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ABSTRACT. We improve by an exponential factor the best known asymptotic upper bound for the density of sets avoiding 1 in Euclidean space. This result is obtained by a combination of an analytic bound that is an analogue of Lovász theta number and of a combinatorial argument involving finite subgraphs of the unit distance graph. In turn, we straightforwardly obtain an asymptotic improvement for the measurable chromatic number of Euclidean space. We also tighten previous results for the dimensions between 4 and 24.

1. INTRODUCTION

In the Euclidean space $\mathbb{R}^n$, a subset S is said to *avoid* 1 if $\|x-y\| \neq 1$ for all x, y in S. For example, one can take the union of open balls of radius $1/2$ with centers in $(2\mathbb{Z})^n$. It is natural to wonder how large S can be, given that it avoids 1, in the sense of the proportion of space that S occupies. To be more precise, if S is a measurable set, its *density* $\delta(S)$ is defined in the usual way by

$$\delta(S) = \limsup_{R\to\infty} \frac{\mathrm{vol}(B_R \cap S)}{\mathrm{vol}(B_R)}, \tag{1}$$

where B_R denotes the ball of center 0 and radius R in $\mathbb{R}^n$ and $\mathrm{vol}(S)$ is the Lebesgue measure of S. We are interested in the supreme density $m_1(\mathbb{R}^n)$ of the measurable sets avoiding 1.

In terms of graphs, a set S avoiding 1 is an independent set of the *unit distance graph*, the graph drawn on $\mathbb{R}^n$ that connects by an edge every pair of points at distance 1, and $m_1(\mathbb{R}^n)$ is a substitute for the *independence number* of this graph.

Larman and Rogers introduced in [10] the number $m_1(\mathbb{R}^n)$ in order to allow for analytic tools in the study of the *chromatic number* $\chi(\mathbb{R}^n)$ of the unit distance graph, i.e. the minimal number of colors needed to color $\mathbb{R}^n$ so that points at distance 1 receive different colors. Indeed, the inequality

$$\chi_m(\mathbb{R}^n) \geq \frac{1}{m_1(\mathbb{R}^n)}. \tag{2}$$

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holds, where $\chi_m(\mathbb{R}^n)$ denotes the *measurable chromatic number* of $\mathbb{R}^n$. In the definition of $\chi_m(\mathbb{R}^n)$, the measurability of the color classes is required, so $\chi_m(\mathbb{R}^n) \geq \chi(\mathbb{R}^n)$. We note that (2) is the exact analogue of the well known relation between the chromatic number $\chi(G)$ and the independence number $\alpha(G)$ of a finite graph $G = (V, E)$:

$$\chi(G) \geq \frac{|V|}{\alpha(G)}. \tag{3}$$

Following (2), in order to lower bound $\chi_m(\mathbb{R}^n)$, it is enough to upper bound $m_1(\mathbb{R}^n)$. As shown in [10], finite configurations of points in $\mathbb{R}^n$ can be used for this purpose. Indeed, if $G = (V, E)$ is a finite induced subgraph of the unit distance graph of $\mathbb{R}^n$, meaning that $V = \{v_1, \dots, v_M\} \subset \mathbb{R}^n$ and $E = \{ij : \|v_i - v_j\| = 1\}$, then

$$m_1(\mathbb{R}^n) \leq \frac{\alpha(G)}{|V|}. \tag{4}$$

Combined with the celebrated Frankl and Wilson intersection theorem [15], this inequality has lead to the asymptotic upper bound of 1.207^{-n}, proving the exponential decrease of $m_1(\mathbb{R}^n)$, later improved to 1.239^{-n} in [16] following similar ideas. However, (4) can by no means result in a lower estimate for $\chi_m(\mathbb{R}^n)$ that would be tighter than that of $\chi(\mathbb{R}^n)$ since the inequalities $\chi(\mathbb{R}^n) \geq \chi(G) \geq \frac{|V|}{\alpha(G)}$ obviously hold. In [21], a more sophisticated configuration principle was introduced that improved the upper estimates of $m_1(\mathbb{R}^n)$ for dimensions $2 \leq n \leq 25$, but didn't move forward to an asymptotic improvement.

A completely different approach was taken in [13], where an analogue of Lovász theta number is defined and computed for the unit distance graph (see also [5] for an earlier approach dealing with the unit sphere of Euclidean space). This number, that will be denoted here $\vartheta(\mathbb{R}^n)$, has an explicit expression in terms of Bessel functions, and satisfies

$$\vartheta(\mathbb{R}^n) \approx (\sqrt{e/2})^{-n} \approx (1.165)^{-n}. \tag{5}$$

Although asymptotically not as good as Frankl and Wilson estimate, for small dimensions, $\vartheta(\mathbb{R}^n)$ did improve the previously known upper bounds of $m_1(\mathbb{R}^n)$. Moreover, this bound was further strengthened in [13] by adding extra inequalities arising from simplicial configurations of points.

In this paper, we step on the results in [13], by considering more general configurations of points. More precisely, a linear program is associated to any finite induced subgraph of the unit distance graph $G = (V, E)$, whose optimal value $\vartheta_G(\mathbb{R}^n)$ satisfies

$$m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n). \tag{6}$$

We prove that $\vartheta_G(\mathbb{R}^n)$ decreases exponentially faster than both $\vartheta(\mathbb{R}^n)$ and the ratio $\alpha(G)/|V|$, when G is taken in the family of graphs considered by Frankl and Wilson, or in the family of graphs defined by Raigorodskii. We obtain the improved estimate

Theorem 1.1.

$$m_1(\mathbb{R}^n) \lessapprox (1.268)^{-n}. \tag{7}$$

Moreover, careful choices of graphs allow us to tighten the upper estimates of $m_1(\mathbb{R}^n)$ in the range of dimensions $4 \leq n \leq 24$ (see Table 2).

This paper is organized as follows: section 2 explains the subgraph trick in the broader context of homogeneous graphs, as this method can be of independent interest. We start with finite graphs, and then extend our approach to graphs defined on compact spaces and finally to the unit distance graph. Section 3 discusses so-called generalized Johnson graphs, which are the graphs associated to a single Hamming distance on binary words of fixed weight. These graphs include the family of graphs considered by Frankl and Wilson. Using semidefinite programming, we compute new upper bounds for their independence number in small dimensions. These new estimates will help us improving the estimates of $m_1(\mathbb{R}^n)$ for small dimensions in section 5. Section 4 gives the proof of Theorem 1 and section 5 presents our numerical results for dimensions 4 to 24.

Notations: Let u_n and $v_n \neq 0$ be two sequences. We denote $u_n \sim v_n$ if u_n and v_n are equivalent, i.e $\lim u_n/v_n = 1$, $u_n \approx v_n$ if there exists $\alpha, \beta \in \mathbb{R}$, $\beta > 0$, such that $u_n/v_n \sim \beta n^\alpha$ and, for positive sequences, $u_n \lessapprox v_n$ if there exists $\alpha, \beta \in \mathbb{R}$, $\beta > 0$ such that $u_n/v_n \leq \beta n^\alpha$.

2. Tightening the theta number with subgraphs

Let $\mathcal{G} = (X, E)$ be a finite graph. We recall that its theta number $\vartheta(\mathcal{G})$, introduced in [11], is the optimal value of a semidefinite program, that satisfies

$$\alpha(\mathcal{G}) \leq \vartheta(\mathcal{G}) \leq \chi(\overline{\mathcal{G}}).$$

Here $\alpha(\mathcal{G})$ denotes as usual the independence number of $\mathcal{G}$, i.e. the maximal number of vertices that are pairwise not connected, $\overline{\mathcal{G}}$ is the complementary graph, and $\chi(\overline{\mathcal{G}})$ is its chromatic number, the least number of colors needed to color all vertices so that connected vertices receive different colors.

Among the many equivalent definitions of $\vartheta(\mathcal{G})$, the most adequate for us follows from the properties of a certain function naturally associated to an independent set $A \subset X$:

$$S_A(x,y) := 1_A(x)1_A(y)/|A|. \tag{8}$$

Here 1_A and $|A|$ denote respectively the characteristic function of A and its cardinality. Then, S_A satisfies a number of linear conditions:

$$\sum_{x\in X} S_A(x,x) = 1, \qquad \sum_{(x,y)\in X^2} S_A(x,y) = |A|, \quad S_A(x,y) = 0 \; xy \in E. \tag{9}$$

Moreover, viewed as a symmetric matrix indexed by the vertex set X, S_A is positive semidefinite and of rank one, so

$$\begin{aligned} \alpha(\mathcal{G}) \leq \vartheta(\mathcal{G}) := \sup\big\{ \textstyle\sum_{(x,y)\in X^2} S(x,y) \;:\; & S \in \mathbb{R}^{X\times X},\ S \succeq 0, \\ & \textstyle\sum_{x\in X} S(x,x) = 1, \\ & S(x,y) = 0 \ (xy \in E) \big\}. \end{aligned} \tag{10}$$

In the definition of $\vartheta(\mathcal{G})$, we leave aside the condition that S_A has rank one because it wouldn't fit into a convex program. The notation $S \succeq 0$ means that S is a symmetric, positive semidefinite matrix.

Now we assume that $\mathcal{G}$ affords the transitive action of a finite group Γ, meaning that Γ acts transitively on X while preserving the edge set E. We choose a base point $p \in X$, and let Γ_p denote the stabilizer of p in Γ, so that X can be identified with the quotient space Γ/Γ_p. For example, $\mathcal{G}$ could be a Cayley graph on Γ; in this case, we can take for p the neutral element e of Γ and $\Gamma_e = \{e\}$.

Going back to the general case, by a standard averaging argument, in the definition of $\vartheta(\mathcal{G})$ the variable matrix S can be assumed to be Γ-invariant, meaning that $S(\gamma x, \gamma y) = S(x, y)$ for all $\gamma \in \Gamma$, $x, y \in X$. Introducing $f(x) = |X| S(x, p)$ leads to the equivalent formulation:

$$\vartheta(\mathcal{G}) = \sup \Big\{ \sum_{x \in X} f(x) \ : \ \begin{array}{l} f \in \mathbb{R}^X,\ f(\gamma x) = f(x)\ (\gamma \in \Gamma_p), \\ f \succeq 0, \\ f(p) = 1, \\ f(x) = 0\ (xp \in E)\ \Big\}. \end{array} \tag{11}$$

Here, $f \succeq 0$ means the following: for all $x \in X$, let $\gamma_x \in \Gamma$ be chosen so that $x = \gamma_x p$. We note that the value of $f(\gamma_y^{-1}\gamma_x p)$ does not depend on this choice. Then, we ask that $(x, y) \to f(\gamma_y^{-1}\gamma_x p)$ is symmetric positive semidefinite. Equivalently, the function $\gamma \to f(\gamma p)$ is a function of positive type on Γ in the sense of [14] (see also [6]).

Now let V be a subset of X; the graph $\mathcal{G}$ induces a graph structure on V that will be denoted G. In other words, $G = (V, E \cap V^2)$. Then we have the obvious inequality

$$\sum_{v \in V} 1_A(v) \leq \alpha(G). \tag{12}$$

Indeed, if $v_1, \dots, v_k \in A$, because A avoids E, the set $\{v_1, \dots, v_k\}$ is an independent set of G and so $k \leq \alpha(G)$. Moreover, for any $\gamma \in \Gamma$, because γA is also an independent set of $\mathcal{G}$, we have as well

$$\sum_{v \in V} 1_A(\gamma v) \leq \alpha(G). \tag{13}$$

We note that summing up the above inequality over $\gamma \in \Gamma$, and taking into account that $\sum_{\gamma \in \Gamma} 1_A(\gamma v) = |\Gamma_p||A|$ and $|X| = |\Gamma|/|\Gamma_p|$, leads to

$$\frac{|A|}{|X|} \leq \frac{\alpha(G)}{|V|}. \tag{14}$$

The inequality (14) although elementary turns to be very useful. For example, in coding theory it is applied to relate the sizes of codes in Hamming and Johnson spaces respectively, following Elias and Bassalygo principle. Also, Larman and Rogers inequality (4) can be seen as an analogue of (14) for the unit distance graph.

It turns out that (13) can be inserted directly in $\vartheta(\mathcal{G})$, providing this way a more efficient use of this inequality. In order to do that, we introduce an averaged form of $f_A(x) = |X| S_A(x,p)$:

$$\overline{f_A}(x) := \frac{1}{|\Gamma_p|} \sum_{\gamma\in\Gamma} 1_A(\gamma x) 1_A(\gamma p)/|A|. \tag{15}$$

From (13) we have $\sum_{v\in V} \overline{f_A}(v) \leq \alpha(G)$, so this inequality can be added to the program defining $\vartheta(G)$. Let us introduce:

$$\begin{aligned} \vartheta_G(\mathcal{G}) = \sup \big\{ \textstyle\sum_{x\in X} f(x) \ : \quad & f \in \mathbb{R}^X,\ f(\gamma x) = f(x)\ (\gamma \in \Gamma_p), \\ & f \succeq 0, \\ & f(p) = 1, \\ & f(x) = 0\ (xp \in E) \\ & \textstyle\sum_{v\in V} f(v) \leq \alpha(G) \ \big\}. \end{aligned} \tag{16}$$

Obviously, $\overline{f_A}$ satisfies the constraints of this program and $\sum_{x\in X} \overline{f_A}(x) = |A|$ so we have

$$\alpha(\mathcal{G}) \leq \vartheta_G(\mathcal{G}) \leq \vartheta(\mathcal{G}). \tag{17}$$

These results can be easily extended to a graph defined on a compact set X endowed with the homogeneous action of a compact group Γ. The Haar measure on Γ induces a measure on X such that for any measurable function φ,

$$\int_X \varphi(x)dx = \frac{1}{|\Gamma_p|} \int_\Gamma \varphi(\gamma p) d\gamma.$$

Volumes for these measures will be denoted $|\ |$. In this setting, the independence number $\alpha(\mathcal{G})$ of $\mathcal{G}$ is by definition the maximum volume of a measurable independent set. The theta number generalizes to:

$$\begin{aligned} \vartheta(\mathcal{G}) = \sup \big\{ \textstyle\int_X f(x)dx \ : \quad & f \in \mathcal{C}(X),\ f(\gamma x) = f(x)\ (\gamma \in \Gamma_p), \\ & f \succeq 0, \\ & f(p) = 1, \\ & f(x) = 0\ (xp \in E) \ \big\} \end{aligned} \tag{18}$$

where $\mathcal{C}(C)$ denotes the space of real valued continuous functions on X. Now let $V \subset X$ be a subset of X together with a finite positive Borel measure λ on V. With previous notations,

$$\alpha_\lambda(G) := \sup\{\lambda(A) \ : \ A \subset V, A^2 \cap E = \emptyset\}.$$

Then the previous reasoning go through, replacing finite sums by integrals in (15) and (13) and applying Fubini theorem. The inequalities (13) and (14) become respectively

$$\int_V 1_A(\gamma v) d\lambda(v) \leq \alpha_\lambda(G) \tag{19}$$

and

$$\frac{|A|}{|X|} \leq \frac{\alpha_\lambda(G)}{\lambda(V)}. \tag{20}$$

With

$$\vartheta_G(\mathcal{G}) = \sup\Big\{ \int_X f(x)dx \ : \ \begin{array}{l} f \in \mathcal{C}(X),\ f(\gamma x) = f(x)\ (\gamma \in \Gamma_p), \\ f \succeq 0, \\ f(p) = 1, \\ f(x) = 0\ (xp \in E) \\ \int_V f(v)d\lambda(v) \leq \alpha_\lambda(G) \Big\} \end{array} \tag{21}$$

we have

$$\alpha(\mathcal{G}) \leq \vartheta_G(\mathcal{G}) \leq \vartheta(\mathcal{G}). \tag{22}$$

Example: Taking $X = S^{n-1}$, the unit sphere of $\mathbb{R}^n$, and $E = \{xy \ : \ \|x - y\| = d\}$, defines a graph homogeneous under the action of the orthogonal group that fits into the above setting. Moreover, up to a suitable rescaling, this graph is an induced subgraph of our main object of study, the unit distance graph. The theta number of this graph was studied in [5].

Remark 2.1. *The introduction of an arbitrary measure λ on the subgraph is of interest even in the case of a finite graph $\mathcal{G}$, for example it allows for counting multiple points in V.*

Now we come to the case of the unit distance graph. This graph can be viewed as a Cayley graph on the group of translations $(\mathbb{R}^n, +)$, but because $\mathbb{R}^n$ is not compact it does not fit in our previous setting. We are anyway not interested in the largest Lebesgue measure of a set avoiding distance 1, which would be infinite, but rather in its largest *density* (1). Here we follow [13] and [12] to which we refer for details. In place of (15) we consider

$$\overline{f_A}(x) = \limsup_{R \to +\infty} \frac{1}{\mathrm{vol}(B_R)} \int_{B_R} 1_A(x+y)1_A(y)/\delta(A)dy. \tag{23}$$

We note that (23) resembles (15) where we would have set $p = 0^n$ and Γ would be the group of translations by $y \in \mathbb{R}^n$, except that summation over Γ is replaced by averaging over larger and larger balls. Accordingly, with

$$\delta(f) := \limsup_{R \to +\infty} \frac{1}{\mathrm{vol}(B_R)} \int_{B_R} f(x)dx, \tag{24}$$

let

$$\vartheta(\mathbb{R}^n) = \sup\big\{\delta(f) \ : \ \begin{array}{l} f \in \mathcal{C}(\mathbb{R}^n), \\ f \succeq 0, \\ f(0) = 1, \\ f(x) = 0\ (\|x\| = 1) \big\}. \end{array} \tag{25}$$

To an induced subgraph $G = (V, E)$ of the unit distance graph, endowed with a finite positive Borel measure λ, we associate

$$\vartheta_G(\mathbb{R}^n) = \sup\big\{\delta(f) \ : \ \begin{array}{l} f \in \mathcal{C}(\mathbb{R}^n), \\ f \succeq 0, \\ f(0) = 1, \\ f(x) = 0\ (\|x\| = 1) \\ \int_V f(v)d\lambda(v) \leq \alpha_\lambda(G) \big\}. \end{array} \tag{26}$$

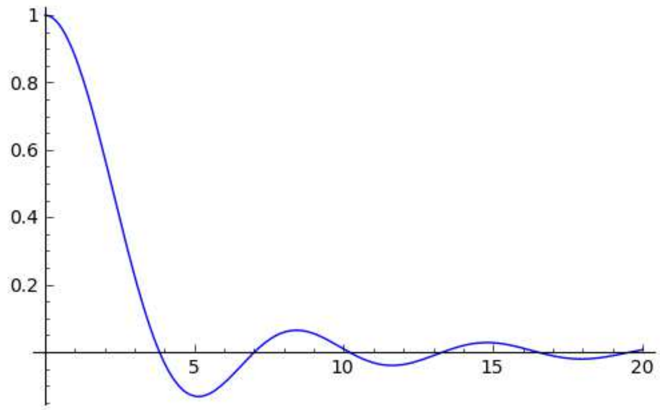

FIGURE 1. $\Omega_4(t)$

Similarly, we have $m_1(\mathbb{R}^n) \leq \vartheta_G(\mathbb{R}^n) \leq \vartheta(\mathbb{R}^n)$.

In the next step the action of the orthogonal group $O(\mathbb{R}^n)$ is exploited in order to simplify the programs (25) and (26). The continuous functions of positive type on $\mathbb{R}^n$ that are $O(\mathbb{R}^n)$-invariant cab be expressed as ([14]):

$$f(x) = \int_0^{+\infty} \Omega_n(t\|x\|)d\mu(t) \tag{27}$$

where μ is a finite positive Borel measure and Ω_n is the Fourier transform of the unit sphere S^{n-1}:

$$\Omega_n(\|u\|) = \frac{1}{\omega_n} \int_{S^{n-1}} e^{iu\cdot\xi} d\omega(\xi). \tag{28}$$

where ω denotes the surface measure on the unit sphere and $\omega_n = \omega(S^{n-1})$. We note that $\Omega_n(0) = 1$. The function Ω_n expresses in terms of the Bessel function of the first kind:

$$\Omega_n(t) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{t}\right)^{\frac{n}{2}-1} J_{\frac{n}{2}-1}(t). \tag{29}$$

Then, (26) becomes

$$\begin{aligned} \vartheta_G(\mathbb{R}^n) = \sup \big\{ \mu(0) \ : \quad & \mu \text{ a positive Borel measure on } [0, +\infty[, \\ & \textstyle\int_0^{+\infty} d\mu(t) = 1, \\ & \textstyle\int_0^{+\infty} \Omega_n(t) d\mu(t) = 0, \\ & \textstyle\int_V \big(\int_0^{+\infty} \Omega_n(t\|v\|)d\mu(t)\big) d\lambda(v) \leq \alpha_\lambda(G) \ \big\} \end{aligned}$$

and, applying weak duality,

$$\begin{aligned} \vartheta_G(\mathbb{R}^n) \leq \inf \big\{ z_0 + z_2 \tfrac{\alpha_\lambda(G)}{\lambda(V)} : \quad & z_2 \geq 0 \\ & z_0 + z_1 + z_2 \geq 1 \\ & z_0 + z_1 \Omega_n(t) + z_2 \tfrac{1}{\lambda(V)} \textstyle\int_V \Omega_n(t\|v\|) d\lambda(v) \geq 0 \ (t > 0) \big\}. \end{aligned} \tag{30}$$

In the next sections we will take for G a finite subgraph on M vertices $V = \{v_1, \dots, v_M\}$ with equal norms $\|v_i\| = r$, and the measure λ will be the counting measure on V; in this case the expression of the dual program simplifies:
(31)

$$\vartheta_G(\mathbb{R}^n) \le \inf \big\{ z_0 + z_2 \tfrac{\alpha(G)}{M} : \quad \begin{array}{l} z_2 \ge 0 \\ z_0 + z_1 + z_2 \ge 1 \\ z_0 + z_1 \Omega_n(t) + z_2 \Omega_n(rt) \ge 0 \ (t > 0) \big\}. \end{array}$$

We also recall from [13] and [12] that $\vartheta(\mathbb{R}^n)$ has an explicit expression:

$$\vartheta(\mathbb{R}^n) = \frac{-\Omega_n(j_{n/2,1})}{1 - \Omega_n(j_{n/2,1})} \tag{32}$$

where $j_{n/2,1}$ is the first positive zero of $J_{n/2}$ and is the value at which the function Ω_n reaches its absolute minimum (see Figure 1 for a plot of $\Omega_4(t)$). This expression can be recovered from (30) if the variable z_2 is set to 0. Unfortunately, the programs (30) or (31) cannot be solved explicitly in a similar fashion. Instead, we will content ourselves with the construction of explicit feasible solutions in section 4 and with numerical solutions in section 5.

Remark 2.2. *In order to tighten the inequality $\alpha(G) \le \vartheta(G)$, it is customary to add the condition $S \ge 0$ (meaning all coefficients of S are non-negative) to the constraints in* (10)*; the new optimal value is denoted $\vartheta'(G)$ and coincides with the linear programming bound introduced earlier by P. Delsarte in the context of association schemes (see* [7] *and* [17]*). Obviously, this condition can be also added to $\vartheta_G(G)$. However, it should be noted that, for the unit distance graph, $\vartheta(\mathbb{R}^n) = \vartheta'(\mathbb{R}^n)$, because the optimal function for $\vartheta(\mathbb{R}^n)$* (25)*, given by:*

$$f(x) := \big(-\Omega_n(j_{n/2,1}) + \Omega_n(j_{n/2,1}\|x\|)\big)/(1 - \Omega_n(j_{n/2,1}))$$

does take non-negative values.

3. THE GENERALIZED JOHNSON GRAPHS

In this section, we introduce certain finite graphs that will play a major role in the next sections.

We denote $J(n, w, i)$ and call *generalized Johnson graph* the graph with vertices the set of n-tuples of 0's and 1's, with w coordinates equal to 1, and with edges connecting pairs of n-tuples having exactly i coordinates in common equal to 1. The coordinates sum to w, and the squared Euclidean distance between two vertices connected by an edge is equal to $2(w - i)$, so, after rescaling, $J(n, w, i)$ is an induced subgraph of the unit distance graph of dimension $n - 1$. A straightforward calculation shows that it lies on a sphere of radius $\sqrt{w(1 - w/n)/(2(w - i))}$.

In view of (31), we need the value of the independence number $\alpha(J(n, w, i))$ of $J(n, w, i)$. It turns out that computing it directly becomes intractable for $n > 10$. For the graphs $J(n, 3, 1)$, there is an explicit formula due to Erdös and Sös (see [10, Lemma 18]), but the number of vertices in this case grows like n^3 and we rather need an exponential number of vertices, which requires that w grows linearly with n.

If, being less demanding, we seek only for an upper estimate of $\alpha(J(n,w,i))$, two essentially different strategies are available. One is provided by Frankl and Wilson intersection theorem [15] and applies for certain values of the parameters w and i:

Theorem 3.1. [15] *If q is a power of a prime number,*

$$\alpha(J(n,2q-1,q-1)) \leq \binom{n}{q-1}. \tag{33}$$

Taking $q \sim an$ leads to

$$\frac{\alpha(J(n,2q-1,q-1))}{|J(n,2q-1,q-1)|} \leq \frac{\binom{n}{q-1}}{\binom{n}{2q-1}} \approx \mathrm{e}^{-(H(2a)-H(a))n}$$

where $H(t) = -t\log(t) - (1-t)\log(1-t)$ denotes the entropy function. The optimal choice of a, i.e. the value of a that maximizes $H(2a) - H(a)$ is $a = (2-\sqrt{2})/4$, from which one obtains the upper estimate $(1.207)^{-n}$. Let us recall that this result gave the first lower estimate of exponential growth for the chromatic number of $\mathbb{R}^n$ [15].

Another possibility would be to upper bound $\alpha(J(n,w,i))$ by the theta number $\vartheta(J(n,w,i))$ of the graph $J(n,w,i)$. The group of permutations of the n coordinates acts transitively on its vertices as well as on its edges so from [11, Theorem 9], its theta number expresses in terms of the largest and smallest eigenvalues of the graph; taking into account that these eigenvalues, being the eigenvalues of the Johnson scheme, are computed in [8] in terms of Hahn polynomials, we have, if $Z_k(i) := Q_k(w-i)/Q_k(0)$ with the notations of [8]:

$$\frac{\vartheta(J(n,w,i))}{|J(n,w,i)|} = \frac{-\min_{k\in[w]} Z_k(i)}{1-\min_{k\in[w]} Z_k(i)}. \tag{34}$$

We note that this expression is completely analogous to (32); indeed, both graphs afford an automorphism group that is edge transitive. We refer to [4] for an interpretation of (32) in terms of eigenvalues of operators.

The bound on $\alpha(J(n,w,i))$ given by (34) turns to be poor. Computing $\vartheta'(J(n,w,i))$ instead of $\vartheta(J(n,w,i))$ (see Remark 2.2) represents an easy way to tighten it. Indeed, one can see that $\vartheta(J(n,w,i)) = \vartheta'(J(n,w,i))$ only if

$$Z_{k_0}(i) = \min_{j\in[w]} Z_{k_0}(j) \tag{35}$$

where k_0 satisfies $Z_{k_0}(i) = \min_{k\in[w]} Z_k(i)$. It turns out that (35) is not always fulfilled and in these cases $\vartheta(J(n,w,i) < \vartheta'(J(n,w,i)$.

A more serious improvement is provided by semidefinite programming following [18, (67)] where constant weight codes with given minimal distance are considered. In order to apply this framework to our setting, we only need to change the range of avoided Hamming distances in [18, (65-iv)].

Table 1 displays the numerical values of the three bounds for certain parameters (n,w,i), selected either because they allow for Frankl and Wilson bound, or because they give the best result in (31) (see section 5). For most of these parameters,

(n, w, i)	$\alpha(J(n, w, i))$	FW bound [15]	$\vartheta'(J(n, w, i))$	SDP bound
$(6, 3, 1)$	4	6	4	4
$(7, 3, 1)$	5	7	5	5
$(8, 3, 1)$	8	8	8	8
$(9, 3, 1)$	8	9	11	8
$(10, 5, 2)$	27	45	30	27
$(11, 5, 2)$	37	55	42	37
$(12, 5, 2)$	57	66	72	57
$(12, 6, 2)$			130	112
$(13, 5, 2)$		78	109	72
$(13, 6, 2)$			191	148
$(14, 7, 3)$		364	290	184
$(15, 7, 3)$		455	429	261
$(16, 7, 3)$		560	762	464
$(16, 8, 3)$			1315	850
$(17, 7, 3)$		680	1215	570
$(17, 8, 3)$			2002	1090
$(18, 9, 4)$		3060	3146	1460
$(19, 9, 4)$		3876	4862	2127
$(20, 9, 3)$			13765	6708
$(20, 9, 4)$		4845	8840	3625
$(21, 9, 4)$		5985	14578	4875
$(21, 10, 4)$			22794	8639
$(22, 9, 4)$		7315	22333	6480
$(22, 11, 5)$			36791	11360
$(23, 9, 4)$		8855	32112	8465
$(23, 11, 5)$			58786	17055
$(24, 9, 4)$		10626	38561	10796
$(24, 12, 5)$			172159	53945
$(25, 9, 4)$		12650	46099	13720
$(26, 13, 6)$		230230	453169	101494
$(27, 13, 6)$		296010	742900	163216

TABLE 1. Bounds for the independence number of $J(n, w, i)$

the semidefinite programming bound turns to be the best one and is significantly better than the theta number. It would be of course very interesting to understand the asymptotic behavior of this bound when n grows to $+\infty$, unfortunately this problem seems to be out of reach to date.

The computations were performed using the solver SDPA [22] available on the NEOS website (`http://www.neos-server.org/neos/`).

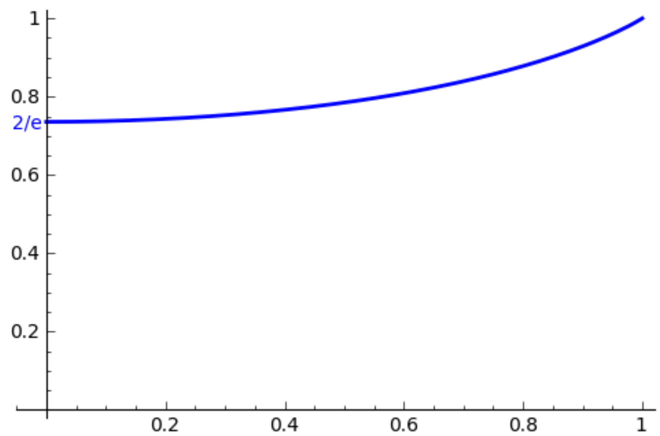

FIGURE 2. $c(r)$

4. THE PROOF OF THEOREM 1

In this section we will show that from (26) an asymptotic improvement of the known upper bounds for $m_1(\mathbb{R}^n)$ can be obtained. For this, we assume that $G_n = (V_n, E_n)$ is a sequence of induced subgraphs of the unit distance graph of dimension n, such that $|V_n| = M_n$ and V_n lies on the sphere of radius $r < 1$, where r is independent of n. We recall from Section 2 that $m_1(\mathbb{R}^n) \le \vartheta_{G_n}(\mathbb{R}^n)$ and that $\vartheta_{G_n}(\mathbb{R}^n)$ is upper bounded by the optimal value of:

$$\inf \Big\{ z_0 + z_2 \tfrac{\alpha(G_n)}{M_n} : \quad \begin{array}{l} z_2 \ge 0 \\ z_0 + z_1 + z_2 \ge 1 \\ z_0 + z_1 \Omega_n(t) + z_2 \Omega_n(rt) \ge 0 \ (t > 0) \Big\}. \end{array} \tag{36}$$

So, in order to upper bound $m_1(\mathbb{R}^n)$, it is enough to construct a suitable feasible solution of (36).

Theorem 4.1. *We assume that, for some* $b < \sqrt{2/\mathrm{e}}$,

$$\frac{\alpha(G_n)}{M_n} \lessapprox b^n.$$

Let

$$c(r) = (1 + \sqrt{1 - r^2})\mathrm{e}^{-\sqrt{1-r^2}} \quad \textit{and} \quad f(r) = \sqrt{2c(r)/\mathrm{e}}.$$

Then, for every $\epsilon > 0$,

$$\vartheta_{G_n}(\mathbb{R}^n) \lessapprox (f(r) + \epsilon)^n.$$

Proof. A feasible solution of (36) is given in the following lemma:

Lemma 4.2. *With the notations of the theorem, let* $\gamma > \sqrt{c(r)}$ *and* $m > \gamma\sqrt{2/\mathrm{e}}$, *then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \ge n_0$,

$$m^n + \Omega_n(t) + \gamma^n \Omega_n(rt) \ge 0, \textit{ for all } t \ge 0. \tag{37}$$

Proof. After having established some preliminary inequalities, we will proceed in three steps. First, we will prove that the inequality (37) holds for "small" t, say $0 \leq t \leq \nu := \frac{n}{2} - 1$, then that it holds for "large" t, say $t \geq \alpha_0 \nu$ where α_0 is an explicit constant, and, at last, we will construct a decreasing sequence $\alpha_0 \geq \alpha_1 \geq \ldots \geq \alpha_k \ldots$ such that the inequality holds for $t \geq \alpha_k \nu$, and prove that $\lim_{k\to\infty} \alpha_k < 1$.

Let j_1 be the first zero of $J_{\nu+1}$, then Ω_n is a decreasing function on $[0, j_1]$ and Ω_n has a global minimum at j_1 (see [2]). So, $\Omega_n(t) \geq \Omega_n(j_1)$. Furthermore, $|J_\nu(t)| \leq 1$ for all $t \in \mathbb{R}$ (see [1] formula 9.1.60), hence

$$|\Omega_n(j_1)| \leq \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{j_1}\right)^{\frac{n}{2}-1}.$$

It follows from [1] formula 9.5.14 that $j_1 \geq \nu$ if n is large enough. From now, we will consider that this inequality holds. Using Stirling formula, we get

$$|\Omega_n(j_1)| \lessapprox \left(\sqrt{\frac{2}{\mathrm{e}}}\right)^n.$$

Let $x \in]0,1[$. Using [1] formula 9.3.2, one gets easily

$$J_\nu(x\nu) \approx \left(\sqrt{\frac{x}{c(x)}}\right)^n.$$

It follows that

$$\Omega_n(x\nu) = \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{x\nu}\right)^\nu J_\nu(x\nu) \approx \left(\sqrt{\frac{2}{\mathrm{e}c(x)}}\right)^n. \tag{38}$$

First step. Suppose that $0 \leq t \leq \nu$. Since $\nu \leq j_1$, $r < 1$ and Ω_n is decreasing on $[0, j_1]$, $\Omega_n(rt) \geq \Omega_n(r\nu)$. From previous results, one gets

$$\Omega_n(t) + \gamma^n \Omega_n(rt) \geq -|\Omega_n(j_1)| + \gamma^n \Omega_n(r\nu) \approx \left(\gamma\sqrt{\frac{2}{\mathrm{e}c(r)}}\right)^n \geq 0$$

if n is large enough.
Second step. Let $\alpha_0 = \frac{1}{\gamma^2}$. For $t \geq \alpha_0\nu$,

$$|\Omega_n(t)| \leq \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{\alpha_0\nu}\right)^{\frac{n}{2}-1} \approx \left(\gamma\sqrt{\frac{2}{\mathrm{e}}}\right)^n.$$

Since $\Omega_n(rt) \geq -|\Omega_n(j_1)|$ and $|\Omega_n(j_1)| \lessapprox \left(\sqrt{\frac{2}{\mathrm{e}}}\right)^n$, it follows from the definition of m that

$$m^n + \Omega_n(t) + \gamma^n \Omega_n(rt) \geq m^n - |\Omega_n(t)| - \gamma^n |\Omega_n(j_1)| \sim m^n \geq 0$$

if n is large enough.
Third step. Let us first study the function c. An elementary computation gives $c'(x) = x\mathrm{e}^{-\sqrt{1-x^2}}$ for $x \in [0,1]$. It implies that $0 \leq c'(x) \leq 1$, hence c is an

increasing function and $c(x) \geq x$ with equality only for $x = 1$. Now, let us define ϕ by

$$\begin{array}{rcl} \phi : [0, \frac{1}{r}] & \longrightarrow & \mathbb{R} \\ x & \mapsto & \frac{1}{2}(\frac{c(rx)}{\gamma^2} + x) \end{array}$$

Since c is increasing, ϕ is also increasing. Furthermore, $\phi(0) = \frac{1}{\mathrm{e}\gamma^2} > 0$ and $\phi(\frac{1}{r}) = \frac{1}{2}(\frac{1}{\gamma^2} + \frac{1}{r}) < \frac{1}{r}$ since $\gamma^2 > c(r) > r$. It follows that the interval $[0, \frac{1}{r}]$ is mapped into itself. One also gets immediately $\phi'(x) = \frac{1}{2}(\frac{rc'(rx)}{\gamma^2} + 1)$. Since $c'(rx) \leq 1$ and $\gamma^2 > r$, $\phi'(x) < 1$. Hence ϕ has only one fixed point, denoted by l. Moreover, $\phi(1) < 1$, so $l < 1$. For any $x_0 \geq l$, the sequence $x_{k+1} = \phi(x_k)$ is a decreasing sequence with limit l (if $x_0 \leq l$, the sequence is increasing).

We now return to the proof of the lemma. We have set $\alpha_0 = \frac{1}{\gamma^2}$ and we assume $\alpha_0 > 1$. Let $\alpha_1 < \alpha_0$ and $t \in [\alpha_1\nu, \alpha_0\nu]$. By construction, $r\alpha_0 = \frac{r}{\gamma^2} < \frac{r}{c(r)} < 1$, hence $rt \leq r\alpha_0\nu < \nu \leq j_1$. Since Ω_n is decreasing on $[0, j_1]$, formula (38) gives

$$\Omega_n(rt) \geq \Omega_n(r\alpha_0\nu) \approx \left(\sqrt{\frac{2}{\mathrm{e}c(r\alpha_0)}}\right)^n.$$

Now $|\Omega_n(t)| \leq \Gamma\left(\frac{n}{2}\right)\left(\frac{2}{\alpha_1\nu}\right)^{\frac{n}{2}-1} \approx \left(\sqrt{\frac{2}{\mathrm{e}\alpha_1}}\right)^n$. Hence, we will have $\Omega_n(t) + \gamma^n\Omega_n(rt) \geq 0$, for n large enough, as soon as $\alpha_1 > \frac{c(r\alpha_0)}{\gamma^2}$ (but we need strict inequality). We can take $\alpha_1 = \phi(\alpha_0)$. Defining the sequence α_k by $\alpha_{k+1} = \phi(\alpha_k)$, we get, using the same method, that for k fixed and all n large enough, $\Omega_n(t) + \gamma^n\Omega_n(rt) \geq 0$ for $t \geq \alpha_k\nu$. Since $\lim \alpha_k = l < 1$, there exists an integer k such that $\alpha_k < 1$. This concludes the proof of the lemma. □

Now we return to the proof Theorem 4.1. Let $\epsilon > 0$; let $\gamma = \sqrt{c(r)} + \epsilon$ and $m = \sqrt{2c(r)/\mathrm{e}} + \epsilon$. The lemma shows that for n sufficiently large, $(z_0, z_1, z_2) = (m^n, 1, \gamma^n)$ is a feasible solution of (36). So, for these values of n, the optimal value of (36) is upper bounded by $m^n + \gamma^n\alpha(G_n)/M_n$, leading to

$$\vartheta_{G_n}(\mathbb{R}^n) \lessapprox \left(\sqrt{2c(r)/\mathrm{e}} + \epsilon\right)^n + \left(b\sqrt{c(r)} + \epsilon\right)^n \lessapprox (f(r) + \epsilon)^n.$$

□

We are now in the position to complete the proof of Theorem 1.1. We will first consider a sequence of generalized Johnson graphs, and will apply Theorem 4.1 combined with Frankl and Wilson estimate of the ratio $\alpha(G)/M$. We will not achieve the strongest result claimed in Theorem 1.1 with these graphs, in return they are quite easy to analyze. Then, we will take the graphs considered in [16].

Let $G_n = J(n, 2p_n - 1, p_n - 1)$ where p_n is a sequence of prime numbers such that $p_n \sim an$ for some constant real number a. The value of $a < 1/4$ will be chosen later in order to optimize the resulting bound. We have (see section 3)

$$\frac{\alpha(G_n)}{M_n} \lessapprox b(a)^n \text{ where } b(a) = \mathrm{e}^{-(H(2a)-H(a))}.$$

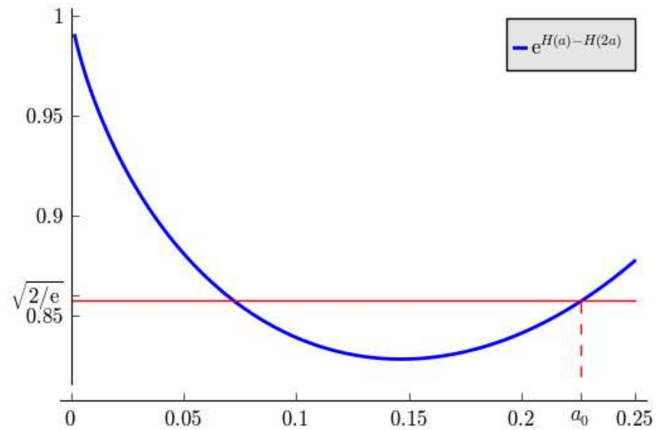

FIGURE 3. $\mathrm{e}^{H(a)-H(2a)}$

These graphs can be realized as unit distance graphs in $\mathbb{R}^n$ in infinitely many ways, depending on the real values chosen for the 0 and 1 coordinates. An easy computation shows that these embeddings realize every radius r such that

$$r \geq r_{\min}(n, p_n) := \sqrt{\frac{(n-2p_n+1)(2p_n-1)}{2np_n}} \sim r(a) := \sqrt{1-2a}.$$

The function $f(r(a)) = \sqrt{2c(r(a))/\mathrm{e}}$ is decreasing with a, so we will take the largest possible value for a, under the constraint $b(a) \leq \sqrt{2/\mathrm{e}}$. Let this value be denoted a_0; then $b(a_0) = \sqrt{2/\mathrm{e}}$ and (see Figure 3)

$$0.2268 \leq a_0 \leq 0.2269.$$

We fix now $a = a_0$. For a given $\epsilon > 0$, because the function $f(r) = \sqrt{2c(r)/\mathrm{e}}$ is continuous, there is a $r > r(a_0)$ such that $f(r) = f(r(a_0)) + \epsilon$, and such that, for n sufficiently large, r is a valid radius for all the graphs G_n. Applying Theorem 4.1 to this value of r and to ϵ, we obtain

$$\vartheta_{G_n}(\mathbb{R}^n) \lessapprox (f(r(a_0)) + \epsilon)^n$$

with

$$f(r(a_0)) = \sqrt{2(1+\sqrt{2a_0})\mathrm{e}^{-(1+\sqrt{2a_0})}} < (1.262)^{-1}.$$

In [16], Raigorodski considers graphs with vertices in $\{-1,0,1\}^n$, where the number of -1, respectively of 1, is growing linearly with n. If the number of 1 is equivalent to $x_1 n$ and the number of -1 to $x_2 n$, with $x_2 \leq x_1$, if $z = (x_1+3x_2)/2$, and $y_1 = (-1+\sqrt{-3z^2+6z+1})/3$, he shows that:

$$\frac{\alpha(G_n)}{M_n} \lessapprox b(x_1,x_2)^n \text{ where } b(x_1,x_2) = \mathrm{e}^{-(H_2(x_1,x_2)-H_2(y_1,(z-y_1/2))} \tag{39}$$

where $H_2(u,v) = -u\log(u) - v\log(v) - (1-u-v)\log(1-u-v)$. The proof of (39) relies on a similar argument as in Frankl-Wilson intersection theorem. These graphs can be realized as subgraphs of the unit distance graph in $\mathbb{R}^n$ with minimal radius

$$r(x_1, x_2) = \sqrt{\frac{(x_1 + x_2) - (x_1 - x_2)^2}{(x_1 + 3x_2)}}.$$

For $x_1 = 0.22$ and $x_2 = 0.20$, the inequality $b(x_1, x_2) < \sqrt{2/\mathrm{e}}$ holds and $f(r(x_1, x_2)) < 1.268^{-1}$, leading to the announced inequality (7).

Remark 4.3. *The possibility to further improve the basis of exponential growth using Theorem 4.1 is rather limited. Indeed,* $f(r) \geq \sqrt{2c(1/2)/\mathrm{e}} > (1.316)^{-1}$. *So, with this method, we cannot reach a better basis that* 1.316.

5. Numerical results for dimensions up to 24

In this range of dimensions, we have tried many graphs in order to improve the known upper estimates of $m_1(\mathbb{R}^n)$ (and, in turn, the lower estimates of the measurable chromatic number). We report here the best we could achieve. For each dimension $4 \leq n \leq 24$, Table 2 displays a feasible solution (z_0, z_1, z_2) of (31) where the notations are those of section 2: G is an induced subgraph of the unit distance graph in dimension n, and it has M vertices at distance r from 0. The number given in the third column is the exact value, or an upper bound, of its independence number $\alpha(G)$, and replaces $\alpha(G)$ in (31). The last column contains the objective value of (31), thus an upper bound for $m_1(\mathbb{R}^n)$. Table 3 gives the corresponding lower bounds for $\chi_m(\mathbb{R}^n)$, compared to the previous best known ones.

The computation of (z_0, z_1, z_2) was performed in a similar way as in [13]: a large interval e.g. $[0, 50]$ is sampled in order to replace the condition $z_0 + z_1\Omega_n(t) + z_2\Omega_n(rt) \geq 0$ for all $t > 0$ by a finite number of inequalities; the resulting linear program is solved leading to a solution (z_0^*, z_1^*, z_2^*). The function $z_0^* + z_1^*\Omega_n(t) + z_2^*\Omega_n(rt)$ is almost feasible for (31); its absolute minimum is reached in the range $[0, 50]$ so it is enough to slightly increase z_0^* in order to obtain a true feasible solution. The computations were performed with the help of the softwares SAGE [19] and lpsolve [3].

A few words about the graphs involved in the computations are in order:

The 600-cell is a regular polytope of dimension 4 with 120 vertices: the sixteen points $(\pm 1/2, \pm 1/2, \pm 1/2, \pm 1/2)$, the eight permutations of $(\pm 1, 0, 0, 0)$ and the 96 points that are even permutations of $(0, \pm 1/(2\phi), \pm 1/2, \pm \phi/2)$, where $\phi = (1 + \sqrt{5})/2$. If d is the distance between two non antipodal vertices, we have $d^2 \in \{(5 \pm \sqrt{5})/2, 3, (3 \pm \sqrt{5})/2, 2, 1\}$. Each value of d gives raise to a graph connecting the vertices that are at distance d apart; these graphs, after rescaling so that the edges have length 1, lie on the sphere of radius $r = 1/d$. Their independence numbers are respectively equal to: $39, 26, 24, 26, 20$. We note that applying the conjugation $\sqrt{5} \to -\sqrt{5}$ will obviously not change the independence number. Among these graphs, the best result in dimension 4, recorded in Table 2, was obtained with $d = \sqrt{3}$. It turned out that the same graph gave the best result we could achieve in dimensions 5 and 6.

The root system E_8 is the following set of 240 points in $\mathbb{R}^8$: the points $((\pm 1)^2, 0^6)$ and all their permutations, and the points $((\pm 1/2)^8)$ with an even number of minus signs. The distances between two non antipodal points take three different values: $d = \sqrt{2}, 2, \sqrt{6}$. The unit distance subgraph associated to a value of d lies on a sphere of radius $r = \sqrt{2}/d$ and has an independence number equal respectively to $16, 16, 36$. The one with the smallest radius $r = \sqrt{1/3}$ gives the best bound in dimension 8 as well as in dimensions 9, 10, and 11 (in these dimensions we have compared with Johnson graphs).

The configuration in dimension 7 is derived from E_8: given $p \in E_8$, we take the set of points in E_8 closest to p. Independently of p, this construction leads to 56 points that lie on a hyperplane. The graph defined by the distance $\sqrt{6}$ after suitable rescaling corresponds to $r = \sqrt{6}/4$ and has independence number 7.

In the other dimensions up to 23, our computations involve generalized Johnson graphs as described in section 3. In dimension 24, we obtained a better result with the so-called orthogonality graph $\Omega(24)$. For $n = 0 \bmod 4$, $\Omega(n)$ denotes the graph with vertices in $\{0,1\}^n$, where the edges connect the points at Hamming distance $n/2$. Using semidefinite programming, an upper bound of its independence number is computed for $n = 16, 20, 24$ in [9].

TABLE 2. Feasible solutions of (31) and corresponding upper bounds for $m_1(\mathbb{R}^n)$

n	G	α	M	r	z_0	z_1	z_2	$z_0 + z_2\alpha/M$
4	600-cell	26	120	$\sqrt{3}/3$	0.0421343	0.690511	0.267355	0.100062
5	600-cell	26	120	$\sqrt{3}/3$	0.023477	0.772059	0.204465	0.0677778
6	600-cell	26	120	$\sqrt{3}/3$	0.0141514	0.830343	0.155506	0.0478444
7	E_8 kissing	7	56	$\sqrt{6}/4$	0.007948	0.834435	0.157617	0.0276502
8	E_8	36	240	$\sqrt{3}/3$	0.0053364	0.899613	0.0950508	0.0195941
9	E_8	36	240	$\sqrt{3}/3$	0.0033303	0.921154	0.0755157	0.0146577
10	E_8	36	240	$\sqrt{3}/3$	0.00209416	0.937453	0.0604529	0.0111621
11	E_8	36	240	$\sqrt{3}/3$	0.00132364	0.949973	0.0487036	0.00862918
12	$J(13,6,2)$	148	1716	$\sqrt{21/52}$	9.002e-04	0.938681	0.0604188	0.00611112
13	$J(14,7,3)$	184	3432	$\sqrt{7/16}$	5.933e-04	0.936921	0.0624857	0.00394335
14	$J(15,7,3)$	261	6435	$\sqrt{7/15}$	3.9393e-04	0.935283	0.0643239	0.00300288
15	$J(16,8,3)$	850	12870	$\sqrt{2/5}$	2.7212e-04	0.967168	0.0325604	0.00242258
16	$J(17,8,3)$	1090	24310	$\sqrt{36/85}$	1.9080e-04	0.968014	0.0317961	0.00161646
17	$J(18,9,4)$	1460	48620	$\sqrt{9/20}$	1.34658e-04	0.967557	0.0323093	0.00110487
18	$J(19,9,4)$	2127	92378	$\sqrt{9/19}$	9.50746e-05	0.96714	0.032765	8.49488e-04
19	$J(20,9,3)$	6708	167960	$\sqrt{33/80}$	5.944e-05	0.98275	0.0171908	7.46008e-04
20	$J(21,10,4)$	8639	352716	$\sqrt{55/126}$	4.44363e-05	0.982618	0.0173381	4.69095e-04
21	$J(22,11,5)$	11360	705432	$\sqrt{11/24}$	3.2936e-05	0.982495	0.0174727	3.1431e-04
22	$J(23,11,5)$	17055	1352078	$\sqrt{11/23}$	2.4315e-05	0.982385	0.0175913	2.46211e-04
23	$J(24,12,5)$	53945	2704156	$\sqrt{3/7}$	1.40898e-05	0.990052	0.00993429	2.12269e-04
24	$\Omega(n)$	183373	2^{24}	$\sqrt{1/2}$	1.30001e-05	0.984309	0.0156786	1.84366e-04

n	previous best lower bound for $\chi_m(\mathbb{R}^n)$	new lower bound for $\chi_m(\mathbb{R}^n)$
4	9 [13]	10
5	14[13]	15
6	20 [13]	21
7	28 [13]	37
8	39[13]	52
9	54[13]	69
10	73 [13]	90
11	97 [13]	116
12	129 [13]	164
13	168[13]	254
14	217[13]	334
15	279[13]	413
16	355 [13]	619
17	448[13]	906
18	563[13]	1178
19	705 [13]	1341
20	879[13]	2132
21	1093[13]	3182
22	1359[13]	4062
23	1690 [13]	4712
24	2106[13]	5424

TABLE 3. Lower bounds for the measurable chromatic number

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CHRISTINE BACHOC, UNIV. BORDEAUX, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, 351, COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE
E-mail address: `christine.bachoc@u-bordeaux.fr`

ALBERTO PASSUELLO, UNIV. BORDEAUX, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, 351, COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE
E-mail address: `alberto.passuello@u-bordeaux.fr`

ALAIN THIERY, UNIV. BORDEAUX, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, 351, COURS DE LA LIBÉRATION, 33405 TALENCE, FRANCE
E-mail address: `alain.thiery@u-bordeaux.fr`