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Note on the sequence A157751

Alain Thiery

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In this note, the following conjecture of Clark Kimberling(?) is proved

Conjecture. *Let $(F_n(X))_{n\in\mathbb{N}}$ be the sequence of polynomials defined by $F_n(X) = (X+1)F_{n-1}(X) + F_{n-1}(-X)$ with initial term $F_0(X) = 1$. If n is even then $F_n(X)$ has no real roots, and if n is odd then $F_n(X)$ has exactly one real root, denoted by r, and if $n \geqslant 5$ then $0 < -r < n$.*

First we give a new formula for F_n.

Lemma 1. *Denote P_n the even part of F_n and I_n its odd part. Let $\sqrt{X^2+1}$ be a square root of X^2+1. Then*

$$P_n(X) = \frac{1}{2\sqrt{X^2+1}}((1+\sqrt{X^2+1})^{n+1} - (1-\sqrt{X^2+1})^{n+1}),$$
$$I_n(X) = \frac{X}{2\sqrt{X^2+1}}((1+\sqrt{X^2+1})^{n} - (1-\sqrt{X^2+1})^{n}).$$

Proof. It is obvious that $P_0 = 1$ and $I_0 = 0$. From the definition of F_n, one gets

$$\begin{aligned} F_n(X) &= (X+1)F_{n-1}(X) + F_{n-1}(-X) \\ &= (X+1)(P_{n-1}(X) + I_{n-1}(X)) + P_{n-1}(X) - I_{n-1}(X) \\ &= (2P_{n-1}(X) + XI_{n-1}(X)) + XP_{n-1}(X). \end{aligned}$$

It follows immediatly that

$$\begin{cases} P_n(X) = 2P_{n-1}(X) + XI_{n-1}(X), \\ I_n(X) = XP_{n-1}(X). \end{cases}$$

Denote $A = \begin{pmatrix} 2 & X \\ X & 0 \end{pmatrix}$. The previous equation can be rewritten as

$$\begin{pmatrix} P_n(X) \\ I_n(X) \end{pmatrix} = A \begin{pmatrix} P_{n-1}(X) \\ I_{n-1}(X) \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is easy to see that the eigenvalues of A are $r_1 = 1+\sqrt{X^2+1}$ and $r_2 = 1-\sqrt{X^2+1}$, the associated eigenvectors are $v_1 = \begin{pmatrix} r_1/X \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} r_2/X \\ 1 \end{pmatrix}$, and

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = X\frac{v_1 - v_2}{r_1 - r_2} = \frac{X}{2\sqrt{X^2+1}}(v_1 - v_2).$$

This implies that

$$\begin{aligned}\begin{pmatrix}P_n(X)\\ I_n(X)\end{pmatrix} &= \frac{X}{2\sqrt{X^2+1}}A^n(v_1-v_2)\\ &= \frac{X}{2\sqrt{X^2+1}}(r_1^n v_1 - r_2^n v_2)\\ &= \frac{X}{2\sqrt{X^2+1}}\begin{pmatrix}\frac{1}{X}(r_1^{n+1}-r_2^{n+1})\\ r_1^n - r_2^n\end{pmatrix}\end{aligned}$$

□

The proof of the conjecture is based on the following formula[1]

Lemma 2. *For all $n \in \mathbb{N}$,*

$$F_n(X)F_n(-X)(X^2+1) = (-1)^n X^{2n+2} + \sum_{k=0}^{n}\binom{2n+1}{2k}(X^2+1)^k.$$

Proof. With the previous notations, one has

$$\begin{aligned}F_n(X)F_n(-X)(X^2+1) &= (P_n(X)+I_n(X))(P_n(X)-I_n(X))(X^2+1)\\ &= ((P_n(X))^2-(I_n(X))^2)(X^2+1)\\ &= \frac{1}{4}((r_1^{n+1}-r_2^{n+1})^2 - X^2(r_1^n-r_2^n)^2).\end{aligned}$$

Using $r_1r_2 = -X^2$, one gets

$$\begin{aligned}F_n(X)F_n(-X)(X^2+1) &= \frac{1}{4}(r_1^{2(n+1)} - 2(-X^2)^{n+1} + r_2^{2(n+1)}\\ &\qquad - X^2(r_1^{2n} - 2(-X^2)^n + r_2^{2n}))\\ &= (-1)^{n+2}X^{2n+2} + \frac{1}{4}(r_1^{2(n+1)} + r_2^{2(n+1)} - X^2(r_1^{2n}+r_2^{2n})).\end{aligned}$$

Let p be an integer. Binomial formula gives $r_i^p = (1\pm\sqrt{X^2+1})^p = \sum_{k=0}^{p}\binom{p}{k}(\pm\sqrt{X^2+1})^k$ ($i = 1$ or 2). It implies immediatly that

$$r_1^{2p} + r_2^{2p} = 2\sum_{k=0}^{p}\binom{2p}{2k}(X^2+1)^k.$$

Using this with $p = n$ or $n+1$, one gets

$$r_1^{2(n+1)}+r_2^{2(n+1)}-X^2(r_1^{2n}+r_2^{2n}) = 2\sum_{k=0}^{n+1}\binom{2(n+1)}{2k}(X^2+1)^k - 2X^2\sum_{k=0}^{n}\binom{2n}{2k}(X^2+1)^k.$$

[1] This formula was discovered working on Tchebychev polynomials

Writing $-X^2 = 1 - (X^2+1)$, the right-hand side becomes

$$2\left(\sum_{k=0}^{n+1}\binom{2(n+1)}{2k}(X^2+1)^k + (1-(X^2+1))\sum_{k=0}^{n}\binom{2n}{2k}(X^2+1)^k\right)$$
$$= 2\left(\sum_{k=0}^{n+1}\binom{2(n+1)}{2k}(X^2+1)^k + \sum_{k=0}^{n}\binom{2n}{2k}(X^2+1)^k - \sum_{k=0}^{n}\binom{2n}{2k}(X^2+1)^{k+1}\right)$$
$$= 2\Bigg((1+1)(X^2+1)^0 + (1-1)(X^2+1)^{n+1}$$
$$+\sum_{k=1}^{n}\binom{2(n+1)}{2k} + \binom{2n}{2k} - \binom{2n}{2(k-1)}(X^2+1)^k\Bigg)$$

To finish the proof, one must show that, for $1 \leqslant k \leqslant n$,

$$\binom{2(n+1)}{2k} + \binom{2n}{2k} - \binom{2n}{2(k-1)} = 2\binom{2n+1}{2k}.$$

We will use the well-known formula $\binom{p}{l} + \binom{p}{l+1} = \binom{p+1}{l+1}$.

$$\begin{aligned}
&\binom{2(n+1)}{2k} + \binom{2n}{2k} - \binom{2n}{2(k-1)} - 2\binom{2n+1}{2k}\\
&= \binom{2(n+1)}{2k} - \binom{2n+1}{2k} + \binom{2n}{2k} - \binom{2n+1}{2k} - \binom{2n}{2(k-1)}\\
&= \binom{2n+1}{2k-1} - \binom{2n}{2k-1} - \binom{2n}{2(k-1)}\\
&= 0.
\end{aligned}$$

□

Theorem. *If n is even then $F_n(X)$ has no real roots, and if n is odd then $F_n(X)$ has exactly one real root, denoted by r, and if $n \geqslant 5$ then $0 < -r < n$.*

Proof. If n is even, $F_n(X)F_n(-X)(X^2+1) = X^{2n+2} + \sum_{k=0}^{n}\binom{2n+1}{2k}(X^2+1)^k$ is a sum of positive terms, hence it is never zero.

If n is odd, then F_n has at least one real root. Since the coefficients of F_n are positive (see `http://oeis.org/A157751`), the roots must be negative. It is then sufficient to prove that $Q_n(X) := F_n(X)F_n(-X)(X^2+1)$ has only one positive root, α, and that $0 < \alpha < n$. Let $R_n(X) = X^{2n+2}Q_n(1/X)$ be the reciprocal polynomial of Q_n. From Lemma 2, R_n has only postive coefficients, except its constant term which is -1. Hence its derivative has only positive coefficients. It follows that R_n is a strictly increasing function on $\mathbb{R}_+$, the set of positive real numbers, and it has at most one real root in $\mathbb{R}_+$. Since the roots of Q_n are the inverse of the roots of R_n, we have proved that R_n has exactly one positive real roots.

It remains to show that $\alpha < n$ if $n \geqslant 5$. Since $Q_n(0) > 0$, it is sufficient to show that $Q_n(n) < 0$.

$$\begin{aligned}
Q_n(n) &= -n^{2n+2} + \sum_{k=0}^{n} \binom{2n+1}{2k} (n^2+1)^k \\
&< -n^{2n+2} + \sum_{k=0}^{n} \binom{2n+1}{2k} (n+1)^{2k} \\
&< -n^{2n+2} + \sum_{l=0}^{2n+1} \binom{2n+1}{l} (n+1)^{l} \\
&= -n^{2n+2} + (n+1)^{2n+1}.
\end{aligned}$$

A sufficient condition to have $Q_n(n) < 0$ is $(n+1)^{2n+1} \leqslant n^{2n+2}$. The latter is equivalent to $(2n+1)\log(n+1) \leqslant (2n+2)\log(n)$. But it is well-known that $\log(n+1) - \log(n) \leqslant \frac{1}{n}$. Hence $(2n+1)\log(n+1) \leqslant (2n+1)(\log(n) + \frac{1}{n}) = (2n+1)\log(n) + 2 + \frac{1}{n}$. For $n = 9$, $2 + \frac{1}{n} = 2.11111\ldots$ and $\log(n) = 2.19722\ldots$ It follows that for all $n \geqslant 9$, $2 + \frac{1}{n} \leqslant \log(n)$ (the left-hand side is decreasing and the right-hand is increasing). We conclude that, for $n \geqslant 9$, $(2n+1)\log(n+1) \leqslant (2n+1)\log(n) + 2 + \frac{1}{n} \leqslant (2n+1)\log(n) + \log(n) = (2n+2)\log(n)$. This proves the theorem for $n \geqslant 9$. Using a computer, one gets, for $n = 5$, $-r = 4.80441\ldots$ and, for $n = 7$, $-r = 5.99442\ldots$ $\square$

Remark. Let $a > 0$. One can compute $Q_n(n/a)$ as above and show that $-r/n \leqslant 1/a$ if $n \geqslant a\exp(3a)$. This implies immediatly $\lim_{n\to\infty} -r/n = 0$.

We also have

$$\begin{aligned}
Q_n(\sqrt{n}) &= -n^{n+1} + \sum_{k=0}^{n} \binom{2n+1}{2k} (n+1)^k \\
&> -n^{n+1} + \binom{2n+1}{2n} (n+1)^n \\
&= -n^{n+1} + (2n+1)(n+1)^n \\
&> -n^{n+1} + (n+1)^{n+1} > 0,
\end{aligned}$$

hence $-r > \sqrt{n}$ and $\lim_{n\to\infty} -r = \infty$.

Using the recursive definition of F_n twice, one easily gets

$$\begin{aligned}
F_n(X) &= (X+1)F_{n-1}(X) + F_{n-1}(-X) \\
&= (X+1)((X+1)F_{n-2}(X) + F_{n-2}(-X)) + (-X+1)F_{n-2}(-X) + F_{n-2}(X) \\
&= ((X+1)^2+1)F_{n-2}(X) + 2F_{n-2}(-X).
\end{aligned}$$

Let r' be the real root of F_{n-2}, then $F_n(r') = 2F_{n-2}(-r') > 0$. It follows that $r < r'$, hence $-r$ is an increasing function of n.