leq Ch\|\psi\|^2.$$

As a consequence, we get:

Proposition 14.3. *For all $n \geq 1$, there exists $h_0 > 0$ such that for $h \in (0, h_0)$, we have:*

$$\lambda_n(h) = \nu(\alpha_0, \zeta_0^{\mathsf{e}})h + O(h^{3/2}).$$

PROOF. We have:

$$\check{\mathfrak{Q}}_h(\check{\psi}) = \langle \check{G}^{-1}\check{\nabla}_h\check{\psi}, \check{\nabla}_h\check{\psi}\rangle_{\mathsf{L}^2(\,\mathrm{d}\check{s}\,\mathrm{d}\check{t}\,\mathrm{d}\check{z})}.$$

With the Taylor expansion of $\check{G}^{-1}$ and $|\check{G}|$ and the estimates of Agmon with respect to $\check{t}$ and $\check{z}$, we infer:

$$\check{\mathfrak{Q}}_h(\check{\psi}) \geq \mathcal{Q}_h^{\mathsf{flat}}(\check{\psi}) - Ch^{3/2}\|\check{\psi}\|^2.$$

where:

$$\check{\mathfrak{Q}}_h^{\mathsf{flat}}(\check{\psi}) = \|hD_{\check{t}}\check{\psi}\|^2 + \|h\mathcal{T}_0\mathcal{T}(\check{s})^{-1}D_{\check{z}}\check{\psi}\|^2 + \|(hD_{\check{s}} + \zeta_0^{\mathsf{e}}h^{1/2} - \check{t})\check{\psi}\|^2.$$

Moreover, we have:

$$\check{\mathfrak{Q}}_h^{\mathsf{flat}}(\check{\psi}) \geq \|hD_{\check{t}}\check{\psi}\|^2 + \|hD_{\check{z}}\check{\psi}\|^2 + \|(hD_{\check{s}} + \zeta_0^{\mathsf{e}}h^{1/2} - \check{t})\check{\psi}\|^2 \geq \nu(\alpha_0, \zeta_0^{\mathsf{e}})h.$$

$\square$

A rough localization estimate is given by the following proposition.

Proposition 14.4. *There exist $\varepsilon_0 > 0, h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$:*

$$\int_\Omega e^{\chi(\mathbf{x})h^{-1/8}|s(\mathbf{x})|}|\psi|^2\,\mathrm{d}\mathbf{x} \leq C\|\psi\|^2,$$
$$\mathfrak{Q}_h(e^{\chi(\mathbf{x})h^{-1/8}|s(\mathbf{x})|}\psi) \leq Ch\|\psi\|^2,$$

where χ is a smooth cutoff function supported in a fixed neighborhood of E.

We use a cutoff function $\chi_h(\mathbf{x})$ near $\mathbf{x}_0$ such that:

$$\chi_h(\mathbf{x}) = \chi_0(h^{1/8-\gamma}\check{s}(\mathbf{x}))\chi_0(h^{1/2-\gamma}\check{t}(\mathbf{x}))\chi_0(h^{1/2-\gamma}\check{z}(\mathbf{x})).$$

For all $N \geq 1$, let us consider L^2-normalized eigenpairs $(\lambda_n(h), \psi_{n,h})_{1\leq n\leq N}$ such that $\langle \psi_{n,h}, \psi_{m,h}\rangle = 0$ when $n \neq m$. We consider the N dimensional space defined by:

$$\mathfrak{E}_N(h) = \operatorname*{span}_{1\leq n\leq N} \tilde{\psi}_{n,h}, \quad \text{where} \quad \tilde{\psi}_{n,h} = \chi_h\psi_{n,h}.$$

Notation 14.5. *We will denote by $\tilde{\psi}(=\chi_h\psi)$ the elements of $\mathfrak{E}_N(h)$.*

Let us state a proposition providing the localization of the eigenfunctions with respect to $D_{\check{s}}$ (the proof is left to the reader as an exercise).

Proposition 14.6. *There exist $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$ and $\check{\psi} \in \check{\mathfrak{E}}_N(h)$, we have:*

$$\|D_{\check{s}}\check{\psi}\| \leq Ch^{-1/4}\|\check{\psi}\|.$$

3. Projection method

The result of Proposition 14.6 implies an approximation result for the eigenfunctions. Let us recall the scaling defined in (14.1.1):

$$\check{s} = h^{1/4}\sigma, \quad \check{t} = h^{1/2}\tau, \quad \check{z} = h^{1/2}\mathbf{z}. \tag{14.3.1}$$

Notation 14.7. *We will denote by $\mathcal{E}_N(h)$ the set of the rescaled elements of $\check{\mathfrak{E}}_N(h)$. The elements of $\mathcal{E}_N(h)$ will be denoted by $\hat{\psi}$. Moreover we will denote by $\mathcal{L}_h$ the operator $h^{-1}\check{\mathcal{L}}_h$ in the rescaled coordinates. The corresponding quadratic form will be denoted by $\mathcal{Q}_h$.*

Lemma 14.8. *There exist $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$ and $\hat{\psi} \in \mathcal{E}_N(h)$, we have:*

(14.3.2)

$$\|\hat{\psi} - \Pi_0\hat{\psi}\| + \|D_\tau(\hat{\psi} - \Pi_0\hat{\psi})\| + \|D_{\mathfrak{z}}(\hat{\psi} - \Pi_0\hat{\psi})\| \leq Ch^{1/8}\|\hat{\psi}\|$$

(14.3.3)

$$\|\sigma(\hat{\psi} - \Pi_0\hat{\psi})\| + \|\sigma D_\tau(\hat{\psi} - \Pi_0\hat{\psi})\| + \|\sigma D_{\mathfrak{z}}(\hat{\psi} - \Pi_0\hat{\psi})\| \leq Ch^{1/8-\gamma}(\|\hat{\psi}\| + (\|\sigma\hat{\psi}\|),$$

where Π_0 is the projection on $u_{\zeta_0^{\mathsf{e}}}$:

$$\Pi_0\hat{\psi} = \langle\hat{\psi}, u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\rangle_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})} u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}.$$

This approximation result allows us to catch the behavior of the eigenfunction with respect to $\check{s}$. In fact, this is the core of the dimension reduction process of the next proposition. Indeed $\sigma^2 D^2_{\mathfrak{z}}$ is not an elliptic operator, but, once projected on $u_{\zeta_0^{\mathsf{e}}}$, it becomes elliptic.

Proposition 14.9. *There exist $h_0 > 0$ and $C > 0$ such that, for $h \in (0, h_0)$ and $\check{\psi} \in \check{\mathfrak{E}}_N(h)$, we have:*

$$\|\check{s}\check{\psi}\| \leq Ch^{1/4}\|\check{\psi}\|.$$

Proof. It is equivalent to prove that:

$$\|\sigma\hat{\psi}\| \leq C\|\hat{\psi}\|.$$

The proof of Proposition 14.3 provides the inequality:

$$\|D_\tau\hat{\psi}\|^2 + \|\mathcal{T}_0\mathcal{T}(h^{1/4}\sigma)^{-1}D_{\mathfrak{z}}\hat{\psi}\|^2 + \|(h^{1/4}D_\sigma + \zeta_0^{\mathsf{e}} - \tau)\hat{\psi}\|^2 \leq (\nu_1^{\mathsf{e}}(\alpha_0, \zeta_0^{\mathsf{e}}) + Ch^{1/2})\|\hat{\psi}\|^2.$$

From the non-degeneracy of the maximum of α, we deduce the existence of $c > 0$ such that:

$$\|\mathcal{T}_0\mathcal{T}(h^{1/4}\sigma)^{-1}D_{\mathfrak{z}}\hat{\psi}\|^2 \geq \|D_{\mathfrak{z}}\hat{\psi}\|^2 + ch^{1/2}\|\sigma D_{\mathfrak{z}}\hat{\psi}\|^2$$

so that we have:

$$ch^{1/2}\|\sigma D_{\mathfrak{z}}\hat{\psi}\|^2 \leq Ch^{1/2}\|\hat{\psi}\|^2$$

and:

$$\|\sigma D_{\mathfrak{z}}\hat{\psi}\| \leq \tilde{C}\|\hat{\psi}\|.$$

It remains to use Lemma 14.8 and especially (14.3.3). In particular, we have:

$$\|\sigma D_{\mathfrak{z}}(\hat{\psi} - \Pi_0\hat{\psi})\| \leq Ch^{1/8-\gamma}(\|\hat{\psi}\| + (\|\sigma\hat{\psi}\|).$$

We infer:

$$\|\sigma D_{\mathfrak{z}}\Pi_0\hat{\psi}\| \leq \tilde{C}\|\hat{\psi}\| + Ch^{1/8-\gamma}(\|\hat{\psi}\| + (\|\sigma\hat{\psi}\|).$$

Let us write
$$\Pi_0\hat\psi = f_h(\sigma)u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}(\tau,\mathsf{z}).$$
We have:
$$\|\sigma D_{\mathsf{z}}\Pi_0\hat\psi\| = \|D_{\mathsf{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}\|\sigma f_h\|_{\mathsf{L}^2(d\sigma)} = \|D_{\mathsf{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}\|\sigma f_h u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\| = \|D_{\mathsf{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}\|\sigma\Pi_0\hat\psi\|.$$
We use again Lemma 14.8 to get:
$$\|\sigma D_{\mathsf{z}}\Pi_0\hat\psi\| = \|D_{\mathsf{z}}u_{\zeta_0^{\mathsf{e}}}\|_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}\|\sigma\hat\psi\| + O(h^{1/8-\gamma})(\|\hat\psi\| + \|\sigma\hat\psi\|).$$
We deduce:
$$\|D_{\mathsf{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}\|\sigma\hat\psi\| \leq \tilde C\|\hat\psi\| + 2Ch^{1/8-\gamma}(\|\hat\psi\| + (\|\sigma\hat\psi\|)$$
and the conclusion follows. □

Proposition 14.10. *There exists $h_0 > 0$ such that for $h \in (0, h_0)$ and $\hat\psi \in \hat{\mathfrak{E}}_N(h)$, we have:*
$$\hat{\mathcal{Q}}_h(\hat\psi) \geq \|D_\tau\hat\psi\|^2 + \|D_{\mathsf{z}}\hat\psi\|^2 + \|(h^{1/4}D_\sigma - \tau + \zeta_0^{\mathsf{e}})\hat\psi\|^2 + h^{1/2}\mathcal{T}_0^{-1}\kappa\|D_{\mathsf{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|^2_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}\sigma^2 + \tilde C_0 h^{1/2}\|\hat\psi\|^2 + o(h^{1/2})\|\hat\psi\|^2,$$
with:

(14.3.4) $\tilde C_0 = \langle(2(\zeta_0^{\mathsf{e}} - \tau)\hat r_1 u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}, u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\rangle_{\mathsf{L}^2(\,\mathrm{d}\tau\,\mathrm{d}\mathsf{z})} + \frac{1}{2}\langle\hat l\hat P u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}, \hat P u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\rangle_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})} + \langle\hat L\hat P u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}, \hat P u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\rangle_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}$,

where $\hat P, \hat l, \hat L$ and $\hat r_j$ are homogeneous polynomials defined in (14.1.5), (14.1.7), (14.1.7) *and* (14.1.9).

Let us introduce the operator:

(14.3.5) $$D_\tau^2 + D_{\mathsf{z}}^2 + (h^{1/4}D_\sigma - \tau + \zeta_0^{\mathsf{e}})^2 + h^{1/2}\mathcal{T}_0^{-1}\kappa\|D_{\mathsf{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|^2\sigma^2 + C_0h^{1/2}.$$

After Fourier transform with respect to σ, the operator (14.3.5) becomes:

(14.3.6) $$D_\tau^2 + D_{\mathsf{z}}^2 + (h^{1/4}\xi - \tau + \zeta_0^{\mathsf{e}})^2 + h^{1/2}\mathcal{T}_0^{-1}\kappa\|D_{\mathsf{z}}u^{\mathsf{e}}_{\zeta_0^{\mathsf{e}}}\|^2_{\mathsf{L}^2(\mathcal{S}_{\alpha_0})}D_\xi^2 + C_0h^{1/2}.$$

Exercise. Use the Born-Oppenheimer approximation to estimate the lowest eigenvalues of this last operator and deduce Theorem 3.13.

CHAPTER 15

Magnetic Birkhoff normal form and low lying spectrum

Μηδείς ἀγεωμέτρητος εἰσίτω μου τὴν στέγην.

This chapter is devoted to the proofs of Theorems 3.19 and 3.18 announced in Chapter 15, Section 4.

1. Magnetic Birkhoff normal form

In this section we prove Theorem 3.19.

1.1. Symplectic normal bundle of Σ. We introduce the submanifold of all particles at rest ($\dot{q} = 0$):

$$\Sigma := H^{-1}(0) = \{(q,p); \qquad p = A(q)\}.$$

Since it is a graph, it is an embedded submanifold of $\mathbb{R}^4$, parameterized by $q \in \mathbb{R}^2$.

Lemma 15.1. *Σ is a symplectic submanifold of $\mathbb{R}^4$. In fact,*

$$j^*\omega_{\restriction\Sigma} = dA \simeq B,$$

where $j : \mathbb{R}^2 \to \Sigma$ is the embedding $j(q) = (q, A(q))$.

Proof. We compute $j^*\omega = j^*(dp_1 \wedge dq_1 + dp_2 \wedge dq_2) = (-\frac{\partial A_1}{\partial q_2} + \frac{\partial A_2}{\partial q_1})dq_1 \wedge dq_2 \neq 0$. $\square$

Since we are interested in the low energy regime, we wish to describe a small neighborhood of Σ in $\mathbb{R}^4$, which amounts to understanding the normal symplectic bundle of Σ. For any $q \in \Omega$, we denote by $T_q\mathbf{A} : \mathbb{R}^2 \to \mathbb{R}^2$ the tangent map of $\mathbf{A}$. Then of course the vectors $(Q, T_q\mathbf{A}(Q))$, with $Q \in T_q\Omega = \mathbb{R}^2$, span the tangent space $T_{j(q)}\Sigma$. It is interesting to notice that the symplectic orthogonal $T_{j(q)}\Sigma^\perp$ is very easy to describe as well.

Lemma 15.2. *For any $q \in \Omega$, the vectors*

$$u_1 := \frac{1}{\sqrt{|B|}}(e_1, {}^tT_q\boldsymbol{A}(e_1)); \quad v_1 := \frac{\sqrt{|B|}}{B}(e_2, {}^tT_q\boldsymbol{A}(e_2))$$

form a symplectic basis of $T_{j(q)}\Sigma^\perp$.

PROOF. Let $(Q_1, P_1) \in T_{j(q)}\Sigma$ and (Q_2, P_2) with $P_2 = {}^tT_q\mathbf{A}(Q_2)$. Then from (3.4.4) we get

$$\begin{aligned}\omega((Q_1, P_1), (Q_2, P_2)) &= \langle T_q\mathbf{A}(Q_1), Q_2\rangle - \langle {}^tT_q\mathbf{A}(Q_2), Q_1\rangle \\ &= 0.\end{aligned}$$

This shows that u_1 and v_1 belong to $T_{j(q)}\Sigma^\perp$. Finally

$$\begin{aligned}\omega(u_1, v_1) &= \frac{1}{B}\left(\langle {}^tT_q\mathbf{A}(e_1), e_2\rangle - \langle {}^tT_q\mathbf{A}(e_2), e_1\rangle\right) \\ &= \frac{1}{B}\langle e_1, (T_q\mathbf{A} - {}^tT_q\mathbf{A})(e_2)\rangle \\ &= \frac{1}{B}\langle e_1, \vec{B} \wedge e_2\rangle = -\frac{B}{B}\langle e_1, e_1\rangle = -1.\end{aligned}$$

$\square$

Thanks to this lemma, we are able to give a simple formula for the transversal Hessian of H, which governs the linearized (fast) motion:

Lemma 15.3. *The transversal Hessian of H, as a quadratic form on $T_{j(q)}\Sigma^\perp$, is given by*

$$\forall q \in \Omega, \forall (Q, P) \in T_{j(q)}\Sigma^\perp, \quad d_q^2H((Q, P)^2) = 2\|Q \wedge \vec{B}\|^2.$$

PROOF. Let $(q, p) = j(q)$. From (3.4.2) we get

$$dH = 2\langle p - A, dp - T_q\mathbf{A} \circ dq\rangle.$$

Thus

$$d^2H((Q, P)^2) = 2\|(dp - T_q\mathbf{A} \circ dq)(Q, P)\|^2 + \langle p - A, M((Q, P)^2)\rangle,$$

and it is not necessary to compute the quadratic form M, since $p - A = 0$. We obtain

$$\begin{aligned}d^2H((Q, P)^2) &= 2\|P - T_q\mathbf{A}(Q)\|^2 \\ &= 2\|({}^tT_q\mathbf{A} - T_q\mathbf{A})(Q)\|^2 = 2\|Q \wedge \vec{B}\|^2.\end{aligned}$$

$\square$

We may express this Hessian in the symplectic basis (u_1, v_1) given by Lemma 15.2:

$$d^2H_{\restriction T_{j(q)}\Sigma^\perp} = \begin{pmatrix} 2\,|B| & 0 \\ 0 & 2\,|B| \end{pmatrix}. \tag{15.1.1}$$

Indeed, $\|e_1 \wedge \vec{B}\|^2 = B^2$, and the off-diagonal term is $\frac{1}{B}\langle e_1 \wedge \vec{B}, e_2 \wedge \vec{B}\rangle = 0$.

1.2. Proof of Theorem 3.19. We use the notation of the previous section. We endow $\mathbb{C}_{z_1} \times \mathbb{R}^2_{z_2}$ with canonical variables $z_1 = x_1 + i\xi_1$, $z_2 = (x_2, \xi_2)$, and symplectic form $\omega_0 := d\xi_1 \wedge dx_1 + d\xi_2 \wedge dx_2$. By Darboux's theorem, there exists a diffeomorphism

$g : \Omega \to g(\Omega) \subset \mathbb{R}^2_{z_2}$ such that $g(q_0) = 0$ and

$$g^*(d\xi_2 \wedge dx_2) = j^*\omega.$$

(We identify g with φ in the statement of the theorem.) In other words, the new embedding $\tilde{\jmath} := j \circ g^{-1} : \mathbb{R}^2 \to \Sigma$ is symplectic. In fact we can give an explicit choice for g by introducing the global change of variables:

$$x_2 = q_1, \quad \xi_2 = \int_0^{q_2} B(q_1, s)\,\mathrm{d}s.$$

Consider the following map $\tilde{\Phi}$ (where we identify Ω and $g(\Omega)$):

$$\mathbb{C} \times \Omega \xrightarrow{\tilde{\Phi}} N\Sigma \tag{15.1.2}$$

$$(x_1 + i\xi_1, z_2) \mapsto x_1 u_1(z_2) + \xi_1 v_1(z_2), \tag{15.1.3}$$

where $u_1(z_2)$ and $v_1(z_2)$ are the vectors defined in Lemma 15.2 with $q = g^{-1}(z_2)$. This is an isomorphism between the normal symplectic bundle of $\{0\} \times \Omega$ and $N\Sigma$, the normal symplectic bundle of Σ: indeed, Lemma 15.2 says that for fixed z_2, the map $z_1 \mapsto \tilde{\Phi}(z_1, z_2)$ is a linear symplectic map. This implies, by a general result of Weinstein [**170**], that there exists a symplectomorphism Φ from a neighborhood of $\{0\} \times \Omega$ to a neighborhood of $\tilde{\jmath}(\Omega) \subset \Sigma$ whose differential at $\{0\} \times \Omega$ is equal to $\tilde{\Phi}$. Let us recall how to prove this.

First, we may identify $\tilde{\Phi}$ with a map into $\mathbb{R}^4$ by

$$\tilde{\Phi}(z_1, z_2) = \tilde{\jmath}(z_2) + x_1 u_1(z_2) + \xi_1 v_1(z_2).$$

Its Jacobian at $z_1 = 0$ in the canonical basis of $T_{z_1}\mathbb{C} \times T_{z_2}\Omega = \mathbb{R}^4$ is a matrix with column vectors $[u_1, v_1, T_{z_2}\tilde{\jmath}(e_1), T_{z_2}\tilde{\jmath}(e_2)]$, which by Lemma 15.2 is a basis of $\mathbb{R}^4$: thus $\tilde{\Phi}$ is a local diffeomorphism at every $(0, z_2)$. Therefore if $\epsilon > 0$ is small enough, $\tilde{\Phi}$ is a diffeomorphism of $B(\epsilon) \times \Omega$ into its image.

($B(\epsilon) \subset \mathbb{C}$ is the open ball of radius ϵ).

Since $\tilde{\jmath}$ is symplectic, Lemma 15.2 implies that the basis $[u_1, v_1, T_{z_2}\tilde{\jmath}(e_1), T_{z_2}\tilde{\jmath}(e_2)]$ is symplectic in $\mathbb{R}^4$; thus the Jacobian of $\tilde{\Phi}$ on $\{0\} \times \Omega$ is symplectic. This in turn can be expressed by saying that the 2-form

$$\omega_0 - \tilde{\Phi}^*\omega_0$$

vanishes on $\{0\} \times \Omega$.

Lemma 15.4. *Let us consider ω_0 and ω_1 two 2-forms on $\mathbb{R}^4$ which are closed and non degenerate. Let us assume that $\omega_1 = \omega_0$ on $\{0\} \times \Omega$ where Ω is a bounded open set. In a neighborhood of $\{0\} \times \Omega$ there exists a change of coordinates ψ_1 such that:*

$$\psi_1^*\omega_1 = \omega_0 \quad \textit{and} \quad \psi_1 = \mathsf{Id} + O(|z_1|^2).$$

PROOF. The proof of this relative Darboux lemma is standard but we recall it for completeness (see [**131**, p. 92]).

Let us begin to recall how we can find a 1-form σ on $\mathbb{R}^2$ such that in a neighborhood of $\{0\} \times \Omega$:

$$\tau := \omega_1 - \omega_0 = d\sigma \quad \text{and} \quad \sigma = O(|z_1|^2).$$

We introduce the family of diffeomorphisms $(\phi_t)_{0<t\leq 1}$ defined by:

$$\phi_t(x_1, x_2, \xi_1, \xi_2) = (tx_1, x_2, t\xi_1, \xi_2)$$

and we let:

$$\phi_0(x_1, x_2, \xi_1, \xi_2) = (0, x_2, 0, \xi_2).$$

We have:

$$\phi_0^* \tau = 0 \quad \text{and} \quad \phi_1^* \tau = \tau; \tag{15.1.4}$$

Let us denote by X_t the vector field associated with ϕ_t:

$$X_t = \frac{d\phi_t}{dt}(\phi_t^{-1}) = (t^{-1}x_1, 0, t^{-1}\xi_1, 0) = t^{-1}x_1 e_1 + t^{-1}\xi_1 e_3.$$

Let us compute the Lie derivative of τ along X_t: $\frac{d}{dt}\phi_t^*\tau = \phi_t^*\mathcal{L}_{X_t}\tau$. From the Cartan formula, we have: $\mathcal{L}_{X_t} = \iota(X_t)d\tau + d(\iota(X_t)\tau)$. Since τ is closed on $\mathbb{R}^4$, we have $d\tau = 0$. Therefore it follows:

$$\frac{d}{dt}\phi_t^*\tau = d(\phi_t^*\iota(X_t)\tau). \tag{15.1.5}$$

We consider the 1-form

$$\sigma_t := \phi_t^*\iota(X_t)\tau = x_1 \tau_{\phi_t(x_1,x_2,\xi_1,\xi_2)}(e_1, \nabla\phi_t(\cdot)) + \xi_1 \tau_{\phi_t(x_1,x_2,\xi_1,\xi_2)}(e_3, \nabla\phi_t(\cdot)) = O(|z_1|^2).$$

We see from (15.1.5) that the map $t \mapsto \phi_t^*\tau$ is smooth on $[0,1]$. To conclude, let σ be the 1-form defined on a neighborhood of $\{0\} \times \Omega$ by $\sigma = \int_0^1 \sigma_t \, dt$; it follows from (15.1.4) and (15.1.5) that:

$$\frac{d}{dt}\phi_t^*\tau = d\sigma_t \quad \text{and} \quad \tau = d\sigma.$$

Finally we use a standard deformation argument due to Moser. For $t \in [0,1]$, we let: $\omega_t = \omega_0 + t(\omega_1 - \omega_0)$. The 2-form ω_t is closed and non degenerate (up to choosing a neighborhood of $\hat{z}_1 = 0$ small enough). We look for ψ_t such that:

$$\psi_t^*\omega_t = \omega_0.$$

For that purpose, let us determine a vector field Y_t such that:

$$\frac{d}{dt}\psi_t = Y_t(\psi_t).$$

By using again the Cartan formula, we get:

$$0 = \frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \iota(Y_t)d\omega_t + d(\iota(Y_t)\omega_t)\right).$$

This becomes:

$$\omega_0 - \omega_1 = d(\iota(Y_t)\omega_t).$$

We are led to solve:

$$\iota(Y_t)\omega_t = -\sigma.$$

By non degeneracy of ω_t, this determines Y_t. Since Y_t vanishes on $\{0\} \times \Omega$, there exists a neighborhood of $\{0\} \times \Omega$ small enough in the $\hat{z}_1$-direction such that ψ_t exists until the time $t = 1$ and satisfies $\psi_t^* \omega_t = \omega_0$. Since $\sigma = O(|z_1|^2)$, we get $\psi_1 = \mathsf{Id} + O(|z_1|^2)$. $\square$

Lemma 15.5. *There exists a smooth and injective map $S : B(\epsilon) \times \Omega \to B(\epsilon) \times \Omega$, which is tangent to the identity along $\{0\} \times \Omega$, such that*

$$S^* \tilde{\Phi}^* \omega = \omega_0.$$

PROOF. It is sufficient to apply Lemma 15.4 to $\omega_1 = \tilde{\Phi}^* \omega_0$. $\square$

We let $\Phi := \tilde{\Phi} \circ S$; this is the claimed symplectic map. We let $(z_1, z_2) = \Phi(\hat{z}_1, \hat{z}_2)$. Let us now analyze how the Hamiltonian H is transformed under Φ. The zero-set $\Sigma = H^{-1}(0)$ is now $\{0\} \times \Omega$, and the symplectic orthogonal $T_{\tilde{\jmath}(0,\hat{z}_2)}\Sigma^\perp$ is canonically equal to $\mathbb{C} \times \{\hat{z}_2\}$. By (15.1.1), the matrix of the transversal Hessian of $H \circ \Phi$ in the canonical basis of $\mathbb{C}$ is simply

$$d^2(H \circ \Phi)_{\upharpoonright \mathbb{C} \times \{\hat{z}_2\}} = d^2_{\Phi(0,\hat{z}_2)} H \circ (d\Phi)^2 = \begin{pmatrix} 2\,|B(g^{-1}(\hat{z}_2))| & 0 \\ 0 & 2\,|B(g^{-1}(\hat{z}_2))| \end{pmatrix}. \tag{15.1.6}$$

Therefore, by Taylor's formula in the $\hat{z}_1$ variable (locally uniformly with respect to the $\hat{z}_2$ variable seen as a parameter), we get

$$\begin{aligned} H \circ \Phi(\hat{z}_1, \hat{z}_2) &= H \circ \Phi_{\upharpoonright \hat{z}_1 = 0} + dH \circ \Phi_{\upharpoonright \hat{z}_1 = 0}(\hat{z}_1) + \frac{1}{2} d^2(H \circ \Phi)_{\upharpoonright \hat{z}_1 = 0}(\hat{z}_1^2) + \mathcal{O}(|\hat{z}_1|^3) \\ &= 0 + 0 + \left|B(g^{-1}(\hat{z}_2))\right| |\hat{z}_1|^2 + \mathcal{O}(|\hat{z}_1|^3). \end{aligned}$$

In order to obtain the result claimed in the theorem, it remains to apply a formal Birkhoff normal form in the $\hat{z}_1$ variable, to simplify the remainder $\mathcal{O}(\hat{z}_1^3)$. This classical normal form is a particular case of the semiclassical normal form that we prove below (Proposition 15.7); therefore we simply refer to this proposition, and this finishes the proof of the theorem, where, for simplicity of notation, the variables (z_1, z_2) actually refer to $(\hat{z}_1, \hat{z}_2)$.

1.3. Semiclassical Birkhoff normal form. We follow the spirit of [**34, 167**] (see also [**166**]). In the coordinates $\hat{x}_1, \hat{\xi}_1, \hat{x}_2, \hat{\xi}_2$ (which are defined in a neighborhood of $\{0\} \times \Omega$), the Hamiltonian takes the form:

$$\hat{H}(\hat{z}_1, \hat{z}_2) = H^0 + O(|\hat{z}_1|^3), \qquad \text{where } H^0 = B(g^{-1}(\hat{z}_2))|\hat{z}_1|^2. \tag{15.1.7}$$

Let us now consider the space of the formal power series in $\hat{x}_1, \hat{\xi}_1, h$ with coefficients smoothly depending on $(\hat{x}_2, \hat{\xi}_2)$: $\mathcal{E} = \mathcal{C}^\infty_{\hat{x}_2, \hat{\xi}_2}[\hat{x}_1, \hat{\xi}_1, h]$. We endow $\mathcal{E}$ with the Moyal product (compatible with the Weyl quantization) denoted by $\star$ and the commutator of two series κ_1 and κ_2 (in all variables $(\hat{x}_1, \hat{\xi}_1, \hat{x}_2, \hat{\xi}_2)$) is defined as:

$$[\kappa_1, \kappa_2] = \kappa_1 \star \kappa_2 - \kappa_2 \star \kappa_1.$$

Explicitly, we have

$$[\kappa_1, \kappa_2](\hat{x}, \hat{\xi}, h) = 2\sinh\Big(\frac{h}{2i}\Box\Big)\big(f(x,\xi,h)g(y,\eta,h)\big)\Big|_{\substack{x=y=\hat{x},\\ \xi=\eta=\hat{\xi}}}$$

where

$$\Box = \sum_{j=1}^{2} \partial_{\xi_j}\partial_{y_j} - \partial_{x_j}\partial_{\eta_j}.$$

Notation 15.6. *The degree of* $\hat{x}_1^\alpha \hat{\xi}_1^\beta h^l$ *is* $\alpha+\beta+2l$. $\mathcal{D}_N$ *denotes the space of the monomials of degree* N. $\mathcal{O}_N$ *is the space of formal series with valuation at least* N. *We notice that* $[\mathcal{O}_{N_1}, \mathcal{O}_{N_2}] \subset \mathcal{O}_{N_1+N_2}$. *For* $\tau, \gamma \in \mathcal{E}$, *we denote* $\mathsf{ad}_\tau \gamma = [\tau, \gamma]$.

Proposition 15.7. *Given* $\gamma \in \mathcal{O}_3$, *there exist formal power series* $\tau, \kappa \in \mathcal{O}_3$ *such that:*

$$e^{ih^{-1}\mathsf{ad}_\tau}(H^0 + \gamma) = H^0 + \kappa,$$

with: $[\kappa, |\hat{z}_1|^2] = 0$.

Proof. Let $N \geq 1$. Assume that we have, for $N \geq 1$ and $\tau_N \in \mathcal{O}_3$:

$$e^{ih^{-1}\mathsf{ad}_{\tau_N}}(H^0 + \gamma) = H^0 + K_3 + \cdots + K_{N+1} + R_{N+2} + \mathcal{O}_{N+3},$$

where $K_i \in \mathcal{D}_i$ commutes with $|\hat{z}_1|^2$ and where $R_{N+2} \in \mathcal{D}_{N+2}$.

Let $\tau' \in \mathcal{D}_{N+2}$. A computation provides:

$$e^{ih^{-1}\mathsf{ad}_{\tau_N+\tau'}}(H^0 + \gamma) = H^0 + K_3 + \cdots + K_{N+1} + K_{N+2} + \mathcal{O}_{N+3},$$

with:

$$K_{N+2} = R_{N+2} + B(g^{-1}(\hat{z}_2))ih^{-1}\mathsf{ad}_{\tau'}|\hat{z}_1|^2 = R_{N+2} - B(g^{-1}(\hat{z}_2))ih^{-1}\mathsf{ad}_{|\hat{z}_1|^2}\tau',$$

where we have used

$$ih^{-1}\mathsf{ad}_{\tau'}H_0 = B(g^{-1}(\hat{z}_2))ih^{-1}\mathsf{ad}_{\tau'}|\hat{z}_1|^2 + \mathcal{O}_{N+4}.$$

We can write:

$$R_{N+2} = K_{N+2} + B(g^{-1}(\hat{z}_2))ih^{-1}\mathsf{ad}_{|\hat{z}_1|^2}\tau'.$$

Since $B(g^{-1}(\hat{z}_2)) \neq 0$, we deduce the existence of τ' and K_{N+2} such that K_{N+2} commutes with $|\hat{z}_1|^2$. Note that $ih^{-1}\mathsf{ad}_{|\hat{z}_1|^2} = \{|\hat{z}_1|^2, \cdot\}$. $\Box$

1.4. Proof of Theorem 3.19. Since the formal series κ given by Proposition 15.7 commutes with $|\hat{z}_1|^2$, we can write it as a polynomial in $|\hat{z}_1|^2$:

$$\kappa = \sum_{k\geq 0}\sum_{l+m=k} h^l c_{l,m}(\hat{z}_2)|\hat{z}_1|^{2m}.$$

This formal series can be reordered by using the monomials $(|\hat{z}_1|^2)^{\star m}$ for the product law $\star$:

$$\kappa = \sum_{k\geq 0}\sum_{l+m=k} h^l c^\star_{l,m}(\hat{z}_2)(|\hat{z}_1|^2)^{\star m}.$$

Thanks to the Borel lemma, there exists a smooth function with compact support $f^\star(h, |\hat{z}_1|^2, \hat{z}_2)$ such that the Taylor expansion with respect to $(h, |\hat{z}_1|^2)$ of $f^\star(h, |\hat{z}_1|^2, \hat{z}_2)$ is given by κ and:

$$\sigma^{\mathsf{T},w}\left(\mathsf{Op}_h^w\left(f^\star(h, \mathcal{I}_h, \hat{z}_2)\right)\right) = \kappa, \tag{15.1.8}$$

where $\sigma^{\mathsf{T},w}$ means that we consider the formal Taylor series of the Weyl symbol with respect to $(h, \hat{z}_1)$. Here, the operator $\mathsf{Op}_h^w\left(f^\star(h, \mathcal{I}_h, \hat{z}_2)\right)$ has to be understood as the Weyl quantization with respect to $\hat{z}_2$ of an operator valued symbol. We can write it in the form:

$$\mathsf{Op}_h^w f^\star(h, \mathcal{I}_h, \hat{z}_2) = C_0 h + \mathcal{H}^0 + \mathsf{Op}_h^w \tilde{f}^\star(h, \mathcal{I}_h, \hat{z}_2),$$

where $\mathcal{H}_h^0 = \mathsf{Op}_h^w(H^0)$ and $\sigma^{\mathsf{T},w}(\mathsf{Op}_h^w(\tilde{f}^\star(h, \mathcal{I}_h, \hat{z}_2)))$ is in $\mathcal{O}_4$. Thus, given any $\eta > 0$, we may choose the support of $f^\star$ small enough in order to have, for all $\psi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$,

$$|\langle \mathsf{Op}_h^w \tilde{f}^\star(h, \mathcal{I}_h, \hat{z}_2)\psi, \psi\rangle| \leq \eta \|\mathcal{I}_h^{1/2}\psi\|^2. \tag{15.1.9}$$

Moreover we can also introduce a smooth symbol a_h with compact support such that $\sigma^{\mathsf{T},w}(a_h) = \tau$. Using (15.1.7) and applying the Egorov theorem (see [**128**, Theorems 5.5.5 and 5.5.9], [**158**] or [**172**]), we can find an invertible Fourier Integral Operator V_h, which is microlocally unitary and such that:

$$V_h^* \mathcal{L}_{h,\mathbf{A}} V_h = C_0 h + \mathcal{H}_h^0 + \mathsf{Op}_h^w(r_h),$$

so that $\sigma^{\mathsf{T},w}\left(\mathsf{Op}_h^w(r_h)\right) = \gamma \in \mathcal{O}_3$. In fact, one can choose V_h such that the subprincipal symbol is preserved by conjugation (see for instance [**101**, Appendix A]), which implies that $C_0 = 0$[1]. It remains to use Proposition 15.7 and again the Egorov theorem to notice that $e^{ih^{-1}\mathsf{Op}_h^w(a_h)} \mathsf{Op}_h^w(r_h) e^{-ih^{-1}\mathsf{Op}_h^w(a_h)}$ is a pseudo-differential operator such that the formal Taylor series of its symbol is κ. Therefore, recalling (15.1.8), we have found a microlocally unitary Fourier Integral Operator U_h such that:

$$U_h^* \mathcal{L}_{h,\mathbf{A}} U_h = \mathcal{H}_h^0 + \mathsf{Op}_h^w\left(\tilde{f}^\star(h, \mathcal{I}_h, \hat{z}_2)\right) + R_h, \tag{15.1.10}$$

where R_h is a pseudo-differential operator such that $\sigma^{\mathsf{T},w}(R_h) = 0$. It remains to prove the division property expressed in the last statement of item (3) of Theorem 3.19. By the Morse Lemma, there exists in a fixed neighborhood of $z_1 = 0$ in $\mathbb{R}^4$ a (non symplectic) change of coordinates $\tilde{z}_1$ such that $d_0 = c(z_2)\,|\tilde{z}_1|^2$. It is enough to prove the result in this microlocal neighborhood. Now, for any $N \geq 1$, we proceed by induction. We assume that we can write R_h in the form:

$$R_h = \mathsf{Op}_h^w\left(s_0 + h s_1 + \cdots + h^k s_k\right) D_h^N + O(h^{k+1}),$$

with symbols s_j which vanish at infinite order with respect to $\hat{z}_1$. We look for s_{k+1} such that:

$$R_h = \mathsf{Op}_h^w\left(s_0 + h s_1 + \cdots + h^k s_k + h^{k+1} s_{k+1}\right) D_h^N + O(h^{k+2}) \tilde{R}_{h,k}.$$

[1]We give another proof of this fact in Remark 15.8 below.

We are reduced to find s_{k+1} such that:

$$\tilde{r}_{0,k} = d_0^N s_{k+1}.$$

Since $\tilde{r}_{0,k}$ vanishes at any order at zero we can find a smooth function ϕ_k such that:

$$\tilde{r}_{0,k} = |\tilde{z}_1|^{2N} \phi.$$

We have $s_{k+1}(\tilde{z}_1, z_2) = \frac{\phi_k(\tilde{z}_1, z_2)}{c(z_2)^N}$.

This ends the proof of Theorem 3.19.

Remark 15.8. *It is well known that (see* [**92**, Theorem 1.1]*), when $B > 0$, the smallest eigenvalue $\lambda_1(h)$ of $\mathcal{H}_{h,A}$ has the following asymptotics*

$$\lambda_1(h) \sim h \min_{q \in \mathbb{R}^2} B(q).$$

We will see in Section 2.1 that the corresponding eigenfunctions are microlocalized on Σ at the minima of B. Therefore the normal form would imply, by a variational argument, that

$$\lambda_1(h) \geq C_0 h + \mu_1(h) + o(h), \tag{15.1.11}$$

where $\mu_1(h)$ is the smallest eigenvalue of $\mathcal{N}_h := \mathcal{H}^0 + \mathsf{Op}_h^w(f^\star(h, \mathcal{I}, z_2))$. Similarly, we will see in 2.2 that the lowest eigenfunctions of $\mathcal{N}_h$ are also microlocalized in $\hat{z}_1$ and $\hat{z}_2$, and therefore

$$\lambda_1(h) \sim C_0 h + \mu_1(h).$$

By Gårding's inequality and point (4) *of Theorem 3.19, $\mu_1(h) \sim h \min B$. Comparing with* (15.1.11)*, we see that $C_0 = 0$.*

2. Spectral theory

This section is devoted to the proof of Theorem 3.18. The main idea is to use the eigenfunctions of $\mathcal{L}_{h,\mathbf{A}}$ and $\mathcal{L}_h^{\mathsf{No}}$ as test functions in the pseudo-differential identity (15.1.10) given in Theorem 3.19 and to apply the variational characterization of the eigenvalues given by the min-max principle. In order to control the remainders we shall prove the microlocalization of the eigenfunctions of $\mathcal{L}_{h,\mathbf{A}}$ and $\mathcal{L}_h^{\mathsf{No}}$ thanks to the confinement assumption (3.4.8).

2.1. Localization and microlocalization of the eigenfunctions of $\mathcal{L}_{h,\mathbf{A}}$. The space localization of the eigenfunctions of $\mathcal{L}_{h,\mathbf{A}}$, which is the quantum analog of Theorem 3.16, is a consequence of the so-called Agmon estimates.

Proposition 15.9. *Let us assume* (3.4.8)*. Let us fix $0 < C_1 < \tilde{C}_1$ and $\alpha \in (0, \frac{1}{2})$. There exist $C, h_0, \varepsilon_0 > 0$ such that for all $0 < h \leq h_0$ and for all eigenpair (λ, ψ) of $\mathcal{L}_{h,\mathbf{A}}$ such that $\lambda \leq C_1 h$, we have:*

$$\int |e^{\chi(q) h^{-\alpha} |q|} \psi|^2 \, \mathrm{d}q \leq C \|\psi\|^2,$$

where χ is zero for $|q| \leq M_0$ and 1 for $|q| \geq M_0+\varepsilon_0$. Moreover, we also have the weighted H^1 estimate:

$$\int |e^{\chi(q)h^{-\alpha}|q|}(-ih\nabla + \boldsymbol{A})\psi|^2 \,\mathrm{d}q \leq Ch\|\psi\|^2.$$

Remark 15.10. *This estimate is interesting when $|x| \geq M_0+\varepsilon_0$. In this region, we deduce by standard elliptic estimates that $\psi = O(h^\infty)$ in suitable norms (see for instance* [**82**, Proposition 3.3.4] *or more recently* [**152**, Proposition 2.6]*). Therefore, the eigenfunctions are localized in space in the ball of center $(0,0)$ and radius $M_0+\varepsilon_0$.*

We shall now prove the microlocalization of the eigenfunctions near the zero set of the magnetic Hamiltonian Σ.

Proposition 15.11. *Let us assume (3.4.8). Let us fix $0 < C_1 < \tilde{C}_1$ and consider $\delta \in \left(0, \frac{1}{2}\right)$. Let (λ, ψ) be an eigenpair of $\mathcal{L}_{h,\boldsymbol{A}}$ with $\lambda \leq C_1 h$. Then, we have:*

$$\psi = \chi_1\left(h^{-2\delta}\mathcal{L}_{h,\boldsymbol{A}}\right)\chi_0(q)\psi + O(h^\infty),$$

where χ_0 is smooth cutoff function supported in a compact set in the ball of center $(0,0)$ and radius $M_0+\varepsilon_0$ and where χ_1 a smooth cutoff function being 1 near 0.

Proof. In view of Proposition 15.9, it is enough to prove that

$$\left(1-\chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\right)\left(\chi_0(q)\psi\right) = O(h^\infty). \tag{15.2.1}$$

By the space localization, we have:

$$\mathcal{L}_{h,\mathbf{A}}(\chi_0(q)\psi) = \lambda\chi_0(q)\psi + O(h^\infty).$$

Then, we get:

$$\left(1-\chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\right)\mathcal{L}_{h,\mathbf{A}}(\chi_0(q)\psi) = \lambda\left(1-\chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\right)(\chi_0(x)\psi) + O(h^\infty).$$

This implies:

$$\begin{aligned} h^{2\delta}\|\left(1-\chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\right)(\chi_0(q)\psi)\|^2 &\leq q_{h\,A}\left(\left(1-\chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\right)\mathcal{L}_{h,\mathbf{A}}(\chi_0(q)\psi)\right) \\ &\leq C_1 h\|\left(1-\chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\right)(\chi_0(q)\psi)\|^2 + O(h^\infty)\|\psi\|^2. \end{aligned}$$

Since $\delta \in \left(0, \frac{1}{2}\right)$, we deduce (15.2.1). ☐

2.2. Microlocalization of the eigenfunctions of $\mathcal{L}_h^{\mathsf{No}}$. The next two propositions state the microlocalization of the eigenfunctions of the normal form $\mathcal{L}_h^{\mathsf{No}}$.

Proposition 15.12. *Let us consider the pseudo-differential operator:*

$$\mathcal{L}_h^{\mathsf{No}} = \mathcal{H}_h^0 + \mathsf{Op}_h^w \tilde{f}^\star(h, \mathcal{I}_h, \hat{z}_2).$$

We assume the confinement assumption (3.4.8). We can consider $\tilde{M}_0 > 0$ such that $B \circ \varphi^{-1}(\hat{z}_2) \geq \tilde{C}_1$ for $|\hat{z}_2| \geq \tilde{M}_0$. Let us consider $C_1 < \tilde{C}_1$ and an eigenpair (λ, ψ) of $\mathcal{L}_h^{\mathsf{No}}$ such that $\lambda \leq C_1 h$. Then, for all $\varepsilon_0 > 0$ and for all smooth cutoff function χ supported *in* $|\hat{z}_2| \geq \tilde{M}_0 + \varepsilon_0$*, we have:*

$$\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi = O(h^\infty).$$

PROOF. We notice that:

$$\mathcal{L}_h^{\mathsf{No}}\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi = \lambda\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi + h\mathcal{R}_h\psi,$$

where the symbol of $\mathcal{R}_h$ is supported in compact slightly smaller than the support of χ. We may consider a cutoff function $\underline{\chi}$ which is 1 on a small neighborhood of this support. We get:

$$\langle\mathcal{L}_h^{\mathsf{No}}\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi, \mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi\rangle \leq \lambda\|\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi\|^2 + Ch\|\mathsf{Op}_h^w\left(\underline{\chi}(\hat{z}_2)\right)\psi\|\|\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi\|$$

Thanks to the Gårding inequality, we have:

$$\begin{aligned}\langle\mathcal{H}_h^0\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi, \mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi\rangle \geq &(\tilde{C}_1 - Ch)\|\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\mathcal{I}_h^{1/2}\psi\|^2\\ \geq &(\tilde{C}_1 - Ch)h\|\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi\|^2.\end{aligned}$$

We can consider $\mathsf{Op}_h^w\tilde{f}^\star(h, \mathcal{I}_h, \hat{z}_2)$ as a perturbation of $\mathcal{H}_h^0$ (see (15.1.9)). Since $C_1 < \tilde{C}_1$ we infer that:

$$\|\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi\| \leq Ch\|\mathsf{Op}_h^w\left(\underline{\chi}(\hat{z}_2)\right)\psi\|.$$

Then a standard iteration argument provides $\mathsf{Op}_h^w\left(\chi(\hat{z}_2)\right)\psi = O(h^\infty)$. ☐

Proposition 15.13. *Keeping the assumptions and the notation of Proposition 15.12, we consider* $\delta \in \left(0, \frac{1}{2}\right)$ *and an eigenpair* (λ, ψ) *of* $\mathcal{L}_h^{\mathsf{No}}$ *with* $\lambda \leq C_1 h$*. Then, we have:*

$$\psi = \chi_1\left(h^{-2\delta}\mathcal{I}_h\right)\mathsf{Op}_h^w\left(\chi_0(\hat{z}_2)\right)\psi + O(h^\infty),$$

for all smooth cutoff function χ_1 *supported in a neighborhood of zero and all smooth cutoff function* χ_0 *being* 1 *near zero and supported in the ball of center* 0 *and radius* $\tilde{M}_0 + \varepsilon_0$*.*

PROOF. The proof follows the same lines as for Proposition 15.12 and Proposition 15.11. ☐

2.2.1. *Proof of Theorem 3.18.* As we proved in the last section, each eigenfunction of $\mathcal{L}_{h,\mathbf{A}}$ or $\mathcal{L}_h^{\mathsf{No}}$ is microlocalized. Nevertheless we do not know yet if all the functions in the range of the spectral projection below $C_1 h$ are microlocalized. This depends on the rank of the spectral projection. The next two lemmas imply that this rank does not increase more than polynomially in h^{-1} (so that the functions lying in the range of the spectral projection are microlocalized). We will denote by $\mathsf{N}(\mathcal{M}, \lambda)$ the number of eigenvalues of $\mathcal{M}$ less than or equal to λ.

Lemma 15.14. *There exists* $C > 0$ *such that for all* $h > 0$*, we have:*

$$\mathsf{N}(\mathcal{L}_{h,\boldsymbol{A}}, C_1 h) \leq Ch^{-1}.$$

PROOF. We notice that:

$$\mathsf{N}(\mathcal{L}_{h,\mathbf{A}}, C_1 h) = \mathsf{N}(\mathcal{H}_{1,h^{-1}A}, C_1 h^{-1})$$

and that, for all $\varepsilon \in (0,1)$:

$$q_{1,h^{-1}\mathbf{A}}(\psi) \geq (1-\varepsilon) q_{1,h^{-1}\mathbf{A}}(\psi) + \varepsilon \int_{\mathbb{R}^2} \frac{B(x)}{h} |\psi|^2 \, \mathrm{d}x$$

so that we infer:

$$\mathsf{N}(\mathcal{L}_{h,\mathbf{A}}, C_1 h) \leq \mathsf{N}(\mathcal{H}_{1,h^{-1}\mathbf{A}} + \varepsilon(1-\varepsilon)^{-1} h^{-1} B, (1-\varepsilon)^{-1} C_1 h^{-1}).$$

Then, the diamagnetic inequality [2] jointly with a Lieb-Thirring estimate (see the original paper [**119**]) provides for all $\gamma > 0$ the existence of $L_{\gamma,2} > 0$ such that for all $h > 0$ and $\lambda > 0$:

$$\sum_{j=1}^{\mathsf{N}(\mathcal{H}_{1,h^{-1}\mathbf{A}} + \varepsilon(1-\varepsilon)^{-1} h^{-1} B, \lambda)} \left| \tilde{\lambda}_j(h) - \lambda \right|^{\gamma} \leq L_{\gamma,2} \int_{\mathbb{R}^2} (\varepsilon(1-\varepsilon)^{-1} h^{-1} B(x) - \lambda)_-^{1+\gamma} \, \mathrm{d}x.$$

We apply this inequality with $\lambda = (1+\eta)(1-\varepsilon)^{-1} C_1 h^{-1}$, for some $\eta > 0$. This implies that:

$$\sum_{j=1}^{N_{\epsilon,h,\eta}} \left| \tilde{\lambda}_j(h) - \lambda \right|^{\gamma} \leq L_{\gamma,2} \int_{B(x) \leq (1+\eta) C_1/\epsilon} (\lambda - \varepsilon(1-\varepsilon)^{-1} h^{-1} B(x))^{1+\gamma} \, \mathrm{d}x$$

with $N_{\epsilon,h,\eta} := \mathsf{N}(\mathcal{H}_{1,h^{-1}\mathbf{A}} + \varepsilon(1-\varepsilon)^{-1} h^{-1} B, (1-\varepsilon)^{-1} C_1 h^{-1})$, so that:

$$(\eta(1-\varepsilon)^{-1} C_1 h^{-1})^{\gamma} N_{\epsilon,h,\eta} \leq L_{\gamma,2} (h(1-\varepsilon))^{-1-\gamma} \int_{B(x) \leq \frac{(1+\eta)C_1}{\varepsilon}} ((1+\eta) C_1 - \varepsilon B(x))^{1+\gamma} \, \mathrm{d}x.$$

For η small enough and ε is close to 1, we have $(1+\eta)\varepsilon^{-1} C_1 < \tilde{C}_1$ so that the integral is finite, which gives the required estimate. $\square$

Lemma 15.15. *There exists $C > 0$ and $h_0 > 0$ such that for all $h \in (0, h_0)$, we have:*

$$\mathsf{N}(\mathcal{N}_h, C_1 h) \leq C h^{-1}.$$

PROOF. Let $\epsilon \in (0,1)$. By point (4) of Theorem 3.19, it is enough to prove that $\mathsf{N}(\mathcal{H}_h^0, \frac{C_1 h}{1-\epsilon}) \leq C h^{-1}$. The eigenvalues and eigenfunctions of $\mathcal{H}_h^0$ can be found by separation of variables: $\mathcal{H}_h^0 = \mathcal{I}_h \otimes \mathsf{Op}_h^w(B \circ \varphi^{-1})$, where $\mathcal{I}_h$ acts on $L^2(\mathbb{R}_{x_1})$ and $\hat{B}_h := \mathsf{Op}_h^w(B \circ \varphi^{-1})$ acts on $L^2(\mathbb{R}_{x_2})$. Thus,

$$\mathsf{N}(\mathcal{H}_h^0, h C_{1,\epsilon}) = \#\{(n,m) \in (\mathbb{N}^*)^2; \quad (2n-1) h \gamma_m(h) \leq h C_{1,\epsilon}\},$$

where $C_{1,\epsilon} := \frac{C_1}{1-\epsilon}$, and $\gamma_1(h) \leq \gamma_2(h) \leq \cdots$ are the eigenvalues of $\hat{B}_h$. A simple estimate gives

$$\mathsf{N}(\mathcal{H}_h^0, C_{1,\epsilon}) \leq \left(1 + \left\lfloor \frac{1}{2} + \frac{C_{1,\epsilon}}{2\gamma_1(h)} \right\rfloor\right) \cdot \#\{m \in \mathbb{N}^*; \quad \gamma_m(h) \leq C_{1,\epsilon}\}.$$

[2]See [**40**, Theorem 1.13] and the link with the control of the resolvent kernel in [**109, 161**].

If ϵ is small enough, $C_{1,\epsilon} < \tilde{C}_1$, and then Weyl asymptotics (see for instance [**49**, Chapter 9]) for $\hat{B}_h$ gives

$$\mathsf{N}(\hat{B}_h, C_{1,\epsilon}) \sim \frac{1}{2\pi h}\mathrm{vol}\{B\circ\varphi^{-1} \leq C_{1,\epsilon}\},$$

and Gårding's inequality implies $\gamma_1(h) \geq \min_{q\in\mathbb{R}^2} B - O(h)$, which finishes the proof. $\square$

Remark 15.16. *With additional hypotheses on the magnetic field, it has been proved that the $O(h^{-1})$ estimate is in fact optimal: see for instance* [**36**] *and* [**165**, Remark 1]. *Actually, it would likely follow from Theorem 3.18 and Theorem 3.19 that these Weyl asymptotics hold in general under the confinement assumption.*

Let us now consider $\lambda_1(h), \cdots, \lambda_{\mathsf{N}(\mathcal{L}_{h,\mathbf{A}},C_1 h)}(h)$ the eigenvalues of $\mathcal{L}_{h,\mathbf{A}}$ below $C_1 h$. We can consider corresponding normalized eigenfunctions ψ_j such that : $\langle\psi_j,\psi_k\rangle = \delta_{kj}$. We introduce the N-dimensional space:

$$V = \chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\chi_0(q)\underset{1\leq j\leq N}{\mathsf{span}}\,\psi_j.$$

Let us bound from above the quadratic form of $\mathcal{L}_h^{\mathsf{No}}$ denoted by $\mathcal{Q}_h^{\mathsf{No}}$. For $\psi \in \underset{1\leq j\leq N}{\mathsf{span}}\,\psi_j$, we let:

$$\tilde{\psi} = \chi_1\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\chi_0(q)\psi$$

and we can write:

$$\mathcal{Q}_h^{\mathsf{No}}(U_h^*\tilde{\psi}) = \langle U_h\mathcal{L}_h^{\mathsf{No}}U_h^*\tilde{\psi},\tilde{\psi}\rangle = \langle U_hU_h^*\mathcal{L}_{h,\mathbf{A}}U_hU_h^*\tilde{\psi},\tilde{\psi}\rangle - \langle U_hR_hU_h^*\tilde{\psi},\tilde{\psi}\rangle.$$

Since U_h is microlocally unitary, the elementary properties of the pseudo-differential calculus yield:

$$\langle U_h\mathcal{L}_h^{\mathsf{No}}U_h^*\tilde{\psi},\tilde{\psi}\rangle = \langle\mathcal{L}_{h,\mathbf{A}}\tilde{\psi},\tilde{\psi}\rangle - \langle U_hR_hU_h^*\tilde{\psi},\tilde{\psi}\rangle + O(h^\infty)\|\tilde{\psi}\|^2.$$

Then, thanks to Proposition 15.11 and Lemma 15.14 we may replace $\tilde{\psi}$ by ψ up to a remainder of order $O(h^\infty)\|\tilde{\psi}\|$:

$$\langle U_h\mathcal{L}_h^{\mathsf{No}}U_h^*\tilde{\psi},\tilde{\psi}\rangle = \langle\mathcal{L}_{h,\mathbf{A}}\psi,\psi\rangle - \langle U_hR_hU_h^*\tilde{\psi},\tilde{\psi}\rangle + O(h^\infty)\|\tilde{\psi}\|^2$$

so that:

$$\langle U_h\mathcal{L}_h^{\mathsf{No}}U_h^*\tilde{\psi},\tilde{\psi}\rangle \leq \lambda_N(h)\|\psi\|^2 + |\langle U_hR_hU_h^*\tilde{\psi},\tilde{\psi}\rangle| + O(h^\infty)\|\tilde{\psi}\|^2$$

and:

$$\langle U_h\mathcal{L}_h^{\mathsf{No}}U_h^*\tilde{\psi},\tilde{\psi}\rangle \leq \lambda_N(h)\|U_h^*\tilde{\psi}\|^2 + |\langle U_hR_hU_h^*\tilde{\psi},\tilde{\psi}\rangle| + O(h^\infty)\|U_h^*\tilde{\psi}\|^2.$$

Let us now estimate the remainder term $U_hR_hU_h^*\tilde{\psi}$. We have:

$$U_hR_hU_h^*\tilde{\psi} = U_hR_hU_h^*\underline{\chi_1}\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)\tilde{\psi} = U_hR_hU_h^*\underline{\chi_1}\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)(U_h^*)^{-1}U_h^*\tilde{\psi} + O(h^\infty)\|U_h^*\tilde{\psi}\|,$$

where $\underline{\chi_1}$ has a support slightly bigger then the one of χ_1. We notice that

$$U_h^*\underline{\chi_1}\left(h^{-2\delta}\mathcal{L}_{h,\mathbf{A}}\right)(U_h^*)^{-1} = \underline{\chi_1}\left(h^{-2\delta}U_h^*\mathcal{L}_{h,\mathbf{A}}(U_h^*)^{-1}\right).$$

Let us now apply (3.4.10) with $D_h = U_h^* \mathcal{L}_{h,\mathbf{A}} (U_h^*)^{-1}$ to get:

$$R_h = S_{h,M}(U_h^* \mathcal{L}_{h,\mathbf{A}} (U_h^*)^{-1})^M + K_N + O(h^\infty)$$

so that:

$$\|U_h R_h U_h^* \underline{\chi_1} \left(h^{-2\delta} \mathcal{L}_{h,\mathbf{A}}\right) \tilde{\psi}\| = O(h^{2M\delta}) \|U_h^* \tilde{\psi}\|^2.$$

We infer that:

$$\mathcal{Q}_h^{\mathsf{No}}(U_h^* \tilde{\psi}) \leq \lambda_N(h) \|U_h^* \tilde{\psi}\|^2 + O(h^{2M\delta}) \|U_h^* \tilde{\psi}\|^2.$$

From the min-max principle, it follows that:

$$\mu_N(h) \leq \lambda_N(h) + O(h^{2M\delta}).$$

The converse inequality follows from a similar proof, using Proposition 15.13 and Lemma 15.15. This ends the proof of Theorem 3.18.

Part 5

Waveguides

CHAPTER 16

Magnetic effects in curved waveguides

Hic, ne deficeret, metuens avidusque videndi
Flexit amans oculos, et protinus illa relapsa est.
Bracchiaque intendens prendique et prendere certans
Nil nisi cedentes infelix arripit auras.
Jamque iterum moriens non est de coniuge quicquam
Questa suo (quid enim nisi se quereretur amatam?)
Supremumque vale, quod iam vix auribus ille
Acciperet, dixit revolutaque rursus eodem est.

Metamorphoses, Liber X, Ovidius

In this chapter we prove Theorem 4.2 and we give the main steps in the proof of Theorem 4.5 which is much more technically involved. In particular we show on this non trivial example how to establish the norm resolvent convergence (see Lemma 4.8).

1. Two dimensional waveguides

1.1. Proof of Theorem 4.2. Let us consider $\delta \leq 1$ and $K \geq 2 \sup \frac{\kappa^2}{4}$.

A first approximation. We let:

$$\mathcal{L}^{[2]}_{\varepsilon,\delta} = \mathcal{L}^{[2]}_{\varepsilon,\varepsilon^{-\delta}\mathcal{A}_\varepsilon} - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K$$

and

$$\mathcal{L}^{\mathsf{app},[2]}_{\varepsilon,\delta} = (i\partial_s + \varepsilon^{1-\delta}\mathbf{B}(s,0)\tau)^2 - \frac{\kappa^2}{4} - \varepsilon^{-2}\partial_\tau^2 - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K.$$

The corresponding quadratic forms, defined on $\mathsf{H}^1_0(\Omega)$, are denoted by $\mathcal{Q}^{[2]}_{\varepsilon,\delta}$ and $\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}$ whereas the sesquilinear forms are denoted by $\mathcal{B}^{[2]}_{\varepsilon,\delta}$ and $\mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,\delta}$. We can notice that:

$$\left| V_\varepsilon(s,\tau) - \left(-\frac{\kappa(s)^2}{4}\right) \right| \leq C\varepsilon$$

so that the operators $\mathcal{L}^{[2]}_{\varepsilon,\delta}$ and $\mathcal{L}^{\mathsf{app},[2]}_{\varepsilon,\delta}$ are invertible for ε small enough. Moreover there exists $c > 0$ such that for all $\varphi \in \mathsf{H}^1_0(\Omega)$:

$$\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\varphi) \geq c\|\varphi\|^2, \quad \mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\varphi) \geq c\|\varphi\|^2.$$

Let $\phi, \psi \in \mathsf{H}^1_0(\Omega)$. We have to analyse the difference of the sesquilinear forms:

$$\mathcal{B}^{[2]}_{\varepsilon,\delta}(\phi,\psi) - \mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi,\psi).$$

We easily get:

$$\left|\langle V_\varepsilon\phi,\psi\rangle-\langle-\frac{\kappa^2}{4}\phi,\psi\rangle\right|\leq C\varepsilon\|\phi\|\|\psi\|\leq\tilde{C}\varepsilon\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}.$$

We must investigate:

$$\langle m_\varepsilon^{-1}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))m_\varepsilon^{-1/2}\phi,(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))m_\varepsilon^{-1/2}\psi\rangle.$$

We notice that:

$$|\partial_s m_\varepsilon^{-1/2}|\leq C\varepsilon,\quad |m_\varepsilon^{-1/2}-1|\leq C\varepsilon.$$

We have:

$$\begin{aligned}&|\langle m_\varepsilon^{-1}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))m_\varepsilon^{-1/2}\phi,(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))(m_\varepsilon^{-1/2}-1)\psi\rangle|\\&\quad\leq C\varepsilon\|m_\varepsilon^{-1/2}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))m_\varepsilon^{-1/2}\phi\|(\|\psi\|+\|m_\varepsilon^{-1/2}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\psi\|)\\&\qquad\leq C\varepsilon(\|(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\phi\|+\|\phi\|)(\|\psi\|+\|m_\varepsilon^{-1/2}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\psi\|).\end{aligned}$$

By the Taylor formula, we get (since $\delta\leq 1$):

$$|\mathcal{A}_1(s,\varepsilon\tau)-\varepsilon b\mathbf{B}(s,0)\tau|\leq Cb\varepsilon^2\leq C\varepsilon. \tag{16.1.1}$$

so that:

$$\|(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\phi\|\leq\|(i\partial_s+\varepsilon b\mathbf{B}(s,0)\tau)\phi\|+Cb\varepsilon^2\|\phi\|.$$

We infer that:

$$\begin{aligned}&|\langle m_\varepsilon^{-1}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))m_\varepsilon^{-1/2}\phi,(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))(m_\varepsilon^{-1/2}-1)\psi\rangle|\\&\quad\leq C\varepsilon\left(\|\phi\|\|\psi\|+\|\phi\|\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}+\|\psi\|\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}+\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}\right)\\&\qquad\leq\tilde{C}\varepsilon\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}.\end{aligned}$$

It remains to analyse:

$$\langle m_\varepsilon^{-1}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))m_\varepsilon^{-1/2}\phi,(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\psi\rangle.$$

With the same kind of arguments, we deduce:

$$\begin{aligned}&|\langle m_\varepsilon^{-1}(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))m_\varepsilon^{-1/2}\phi,(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\psi\rangle-\langle(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\phi,(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\psi\rangle|\\&\qquad\leq\tilde{C}\varepsilon\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}.\end{aligned}$$

We again use (16.1.1) to infer:

$$\begin{aligned}&\langle(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\phi,(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\psi\rangle-\langle(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\phi,(i\partial_s+b\varepsilon\mathbf{B}(s,0)\tau)\psi\rangle|\\&\qquad\leq C\varepsilon\|(i\partial_s+b\mathcal{A}_1(s,\varepsilon\tau))\phi\|\|\psi\|.\leq\tilde{C}\varepsilon\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}.\end{aligned}$$

In the same way, we deduce:

$$\langle (i\partial_s + b\mathcal{A}_1(s,\varepsilon\tau))\phi, (i\partial_s + b\mathcal{A}_1(s,\varepsilon\tau))\psi\rangle - \langle (i\partial_s + b\varepsilon\mathbf{B}(s,0)\tau)\phi, (i\partial_s + b\varepsilon\mathbf{B}(s,0)\tau)\psi\rangle| \leq \tilde{C}\varepsilon\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}.$$

We get:

$$\left|\mathcal{B}^{[2]}_{\varepsilon,\delta}(\phi,\psi) - \mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi,\psi)\right| \leq C\varepsilon\sqrt{\mathcal{Q}^{[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi)}.$$

By Lemma 4.8, we infer that:

$$\left\|\left(\mathcal{L}^{[2]}_{\varepsilon,\delta}\right)^{-1} - \left(\mathcal{L}^{\mathsf{app},[2]}_{\varepsilon,\delta}\right)^{-1}\right\| \leq \tilde{C}\varepsilon. \tag{16.1.2}$$

Case $\delta < 1$. The same kind of arguments provides:

$$\left|\mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\phi,\psi) - \mathcal{B}^{\mathsf{eff},[2]}_{\varepsilon,\delta}(\phi,\psi)\right| \leq C\varepsilon^{1-\delta}\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,\delta}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{eff},[2]}_{\varepsilon,\delta}(\phi)}$$

By Lemma 4.8, we get that:

$$\left\|\left(\mathcal{L}^{\mathsf{app},[2]}_{\varepsilon,\delta}\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[2]}_{\varepsilon,\delta}\right)^{-1}\right\| \leq \tilde{C}\varepsilon^{1-\delta}.$$

Case $\delta = 1$. This case is slightly more complicated to analyse. We must estimate the difference the sesquilinear forms:

$$\mathcal{D}_\varepsilon(\phi,\psi) = \mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,1}(\phi,\psi) - \mathcal{B}^{\mathsf{eff},[2]}_{\varepsilon,1}(\phi,\psi).$$

We have:

$$\mathcal{D}_\varepsilon(\phi,\psi) = \langle i\partial_s\phi, \mathbf{B}(s,0)\tau\psi\rangle + \langle \mathbf{B}(s,0)\tau\phi, i\partial_s\psi\rangle + \langle \mathbf{B}(s,0)^2\tau^2\phi,\psi\rangle - \|\tau J_1\|^2_\omega\langle \mathbf{B}(s,0)^2\phi,\psi\rangle.$$

We introduce the projection defined for $\varphi \in \mathsf{H}^1_0(\Omega)$:

$$\Pi_0\varphi = \langle \varphi, J_1\rangle_\omega\, J_1$$

and we let, for all $\varphi \in \mathsf{H}^1_0(\Omega)$:

$$\varphi^\parallel = \Pi_0\varphi, \quad \varphi^\perp = (\mathsf{Id} - \Pi_0)\varphi.$$

We can write:

$$\mathcal{D}_\varepsilon(\phi,\psi) = \mathcal{D}_\varepsilon(\phi^\parallel,\psi^\parallel) + \mathcal{D}_\varepsilon(\phi^\parallel,\psi^\perp) + \mathcal{D}_\varepsilon(\phi^\perp,\psi^\parallel) + \mathcal{D}_\varepsilon(\phi^\perp,\psi^\perp).$$

By using that $\langle \tau J_1, J_1\rangle_\omega = 0$, we get:

$$\mathcal{D}_\varepsilon(\phi^\parallel,\psi^\parallel) = 0.$$

Then we have:

$$\|\tau J_1\|^2_\omega\langle \mathbf{B}(s,0)^2\phi^\parallel,\psi^\perp\rangle = 0, \quad |\langle \mathbf{B}(s,0)^2\tau^2\phi^\parallel,\psi^\perp\rangle| \leq C\|\phi^\parallel\|\|\psi^\perp\|. \tag{16.1.3}$$

Thanks to the min-max principle, we deduce:
(16.1.4)
$$\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\perp}) \geq \frac{\lambda_2^{\mathsf{Dir}}(\omega)-\lambda_1^{\mathsf{Dir}}(\omega)}{\varepsilon^2}\|\psi^{\perp}\|^2, \quad \mathcal{Q}^{\mathsf{eff},[2]}_{\varepsilon,1}(\phi^{\perp}) \geq \frac{\lambda_2^{\mathsf{Dir}}(\omega)-\lambda_1^{\mathsf{Dir}}(\omega)}{\varepsilon^2}\|\phi^{\perp}\|^2.$$
Therefore we get:
$$|\langle \mathbf{B}(s,0)^2\tau^2\phi^{\parallel}, \psi^{\perp}\rangle| \leq C\varepsilon\|\phi\|\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\perp})}.$$
We have:
$$\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi) = \mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\parallel}) + \mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\perp}) + \mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\parallel},\psi^{\perp}) + \mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\perp},\psi^{\parallel}).$$
We can write:
$$\mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\parallel},\psi^{\perp}) = \langle (i\partial_s + \mathbf{B}(s,0)\tau)\psi^{\parallel}, (i\partial_s + \mathbf{B}(s,0)\tau)\psi^{\perp}\rangle.$$
We notice that:
$$(16.1.5) \quad \langle (i\partial_s)\psi^{\parallel}, (i\partial_s)\psi^{\perp}\rangle = 0, \quad |\langle \mathbf{B}(s,0)\tau\psi^{\parallel}, \mathbf{B}(s,0)\tau\psi^{\perp}\rangle| \leq C\|\psi^{\parallel}\|\|\psi^{\perp}\| \leq C\|\psi\|^2.$$
Moreover we have:
$$|\langle (i\partial_s)\psi^{\parallel}, \mathbf{B}(s,0)\tau\psi^{\perp}\rangle| \leq C\|(i\partial_s\psi)^{\parallel}\|\|\psi^{\perp}\| \leq C\|i\partial_s\psi\|\|\psi\| \leq \tilde{C}\|\psi\|^2 + \tilde{C}\|\psi\|\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi)}.$$
The term $\mathcal{B}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\perp},\psi^{\parallel})$ can be analysed in the same way so that:
$$\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi^{\perp}) \leq \mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi) + C\|\psi\|^2 + C\|\psi\|\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi)} \leq \tilde{C}(\|\psi\|^2 + \mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi)).$$
We infer:
$$(16.1.6) \qquad |\langle \mathbf{B}(s,0)^2\tau^2\phi^{\parallel}, \psi^{\perp}\rangle| \leq C\varepsilon\|\phi\|\left(\|\psi\| + \sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi)}\right).$$
We must now deal with the term
$$\langle i\partial_s\phi^{\parallel}, \mathbf{B}(s,0)\tau\psi^{\perp}\rangle.$$
We have:
$$|\langle i\partial_s\phi^{\parallel}, \mathbf{B}(s,0)\tau\psi^{\perp}\rangle| \leq C\|i\partial_s\phi\|\|\psi^{\perp}\|$$
and we easily deduce that:
$$(16.1.7) \qquad |\langle i\partial_s\phi^{\parallel}, \mathbf{B}(s,0)\tau\psi^{\perp}\rangle| \leq C\varepsilon\sqrt{\mathcal{Q}^{\mathsf{eff},[2]}_{\varepsilon,1}(\phi)}\left(\|\psi\| + \sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi)}\right),$$
We also get the same kind of estimate by exchanging ψ and ϕ. Gathering (16.1.3), (16.1.5), (16.1.6) and (16.1.7), we get the estimate:
$$|\mathcal{D}_\varepsilon(\phi^{\parallel},\psi^{\perp})| \leq C\varepsilon\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{eff},[2]}_{\varepsilon,1}(\phi)}.$$
By exchanging the roles of ψ and ϕ, we can also prove:
$$|\mathcal{D}_\varepsilon(\phi^{\perp},\psi^{\parallel})| \leq C\varepsilon\sqrt{\mathcal{Q}^{\mathsf{app},[2]}_{\varepsilon,1}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{eff},[2]}_{\varepsilon,1}(\phi)}.$$

We must estimate $\mathcal{D}_\varepsilon(\phi^\perp, \psi^\perp)$. With (16.1.4), we immediately deduce that:

$$|\langle \mathbf{B}(s,0)^2\tau^2\phi^\perp, \psi^\perp\rangle - \|\tau J_1\|_\omega^2 \langle \mathbf{B}(s,0)^2\phi^\perp, \psi^\perp\rangle| \leq C\varepsilon^2\|\phi\|\|\psi\|.$$

We find that:

$$|\langle i\partial_s\phi^\perp, \mathbf{B}(s,0)\tau\psi^\perp\rangle| \leq C\|\psi^\perp\|\|i\partial_s\phi\|$$

and this term can treated as the others. Finally we deduce the estimate:

$$|\mathcal{D}_\varepsilon(\phi,\psi)| \leq C\varepsilon\sqrt{\mathcal{Q}_{\varepsilon,1}^{\mathsf{app},[2]}(\psi)}\sqrt{\mathcal{Q}_{\varepsilon,1}^{\mathsf{eff},[2]}(\phi)}.$$

We apply Lemma 4.8 and the estimate (16.1.2) to obtain Theorem 4.2.

1.2. Proof of Corollary 4.3. Let us expand the operator $\mathcal{L}_{\varepsilon,b\mathcal{A}_\varepsilon}^{[2]}$ in formal power series:

$$\mathcal{L}_{\varepsilon,b\mathcal{A}_\varepsilon}^{[2]} \sim \sum_{j=0} \varepsilon^{j-2} L_j,$$

where

$$L_0 = -\partial_\tau^2, \quad L_1 = 0, \quad L_2 = (i\partial_s + \tau\mathbf{B}(s,0))^2 - \frac{\kappa(s)^2}{4}.$$

We look for a quasimode in the form of a formal power series:

$$\psi \sim \sum_{j\geq 0} \varepsilon^j \psi_j$$

and a quasi-eigenvalue:

$$\gamma \sim \sum_{j\geq 0} \gamma_j \varepsilon^{j-2}.$$

We must solve:

$$(L_0 - \gamma_0)u_0 = 0.$$

We choose $\gamma_0 = \frac{\pi^2}{4}$ and we take:

$$\psi_0(s,t) = f_0(s)J_1(\tau),$$

with $J_1(\tau) = \cos\left(\frac{\pi\tau}{2}\right)$. Then, we must solve:

$$(L_0 - \gamma_0)\psi_1 = \gamma_1\psi_0.$$

We have $\gamma_1 = 0$ and $\psi_1 = f_1(s)J_1(\tau)$. Then, we solve:

$$(L_0 - \gamma_0)\psi_2 = \gamma_2 u_0 - L_2 u_0. \tag{16.1.8}$$

The Fredholm condition implies the equation:

$$-\partial_s^2 f + \left(\left(\frac{1}{3} + \frac{2}{\pi^2}\right)\mathbf{B}(s,0)^2 - \frac{\kappa(s)^2}{4}\right) f_0 = \mathcal{T}^{[2]} f_0 = \gamma_2 f_0$$

and we take for $\gamma_2 = \gamma_{2,n} = \mu_n$ a negative eigenvalue of $\mathcal{T}^{[2]}$ and for f_0 a corresponding normalized eigenfunction (which has an exponential decay).

This leads to the choice:

$$\psi_2 = \psi_2^\perp(s,\tau) + f_2(s)J_1(\tau),$$

where $\psi_2^\perp$ is the unique solution of (16.1.8) which satisfies $\langle \psi_2^\perp, J_1\rangle_\tau = 0$. We can continue the construction at any order where this formal series method is used in a semiclassical context). We write $(\gamma_{j,n}, \psi_{j,n})$ instead of (γ_j, ψ_j) to emphasize the dependence on n (determined in the choice of γ_2). We let:

$$\Psi_{J,n}(\varepsilon) = \sum_{j=0}^{J} \varepsilon^j \psi_{j,n}, \text{ and } \Gamma_{J,n}(\varepsilon) = \sum_{j=0}^{J} \varepsilon^{-2+j}\gamma_{j,n}. \tag{16.1.9}$$

A computation provides:

$$\|(\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon} - \Gamma_{J,n}(\varepsilon))\Psi_{J,n}(\varepsilon)\| \leq C\varepsilon^{J+1}.$$

The spectral theorem implies that:

$$\mathrm{dist}(\Gamma_{J,n}(\varepsilon), \mathsf{sp}_{\mathsf{dis}}(\mathcal{L}^{[2]}_{\varepsilon,b\mathcal{A}_\varepsilon})) \leq C\varepsilon^{J+1}.$$

It remains to use the spectral gap given by the approximation of the resolvent in Theorem 4.2 and Corollary 4.3 follows.

2. Three dimensional waveguides

2.1. Preliminaries. We will adopt the following notation:

Notation 16.1. *Given an open set $U \subset \mathbb{R}^d$ and a vector field $\mathsf{F} = \mathsf{F}(y_1, \cdots, y_d) : U \to \mathbb{R}^d$ in dimension $d = 2, 3$, we will use in our computations the following notation:*

$$\mathsf{curl}\,\mathsf{F} = \begin{cases} \partial_{y_1}\mathsf{F}_2 - \partial_{y_2}\mathsf{F}_1 & \text{if} \quad d = 2,\\ (\partial_{y_2}\mathsf{F}_3 - \partial_{y_3}\mathsf{F}_2, \partial_{y_3}\mathsf{F}_1 - \partial_{y_1}\mathsf{F}_3, \partial_{y_1}\mathsf{F}_2 - \partial_{y_2}\mathsf{F}_1) & \text{if} \quad d = 3.\end{cases}$$

The reader is warned that, if $(y_1, \cdots, y_d)$ represent curvilinear coordinates, the outcome will differ from the usual (invariant) definition of curl.

We recall the relations between $\mathcal{A}$, $\mathcal{B}$ and $\mathbf{A}$, $\mathbf{B}$. This can be done in terms of differential forms. Let us consider the 1-form:

$$\xi_{\mathbf{A}} = \mathbf{A}_1\,\mathrm{d}x_1 + \mathbf{A}_2\,\mathrm{d}x_2 + \mathbf{A}_3\,\mathrm{d}x_3.$$

We consider Φ the diffeomorphism defined in (4.1.5). The pull-back of $\xi_{\mathbf{A}}$ by Φ is given by:

$$\Phi^*\xi_{\mathbf{A}} = \mathcal{A}_1\,\mathrm{d}t_1 + \mathcal{A}_2\,\mathrm{d}t_2 + \mathcal{A}_3\,\mathrm{d}t_3.$$

where $\mathcal{A} = {}^tD\Phi\mathbf{A}(\Phi)$ since we have $x = \Phi(t)$ and we can write:

$$\mathrm{d}x_i = \sum_{j=1}^{3} \partial_j x_i\,\mathrm{d}t_j. \tag{16.2.1}$$

We can compute the exterior derivatives:

$$\mathrm{d}\xi_{\mathbf{A}} = \mathbf{B}_{23}\,\mathrm{d}x_2 \wedge \mathrm{d}x_3 + \mathbf{B}_{13}\,\mathrm{d}x_1 \wedge \mathrm{d}x_3 + \mathbf{B}_{12}\,\mathrm{d}x_1 \wedge \mathrm{d}x_2$$

and

$$d(\Phi^*\xi_{\mathbf{A}}) = \mathcal{B}_{23}\,\mathrm{d}t_2 \wedge \mathrm{d}t_3 + \mathcal{B}_{13}\,\mathrm{d}t_1 \wedge \mathrm{d}t_3 + \mathcal{B}_{12}\,\mathrm{d}t_1 \wedge \mathrm{d}t_2,$$

with $\mathcal{B} = \operatorname{curl}\mathcal{A}$ and $\mathbf{B} = \operatorname{curl}\mathbf{A}$ (see Notation 16.1). It remains to notice that the pull-back and the exterior derivative commute to get:

$$\Phi^* d\xi_{\mathbf{A}} = d(\Phi^*\xi_{\mathbf{A}})$$

and, using again (16.2.1), it provides the relation:

$$\mathcal{B} = {}^t Com(D\Phi)\mathbf{B} = \det(D\Phi)(D\Phi)^{-1}\mathbf{B},$$

where ${}^t Com(D\Phi)$ denotes the transpose of the comatrix of $D\Phi$. Let us give an interpretation of the components of $\mathcal{B}$. A straightforward computation provides the following expression for $D\Phi$:

$$[hT(s)+h_2(\sin\theta M_2-\cos\theta M_3)+h_3(-\cos\theta M_2-\sin\theta M_3), \cos\theta M_2+\sin\theta M_3, -\sin\theta M_2+\cos\theta M_3]$$

so that $\det D\Phi = h$ and

$\mathcal{B}_{23} = h(h^2+h_2^2+h_3^2)^{-1/2}\mathbf{B}\cdot T(s)$, $\mathcal{B}_{13} = -h\mathbf{B}\cdot(-\cos\theta M_2-\sin\theta M_3)$, $\mathcal{B}_{12} = h\mathbf{B}\cdot(-\sin\theta M_2+\cos\theta M_3)$.

Let us check that $\mathfrak{L}^{[3]}_{\varepsilon,b\mathbf{A}}$ (whose quadratic form is denoted by $\mathfrak{Q}^{[3]}_{\varepsilon,b\mathbf{A}}$) is unitarily equivalent to $\mathfrak{L}^{[3]}_{\varepsilon,b\mathcal{A}}$ given in (4.1.7). For that purpose we let:

$$G = {}^t D\Phi D\Phi$$

and a computation provides:

$$G = \begin{pmatrix} h^2+h_2^2+h_3^2 & -h_3 & -h_2 \\ -h_3 & 1 & 0 \\ -h_2 & 0 & 1 \end{pmatrix}$$

and:

$$G^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + h^{-2}\begin{pmatrix} 1 \\ h_3 \\ h_2 \end{pmatrix}\begin{pmatrix} 1 & h_3 & h_2 \end{pmatrix}.$$

We notice that $|G| = h^2$. In terms of quadratic form we write:

$$\mathfrak{Q}^{[3]}_{\varepsilon,b\mathbf{A}}(\psi) = \int_{\mathbb{R}\times(\varepsilon\omega)} |{}^t D\Phi^{-1}(-i\nabla_t + {}^t D\Phi\mathbf{A}(\Phi))|^2\, h\,\mathrm{d}t$$

and

$$\begin{aligned}\mathfrak{Q}^{[3]}_{\varepsilon,b\mathbf{A}}(\psi) = & \int_{\mathbb{R}\times(\varepsilon\omega)} \left(|(-i\partial_{t_2} + b\mathcal{A}_2)\psi|^2 + |(-i\partial_{t_3} + b\mathcal{A}_3)\psi|^2\right)\, h\,\mathrm{d}t \\ & + \int_{\mathbb{R}\times(\varepsilon\omega)} h^{-2}|\left(-i\partial_s + b\mathcal{A}_1 + h_3(-i\partial_{t_2} + b\mathcal{A}_2) + h_2(-i\partial_{t_3} + b\mathcal{A}_3)\right)\psi|^2\, h\,\mathrm{d}t\end{aligned}$$

so that:

$$\begin{aligned}&\mathfrak{Q}^{[3]}_{\varepsilon,b\mathbf{A}}(\psi)\\&=\int_{\mathbb{R}\times(\varepsilon\omega)}\left(|(-i\partial_{t_2}+b\mathcal{A}_2)\psi|^2+|(-i\partial_{t_3}+b\mathcal{A}_3)\psi|^2+h^{-2}|(-i\partial_s+b\mathcal{A}_1-i\theta'\partial_\alpha+\mathcal{R})\psi|^2\right)\,h\,\mathrm{d}t.\end{aligned}$$

Since ω is simply connected (and so is Ω_ε) we may change the gauge and assume that the vector potential is given by:

$$\begin{aligned}\mathcal{A}_1(s,t_2,t_3)&=-\frac{t_2t_3\partial_s\mathcal{B}_{23}(s,0,0)}{2}-\int_0^{t_2}\mathcal{B}_{12}(s,\tilde{t}_2,t_3)\,\mathrm{d}\tilde{t}_2-\int_0^{t_3}\mathcal{B}_{13}(s,0,\tilde{t}_3)\,\mathrm{d}\tilde{t}_3,\\ \mathcal{A}_2(s,t_2,t_3)&=-\frac{t_3\mathcal{B}_{23}(s,0,0)}{2},\\ \mathcal{A}_3(s,t_2,t_3)&=-\frac{t_2\mathcal{B}_{23}(s,0,0)}{2}+\int_0^{t_2}\mathcal{B}_{23}(s,\tilde{t}_2,t_3)\,\mathrm{d}\tilde{t}_2.\end{aligned}\tag{16.2.2}$$

In other words, thanks to the Poincaré lemma, there exists a (smooth) phase function ρ such that $D\Phi\mathbf{A}(\Phi)+\nabla_t\rho=\mathcal{A}$. In particular, we have: $\mathcal{A}_j(s,0)=0,\partial_j\mathcal{A}_j(s,0)=0$ for $j\in\{1,2,3\}$.

2.2. Proof of Theorem 4.5. Let us consider $\delta\leq 1$ and $K\geq 2\sup\frac{\kappa^2}{4}$.

A first approximation. We let:

$$\mathcal{L}^{[3]}_{\varepsilon,\delta}=\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-\delta}\mathcal{A}_\varepsilon}-\varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega)+K$$

and

$$\mathcal{L}^{\mathsf{app},[3]}_{\varepsilon,\delta}=\sum_{j=2,3}(-i\varepsilon^{-1}\partial_{\tau_j}+b\mathcal{A}^{\mathsf{lin}}_{j,\varepsilon})^2+(-i\partial_s+b\mathcal{A}^{\mathsf{lin}}_{1,\varepsilon}-i\theta'\partial_\alpha)^2-\frac{\kappa^2}{4}-\varepsilon^{-2}\partial_\tau^2-\varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega)+K,$$

where:

$$\mathcal{A}^{\mathsf{lin}}_{j,\varepsilon}(s,\tau)=\mathcal{A}_j(s,0)+\varepsilon\tau_2\partial_2\mathcal{A}_j(s,0)+\varepsilon\tau_3\partial_3\mathcal{A}_j(s,0).$$

We recall that $\mathcal{A}$ is given by (16.2.2) and that $\mathcal{L}^{[3]}_{\varepsilon,\varepsilon^{-\delta}\mathcal{A}_\varepsilon}$ is defined in (4.1.9). We have to analyse the difference of the corresponding sesquilinear forms:

$$\mathcal{B}^{[3]}_{\varepsilon,\delta}(\phi,\psi)-\mathcal{B}^{\mathsf{app},[3]}_{\varepsilon,\delta}(\phi,\psi).$$

We leave as an exercise the following estimate:

$$\left\|(\mathcal{L}^{[3]}_{\varepsilon,\delta})^{-1}-(\mathcal{L}^{\mathsf{app},[3]}_{\varepsilon,\delta})^{-1}\right\|\leq\tilde{C}\varepsilon.\tag{16.2.3}$$

2.2.1. *Case* $\delta<1$. This case is similar to the case in dimension 2 since $|b\mathcal{A}^{\mathsf{lin}}_{j,\varepsilon}|\leq C\varepsilon^{1-\delta}$. If we let:

$$\mathcal{L}^{\mathsf{app2},[3]}_{\varepsilon,\delta}=\sum_{j=2,3}(-i\varepsilon^{-1}\partial_{\tau_j})^2+(-i\partial_s-i\theta'\partial_\alpha)^2-\frac{\kappa^2}{4}-\varepsilon^{-2}\partial_\tau^2-\varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega)+K,$$

we easily get:

$$\left\|(\mathcal{L}^{\mathsf{app2},[3]}_{\varepsilon,\delta})^{-1}-(\mathcal{L}^{\mathsf{app},[3]}_{\varepsilon,\delta})^{-1}\right\|\leq\tilde{C}\varepsilon^{1-\delta}.$$

It remains to decompose the sesquilinear form associated with $\mathcal{L}^{\mathsf{app2},[3]}_{\varepsilon,\delta}$ by using the orthogonal projection Π_0 and the analysis follows the same lines as in dimension 2.

2.2.2. *Case $\delta = 1$.* This case cannot be analysed in the same way as in dimension 2. Using the explicit expression of the vector potential (16.2.2), we can write our approximated operator in the form:

$$\begin{aligned}\mathcal{L}^{\mathsf{app2},[3]}_{\varepsilon,1} = \quad & \left(-\varepsilon^{-1} i\partial_{\tau_2} - \tfrac{\mathcal{B}_{23}(s,0,0)}{2}\tau_3\right)^2 + \left(-\varepsilon^{-1} i\partial_{\tau_3} + \tfrac{\mathcal{B}_{23}(s,0,0)}{2}\tau_2\right)^2 \\ & + (-i\partial_s - i\theta'\partial_\alpha - \tau_2\mathcal{B}_{12}(s,0,0) - \tau_3\mathcal{B}_{13}(s,0,0))^2 - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + K.\end{aligned}$$

2.2.3. *Perturbation theory.* Let us introduce the operator on $\mathsf{L}^2(\omega)$ (with Dirichlet boundary condition) and depending on s:

$$\mathcal{P}^2_\varepsilon = \left(-\varepsilon^{-1} i\partial_{\tau_2} - \frac{\mathcal{B}_{23}(s,0,0)}{2}\tau_3\right)^2 + \left(-\varepsilon^{-1} i\partial_{\tau_3} + \frac{\mathcal{B}_{23}(s,0,0)}{2}\tau_2\right)^2.$$

Thanks to perturbation theory the lowest eigenvalue $\nu_{1,\varepsilon}(s)$ of $\mathcal{P}^2_\varepsilon$ is simple and we may consider an associated L^2 normalized eigenfunction $u_\varepsilon(s)$. Let us provide a estimate for the eigenpair $(\nu_{1,\varepsilon}(s), u_\varepsilon(s))$. We have to be careful with the dependence on s in the estimates. Firstly, we notice that there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all s, $\varepsilon \in (0, \varepsilon_0)$ and all $\psi \in \mathsf{H}^1_0(\omega)$:

$$\begin{aligned}(16.2.4) \quad \int_\omega \left|\left(-\varepsilon^{-1} i\partial_{\tau_2} - \frac{\mathcal{B}_{23}(s,0,0)}{2}\tau_3\right)\psi\right|^2 + \left|\left(-\varepsilon^{-1} i\partial_{\tau_3} + \frac{\mathcal{B}_{23}(s,0,0)}{2}\tau_2\right)\psi\right|^2 \mathrm{d}\tau \\ \geq \varepsilon^{-2}\int_\omega |\partial_{\tau_2}\psi|^2 + |\partial_{\tau_3}\psi|^2 \,\mathrm{d}\tau - C\varepsilon^{-1}\|\psi\|^2.\end{aligned}$$

From the min-max principle we infer that:

$$(16.2.5) \qquad \nu_{n,\varepsilon}(s) \geq \varepsilon^{-2}\lambda_n^{\mathsf{Dir}}(\omega) - C\varepsilon^{-1}.$$

Let us analyse the corresponding upper bound. Thanks to the Fredholm alternative, we may introduce R_ω the unique function such that:

$$(16.2.6) \qquad (-\Delta^{\mathsf{Dir}}_\omega - \lambda_1^{\mathsf{Dir}}(\omega))R_\omega = D_\alpha J_1, \quad \langle R_\omega, J_1\rangle_\omega = 0.$$

We use $v_\varepsilon = J_1 + \varepsilon\mathcal{B}_{23}(s,0,0)R_\omega$ as test function for $\mathcal{P}^2_\varepsilon$ and an easy computation provides that there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all s, $\varepsilon \in (0, \varepsilon_0)$:

$$\left\|\left(\mathcal{P}^2_\varepsilon - \left(\varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) + \mathcal{B}^2_{23}(s,0,0)\left(\frac{\|\tau J_1\|^2_\omega}{4} - \langle D_\alpha R_\omega, J_1\rangle_\omega\right)\right)\right)v_\varepsilon\right\|_\omega \leq C\varepsilon.$$

The spectral theorem implies that there exists $n(\varepsilon, s) \geq 1$ such that:

$$\left|\nu_{n(\varepsilon,s),\varepsilon}(s) - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) - \mathcal{B}^2_{23}(s,0,0)\left(\frac{\|\tau J_1\|^2_\omega}{4} - \langle D_\alpha R_\omega, J_1\rangle_\omega\right)\right| \leq C\varepsilon.$$

Due to the spectral gap uniform in s given by (16.2.5) we deduce that there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all s, $\varepsilon \in (0, \varepsilon_0)$:

$$\left|\nu_{1,\varepsilon}(s) - \varepsilon^{-2}\lambda_1^{\mathsf{Dir}}(\omega) - \mathcal{B}_{23}^2(s,0,0)\left(\frac{\|\tau J_1\|^2}{4} - \langle D_\alpha R_\omega, J_1\rangle_\omega\right)\right| \leq C\varepsilon.$$

This new information provides:

$$\left\|\left(\mathcal{P}_\varepsilon^2 - \nu_{1,\varepsilon}(s)\right) v_\varepsilon\right\|_\omega \leq \tilde{C}\varepsilon$$

and thus:

$$\left\|\left(\mathcal{P}_\varepsilon^2 - \nu_{1,\varepsilon}(s)\right)(v_\varepsilon - \langle v_\varepsilon, u_\varepsilon\rangle_\omega u_\varepsilon)\right\|_\omega \leq \tilde{C}\varepsilon.$$

so that, with the spectral theorem and the uniform gap between the eigenvalues:

$$\|v_\varepsilon - \langle v_\varepsilon, u_\varepsilon\rangle_\omega u_\varepsilon\|_\omega \leq C\varepsilon^3.$$

Up to changing u_ε in $-u_\varepsilon$, we infer that :

$$||\langle v_\varepsilon, u_\varepsilon\rangle_\omega| - \|v_\varepsilon\|_\omega| \leq C\varepsilon^3, \quad \|v_\varepsilon - \|v_\varepsilon\|_\omega u_\varepsilon\|_\omega \leq \tilde{C}\varepsilon^3.$$

Therefore we get:

$$\|u_\varepsilon - \tilde{v}_\varepsilon\|_\omega \leq C\varepsilon^3, \quad \tilde{v}_\varepsilon = \frac{v_\varepsilon}{\|v_\varepsilon\|_\omega}$$

and this is easy to deduce:

$$\|\nabla_{\tau_2,\tau_3}(u_\varepsilon - \tilde{v}_\varepsilon)\|_\omega \leq C\varepsilon^3. \tag{16.2.7}$$

2.2.4. *Projection arguments.* We shall analyse the difference of the sesquilinear forms:

$$\mathcal{D}_\varepsilon(\phi,\psi) = \mathcal{L}_{\varepsilon,1}^{\mathsf{app2},[3]}(\phi,\psi) - \mathcal{L}_{\varepsilon,1}^{\mathsf{eff},[3]}(\phi,\psi).$$

We write:

$$\mathcal{D}_\varepsilon(\phi,\psi) = \mathcal{D}_{\varepsilon,1}(\phi,\psi) + \mathcal{D}_{\varepsilon,2}(\phi,\psi),$$

where

$$\mathcal{D}_{\varepsilon,1}(\phi,\psi) = \langle \mathcal{P}_\varepsilon\phi, \mathcal{P}_\varepsilon\psi\rangle - \left\langle\left(-\varepsilon^{-2}\Delta_\omega^{\mathsf{Dir}} + \mathcal{B}_{23}^2(s,0,0)\left(\frac{\|\tau J_1\|_\omega^2}{4} - \langle D_\alpha R_\omega, J_1\rangle_\omega\right)\right)\phi, \psi\right\rangle$$

and

$$\mathcal{D}_{\varepsilon,2}(\phi,\psi) = \langle \mathcal{M}\phi,\psi\rangle - \langle \mathcal{M}^{\mathsf{eff}}\phi,\psi\rangle,$$

with:

$$\mathcal{M} = \left(-i\partial_s - i\theta'\partial_\alpha - \tau_2\mathcal{B}_{12}(s,0,0) - \tau_3\mathcal{B}_{13}(s,0,0)\right)^2,$$

$$\mathcal{M}^{\mathsf{eff}} = \langle(-i\partial_s - i\theta'\partial_\alpha - \mathcal{B}_{12}(s,0,0)\tau_2 - \mathcal{B}_{13}(s,0,0)\tau_3)^2\mathsf{Id}(s)\otimes J_1, \mathsf{Id}(s)\otimes J_1\rangle_\omega.$$

We introduce the projection on $u_\varepsilon(s)$:

$$\Pi_{\varepsilon,s}\varphi = \langle\varphi, u_\varepsilon\rangle_\omega\, u_\varepsilon(s)$$

and, for $\varphi \in \mathsf{H}_0^1(\Omega)$, we let:

$$\varphi^{\|_\varepsilon} = \Pi_{\varepsilon,s}\varphi, \quad \varphi^{\perp_\varepsilon} = \varphi - \Pi_{\varepsilon,s}\varphi.$$

We can write the formula:

$$\mathcal{D}_{\varepsilon,1}(\phi,\psi) = \mathcal{D}_{\varepsilon,1}(\phi^{\|_\varepsilon},\psi^{\|}) + \mathcal{D}_{\varepsilon,1}(\phi^{\|_\varepsilon},\psi^{\perp}) + \mathcal{D}_{\varepsilon,1}(\phi^{\perp_\varepsilon},\psi^{\|}) + \mathcal{D}_{\varepsilon,1}(\phi^{\perp_\varepsilon},\psi^{\perp}),$$

where $\psi^{\|} = \Pi_0\psi = \langle \psi, J_1\rangle_\omega J_1$ and $\psi^{\perp} = \psi - \psi^{\|}$. Using our mixed decomposition, we can get the following bound on $\mathcal{D}_{\varepsilon,1}(\phi,\psi)$:

$$|\mathcal{D}_{\varepsilon,1}(\phi,\psi)| \leq C\varepsilon \sqrt{\mathcal{Q}^{\mathsf{app2},[3]}_{\varepsilon,1}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{eff},[3]}_{\varepsilon,1}(\phi)}. \tag{16.2.8}$$

Moreover we easily get:

$$|\mathcal{D}_{\varepsilon,2}(\phi,\psi)| \leq C\varepsilon \sqrt{\mathcal{Q}^{\mathsf{app2},[3]}_{\varepsilon,1}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{eff},[3]}_{\varepsilon,1}(\phi)}. \tag{16.2.9}$$

Combining (16.2.8) and (16.2.9), we infer that:

$$|\mathcal{D}_{\varepsilon}(\phi,\psi)| \leq C\varepsilon \sqrt{\mathcal{Q}^{\mathsf{app2},[3]}_{\varepsilon,1}(\psi)}\sqrt{\mathcal{Q}^{\mathsf{eff},[3]}_{\varepsilon,1}(\phi)}.$$

With Lemma 4.8 we infer:

$$\left\| \left(\mathcal{L}^{\mathsf{app2},[3]}_{\varepsilon,1}\right)^{-1} - \left(\mathcal{L}^{\mathsf{eff},[3]}_{\varepsilon,1}\right)^{-1} \right\| \leq C\varepsilon. \tag{16.2.10}$$

Finally we deduce Theorem 4.5 from (16.2.3) and (16.2.10).

2.3. Proof of Corollary 4.6. For the asymptotic expansions of the eigenvalues claimed in Corollary 4.6, we leave the proof to the reader since it is a slight adaptation of the proof of Corollary 4.3.

CHAPTER 17

Spectrum of thin triangles

> O egregiam artem! Scis rotunda metiri, in quadratum redigis quamcumque acceperis formam, interualla siderum dicis, nihil est quod in mensuram tuam non cadat: si artifex es, metire hominis animum, dic quam magnus sit, dic quam pusillus sit.
>
> *Epistulae morales ad Lucilium*, LXXXVIII, Seneca

This chapter is devoted to the proof of Theorem 4.13.

1. Quasimodes and boundary layer

1.1. From the triangle to the rectangle. We first perform a change of variables to transform the triangle into a rectangle:

$$u = x \in (-\pi\sqrt{2}, 0), \quad t = \frac{y}{x+\pi\sqrt{2}} \in (-1,1). \tag{17.1.1}$$

so that Tri is transformed into

$$\mathsf{Rec} = (-\pi\sqrt{2}, 0) \times (-1,1). \tag{17.1.2}$$

The operator $\mathcal{L}_{\mathsf{Tri}}(h)$ becomes:

$$\mathcal{L}_{\mathsf{Rec}}(h)(u,t;\partial_u,\partial_t) = -h^2\Big(\partial_u - \frac{t}{u+\pi\sqrt{2}}\,\partial_t\Big)^2 - \frac{1}{(u+\pi\sqrt{2})^2}\,\partial_t^2, \tag{17.1.3}$$

with Dirichlet boundary conditions on $\partial\mathsf{Rec}$. The equation $\mathcal{L}_{\mathsf{Tri}}(h)\psi_h = \beta_h\psi_h$ is transformed into the equation

$$\mathcal{L}_{\mathsf{Rec}}(h)\hat{\psi}_h = \beta_h\hat{\psi}_h \quad \text{with} \quad \hat{\psi}_h(u,t) = \psi_h(x,y).$$

1.2. Quasimodes. We want to construct quasimodes (β_h, ψ_h) for the operator $\mathcal{L}_{\mathsf{Tri}}(h)(\partial_x,\partial_y)$. It will be more convenient to work on the rectangle Rec with the operator $\mathcal{L}_{\mathsf{Rec}}(h)(u,t;\partial_u,\partial_t)$. We introduce the new scales

$$s = h^{-2/3}u \quad \text{and} \quad \sigma = h^{-1}u, \tag{17.1.4}$$

and we look quasimodes $(\beta_h, \hat{\psi}_h)$ in the form of series

$$\beta_h \sim \sum_{j\geq 0} \beta_j h^{j/3} \quad \text{and} \quad \hat{\psi}_h(u,t) \sim \sum_{j\geq 0} \big(\Psi_j(s,t) + \Phi_j(\sigma,t)\big) h^{j/3} \tag{17.1.5}$$

in order to solve $\mathcal{L}_{\mathsf{Rec}}(h)\hat{\psi}_h = \beta_h \hat{\psi}_h$ in the sense of formal series. As will be seen hereafter, an Ansatz containing the scale $h^{-2/3}u$ alone (like for the Born-Oppenheimer operator $\mathcal{H}_{\mathsf{BO,Tri}}(h)$) is not sufficient to construct quasimodes for $\mathcal{L}_{\mathsf{Rec}}(h)$. Expanding the operator in powers of $h^{2/3}$, we obtain the formal series:

$$\mathcal{L}_{\mathsf{Rec}}(h)(h^{2/3}s, t; h^{-2/3}\partial_s, \partial_t) \sim \sum_{j\geq 0} \mathcal{L}_{2j} h^{2j/3} \quad \text{with leading term} \quad \mathcal{L}_0 = -\frac{1}{2\pi^2}\partial_t^2 \tag{17.1.6}$$

and in powers of h:

$$\mathcal{L}_{\mathsf{Rec}}(h)(h\sigma, t; h^{-1}\partial_\sigma, \partial_t) \sim \sum_{j\geq 0} \mathcal{N}_{3j} h^{j} \quad \text{with leading term} \quad \mathcal{N}_0 = -\partial_\sigma^2 - \frac{1}{2\pi^2}\partial_t^2. \tag{17.1.7}$$

In what follows, in order to finally ensure the Dirichlet conditions on the triangle Tri, we will require for our Ansatz the boundary conditions, for any $j \in \mathbb{N}$:

$$\Psi_j(0,t) + \Phi_j(0,t) = 0, \quad -1 \leq t \leq 1 \tag{17.1.8}$$

$$\Psi_j(s,\pm 1) = 0, \quad s < 0 \quad \text{and} \quad \Phi_j(\sigma, \pm 1) = 0, \quad \sigma \leq 0. \tag{17.1.9}$$

More specifically, we are interested in the ground energy $\lambda = \frac{1}{8}$ of the Dirichlet problem for $\mathcal{L}_0$ on the interval $(-1,1)$. Thus we have to solve Dirichlet problems for the operators $\mathcal{N}_0 - \frac{1}{8}$ and $\mathcal{L}_0 - \frac{1}{8}$ on the half-strip

$$\mathsf{Hst} = \mathbb{R}_- \times (-1,1), \tag{17.1.10}$$

and look for *exponentially decreasing solutions*. The situation is similar to that encountered in thin structure asymptotics with Neumann boundary conditions. The following lemma shares common features with the Saint-Venant principle, see for example [**43**, §2].

Lemma 17.1. *We denote the first normalized eigenvector of $\mathcal{L}_0$ on $\mathsf{H}_0^1((-1,1))$ by c_0:*

$$c_0(t) = \cos\left(\frac{\pi t}{2}\right).$$

Let $F = F(\sigma,t)$ be a function in $\mathsf{L}^2(\mathsf{Hst})$ with exponential decay with respect to σ and let $G \in \mathsf{H}^{3/2}((-1,1))$ be a function of t with $G(\pm 1) = 0$. Then there exists a unique $\gamma \in \mathbb{R}$ such that the problem

$$\left(\mathcal{N}_0 - \frac{1}{8}\right)\Phi = F \quad \textit{in} \quad \mathsf{Hst}, \quad \Phi(\sigma,\pm 1) = 0, \quad \Phi(0,t) = G(t) + \gamma c_0(t),$$

admits a (unique) solution in $\mathsf{H}^2(\mathsf{Hst})$ with exponential decay. There holds

$$\gamma = -\int_{-\infty}^{0}\int_{-1}^{1} F(\sigma,t)\,\sigma c_0(t)\,d\sigma dt - \int_{-1}^{1} G(t)\,c_0(t)\,dt.$$

The following two lemmas are consequences of the Fredholm alternative.

Lemma 17.2. *Let $F = F(s,t)$ be a function in $\mathsf{L}^2(\mathsf{Hst})$ with exponential decay with respect to s. Then, there exist solution(s) Ψ such that:*

$$\left(\mathcal{L}_0 - \frac{1}{8}\right)\Psi = F \quad \text{in} \quad \mathsf{Hst}, \qquad \Psi(s,\pm 1) = 0$$

if and only if $\left\langle F(s,\cdot), c_0\right\rangle_t = 0$ for all $s < 0$. In this case, $\Psi(s,t) = \Psi^{\perp}(s,t) + g(s)c_0(t)$ where $\Psi^{\perp}$ satisfies $\left\langle \Psi(s,\cdot), c_0\right\rangle_t \equiv 0$ and has also an exponential decay.

Lemma 17.3. *Let $n \geq 1$. We recall that $z_{\mathsf{Ai}^{\mathsf{rev}}}(n)$ is the n-th zero of the reverse Airy function, and we denote by*

$$g_{(n)} = \mathsf{Ai}^{\mathsf{rev}}\big((4\pi\sqrt{2})^{-1/3}s + z_{\mathsf{Ai}^{\mathsf{rev}}}(n)\big)$$

the eigenvector of the operator $-\partial_s^2 - (4\pi\sqrt{2})^{-1}s$ with Dirichlet condition on $\mathbb{R}_-$ associated with the eigenvalue $(4\pi\sqrt{2})^{-2/3}z_{\mathsf{Ai}^{\mathsf{rev}}}(n)$. Let $f = f(s)$ be a function in $\mathsf{L}^2(\mathbb{R}_-)$ with exponential decay and let $c \in \mathbb{R}$. Then there exists a unique $\beta \in \mathbb{R}$ such that the problem:

$$\left(-\partial_s^2 - \frac{s}{4\pi\sqrt{2}} - (4\pi\sqrt{2})^{-2/3}z_{\mathsf{Ai}^{\mathsf{rev}}}(n)\right) g = f + \beta g_{(n)} \quad \text{in} \quad \mathbb{R}_-, \ \text{with } g(0) = c,$$

has a solution in $\mathsf{H}^2(\mathbb{R}_-)$ with exponential decay.

Now we can start the construction of the terms of our Ansatz (17.1.5).

The equations provided by the constant terms are:

$$\mathcal{L}_0\Psi_0 = \beta_0\Psi_0(s,t), \quad \mathcal{N}_0\Phi_0 = \beta_0\Phi_0(s,t)$$

with boundary conditions (17.1.8)-(17.1.9) for $j = 0$, so that we choose $\beta_0 = \frac{1}{8}$ and $\Psi_0(s,t) = g_0(s)c_0(t)$. The boundary condition (17.1.8) provides: $\Phi_0(0,t) = -g_0(0)c_0(t)$ so that, with Lemma 17.1, we get $g_0(0) = 0$ and $\Phi_0 = 0$. The function $g_0(s)$ will be determined later. Collecting the terms of order $h^{1/3}$, we are led to:

$$(\mathcal{L}_0 - \beta_0)\Psi_1 = \beta_1\Psi_0 - \mathcal{L}_1\Psi_1 = \beta_1\Psi_0, \quad (\mathcal{N}_0 - \beta_0)\Phi_1 = \beta_1\Phi_0 - \mathcal{N}_1\Phi_1 = 0$$

with boundary conditions (17.1.8)-(17.1.9) for $j = 1$. Using Lemma 17.2, we find $\beta_1 = 0$, $\Psi_1(s,t) = g_1(s)c_0(t)$, $g_1(0) = 0$ and $\Phi_1 = 0$. Then, we get:

$$(\mathcal{L}_0 - \beta_0)\Psi_2 = \beta_2\Psi_0 - \mathcal{L}_2\Psi_0, \quad (\mathcal{N}_0 - \beta_0)\Phi_2 = 0,$$

where $\mathcal{L}_2 = -\partial_s^2 + \frac{s}{\pi^3\sqrt{2}}\partial_t^2$ and with boundary conditions (17.1.8)-(17.1.9) for $j = 2$. Lemma 17.2 provides the equation in s variable

$$\left\langle (\beta_2\Psi_0 - \mathcal{L}_2\Psi_0(s,\cdot)), c_0\right\rangle_{\mathsf{L}^2(\,dt)} = 0, \quad s < 0.$$

Taking the formula $\Psi_0 = g_0(s)c_0(t)$ into account this becomes

$$\beta_2 g_0(s) = \left(-\partial_s^2 - \frac{s}{4\pi\sqrt{2}}\right) g_0(s).$$

This equation leads to take $\beta_2 = (4\pi\sqrt{2})^{-2/3} z_{\mathbf{A}}(n)$ and for g_0 the corresponding eigenfunction $g_{(n)}$. We deduce $(\mathcal{L}_0 - \beta_0)\Psi_2 = 0$, then get $\Psi_2(s,t) = g_2(s)c_0(t)$ with $g_2(0) = 0$ and $\Phi_2 = 0$.

We find:

$$(\mathcal{L}_0 - \beta_0)\Psi_3 = \beta_3\Psi_0 + \beta_2\Psi_1 - \mathcal{L}_2\Psi_1, \quad (\mathcal{N}_0 - \beta_0)\Phi_3 = 0,$$

with boundary conditions (17.1.8)-(17.1.9) for $j = 3$. The scalar product with c_0 (Lemma 17.2) and then the scalar product with g_0 (Lemma 17.3) provide $\beta_3 = 0$ and $g_1 = 0$. We deduce: $\Psi_3(s,t) = g_3(s)c_0(t)$, and $g_3(0) = 0$, $\Phi_3 = 0$. Finally we get the equation:

$$(\mathcal{L}_0 - \beta_0)\Psi_4 = \beta_4\Psi_0 + \beta_2\Psi_2 - \mathcal{L}_4\Psi_0 - \mathcal{L}_2\Psi_2, \quad (\mathcal{N}_0 - \beta_0)\Phi_4 = 0,$$

where

$$\mathcal{L}_4 = \frac{\sqrt{2}}{\pi}\, t\partial_t\partial_s - \frac{3}{4\pi^4} s^2\partial_t^2,$$

and with boundary conditions (17.1.8)-(17.1.9) for $j = 4$. The scalar product with c_0 provides an equation for g_2 and the scalar product with g_0 determines β_4. By Lemma 17.2 this step determines $\Psi_4 = \Psi_4^{\perp} + c_0(t)g_4(s)$ with a non-zero $\Psi_4^{\perp}$ and $g_4(0) = 0$. Since by construction $\left\langle \Psi_4^{\perp}(0,\cdot),\, c_0 \right\rangle_{\mathsf{L}^2(\,dt)} = 0$, Lemma 17.1 yields a solution Φ_4 with exponential decay. Note that it also satisfies $\left\langle \Phi_4(\sigma,\cdot),\, c_0 \right\rangle_{\mathsf{L}^2(\,dt)} = 0$ for all $\sigma < 0$.

We leave the obtention of the other terms as an exercise.

2. Agmon estimates and projection method

Let us provide the estimates of Agmon which can be proved.

Proposition 17.4. *Let $\Gamma_0 > 0$. There exist $h_0 > 0$, $C_0 > 0$ and $\eta_0 > 0$ such that for $h \in (0, h_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ satisfying $|\lambda - \frac{1}{8}| \le \Gamma_0 h^{2/3}$, we have:*

$$\int_{\mathsf{Tri}} e^{\eta_0 h^{-1}|x|^{3/2}} \left(|\psi|^2 + |h^{2/3}\partial_x\psi|^2 \right) dxdy \le C_0\|\psi\|^2.$$

Proposition 17.5. *Let $\Gamma_0 > 0$. There exist $h_0 > 0$, $C_0 > 0$ and $\rho_0 > 0$ such that for $h \in (0, h_0)$ and all eigenpair (λ, ψ) of $\mathcal{L}_{\mathsf{Tri}}(h)$ satisfying $|\lambda - \frac{1}{8}| \le \Gamma_0 h^{2/3}$, we have:*

$$\int_{\mathsf{Tri}} (x + \pi\sqrt{2})^{-\rho_0/h} \left(|\psi|^2 + |h\,\partial_x\psi|^2 \right) dxdy \le C_0\|\psi\|^2.$$

Let us consider the first N_0 eigenvalues of $\mathcal{L}_{\mathsf{Rec}}(h)$ (shortly denoted by λ_n). In each corresponding eigenspace, we choose a normalized eigenfunction $\hat{\psi}_n$ so that $\langle \hat{\psi}_n, \hat{\psi}_m \rangle = 0$ if $n \neq m$. We introduce:

$$\mathfrak{E}_{N_0}(h) = \mathsf{span}(\hat{\psi}_1, \ldots, \hat{\psi}_{N_0}).$$

Let us define Q^0_{Rec} the following quadratic form:

$$Q^0_{\mathsf{Rec}}(\hat{\psi}) = \int_{\mathsf{Rec}} \left(\frac{1}{2\pi^2}|\partial_t\hat{\psi}|^2 - \frac{1}{8}|\hat{\psi}|^2 \right) (u + \pi\sqrt{2})\, dudt,$$

associated with the operator $\mathcal{L}^0_{\mathsf{Rec}} = \mathsf{Id}_u \otimes \left(-\frac{1}{2\pi^2}\partial_t^2 - \frac{1}{8}\right)$ on $\mathsf{L}^2(\mathsf{Rec}, (u+\pi\sqrt{2})dudt)$. We consider the projection on the eigenspace associated with the eigenvalue 0 of $-\frac{1}{2\pi^2}\partial_t^2 - \frac{1}{8}$:

$$\Pi_0\hat{\psi}(u,t) = \left\langle \hat{\psi}(u,\cdot), c_0\right\rangle_t c_0(t), \tag{17.2.1}$$

where we recall that $c_0(t) = \cos\left(\frac{\pi}{2}t\right)$. We can now state a first approximation result:

Proposition 17.6. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\hat{\psi} \in \mathfrak{E}_{N_0}(h)$:*

$$0 \leq Q^0_{\mathsf{Rec}}(\hat{\psi}) \leq Ch^{2/3}\|\hat{\psi}\|^2$$

and

$$\|(\mathsf{Id} - \Pi_0)\hat{\psi}\| + \|\partial_t(\mathsf{Id} - \Pi_0)\hat{\psi}\| \leq Ch^{1/3}\|\hat{\psi}\|.$$

Moreover, $\Pi_0 : \mathfrak{E}_{N_0}(h) \to \Pi_0(\mathfrak{E}_{N_0}(h))$ is an isomorphism.

We have already noticed that the quadratic form of the Dirichlet Laplacian on Tri is bounded from below by the Born-Oppenheimer approximation:

$$\mathcal{Q}_{\mathsf{Tri},h}(\psi) \geq \int_{\mathsf{Tri}} h^2|\partial_x\psi|^2 + \frac{\pi^2}{4(u+\pi\sqrt{2})^2}|\psi|^2 \,\mathrm{d}x,$$

so that, by convexity

$$\mathcal{Q}_{\mathsf{Tri},h}(\psi) \geq \int_{\mathsf{Tri}} h^2|\partial_x\psi|^2 + \frac{1}{8}\left(1 - \frac{2x}{\pi\sqrt{2}}\right)|\psi|^2 \,\mathrm{d}x.$$

It remains to change the variables and replace ψ by $\Pi_0\psi$ when ψ is in the span generated by the first eigenfunctions and this is then enough to deduce Theorem 4.13.

CHAPTER 18

Spectrum of broken waveguides

In spite of all this, however, he did not lose sight of his raft, but swam as fast as he could towards it, got hold of it, and climbed on board again so as to escape drowning.

Odyssey, Book V, Homer

In this chapter we present the main ingredients in the proof of Theorem 4.16.

1. Quasimodes

As usual we shall introduce appropriate quasimodes. As we will see, we will have to introduce the notion of Dirichlet-to-Neumann operators to analyze the transmission between the corner and the "guiding part" of the waveguide.

1.1. Preliminaries. In order to construct quasimodes for $\mathcal{L}_{\mathsf{Gui}}(h)$ of the form (γ_h, ψ_h), we use the coordinates (u, t) on the left and (u, τ) on the right and look for quasimodes $\hat{\psi}_h(u, t, \tau) = \psi_h(x, y)$. Such quasimodes will have the form on the left:

$$\psi_{\mathsf{lef}}(u, t) \sim \sum_{j \geq 0} h^{j/3} \left(\Psi_{\mathsf{lef},j}(h^{-2/3}u, t) + \Phi_{\mathsf{lef},j}(h^{-1}u, t) \right), \tag{18.1.1}$$

and on the right:

$$\psi_{\mathsf{rig}}(u, \tau) \sim \sum_{j \geq 0} h^{j/3} \Phi_{\mathsf{rig},j}(h^{-1}u, \tau) \tag{18.1.2}$$

associated with quasi-eigenvalues:

$$\gamma_h \sim \sum_{j \geq 0} \gamma_j h^{j/3}.$$

We will denote $s = h^{-2/3}u$ and $\sigma = h^{-1}u$. Since ψ_h has no jump across the line $x = 0$, we find that ψ_{lef} and ψ_{rig} should satisfy two transmission conditions on the line $u = 0$:

$$\psi_{\mathsf{lef}}(0, t) = \psi_{\mathsf{rig}}(0, t) \quad \text{and} \quad \left(\partial_u - \frac{t}{\pi\sqrt{2}} \partial_t \right) \psi_{\mathsf{lef}}(0, t) = \left(\partial_u - \frac{\partial_\tau}{\pi\sqrt{2}} \right) \psi_{\mathsf{rig}}(0, t), \tag{18.1.3}$$

for all $t \in (0,1)$. For the Ansätze (18.1.1)-(18.1.2) these conditions write for all $j \geq 0$

$$\Psi_{\mathsf{lef},j}(0,t) + \Phi_{\mathsf{lef},j}(0,t) = \Phi_{\mathsf{rig},j}(0,t) \tag{18.1.4}$$

$$\begin{aligned}\partial_\sigma \Phi_{\mathsf{lef},j}(0,t) + \partial_s \Psi_{\mathsf{lef},j-1}(0,t) - \frac{t\partial_t}{\pi\sqrt{2}}\Phi_{\mathsf{lef},j-3}(0,t) - \frac{t\partial_t}{\pi\sqrt{2}}\Psi_{\mathsf{lef},j-3}(0,t)& \\ = \partial_\sigma \Phi_{\mathsf{rig},j}(0,t) - \frac{\partial_\tau}{\pi\sqrt{2}}\Phi_{\mathsf{rig},j-3}(0,t)&,\end{aligned} \tag{18.1.5}$$

where we understand that the terms associated with a negative index are 0.

Notation 18.1. *We still set $s = h^{-2/3}u$ and $\sigma = h^{-1}u$. Like in the case of the triangle* Tri*, the operators $\mathcal{L}^{\mathsf{lef}}_{\mathsf{Gui}}$ and $\mathcal{L}^{\mathsf{rig}}_{\mathsf{Gui}}$, written in variables (s,t) and (σ,t) expand in powers of $h^{2/3}$ and h, respectively. Now we have three operator series:*

- $\mathcal{L}^{\mathsf{lef}}_{\mathsf{Gui}}(h)(h^{2/3}s, t; h^{-2/3}\partial_s, \partial_t) \sim \sum_{j\geq 0} \mathcal{L}_{2j} h^{2j/3}$. *The operators are the same as for* Tri*, but they are defined now on the half-strip* $\mathsf{Hlef} := (-\infty, 0) \times (0,1)$.
- $\mathcal{L}^{\mathsf{lef}}_{\mathsf{Gui}}(h)(h\sigma, t; h^{-1}\partial_\sigma, \partial_t) \sim \sum_{j\geq 0} \mathcal{N}^{\mathsf{lef}}_{3j} h^j$ *defined on* Hlef.
- $\mathcal{L}^{\mathsf{rig}}_{\mathsf{Gui}}(h)(h\sigma, \tau; h^{-1}\partial_\sigma, \partial_\tau) \sim \sum_{j\geq 0} \mathcal{N}^{\mathsf{rig}}_{3j} h^j$ *defined on* $\mathsf{Hrig} := (0,\infty) \times (0,1)$.

We agree to incorporate the boundary conditions on the horizontal sides of Hlef *in the definition of the operators $\mathcal{L}_j$, $\mathcal{N}^{\mathsf{lef}}_j$, and $\mathcal{N}^{\mathsf{rig}}_j$:*

- *Neumann-Dirichlet $\partial_t \Psi(s,0) = 0$ and $\Psi(s,1) = 0$ $(s < 0)$ for $\mathcal{L}_j$,*
- *Neumann-Dirichlet $\partial_t \Phi(\sigma,0) = 0$ and $\Psi(\sigma,1) = 0$ $(\sigma < 0)$ for $\mathcal{N}^{\mathsf{lef}}_j$,*
- *Pure Dirichlet $\Phi(\sigma,0) = 0$ and $\Psi(\sigma,1) = 0$ $(\sigma > 0)$ for $\mathcal{N}^{\mathsf{rig}}_j$.*

Note that

$$\mathcal{N}^{\mathsf{lef}}_0 = -\partial^2_\sigma - \frac{1}{2\pi^2}\partial^2_t \quad \textit{and} \quad \mathcal{N}^{\mathsf{rig}}_0 = -\partial^2_\sigma - \frac{1}{2\pi^2}\partial^2_\tau\,. \tag{18.1.6}$$

1.2. Dirichlet-to-Neumann operators. Here we introduce the Dirichlet-to-Neumann operators T^{rig} and T^{lef} which we use to solve the problems in the variables (σ,t). We denote by I the interface $\{0\} \times (0,1)$ between Hrig and Hlef.

On the right, and with Notation 18.1, we consider the problem:

$$\left(\mathcal{N}^{\mathsf{rig}}_0 - \frac{1}{8}\right)\Phi_{\mathsf{rig}} = 0 \ \text{ in } \ \mathsf{Hrig} \quad \text{and} \quad \Phi_{\mathsf{rig}}(0,t) = G(t)$$

where $G \in H^{1/2}_{00}(I)$. Since the first eigenvalue of the transverse part of $\mathcal{N}^{\mathsf{rig}}_0 - \frac{1}{8}$ is positive, this problem has a unique exponentially decreasing solution Φ_{rig}. Its exterior normal derivative $-\partial_\sigma \Phi_{\mathsf{rig}}$ on the line I is well defined in $H^{-1/2}(I)$. We define:

$$T^{\mathsf{rig}}G = \partial_n \Phi_{\mathsf{rig}} = -\partial_\sigma \Phi_{\mathsf{rig}}.$$

We have:

$$\langle T^{\mathsf{rig}}G, G\rangle = Q_{\mathsf{rig}}(\Phi_{\mathsf{rig}}) \geq C\|G\|^2_{H^{1/2}_{00}(I)}.$$

On the left, we consider the problem:

$$\Big(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\Big)\Phi_{\mathsf{lef}} = 0 \ \text{ in } \ \mathsf{Hlef} \quad \text{and} \quad \Phi_{\mathsf{lef}}(0,t) = G(t)$$

where $G \in H^{1/2}_{00}(I)$.

For all $G \in H^{1/2}_{00}(I)$ such that $\Pi_0 G = 0$ (where Π_0 is defined in (17.2.1)), this problem has a unique exponentially decreasing solution Φ_{lef}. Its exterior normal derivative $\partial_\sigma \Phi_{\mathsf{lef}}$ on the line I is well defined in $H^{-1/2}(I)$. We define:

$$T^{\mathsf{lef}} G = \partial_n \Phi_{\mathsf{lef}} = \partial_\sigma \Phi_{\mathsf{lef}}.$$

We have:

$$\langle T^{\mathsf{lef}} G, G\rangle = Q_{\mathsf{lef}}(\Phi_{\mathsf{lef}}) \geq 0.$$

Proposition 18.2. *The operator $T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1$ is coercive on $H^{1/2}_{00}(I)$ with $\Pi_1 = \mathsf{Id} - \Pi_0$. In particular, it is invertible from $H^{1/2}_{00}(I)$ onto $H^{-1/2}(I)$.*

This proposition allows to prove the following lemma which is in the same spirit as Lemma 17.1, but now for transmission problems on $\mathsf{Hlef} \cup \mathsf{Hrig}$ (we recall that $c_0(t) = \cos(\frac{\pi}{2}t)$):

Lemma 18.3. *Let $F_{\mathsf{lef}} = F_{\mathsf{lef}}(\sigma, t)$ and $F_{\mathsf{rig}} = F_{\mathsf{rig}}(\sigma,\tau)$ be real functions defined on Hlef and Hrig, respectively, with exponential decay with respect to σ. Let $G^0 \in H^{1/2}_{00}(I)$ and $H \in H^{-1/2}(I)$ be data on the interface $I = \partial\mathsf{Hlef} \cap \partial\mathsf{Hrig}$. Then there exists a unique coefficient $\zeta \in \mathbb{R}$ and a unique trace $G \in H^{1/2}_{00}(I)$ such that the transmission problem*

$$\begin{cases} \big(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\big)\Phi_{\mathsf{lef}} = F_{\mathsf{lef}} \quad \textit{in} \ \ \mathsf{Hlef}, & \Phi_{\mathsf{lef}}(0,t) = G(t) + G^0(t) + \zeta c_0(t), \\ \big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\big)\Phi_{\mathsf{rig}} = F_{\mathsf{rig}} \quad \textit{in} \ \ \mathsf{Hrig}, & \Phi_{\mathsf{rig}}(0,t) = G(t), \\ \partial_\sigma \Phi_{\mathsf{lef}}(0,t) - \partial_\sigma \Phi_{\mathsf{rig}}(0,t) = H(t) \quad \textit{on} \ \ I, \end{cases}$$

admits a (unique) solution $(\Phi_{\mathsf{lef}}, \Phi_{\mathsf{rig}})$ with exponential decay.

Proof. Let $(\Phi^0_{\mathsf{lef}}, \zeta_0)$ be the solution provided by Lemma 17.1 for the data $F = F_{\mathsf{lef}}$ and $G = 0$. Let Φ^0_{rig} be the unique exponentially decreasing solution of the problem

$$\Big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\Big)\Phi^0_{\mathsf{rig}} = F_{\mathsf{rig}} \quad \text{in} \quad \mathsf{Hrig}, \quad \Phi^0_{\mathsf{rig}}(0,t) = 0.$$

Let H^0 be the jump $\partial_\sigma \Phi^0_{\mathsf{rig}}(0,t) - \partial_\sigma \Phi^0_{\mathsf{lef}}(0,t)$. If we define the new unknowns $\Phi^1_{\mathsf{rig}} = \Phi_{\mathsf{rig}} - \Phi^0_{\mathsf{rig}}$ and $\Phi^1_{\mathsf{lef}} = \Phi_{\mathsf{lef}} - \Phi^0_{\mathsf{lef}}$, the problem we want to solve becomes

$$\Big(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\Big)\Phi^1_{\mathsf{lef}} = 0 \quad \text{in} \quad \mathsf{Hlef}, \quad \Phi^1_{\mathsf{lef}}(0,t) = G(t) + (\zeta - \zeta_0)c_0(t),$$

$$\Big(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\Big)\Phi^1_{\mathsf{rig}} = 0 \quad \text{in} \quad \mathsf{Hrig}, \quad \Phi^1_{\mathsf{rig}}(0,t) = G(t),$$

$$\partial_\sigma \Phi^1_{\mathsf{rig}}(0,t) - \partial_\sigma \Phi^1_{\mathsf{lef}}(0,t) = H(t) - H^0(t) \quad \text{on} \quad I.$$

Using Proposition 18.2 we can set $G = (T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}(H - H_0)$, which ensures the solvability of the above problem. $\square$

1.3. Construction of quasimodes.

1.3.1. *Terms of order h^0.* Let us write the "interior" equations:

$$\begin{aligned}
\mathsf{lef}_s : &\qquad \mathcal{L}_0\Psi_{\mathsf{lef},0} = \gamma_0\Psi_{\mathsf{lef},0}\\
\mathsf{lef}_\sigma : &\qquad \mathcal{N}_0^{\mathsf{lef}}\Phi_{\mathsf{lef},0} = \gamma_0\Phi_{\mathsf{lef},0}\\
\mathsf{rig}\ \ : &\qquad \mathcal{N}_0^{\mathsf{rig}}\Phi_{\mathsf{rig},0} = \gamma_0\Phi_{\mathsf{rig},0}\,.
\end{aligned}$$

The boundary conditions are:

$$\Psi_{\mathsf{lef},0}(0,t) + \Phi_{\mathsf{lef},0}(0,t) = \Phi_{\mathsf{rig},0}(0,t),$$
$$\partial_\sigma\Phi_{\mathsf{lef},0}(0,t) = \partial_\sigma\Phi_{\mathsf{rig},0}(0,t).$$

We get:

$$\gamma_0 = \frac{1}{8}, \quad \Psi_{\mathsf{lef},0} = g_0(s)c_0(t).$$

We now apply Lemma 18.3 with $F_{\mathsf{lef}} = 0$, $F_{\mathsf{rig}} = 0$, $G_0 = 0$, $H = 0$ to get

$$G = 0 \quad \text{and} \quad \zeta = 0.$$

We deduce: $\Phi_{\mathsf{lef},0} = 0$, $\Phi_{\mathsf{rig},0} = 0$ and, since $\zeta = -g_0(0)$, $g_0(0) = 0$. At this step, we do not have determined g_0 yet.

1.3.2. *Terms of order $h^{1/3}$.* The interior equations read:

$$\begin{aligned}
\mathsf{lef}_s : &\qquad \mathcal{L}_0\Psi_{\mathsf{lef},1} = \gamma_0\Psi_{\mathsf{lef},1} + \gamma_1\Psi_{\mathsf{lef},0}\\
\mathsf{lef}_\sigma : &\qquad \mathcal{N}_0^{\mathsf{lef}}\Phi_{\mathsf{lef},1} = \gamma_0\Phi_{\mathsf{lef},1} + \gamma_1\Phi_{\mathsf{lef},0}\\
\mathsf{rig} : &\qquad \mathcal{N}_0^{\mathsf{rig}}\Phi_{\mathsf{rig},1} = \gamma_0\Phi_{\mathsf{rig},1} + \gamma_1\Phi_{\mathsf{rig},0}.
\end{aligned}$$

Using Lemma 17.2, the first equation implies:

$$\gamma_1 = 0, \quad \Psi_{\mathsf{lef},1}(s,t) = g_1(s)c_0(t).$$

The boundary conditions are:

$$g_1(0)c_0(t) + \Phi_{\mathsf{lef},1}(0,t) = \Phi_{\mathsf{rig},1}(0,t),$$
$$g_0'(0)c_0(t) + \partial_\sigma\Phi_{\mathsf{lef},1}(0,t) = \partial_\sigma\Phi_{\mathsf{rig},1}(0,t).$$

The system becomes:

$$\begin{aligned}
\mathsf{lef}_\sigma : &\qquad \left(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\right)\Phi_{\mathsf{lef},1} = 0,\\
\mathsf{rig}\ \ : &\qquad \left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi_{\mathsf{rig},1} = 0.
\end{aligned}$$

We apply Lemma 18.3 with $F_{\mathsf{lef}} = 0$, $F_{\mathsf{rig}} = 0$, $G_0 = 0$, $H = -g_0'(0)c_0(t)$ to get:

$$G = -g_0'(0)(T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}c_0.$$

Since $G = \Phi_{\mathsf{rig},1}$ and $\zeta = -g_1(0)$, this determines $\Phi_{\mathsf{lef},1}$, $\Phi_{\mathsf{rig},1}$ and $g_1(0)$.

1.3.3. *Terms of order $h^{2/3}$.* The interior equations write:

$$\begin{aligned}
\mathsf{lef}_s : \qquad & \mathcal{L}_2\Psi_{\mathsf{lef},0} + \mathcal{L}_0\Psi_{\mathsf{lef},2} = \sum_{l+k=2} \gamma_l \Psi_{\mathsf{lef},k} \\
\mathsf{lef}_\sigma : \qquad & \mathcal{N}_0^{\mathsf{lef}}\Phi_{\mathsf{lef},2} = \sum_{l+k=2} \gamma_l \Phi_{\mathsf{lef},k} \\
\mathsf{rig} : \qquad & \mathcal{N}_0^{\mathsf{rig}}\Phi_{\mathsf{rig},2} = \frac{1}{8}\Phi_{\mathsf{rig},2},
\end{aligned}$$

with

$$\mathcal{L}_2\Psi_{\mathsf{lef},0} = -g_0''(s)c_0(t) + \frac{1}{\pi^3\sqrt{2}} s g_0(s)\partial_t^2(c_0).$$

Lemma 17.2 and then Lemma 17.3 imply:

$$-g_0'' - \frac{1}{4\pi\sqrt{2}} s g_0 = \gamma_2 g_0. \tag{18.1.7}$$

Thus, γ_2 is one of the eigenvalues of the Airy operator and g_0 an associated eigenfunction. In particular, this determines the unknown functions of the previous steps. We are led to take:

$$\Psi_{\mathsf{lef},2}(s,t) = \Psi_{\mathsf{lef},2}^{\perp} + g_2(s)c_0(t), \text{ with } \Psi_{\mathsf{lef},2}^{\perp} = 0$$

and to the system:

$$\begin{aligned}
\mathsf{lef}_\sigma : \qquad & \left(\mathcal{N}_0^{\mathsf{lef}} - \frac{1}{8}\right)\Phi_{\mathsf{lef},2} = 0 \\
\mathsf{rig} : \qquad & \left(\mathcal{N}_0^{\mathsf{rig}} - \frac{1}{8}\right)\Phi_{\mathsf{rig},2} = 0.
\end{aligned}$$

Using Lemma 18.3, we find

$$G = -g_1'(0)(T^{\mathsf{rig}} + T^{\mathsf{lef}}\Pi_1)^{-1}c_0.$$

This determines $\Phi_{\mathsf{rig},2}$, $\Phi_{\mathsf{lef},2}$ and $g_2(0)$. The function g_1 is still unknown at this step.

The next steps are left to the reader and we apply the spectral theorem as usual (after adding a small correction term in order to exactly satisfy the transmission condition).

2. Reduction to triangles

In this last section, we prove Theorem 4.16. For that purpose, we first state Agmon estimates to show that the first eigenfunctions are essentially living in the triangle Tri so that we can compare the problem in the whole guide with the triangle.

Proposition 18.4. *Let (λ, ψ) be an eigenpair of $\mathcal{L}_{\mathsf{Gui}}(h)$ such that $|\lambda - \frac{1}{8}| \leq Ch^{2/3}$. There exist $\alpha > 0$, $h_0 > 0$ and $C > 0$ such that for all $h \in (0, h_0)$, we have:*

$$\int_{x\geq 0} e^{\alpha h^{-1}x}\left(|\psi|^2 + |h\partial_x\psi|^2\right) \mathrm{d}x\,\mathrm{d}y \leq C\|\psi\|^2.$$

PROOF. The proof is left to the reader, the main ingredients being the IMS formula and the fact that $\mathcal{H}_{\mathsf{BO,Gui}}$ is a lower bound of $\mathcal{L}_{\mathsf{Gui}}(h)$ in the sense of quadratic forms. See also [**45**, Proposition 6.1] for a more direct method. $\square$

We can now achieve the proof of Theorem 4.16. Let ψ_n^h be an eigenfunction associated with $\lambda_{\mathsf{Gui},n}(h)$ and assume that the ψ_n^h are orthogonal in $\mathsf{L}^2(\Omega)$, and thus for the bilinear form $\mathcal{B}_{\mathsf{Gui},h}$ associated with the operator $\mathcal{L}_{\mathsf{Gui}}(h)$.

We choose $\varepsilon \in (0, \frac{1}{3})$ and introduce a smooth cutoff χ^hat the scale $h^{1-\varepsilon}$ for positive x

$$\chi^h(x) = \chi(xh^{\varepsilon-1}) \quad \text{with} \quad \chi \equiv 1 \ \text{ if } \ x \leq \tfrac{1}{2}, \quad \chi \equiv 0 \ \text{ if } \ x \geq 1$$

and we consider the functions $\chi^h\psi_n^h$. We denote:

$$\mathfrak{E}_{N_0}(h) = \mathsf{span}(\chi^h\psi_1^h, \ldots, \chi^h\psi_{N_0}^h).$$

We have:

$$\mathcal{Q}_{\mathsf{Gui},h}(\psi_n^h) = \lambda_{\mathsf{Gui},n}(h)\|\psi_n^h\|^2$$

and deduce by the Agmon estimates of Proposition 18.4:

$$\mathcal{Q}_{\mathsf{Gui},h}(\chi^h\psi_n^h) = \big(\lambda_{\mathsf{Gui},n}(h) + O(h^\infty)\big)\|\chi^h\psi_n^h\|^2.$$

In the same way, we get the "almost"-orthogonality, for $n \neq m$:

$$\mathcal{B}_{\mathsf{Gui},h}(\chi^h\psi_n^h, \chi^h\psi_m^h) = O(h^\infty).$$

We deduce, for all $v \in \mathfrak{E}_{N_0}(h)$:

$$\mathcal{Q}_{\mathsf{Gui},h}(v) \leq \big(\lambda_{\mathsf{Gui},N_0}(h) + O(h^\infty)\big)\|v\|^2.$$

We can extend the elements of $\mathfrak{E}_{N_0}(h)$ by zero so that $\mathcal{Q}_{\mathsf{Gui},h}(v) = \mathcal{Q}_{\mathsf{Tri}_{\varepsilon,h}}(v)$ for $v \in \mathfrak{E}_{N_0}(h)$ where $\mathsf{Tri}_{\varepsilon,h}$ is the triangle with vertices $(-\pi\sqrt{2}, 0)$, $(h^{1-\varepsilon}, 0)$ and $(h^{1-\varepsilon}, h^{1-\varepsilon} + \pi\sqrt{2})$. A dilation reduces us to:

$$\left(1 + \frac{h^{1-\varepsilon}}{\pi\sqrt{2}}\right)^{-2}(-h^2\partial_{\tilde{x}}^2 - \partial_{\tilde{y}}^2)$$

on the triangle Tri. The lowest eigenvalues of this new operator admits the lower bounds $\frac{1}{8} + z_{\mathbf{A}}(n)h^{2/3} - Ch^{1-\varepsilon}$; in particular, we deduce:

$$\lambda_{\mathsf{Gui},N_0}(h) \geq \frac{1}{8} + z_{\mathbf{A}}(N_0)h^{2/3} - Ch^{1-\varepsilon}.$$

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