Tamari lattice, m-Tamari, (a,b)-Tamari and beyond

Xavier Gérard Viennot

Tamari lattice, m-Tamari, (a, b)-Tamari and beyond

Xavier Viennot
(joint work with Louis-Francois Preville-Ratelle)

In recent years, much work has been done on the associahedra and its underlying lattice called the Tamari lattice. The vertices of this lattice are binary trees or equivalently Dyck paths (or ballot paths), enumerated by the Catalan numbers C_n. The order relation is generated by the covering relation defined with the so-called rotation on binary trees, or equivalently a certain elementary transformation on ballot paths. The number of intervals in the Tamari lattice is given by $\frac{2}{n(n+1)}\binom{4n+1}{n-1}$ (Chapoton, [4]), and this number also counts the rooted triangulations in the plane.

Motivated by the higher diagonal coinvariant spaces of the symmetric group, Bergeron [2] introduced the m-Tamari lattice for every integer m by mimicking the construction defining the covering relation on ballot paths, corresponding to $m = 1$. The elements of this lattice are m-ballot paths, that is paths located above the diagonal with slope $1/m$. Much work has been done in algebra and in combinatorics around these diagonal coinvariant spaces, which are representations of the symmetric group indexed by a number of sets of variables. The topic was initiated 20 years ago by Garsia and Haiman, in relation with Macdonald polynomials, and many deep conjectures are still open even in the bivariate case (i.e. with two sets of variables). Bergeron conjectured that the number of intervals in the m-Tamari lattice is the dimension of the alternants of the trivariate higher diagonal coinvariant spaces. He also conjectured that the number of such intervals is given by $\frac{m+1}{n(mn+1)}\binom{(m+1)^2n+m}{n-1}$, extending the formula of Chapoton. This last conjecture was proved by Bousquet-Melou, Fusy and Preville-Ratelle [3].

Some authors [6, 1] extended such combinatorics to paths above a line of rational slope a/b (where a and b are relatively prime). This is the so-called "rational Catalan combinatorics" [1]. The problem to find a generalization of the Tamari lattice for (a, b) is open and the question is posed by Elizalde in his research report for this workshop.

In the talk of this workshop, we propose a solution to this problem, and a much more general extension of the notion of Tamari lattice [10], containing m-Tamari and (a, b)-Tamari. For any path v, made of elementary north and east steps on the square grid, we define a poset $\mathrm{Tam}(v)$. The elements are the paths u with the same endpoints as v and weakly above v. Its covering relations are defined by certain transformations on the paths u that depend on v. If the path v is the path just above the line $ax = by$, we get the (a, b)-Tamari lattice, and recover the particular cases of the ordinary Tamari and m-Tamari lattices.

Then we prove the following three propositions. 1) $\mathrm{Tam}(v)$ is a lattice. 2) The lattice $\mathrm{Tam}(\overleftarrow{v})$, where $\overleftarrow{v}$ is the mirror image of v, exchanging east and north steps, is isomorphic to the dual of $\mathrm{Tam}(v)$. Thus (b, a)-Tamari is dual to (a, b)-Tamari. This property was already briefly mentioned in the thesis of Preville-Ratelle [8], and it is proved here in full generality. 3) We define a partition of the ordinary Tamari lattice (on binary trees with n vertices) into 2^{n-1} intervals $\mathrm{I}(v)$, where each interval is isomorphic to the lattice $\mathrm{Tam}(v)$. Thus all the lattices m-Tamari, (a,b)-Tamari and extensions, are simply contained in the ordinary Tamari lattice. Note that this embedding of the m-Tamari lattice in the ordinary Tamari lattice is not the same as some previous embedding already given in the literature.

The key idea is in defining a bijection between binary trees and pair of paths (u,v), where u is a path above the path v, and where the path v is called the canopy of the binary tree. This notion of canopy was introduced by Loday and Ronco (without giving a name) [7], in some algebraic considerations about the 3 Hopf algebra associated to the trilogy: hypercube, associahedron, permutahedron, with respective dimensions $2^{n-1}, C_n$ and $n!$. The bijection between binary trees and pair of paths (u,v) was introduced in a different form by Delest and Viennot [5]. We describe here a new version of the bijection which involves a "push-gliding" algorithm, and fits to our purpose.

By studying the behavior of the canopy via the rotation on binary trees, we first prove that the set $\mathrm{I}(v)$ of binary trees having a given canopy v is an interval of the (ordinary) Tamari lattice. Then we prove that this interval $\mathrm{I}(v)$ is isomorphic to the poset $\mathrm{Tam}(v)$, using various equivalent definitions of the canopy, and some combinatorics of binary trees and of the "push-gliding" bijection. The three propositions above follow immediately. The duality between the lattices $\mathrm{Tam}(v)$ and $\mathrm{Tam}(\overleftarrow{v})$ follows from the simple fact that the mirror image of a binary tree exchanges the "right" rotation and the "left" rotation defining the covering relation in each of theses lattices.

We mention that in a forthcoming paper [9], Preville-Ratelle has proved that the total number of intervals in the lattices $\mathrm{Tam}(v)$, for all the paths v of length n, is given by $\frac{2(3n+3)!}{(n+2)!(2n+3)!}$, which is the same as the number of rooted non-separable planar maps with $n+2$ edges. This gives an answer to some questions posed by the audience after our talk. Also in his talk at this workshop, Armstrong gave a construction of a simplicial complex for any pair of relatively prime integers (a,b), called the rational associahedra $\mathrm{Ass}(a,b)$ (see [1]). It will be interesting to compare the constructions $\mathrm{Ass}(a,b)$ and (a,b)-Tamari.

References

[1] D. Armstrong, B. Rhoades and N. Williams, *Rational associahedra and non-crossing partitions*, ArXiv:1305.7286.

[2] F. Bergeron and L.-F. Preville-Ratelle, *Higher trivariate diagonal harmonics via generalized Tamari posets*, J. Combinatorics **3**(3) (2012), 317-341.

[3] M. Bousquet-Melou, E. Fusy and L.-F. Preville-Ratelle, *The number of intervals in the m-Tamari lattices*, Electronic J. Combinatorics **18**(2) (2011), 26 pp.

[4] F. Chapoton, *Sur le nombre d'intervalles dans les treillis de Tamari*, Sem. Lothar. Combin B**55**f (2006), 18pp.

[5] M. Delest and X.G. Viennot, *Algebraic languages and polyominos enumeration*, Theoretical Comp. Science **34** (1984), 169-206.

[6] T. Hikita, *Affine Springer fibers of type A and combinatorics of diagonal coinvariants*, (2012), arXiv:1203.5878.

[7] J.-L. Loday and M. Ronco, *Hopf algebra of the planar binary trees*, Adv. Math. **139**(2) (1998), 293-309.

[8] L.-F. Preville-Ratelle, *Combinatoire des espaces coinvariants trivaries du groupe symetrique*, PhD thesis, UQAM, Montreal, Canada, 2012.
[9] L.-F. Preville-Ratelle, *On the number of canopy intervals in the Tamari lattices* , in preparation.
[10] L.-F. Preville-Ratelle and X.G. Viennot, *An extension of Tamari lattices*, in preparation.

LFPR: Universidad de Talca, Talca, Chile
E-mail address: `preville-ratelle@inst-mat.utalca.cl`

XV: CNRS, LABRI, Universite de Bordeaux, Bordeaux, France
E-mail address: `viennot@xavierviennot.org`