Standardness and nonstandardness of next-jump time filtrations

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Abstract

The value of the next-jump time process at each time is the date of its the next jump. We characterize the standardness of the filtration generated by this process in terms of the asymptotic behavior at $n = -\infty$ of the probability that the process jumps at time n. In the case when the filtration is not standard we characterize the standardness of its extracted filtrations.

1 Introduction

This paper provides a complete case study of standardness for a certain family of filtrations. These filtrations are those generated by the next-jump time processes $(Z_n)_{n\leqslant 0}$ defined as follows. For a given sequence $(p_n)_{n\leqslant 0}$ of numbers in $[0,1]$ with $p_0 = 1$, let $(\varepsilon_n)_{n\leqslant 0}$ be a sequence of independent Bernoulli random variables with $\Pr(\varepsilon_n = 1) = p_n$. Define $Z_0 = 0$ and $Z_n = \min\{k \mid n+1 \leqslant k \leqslant 0 \text{ and } \varepsilon_k = 1\}$ for $n \leqslant -1$. Thus $Z_{-1} = Z_0 = 0$ almost surely and denoting by $\Delta Z_n = Z_n - Z_{n-1}$ the size of the jump at time n the two following trajectorial properties of the process $(Z_n)_{n\leqslant 0}$ straightforwardly hold (see figure 1):

- $\{\varepsilon_n = 1\} = \{Z_{n-1} = n\} = \{\Delta Z_n > 0\}$;
- saying that the process $(Z_n)_{n\leqslant 0}$ jumps at time n when $\Delta Z_n > 0$, then the value Z_n of the process at time $n \leqslant -2$ is the date of the next jump.

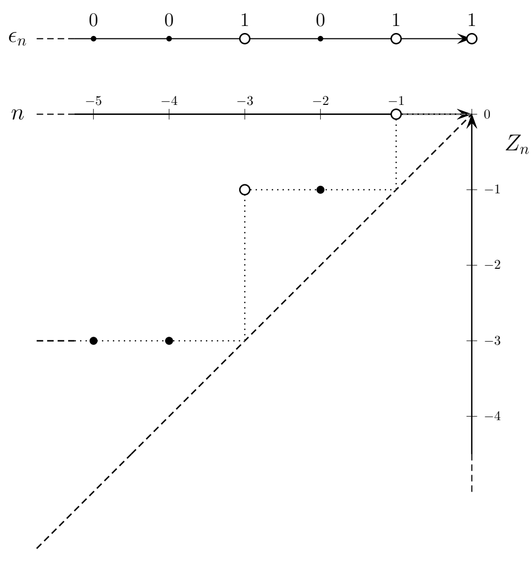

Figure 1: The next-jump time process

The object of interest of our study is the filtration in discrete negative time generated by the next-jump time process $(Z_n)_{n\leqslant 0}$, denoted by $\mathcal{F} = (\mathcal{F}_n)_{n\leqslant 0}$ throughout the paper (thus our notation does not show the dependence on the sequence (p_n) which uniquely defines $\mathcal{F}$ up to isomorphism). The following lemma gives the stochastic properties of the next-jump time process.

Lemma 1.1. *(a) The next-jump time process $(Z_n)_{n\leqslant 0}$ is a Markov process whose Markovian dynamics is described as follows:*

- *(instantaneous distributions) $Z_0 = Z_{-1} = 0$ and for each time $n \leqslant -1$, the law of Z_{n-1} is given by $\Pr(Z_{n-1} = n) = \Pr(\Delta Z_n > 0) = p_n$ and*

$$\Pr(Z_{n-1} = k) = (1 - p_n)\cdots(1 - p_{k-1})p_k$$

for every $k \in \{n+1, \dots, 0\}$;

- *(Markovian transitions) for each time $n \leqslant -1$, the conditional law $\mathcal{L}(Z_n \,|\, Z_{n-1} = k)$ of Z_n given $Z_{n-1} = k$ is the Dirac mass at k for every $k \in \{n+1, \dots, 0\}$ else if $k = n$ it equals the unconditional law $\mathcal{L}(Z_n)$ of Z_n.*

(b) For all integers $n \leqslant 0$ and $m \leqslant n-1$, the equality

$$\mathcal{L}(Z_n, \dots, Z_0 \,|\, \mathcal{F}_m) = \mathcal{L}(Z_n, \dots, Z_0)$$

occurs on the event $\{Z_m \leqslant n\} \supset \{\Delta Z_n > 0\}$.

Proof. We have already seen that $\{\varepsilon_n = 1\} = \{Z_{n-1} = n\} = \{\Delta Z_n > 0\}$, hence one has $\Pr(Z_{n-1} = n) = p_n$. For every integers $n \leqslant -1$ and $k \in [n+1, 0]$ one has $\{Z_{n-1} = k\} = \{\varepsilon_n = 0, \dots, \varepsilon_{k-1} = 0, \varepsilon_k = 1\}$, thereby giving the announced value of $\Pr(Z_{n-1} = k)$ and the equality $\mathcal{L}(Z_n \,|\, Z_{n-1} = k) = \delta_k$, and also showing that $\mathcal{L}(Z_n \,|\, \mathcal{F}_{n-1}) = \mathcal{L}(Z_n \,|\, Z_{n-1} = k)$ on the event $\{Z_{n-1} = k\}$. To finish to check the Markov property and to prove *(a)* it remains to show that $\mathcal{L}(Z_n \,|\, \mathcal{F}_{n-1}) = \mathcal{L}(Z_n \,|\, Z_{n-1} = n)$ on the event $\{Z_{n-1} = n\}$ and $\mathcal{L}(Z_n \,|\, Z_{n-1} = n) = \mathcal{L}(Z_n)$. This is a particular case of point *(b)* since $\{Z_{n-1} = n\} = \{\Delta Z_n > 0\}$. From the definition of the process $(Z_n)_{n\leqslant 0}$ it is clear that $(Z_n, \dots, Z_0)$ is $\sigma(\varepsilon_{n+1}, \dots, \varepsilon_0)$-measurable and it is easy to see that $\{Z_m \leqslant n\} = \cup_{k=m+1}^{k=n}\{\varepsilon_k = 1\}$, thereby showing *(b)*. □

Note that Z_n has the uniform law on $\{n+1, \dots, 0\}$ for every $n \leqslant -1$ in the case when $p_n = (|n|+1)^{-1}$.

It is worth focusing a minute about the kinematic properties of the process $(Z_n)_{n\leqslant 0}$. The last property of lemma 1.1 says that for each time n, the future $(Z_n, \dots, Z_0)$ of the process is conditionally independent of the past σ-field $\mathcal{F}_{n-1}$ given the event $\{\Delta Z_n > 0\} = \{Z_{n-1} = n\}$, that is, on this event the process jumps at time n and the stochastic behavior of $(Z_n, \dots, Z_0)$ is independent of the past of the process up to time $n-1$. On the complementary event $\{Z_{n-1} > n\} \in \mathcal{F}_{n-1}$ the process does not move from time $n-1$ until the next jump time Z_{n-1}. In any case the value of the process at time n is the date of the next jump. Thus, from the last property of lemma 1.1, the information about Z_n available at time $m \leqslant n-1$ is “all or nothing”: on the event $\{Z_m = k\} \in \mathcal{F}_m$ one knows that $Z_n = k$ if $k > n$, whereas $\mathcal{F}_m$ does not provide any information about Z_n if $k \leqslant n$.

The first goal of this paper is to characterize *standardness* of $\mathcal{F}$ in terms of the asymptotic behavior of the probability p_n that the procees jumps at time n. To do so, we will use the *I-cosiness criterion*, which is known to characterize standardness. A filtration is said to be *standard* when it is immersible in the filtration generated by a sequence of independent random variables. We refer to [3], [4] and [6] for details about the notion of standard filtrations. This notion has first been introduced by Vershik ([7],

[8]). This is a property at $n = -\infty$ stronger than the degeneracy of the tail σ-field $\mathcal{F}_{-\infty} := \cap_n \mathcal{F}_n$, called the *Kolmogorov* property in the present paper. It is intuitively expected that both the Kolmogorov property and the standardness property of the next-jump time filtration $\mathcal{F}$ should be related to the asymptotic behavior of $p_n = \Pr(\Delta Z_n > 0)$.

The definition of I-cosiness is presented in section 2. The study of standardness of $\mathcal{F}$ is the object of section 3. The cases when $\mathcal{F}$ is Kolmogorovian but not standard are deeper studied in section 4, where we characterize standardness of the extracted filtrations of $\mathcal{F}$. The study of section 4 is motivated by Vershik's theorem on lacunary isomorphism, which asserts than one can always extract a standard filtration from a non-standard filtrations as long as it is Kolmogorovian.

There are few known examples of families of filtrations for which such a complete standardness study has been achieved. The example of the present paper is by far the easiest one. The standardness characterizations are mainly derived from the I-cosiness criterion and Borel-Cantelli's lemmas, without involving any difficult calculations.

2 Cosiness

The I-cosiness is shortly termed as *cosiness* hereafter. The cosiness property is known to be equivalent to standardness for filtrations $\mathcal{F} = (\mathcal{F}_n)_{n \leqslant 0}$ whose final σ- field $\mathcal{F}_0$ is essentially separable. The cosiness property for a filtration $\mathcal{F}$ is defined with the help of joinings of $\mathcal{F}$. A *joining* of $\mathcal{F}$ is a pair $(\mathcal{F}', \mathcal{F}'')$ of two *jointly immersed* copies $\mathcal{F}'$ and $\mathcal{F}''$ of $\mathcal{F}$. When $\mathcal{F}$ is the filtration generated by a Markov process $(X_n)_{n \leqslant 0}$, then $(\mathcal{F}', \mathcal{F}'')$ is a joining of $\mathcal{F}$ if and only if $\mathcal{F}'$ and $\mathcal{F}''$ respectively are the filtrations generated by two copies $(X'_n)_{n \leqslant 0}$ and $(X''_n)_{n \leqslant 0}$ of $(X_n)_{n \leqslant 0}$ that are both Markovian with respect to the filtration generated by the process $(X'_n, X''_n)_{n \leqslant 0}$. In other words each of the two processes $(X'_n)_{n \leqslant 0}$ and $(X''_n)_{n \leqslant 0}$ is a copy of $(X_n)_{n \leqslant 0}$ but moreover the Markovian dynamics is not altered for one who observes over time both the processes.

An $\mathcal{F}_0$-measurable random variable X taking finitely many values is said to be *cosy* (with respect to $\mathcal{F}$) if for every $\delta > 0$, there exists a joining $(\mathcal{F}', \mathcal{F}'')$ of $\mathcal{F}$ *independent in small time*, that is, the σ- fields $\mathcal{F}'_{n_0}$ and $\mathcal{F}''_{n_0}$ are independent for some integer n_0, and for which the respective copies X' and X'' of X in $\mathcal{F}'$ and $\mathcal{F}''$ are δ-close in the sense that $\Pr(X' \neq X'') < \delta$.

This definition is then extended to σ- fields $\mathcal{E}_0 \subset \mathcal{F}_0$ by saying that the σ- field $\mathcal{E}_0$ is cosy when every $\mathcal{E}_0$-measurable random variable taking finitely many values is cosy. Cosiness of an $\mathcal{F}_0$-measurable random variable X is then equivalent to cosiness of the σ- field $\sigma(X)$ (see [4]). Finally we say the filtration $\mathcal{F}$ is cosy when the final σ- field $\mathcal{F}_0$ is cosy. It is easy to prove that every cosy filtration is Kolmogorovian; on the other hand, proving the equivalence between cosiness and standardness is not so easy.

In lemma 2.2 we state a simple criterion for cosiness. We firstly give a preliminary lemma which we will also use in section 4. By "Markov process" we mean any stochastic process $(X_n)_{n \leqslant 0}$ of Polish-valued random variables satisfying the Markov property.

Lemma 2.1. *Let $(X_n)_{n \leqslant 0}$ be a Markov process and $\mathcal{F}$ the filtration it generates. Let $(X'_n)_{n \leqslant 0}$ and $(X^*_n)_{n \leqslant 0}$ be two independent copies of $(X_n)_{n \leqslant 0}$ and let $\mathcal{F}'$ and $\mathcal{F}^*$ the filtrations they respectively generate. Let T be a bounded from below $\mathcal{F}' \vee \mathcal{F}^*$-stopping time taking its values in $-\mathbb{N} \cup \{+\infty\}$ and such that the equality $\mathcal{L}(X'_{n+1} \,|\, \mathcal{F}'_n) = \mathcal{L}(X^*_{n+1} \,|\, \mathcal{F}^*_n)$ occurs on the event $\{T = n\}$ for every $n \leqslant -1$. Define the process $(X''_n)_{n \leqslant 0}$ by putting*

$$\begin{cases} X''_n = X^*_n & \text{for } n \leqslant T, \\ X''_n = X'_n & \text{for } n > T. \end{cases}$$

Then $(X''_n)_{n \leqslant 0}$ is a copy of $(X_n)_{n \leqslant 0}$ and the two filtrations $\mathcal{F}'$ and $\mathcal{F}''$ provide a joining of $\mathcal{F}$.

Proof. It is easy to check that both processes $(X'_n)_{n\leqslant 0}$ and $(X^*_n)_{n\leqslant 0}$ are Markovian with respect to $\mathcal{F}'\vee\mathcal{F}^*$, and that means that $\mathcal{F}'$ and $\mathcal{F}^*$ are immersed in $\mathcal{F}'\vee\mathcal{F}^*$.

To show the result it suffices to show that the process $(X''_n)_{n\leqslant 0}$ is Markovian with respect to filtration $\mathcal{F}'\vee\mathcal{F}^*$ and has the same Markov kernels as the process $(X_n)_{n\leqslant 0}$ (this implies that this process has the same law as $(X_n)_{n\leqslant 0}$ because we assume that T is bounded from below).

For each $n\leqslant 0$ we denote by $\{P^n_x\}_x$ the n-th Markov kernel of $(X_n)_{n\leqslant 0}$, that is, $\{P^n_x\}_x$ is a regular version of the conditional law of X_n given $X_{n-1}=x$, for x varying in the Polish state space of X_{n-1}. Let $n\leqslant -1$ and B be a Borel set. By immersion of $\mathcal{F}^*$ in $\mathcal{F}'\vee\mathcal{F}^*$, one gets

$$\mathbb{1}_{T>n}\Pr(X''_{n+1}\in B\,|\,\mathcal{F}'_n\vee\mathcal{F}^*_n)=\mathbb{1}_{T>n}P^{n+1}_{X''_n}(B).$$

By immersion of $\mathcal{F}'$ in $\mathcal{F}'\vee\mathcal{F}^*$, one gets

$$\mathbb{1}_{T<n}\Pr(X''_{n+1}\in B\,|\,\mathcal{F}'_n\vee\mathcal{F}^*_n)=\mathbb{1}_{T<n}P^{n+1}_{X''_n}(B).$$

To finish the proof of the lemma, one has to check the equality

$$\mathbb{1}_{T=n}\Pr(X''_{n+1}\in B\,|\,\mathcal{F}'_n\vee\mathcal{F}^*_n)=\mathbb{1}_{T=n}P^{n+1}_{X''_n}(B).$$

Firstly note that $\mathcal{L}(X'_{n+1}\,|\,\mathcal{F}'_n\vee\mathcal{F}^*_n)=\mathcal{L}(X'_{n+1}\,|\,\mathcal{F}'_n)$ and $\mathcal{L}(X^*_{n+1}\,|\,\mathcal{F}'_n\vee\mathcal{F}^*_n)=\mathcal{L}(X^*_{n+1}\,|\,\mathcal{F}^*_n)$ because of the joint immersion of $\mathcal{F}'$ and $\mathcal{F}^*$. Then the desired final equality follows from the assumption of equality of the conditional laws $\mathcal{L}(X'_{n+1}\,|\,\mathcal{F}'_n)$ and $\mathcal{L}(X^*_{n+1}\,|\,\mathcal{F}^*_n)$ on the event $\{T=n\}$. ◻

We say that a process $(X_n)_{n\leqslant 0}$ has the *independent self-meeting property* if $\Pr(X'_n=X^*_n\quad\text{i.o.})=1$ whenever $(X'_n)_{n\leqslant 0}$ and $(X^*_n)_{n\leqslant 0}$ are two independent copies of this process.

Lemma 2.2. *Let $\mathcal{F}$ be the filtration generated by a Markov process $(X_n)_{n\leqslant 0}$ such that X_n takes its values in a finite set for every $n\leqslant 0$. If the Markov process $(X_n)_{n\leqslant 0}$ has the independent self-meeting property then $\mathcal{F}$ is cosy.*

Proof. By lemma 3.33 in [4] it suffices to show that the σ-field $\sigma(X_m,\ldots,X_0)$ is cosy for every $m\leqslant 0$.

Let $(X'_n)_{n\leqslant 0}$ and $(X^*_n)_{n\leqslant 0}$ be two independent copies of $(X_n)_{n\leqslant 0}$, and denote by $\mathcal{F}'$ and $\mathcal{F}^*$ the filtrations they respectively generate. For every integer $m\leqslant 0$ and every $\delta>0$, because of the self-meeting property, one can find $n_0\leqslant m$ such that the probability of the meeting event $A:=\{X'_n=X^*_n\quad\text{for some } n\in[n_0,m]\}$ is larger than $1-\delta$. Using lemma 2.1, define a copy $(X''_n)_{n\leqslant 0}$ of $(X_n)_{n\leqslant 0}$ by putting $X''_n=X^*_n$ for $n\leqslant T$ and put $X''_n=X'_n$ for $n>T$ where T is the stopping time defined by $T=\min\{n\in[n_0,m]\mid X'_n=X^*_n\}$ on the event A and $T=+\infty$ elsewhere. By lemma 2.1 the filtrations $\mathcal{F}'$ and $\mathcal{F}''$ generated by the processes $(X'_n)_{n\leqslant 0}$ and $(X''_n)_{n\leqslant 0}$ provide a joining of $\mathcal{F}$ independent in small time. Furthermore, it is clear from the construction that $\Pr(X'_m=X''_m,\ldots,X'_0=X''_0)>1-\delta$, thereby showing that the σ-field $\sigma(X_m,\ldots,X_0)$ is cosy. ◻

In lemma below we give a simple criterion for the filtration of a Markov process to be Kolmogorovian. Later, in the proof of proposition 3.1, it will be applied to the next-jump time process with the events $A_{m,n}=\{Z_m\leqslant n\}$.

Lemma 2.3. *Let $(X_n)_{n\leqslant 0}$ be a stochastic process and $\mathcal{F}$ the filtration it generates. Let $\{A_{m,n}\}_{n\leqslant 0,m\leqslant n-1}$ be a family of events such that $A_{n-1,n}\subset A_{m,n}\in\mathcal{F}_m$ and*

$$\mathcal{L}(X_n,\ldots,X_0\,|\,\mathcal{F}_m)=\mathcal{L}(X_n,\ldots,X_0)\ \text{ on the event } A_{m,n}$$

for every $n\leqslant 0$ and $m\leqslant n-1$. If $\Pr(A_{n-1,n}\ \text{i.o.})=1$ then $(X_n)_{n\leqslant 0}$ generates a Kolmogorovian filtration.

Proof. By Lévy's reverse martingale convergence theorem, a filtration $\mathcal{F} = (\mathcal{F}_n)_{n\leqslant 0}$ is Kolmogorovian if and only if $\Pr(B \mid \mathcal{F}_n) \to \Pr(B)$ in L^1 for every event $B \in \mathcal{F}_0$. By Dynkin's π-λ theorem, it suffices to show that for each $k \leqslant 0$ every event $B \in \sigma(X_k, \ldots, X_0)$ fulfills this property since the set of all events satisfying this property is a λ-system. Let $k \leqslant 0$. The random time $T = \max\{n \leqslant k \mid A_{n-1,n} \text{ occurs}\}$ is a well-defined $(-\mathbb{N})$-valued random variable under the assumption of the lemma, and one easily checks that $\Pr(B \mid \mathcal{F}_{n_0}) = \Pr(B)$ on the event $\{T > n_0\}$ for every event $B \in \sigma(X_k, \ldots, X_0)$ and every $n_0 \leqslant k$. □

3 Next-jump time processes

We consider the next-jump time processes $(Z_n)_{n\leqslant 0}$ defined in the introduction. Recall that the law of this Markov process is given by the probabilities $p_n = \Pr(\Delta Z_n > 0)$ for $n \leqslant -1$. The filtration generated by $(Z_n)_{n\leqslant 0}$ is denoted by $\mathcal{F}$ and the object of this section is to characterize the Kolmogorovian property and the standardness property for $\mathcal{F}$ in terms of the asymptotic behaviour of the sequence $(p_n)_{n\leqslant 0}$. Our study will be more convenient by tagging the following particular case:

$$p_n = 1 \text{ and } p_k = 0 \text{ for every } k < n \text{ for some } n \leqslant 0 \qquad (*)$$

in which case $\mathcal{F}_m = \{\varnothing, \Omega\}$ for every $m \leqslant n-1$, therefore $\mathcal{F}$ is Kolmogorovian and even standard.

We firstly give a precise statement about the tail σ- field $\mathcal{F}_{-\infty}$ in proposition below.

Proposition 3.1. *The tail σ-field $\mathcal{F}_{-\infty}$ is the σ-field generated by the random variable $N := \inf\{n \leqslant 0 \mid \varepsilon_n = 1\}$. There are three possible situations:*

1) if $\sum p_k = \infty$ then the process $(Z_n)_{n\leqslant 0}$ almost surely jumps infinitely many times, thus $N = -\infty$ almost surely and $\mathcal{F}$ is Kolmogorovian;

2) if $\sum p_k < \infty$ then $(Z_n)_{n\leqslant 0}$ almost surely jumps only finitely many times, thus $N > -\infty$ almost surely and

- *(a) either N is not degenerate and $\mathcal{F}$ is therefore not Kolmogorovian,*
- *(b) or we are in case $(*)$ when $N = n$ almost surely and, as already noted above, $\mathcal{F}$ is Kolmogorovian and even standard.*

Proof. If $\sum p_k = \infty$ then $N = -\infty$ almost surely by Borel-Cantelli's second lemma and $\mathcal{F}$ is Kolmogorovian by lemma 2.3 applied with $A_{m,n} = \{Z_m \leqslant n\}$ and by lemma 1.1. Consequently the equality $\mathcal{F}_{-\infty} = \sigma(N)$ obviously holds in this case. If $\sum p_k < \infty$ then $N > -\infty$ almost surely by Borel-Cantelli's first lemma. Moreover N is $\mathcal{F}_{-\infty}$-measurable and every $\mathcal{F}_{-\infty}$-measurable random variable is constant on the events $\{N = n\}$, $n \in -\mathbb{N}$, because $Z_m = n$ for every $m \leqslant n-1$ on the event $\{N = n\}$. Thus the equality $\mathcal{F}_{-\infty} = \sigma(N)$ also holds in this case. □

Now we study the cosiness property for $\mathcal{F}$. We will see in lemma 3.3 that the converse of lemma 2.2 holds for the next-jump time process: the process $(Z_n)_{n\leqslant 0}$ has the independent self-meeting property if it generates a cosy filtration. Then we will characterize the independent self-meeting property in lemma 3.4 in terms of the sequence $(p_n)_{n\leqslant 0}$, and we will conclude in theorem 3.5.

The idea of the proof of lemma 3.3 runs as follows. Consider the exercise of checking the cosiness criterion for the random variable $(Z_m, \ldots, Z_0)$. Roughly speaking, given an integer $n_0 \leqslant 0$ and two independent copies $(Z'_n)_{n\leqslant n_0}$ and $(Z''_n)_{n\leqslant n_0}$ of the truncated process $(Z_n)_{n\leqslant n_0}$, we have to find how to extend these two copies to two copies $(Z'_n)_{n\leqslant 0}$ and $(Z''_n)_{n\leqslant 0}$ of the whole process $(Z_n)_{n\leqslant 0}$ in a jointly immersed way to reach as probably as possible the meeting event $\{Z'_m = Z''_m, \ldots, Z'_0 = Z''_0\}$. It is clear that the best we can do is to let the two copies behave independently until they meet at some time and then to keep them equal after this time. The independent self-meeting property is then equivalent to the probability of the meeting event being as high as desired when $n_0 \to -\infty$. The proof of lemma 3.3 uses the following lemma about the joinings of $\mathcal{F}$ involved in the cosiness criterion (independent in small time).

Lemma 3.2. *Let $(Z'_n)_{n\leqslant 0}$ and $(Z''_n)_{n\leqslant 0}$ be two copies of $(Z_n)_{n\leqslant 0}$ whose generated filtrations $\mathcal{F}'$ and $\mathcal{F}''$ provide a joining of $\mathcal{F}$ independent up to an integer $n_0 \leqslant 0$. Then the processes $(Z'_n)_{n\leqslant 0}$ and $(Z''_n)_{n\leqslant 0}$ behave independently up to the stopping time $T := \min\{n \mid n_0 \leqslant n \leqslant -1 \text{ and } Z'_n = Z''_n = n+1\}$, which rigorously means that the stopped joint process $(Z'_n \mathbb{1}_{T\geqslant n}, Z''_n \mathbb{1}_{T\geqslant n})_{n\leqslant 0}$ has the same law as the stopped joint process $(Z^*_n \mathbb{1}_{\widetilde{T}\geqslant n}, Z^{**}_n \mathbb{1}_{\widetilde{T}\geqslant n})_{n\leqslant 0}$ for any pair $(Z^*_n)_{n\leqslant 0}$ and $(Z^{**}_n)_{n\leqslant 0}$ of independent copies of $(Z_n)_{n\leqslant 0}$, defining $\widetilde{T}$ similarly to T by $\widetilde{T} := \min\{n \mid n_0 \leqslant n \leqslant -1 \text{ and } Z^*_n = Z^{**}_n = n+1\}$.*

Proof. The processes $(Z'_n)_{n\leqslant 0}$ and $(Z''_n)_{n\leqslant 0}$ are independent up to n_0 and each of them has the same law as $(Z_n)_{n\leqslant 0}$. Their possible joint distributions up to time 0 are obtained by choosing the joint conditional law $\mathcal{L}(Z'_n, Z''_n \mid Z'_{n-1}, Z''_{n-1})$ for n going from n_0+1 to 0. For each n the margins of this law are $\mathcal{L}(Z'_n \mid Z'_{n-1})$ and $\mathcal{L}(Z''_n \mid Z''_{n-1})$ by the immersion property. On the event $\{Z'_{n-1} = Z''_{n-1} = n\}$ the joint conditional law can be any joining of the distribution of Z_n. On the complementary event $\{Z'_{n-1} \neq n \quad \text{or} \quad Z''_{n-1} \neq n\}$ there is nothing to choose because at least one of the margins is a Dirac distribution and hence there is only one possible joint law. □

Lemma 3.3. *The process $(Z_n)_{n\leqslant 0}$ generates a cosy filtration if and only if it has the independent self-meeting property.*

Proof. If $(Z_n)_{n\leqslant 0}$ has the self-meeting property then it generates a cosy filtration by lemma 2.2. Now assume that $(Z_n)_{n\leqslant 0}$ has not the self-meeting property. Let $(Z^*_n)_{n\leqslant 0}$ and $(Z^{**}_n)_{n\leqslant 0}$ be two independent copies of $(Z_n)_{n\leqslant 0}$ and let $\widetilde{M}$ be the random time defined by $\widetilde{M} = \inf\{n \leqslant 0 \mid Z^*_n = Z^{**}_n\}$. Then there exists $m \in -\mathbb{N}$ such that $\varepsilon := \Pr(\widetilde{M} > m) > 0$. We will show that the random variable Z_m is not cosy. Let $(Z'_n)_{n\leqslant 0}$ and $(Z''_n)_{n\leqslant 0}$ be two copies of $(Z_n)_{n\leqslant 0}$ whose generated filtrations $\mathcal{F}'$ and $\mathcal{F}''$ provide a joining of $\mathcal{F}$ independent up to an integer $n_0 \leqslant 0$. Since $\widetilde{M} \leqslant \widetilde{T}$ with the notations of lemma 3.2, the random time $\widetilde{M}$ has the same law as $M := \min\{n \mid Z'_n = Z''_n\}$ by lemma 3.2. Consequently $\Pr(Z'_m = Z''_m) \leqslant \Pr(M \leqslant m) = \Pr(\widetilde{M} \leqslant m) = 1 - \varepsilon$. □

Lemma 3.4. *The process $(Z_n)_{n\leqslant 0}$ has the independent self-meeting property if and only if $(*)$ holds or $\sum p_k^2 = \infty$. When $\sum p_k^2 = \infty$ the process more precisely satisfies $\Pr(Z'_{n-1} = Z^*_{n-1} = n \text{ i.o.}) = 1$ for any pair $(Z'_n)_{n\leqslant 0}$ and $(Z^*_n)_{n\leqslant 0}$ of independent copies of $(Z_n)_{n\leqslant 0}$.*

Proof. In case $(*)$ the independent self-meeting property holds as an obvious consequence of point *2(b)* of proposition 3.1. The property $\Pr(Z'_{n-1} = Z^*_{n-1} = n \quad \text{i.o.}) = 1$ is equivalent to the condition $\sum p_k^2 = \infty$ by Borel-Cantelli's lemmas and then this condition ensures the independent self-meeting property. Conversely, if the process $(Z_n)_{n\leqslant 0}$ has the independent self-meeting property then its filtration $\mathcal{F}$ is cosy by lemma 3.3 and a fortiori it is Kolmogorovian. Discarding case $(*)$, it jumps infinitely many times by proposition 3.1 and consequently $\Pr(Z'_{n-1} = Z^*_{n-1} = n \quad \text{i.o.}) = 1$ because when the two processes meet at some time $n < 0$ but do not equal n, then at time $T-1$ they will meet and equal the next time jump $T = Z'_n = Z^*_n$. □

Theorem 3.5. *For case $(*)$ the filtration of the process $(Z_n)_{n\leqslant 0}$ is standard. For the other cases it is Kolmogorovian if and only if $\sum p_k = \infty$ and it is standard if and only if $\sum p_k^2 = \infty$.*

Proof. Straightforward from the two previous lemmas and proposition 3.1, and from the equivalence between standardness and cosiness. □

For instance $\mathcal{F}$ is Kolmogorovian but not standard when each Z_n has the uniform law on $\{n+1,\ldots,0\}$, which is the case when $p_n = (1+|n|)^{-1}$.

4 Standard subsequences

We keep all notations of the previous section. A filtration *extracted* from $\mathcal{F}$ is a filtration $\mathcal{F}_{\phi(\cdot)} = (\mathcal{F}_{\phi(n)})_{n\leqslant 0}$ for some strictly increasing function $\phi\colon -\mathbb{N} \to -\mathbb{N}$. It is easy to prove that every filtration extracted from a standard filtration is itself standard by using either the definition of standardness or the cosiness criterion. This section is motivated by Vershik's theorem on lacunary isomorphism which asserts the existence of a standard extracted filtration from any Kolmogorovian filtration whose final σ-field is essentially separable (see [5] for a short probabilistic proof of this theorem with Vershik's standardness criterion).

In this section we characterize standardness of the filtrations $\mathcal{F}_{\phi(\cdot)}$ extracted from $\mathcal{F}$ in terms of the speed of the extracting function $\phi\colon -\mathbb{N} \to -\mathbb{N}$. Throughout the section it is understood that we only consider the case when $\mathcal{F}$ is Kolmogorovian (see proposition 3.1), and our study is of interest only when $\mathcal{F}$ is not standard (see theorem 3.5). For instance we cover the case $p_n = (1+|n|)^{-1}$ for which each Z_n has the uniform law on $\{n+1,\ldots,0\}$.

The filtration generated by the Markov process $(Z_{\phi(n)})_{n\leqslant 0}$ is not the filtration $\mathcal{F}_{\phi(\cdot)}$. It is only immersed in $\mathcal{F}_{\phi(\cdot)}$, that is, the process $(Z_{\phi(n)})_{n\leqslant 0}$ is Markovian with respect to the filtration $\mathcal{F}_{\phi(\cdot)}$. Nevertheless, according to the following proposition from [4], cosiness of $\mathcal{F}_{\phi(\cdot)}$ is equivalent to cosiness of the filtration generated by the extracted process $(Z_{\phi(n)})_{n\leqslant 0}$.

Proposition 4.1. *Let $\mathcal{F}_{\phi(\cdot)}$ be a filtration extracted from the filtration $\mathcal{F}$ generated by a Markov process $(Z_n)_{n\leqslant 0}$. Then $\mathcal{F}_{\phi(\cdot)}$ is cosy if and only if the process $(Z_{\phi(n)})_{n\leqslant 0}$ generates a cosy filtration.*

Hereafter we only consider functions ϕ satisfying $\phi(0) = 0$ and $\phi(-1) = -1$ for more convenience. The main result of this section is theorem 4.6 and it remains to be true without this restriction because standardness is an asymptotic property at $n \to -\infty$.

Now we show how a certain next-jump time process $(Y_n)_{n\leqslant 0}$ can be constructed "in real time" from the process $(Z_{\phi(n)})_{n\leqslant 0}$. This process is defined by

$$Y_n = \min\{k \leqslant 0 \mid Z_{\phi(n)} \leqslant \phi(k)\},$$

thus $Y_0 = 0$ and for $n \leqslant -1$ the random variable Y_n takes its values in $\{n+1,\ldots,0\}$ and it is determined by the relation

$$\phi(Y_n - 1) < Z_{\phi(n)} \leqslant \phi(Y_n).$$

Lemma below enumerates some properties of $(Y_n)_{n\leqslant 0}$. Point *(c)* is what we meant when we said that this process is constructed "in real time" from the process $(Z_{\phi(n)})_{n\leqslant 0}$. Figure 2 is helpful to read the proof of this lemma.

Lemma 4.2. *(a) The process $(Y_n)_{n\leqslant 0}$ is the next-jump time process whose law is defined by the probabilities*

$$p_n^{\phi} := \Pr(\Delta Y_n > 0) = 1 - \prod_{k\in]\phi(n-1),\phi(n)]} (1-p_k)$$

for every $n \leqslant -1$.

(b) For each time $n \leqslant 0$, the future $(Z_{\phi(n)}, Z_{\phi(n+1)},\ldots,Z_{\phi(0)})$ of the process $(Z_{\phi(n)})_{n\leqslant 0}$ is conditionally independent of the past σ-field $\mathcal{F}_{\phi(n-1)}$ on the jump event $\{\Delta Y_n > 0\} \in \mathcal{F}_{\phi(n-1)}$.

(c) The filtration generated by $(Y_n)_{n\leqslant 0}$ is immersed in the filtration generated by $(Z_{\phi(n)})_{n\leqslant 0}$. In other words, the process $(Y_n)_{n\leqslant 0}$ is Markovian with respect to the filtration of the process $(Z_{\phi(n)})_{n\leqslant 0}$.

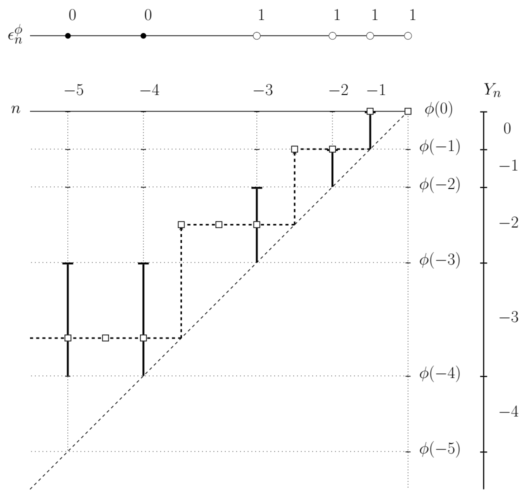

Figure 2: The next-jump time process $(Y_n)_{n\leqslant 0}$. The value of Y_n is shown by the bold interval.

Proof. Point *(a)* results from the equality $Y_n = \min\{k \mid n+1 \leqslant k \leqslant 0 \text{ and } \varepsilon_k^\phi = 1\}$ for every $n \leqslant -1$ where $(\varepsilon_n^\phi)_{n\leqslant 0}$ is the Bernoulli process defined by $\varepsilon_0^\phi = 1$ and $\varepsilon_n^\phi = \max\{\varepsilon_m \mid \phi(n-1) < m \leqslant \phi(n)\}$ for $n \leqslant -1$.

Point *(b)* is deduced from lemma 1.1*(b)* by noting the equality $\{\Delta Y_n > 0\} = \{Z_{\phi(n-1)} \leqslant \phi(n)\}$.

Denoting by $\mathcal{E}$ the filtration generated by $(Y_n)_{n\leqslant 0}$, point *(c)* amounts to say that $\mathcal{L}(Y_n \,|\, \mathcal{F}_{\phi(n-1)}) = \mathcal{L}(Y_n \,|\, \mathcal{E}_{n-1})$ for every $n \leqslant 0$. We know that $\mathcal{L}(Y_n \,|\, \mathcal{F}_{\phi(n-1)}) = \delta_{Y_{n-1}}$ on the event $\{\Delta Y_n = 0\} \in \mathcal{F}_{\phi(n-1)}$ and the equality $\mathcal{L}(Y_n \,|\, \mathcal{F}_{\phi(n-1)}) = \mathcal{L}(Y_n)$ occurs on the event $\{\Delta Y_n > 0\}$ as a consequence of point *(b)*. □

Note that the jumping probabilities of $(Y_n)_{n\leqslant 0}$ are given by

$$p_n^\phi = 1 - \frac{\phi(n)}{\phi(n-1)}$$

in the case when each Z_n has the uniform law on $\{n+1, \ldots, 0\}$.

Standardness ($\iff$ cosiness) of the filtration generated by $(Y_n)_{n\leqslant 0}$ has been characterized in the previous section. We will see (theorem 4.6) that this criterion also characterizes standardness of the extracted filtration $\mathcal{F}_{\phi(\cdot)}$, up to the particular case $(*)$ for the next-jump time process $(Y_n)_{n\leqslant 0}$, and which we treat in lemma below for more convenience.

Lemma 4.3. *If the next-jump time process $(Y_n)_{n\leqslant 0}$ satisfies $(*)$ with p_n^ϕ and p_k^ϕ instead of p_n and p_k then either $\mathcal{F}$ is standard or $\mathcal{F}$ is not Kolmogorovian.*

Proof. Under $(*)$ one has $Y_m = n$ for every $m \leqslant n-1$ and therefore $Z_k \in]\phi(n-1), \phi(n)]$ for every $k \leqslant \phi(n-1)$. Thus we are in case 2) of proposition 3.1. □

Theorem 4.6 will be derived from the results of the previous section and the two following lemmas.

Lemma 4.4. *If the filtration $\mathcal{F}_{\phi(\cdot)}$ is cosy then the filtration of the next-jump time process $(Y_n)_{n\leqslant 0}$ is cosy too.*

Proof. From lemma 4.2*(c)* and since immersion is a transitive relation, the filtration of the next-jump time process $(Y_n)_{n\leqslant 0}$ is immersed in $\mathcal{F}_{\phi(\cdot)}$. The result follows from the elementary fact that cosiness is hereditary for immersion (see [3] or [4]). □

The proof of the following lemma is a slight variant of the proof of lemma 2.2.

Lemma 4.5. *Assume that $\mathcal{F}$ is Kolmogorovian and the next-jump time process $(Y_n)_{n\leqslant 0}$ has the independent self-meeting property. Then the process $\big(Z_{\phi(n)}\big)_{n\leqslant 0}$ generates a cosy filtration.*

Proof. If the next-jump time process $(Y_n)_{n\leqslant 0}$ satisfies $(*)$ (with p_n^ϕ and p_k^ϕ) then we know by lemma 4.3 that $\mathcal{F}$ is standard or $\mathcal{F}$ is not Kolmogorovian. If $\mathcal{F}$ is standard, then $\mathcal{F}_{\phi(\cdot)}$ is standard as well as the filtration of the process $\big(Z_{\phi(n)}\big)_{n\leqslant 0}$ because it is immersed in $\mathcal{F}_{\phi(\cdot)}$.

Now we discard case $(*)$. Under the assumption that $(Y_n)_{n\leqslant 0}$ has the independent self-meeting property we will prove that the σ- field $\sigma(Z_{\phi(m)}, \ldots, Z_{\phi(0)})$ is cosy for every $m < 0$ by mimicking the proof of lemma 2.2. Let $\big(Z'_{\phi(n)}\big)_{n\leqslant 0}$ and $\big(Z^*_{\phi(n)}\big)_{n\leqslant 0}$ be two independent copies of $\big(Z_{\phi(n)}\big)_{n\leqslant 0}$ and call $(Y'_n)_{n\leqslant 0}$ and $(Y^*_n)_{n\leqslant 0}$ the copies of the process $(Y_n)_{n\leqslant 0}$ constructed from $\big(Z'_{\phi(n)}\big)_{n\leqslant 0}$ and $\big(Z^*_{\phi(n)}\big)_{n\leqslant 0}$ in the same way as $(Y_n)_{n\leqslant 0}$ is constructed from $\big(Z_{\phi(n)}\big)_{n\leqslant 0}$.

By lemma 3.4 the event $\{Y'_{n-1} = Y^*_{n-1} = n\}$ almost surely occurs infinitely many times. Hence, for any integer $m < 0$ and any $\delta > 0$, one can find $n_0 < m$ such that the probability of the meeting event $A := \{Y'_n = Y^*_n = n+1 \quad \text{for some } n \in [n_0, m]\}$ is larger than $1-\delta$. In such a situation, define the process $\big(Z''_{\phi(n)}\big)_{n\leqslant 0}$ by putting $Z''_{\phi(n)} = Z^*_{\phi(n)}$ for $n \leqslant T$ and $Z''_{\phi(n)} = Z'_{\phi(n)}$ for $n > T$ where T is the stopping time defined by $T = \min\{n \in [n_0, m] \mid Y'_n = Y^*_n = n+1\}$ on the event A and $T = +\infty$ otherwise. By lemma 4.2*(b)*, the equality $\mathcal{L}(Z'_{\phi(n)} \,|\, \mathcal{F}'_{\phi(n-1)}) = \mathcal{L}(Z^*_{\phi(n)} \,|\, \mathcal{F}^*_{\phi(n-1)})$ holds on the event $\{\Delta Y'_{n+1} > 0\} \cap \{\Delta Y^*_{n+1} > 0\} = \{T = n\}$, therefore lemma 2.1 applies and shows that the so constructed process $\big(Z''_{\phi(n)}\big)_{n\leqslant 0}$ is a copy of $\big(Z_{\phi(n)}\big)_{n\leqslant 0}$, and that the filtrations generated by $\big(Z'_{\phi(n)}\big)_{n\leqslant 0}$ and $\big(Z''_{\phi(n)}\big)_{n\leqslant 0}$ provide a joining of the filtration generated by $\big(Z_{\phi(n)}\big)_{n\leqslant 0}$. Then conclude as in the proof of lemma 2.2 that the σ-field $\sigma(Z_{\phi(m)}, \ldots, Z_{\phi(0)})$ is cosy and finally that $\big(Z_{\phi(n)}\big)_{n\leqslant 0}$ generates a cosy filtration. $\square$

Theorem 4.6. *Assume $\mathcal{F}$ is Kolmogorovian (see theorem 3.5). If $(*)$ holds with p_n^ϕ and p_k^ϕ then $\mathcal{F}_{\phi(\cdot)}$ is standard. Otherwise $\mathcal{F}_{\phi(\cdot)}$ is standard if and only if $\sum \big(p_n^\phi\big)^2 = \infty$.*

Proof. Case $(*)$ is shown by lemma 4.3 and is discarded now. If $\mathcal{F}_{\phi(\cdot)}$ is standard then $\sum \big(p_n^\phi\big)^2 = \infty$ by lemma 4.4 and theorem 3.5. Conversely, if $\sum \big(p_n^\phi\big)^2 = \infty$ then $\mathcal{F}_{\phi(\cdot)}$ is cosy by theorem 3.5, lemma 3.3, lemma 4.5, and proposition 4.1. $\square$

Using the standardness criterion of theorem 4.6 it is easy to check that for any Kolmogorovian next-jump time filtration $\mathcal{F}$, the extracted filtration $(\mathcal{F}_{2n})_{n\leqslant 0}$ is not standard whenever $\mathcal{F}$ is not standard. Thus, the present paper does not provide any example of filtration at the threshold of standardness (see [1]).

5 Other next-jump time processes

Consider the definition of the next-jump time process $(Z_n)_{n\leqslant 0}$ given in the introduction. We could have equivalently formulated this definition by defining the random set $E = \{k \in -\mathbb{N} \mid \varepsilon_k = 1\}$ and then by setting $Z_0 = 0$ and $Z_n = \min\big(E \cap [n+1, 0]\big)$ for $n \leqslant -1$. Starting from any other random subset E of $-\mathbb{N}$ almost surely containing 0, we would define in this way the general next-jump time process. Other conditions on the random set E could be added, such as one guaranteeing that the next-jump time process $(Z_n)_{n\leqslant 0}$ is Markovian; but providing a precise general definition is not our purpose here.

We only wish to mention that such a process generating a Kolmogorovian but not standard filtration appears in [2] but is a little hidden in a bigger filtration. The mathematics of [2] could be a little more clear by using this filtration instead of the bigger filtration in which it is hidden. In [2] the random set E is obtained by beforehand considering a partition $\mathcal{P}$ of $-\mathbb{N}$ made of infinitely many intervals and containing in particular the singleton interval $\{0\}$, and then by choosing at random a point in each of these intervals. The corresponding next-jump time process $(Z_n)_{n\leqslant 0}$ is Markovian. The partition $\mathcal{P}$ used in [2] has been chosen in order that the filtration is Kolmogorovian but not standard. It would not be difficult to study standardness of this filtration for a general choice of the partition $\mathcal{P}$, by using an approach similar to the one used in the present paper.

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