How can a twisted thread correct itself?

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Abstract. The question addressed here is ”How can a twisted thread correct itself?”. We consider a theoretical model where the studied mathematical object represents a twisted thread linking two points. We introduce a random process such that the thread reorganizes itself to become a line between these two points. The modifications are local and also the decision to make a modification is based only on local arguments. Thus, this paper presents a self-organization process which constructs a line between two points. From the practical side, this paper is a generalization of [2] which is part of a study of cooling processes in crystallography [1, 2, 5]. From the theoretical side, this paper improves mathematical tools used to analyze random processes in cellular automata [3, 4]. Note that, the random processes studied on cellular automata were used in the conception of the model of cooling processes.

1 Introduction

We present here a self organizing process which untwists a thread by making only local modifications (called flips) of the thread. These modifications are decided locally with only the knowledge of the close neighborhood. The theoretical model is made of two points linked by a twisted line. This twisted line starts from the bottom left corner and is discrete: it can only move up and right to reach the upper right corner. This line must be able to reorganize itself such that it becomes as close as possible to the continuous line linking the two points. We design here a random process which achieves this goal in polynomial time according to the distance between the two points.

This work is part of a wider study of cooling processes in crystallography [1, 2, 5]. During a cooling, a crystal turns from a chaotic structure into an ordered one. Atoms manage to organize themselves. Of course there is no central

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intelligence in a crystal and all the movements and decisions are made locally. This is what we want to simulate with our twisted thread. Among the previous results cited before, we present two of them. The first one [2] solves our problem when the gradient of the line linking the two points is $\frac{1}{2}$, we generalize this result for any rational gradient. The second one [5], studied this problem in a $2D$ setting: instead of a line, the authors consider a disturbed $2D$ tiling which is able to self-organize into a periodic one. The $2D$ case is a more rational case for crystallography. Nevertheless, all these results are mainly done for developing tools to analyze a more complex model: a disturbed $2D$ tiling which is able to self-organize into an aperiodic one such as Penrose tiling. Practically, aperiodic tilings are a model of quasicrytals which are actually hard to synthesize and whose cooling process is not understood. We aim to bring some lights in this domain.

This work is also part of another set of studies on stochastic cellular automata [3, 4]. In fact, we show that the random process developed here encodes the behavior of the fully asynchronous cellular automaton ECA 178 [3]. Studies on stochastic cellular automata aim to expose and to classify different kinds of stochastic processes. One important part of these works is to exhibit a hierarchy of behaviors and to show that some random processes can encode several other ones. The proof presented here is of particular interest because it can be seen as a reduction from one random process to another one: we use the convergence time of ECA 178 to deduce a convergence time for our random process. This is similar to a reduction proof for classical complexity classes.

The paper is divided as follows. In section 2, we give some precise definitions, and state our main result. In section 3, we recall some results about probability, which are used in section 4, where we prove our result. Section 5 presents some open questions and future works.

2 Notations and definitions

We want to introduce a random process which reorganizes a twisted thread into a line. This section is mainly devoted to definitions and notations but we will end by one proposition and one theorem. The proposition is the characterization of the stable configurations, *i.e.* the structures obtained by the random process defined here and the theorem will be our main result: the random process converges quickly to a stable configuration. The rest of the article is dedicated to the proof of the main theorem. Now, since the definition of our random process is long, we will proceed into six steps. First, we define the *configurations* which are static objects representing the "twisted thread". Secondly, we introduce *flips* which are a way to modify locally a configuration. Thirdly, we introduce *the height* which is a parameter useful to have a better insight of the position occupied by a configuration. Fourthly, we present the *flat lines* which are the best possible discrete approximations of a continuous line and thus the configurations that we are aiming to obtain with our random process. Fifthly, we introduce the *active sites* which are the sites to flip in order to reorganize a configuration into a flat line. And finally, we introduce our random process where time is discrete and where at each time step the configuration is modified locally by a "well chosen" flip.

Consider u, v, n, m positive integers such that $u \leq v$, $gcd(u, v) = 1$, $m = n(u + v)$, the ratio $\frac{v}{u}$ represents the gradient of the continuous line we wish to approximate. To achieve this aim, we will use a discrete line with a period of $u + v$ and thus n represents the number of periods and m represents the total length of the desired line. Consider an oriented graph $G = (V, E)$ where $V = [0, \ldots, nu] \times [0, \ldots, nv]$ and an arc $((i, j), (k, l))$ belong to E if and only if :

- $k = i + 1$ and $l = j$, such arcs are labelled a or
- $k = i$ and $l = j + 1$ such arcs are labelled b.

Thus, an arc labeled a represents moving right in the graph and an arc labeled b represents moving up. In this paper, elements of V are called *sites*. We denote by $\mathbb{P}$ the set of all paths from site $(0, 0)$ to site (nu, nv) and a *configuration* c is $(c_i)_{0 \leq i \leq m} \in \mathbb{P}$ ($c_0 = (0, 0)$ and $c_m = (nu, nv)$). Here, the configurations are static objects but we will soon introduce a dynamics such that they reorganize themselves into a good approximation of the continuous line linking site $(0, 0)$ to site (nu, nv). Note that all paths from $(0, 0)$ to (nu, nv) are of length m and the labels of their arcs form a word w of Σ^m with $\Sigma = \{a, b\}$, where $|w|_a = nu$; $|w|_b = nv$ and for all $0 \leq i \leq m - 1$, w_i is the label of the arc between i and $i+1$. Also note that, this word is unique for each path and that for each word of Σ^m with nu letters a and nv letters b corresponds an unique configuration of $\mathbb{P}$, *i.e.* $\mathbb{P}$ and $\{w \in \Sigma^m : |w|_a = nu \text{ and } |w|_b = nv\}$ are in bijection. By considering a as the vector $(1, 0)$ and b as the vector $(0, 1)$, we have for all $i \in \{0, \ldots, m-1\}$ either $c_{i+1} = c_i + a$ or $c_{i+1} = c_i + b$ and more precisely $c_{i+1} = c_i + w_i$. A *segment* $[c_i, c_j]$ of a configuration c is the subsequence of sites starting in c_i and finishing in c_j. The word $w_1 w_2 ..., w_{j-i}$, on the alphabet $\{a, b\}$, associated to the segment $[c_i, c_j]$, is defined as the labels of the arcs of this segment. The length of the segment $[c_i, c_j]$ is $j - i - 1$, the length of its associated word.

Configurations are static objects, now we introduce a way to modify them locally. Consider a configuration c and $0 \leq i \leq m - 2$, the configuration c' obtained by *flipping* letters i and $i + 1$ in c is defined as followed: consider the words w, w' of Σ^m where w is the word associated to configuration c and w' is obtained by flipping letters i and $i + 1$ in w, *i.e.* $w'_i = w_{i+1}$, $w'_{i+1} = w_i$ and for all $j \in \{0, \ldots, m-1\} \setminus \{i, i+1\}$, $w'_j = w_j$, then c' is the configuration associated to w'. Note that for all $0 \leq j \leq m$ such that $j \neq i + 1$ we have $c_j = c'_j$ and:

- if $w_i = a$ and $w_{i+1} = b$ then $c'_j = c_j - a + b$ and $h(c'_j) = h(c) + u + v$;
- if $w_i = b$ and $w_{i+1} = a$ then $c'_j = c_j + a - b$ and $h(c'_j) = h(c) - u - v$.

We will also denote this operation as "flipping site $i + 1$". For a configuration c, for $1 \leq i \leq m - 1$, site i is *increasing* if $w_i = a$ and $w_{i+1} = b$ (if flipping this site increases its height) otherwise site i is *decreasing* if $w_i = b$ and $w_{i+1} = a$ (if flipping this site decreases its height). Remember that our aim is to introduce a dynamics such that the configuration reorganizes itself into a good discrete approximation of the continuous line linking site $(0,0)$ to site (nu,nv), flipping any site of a configuration at random will not achieve this goal so we have to consider some special sites. For identifying these sites, we have to look at the disposition of the sites more in details.

Now, we focus on V and gives some remarks and tools to better understand the space occupied by the configurations. First, note that a site $(x,y) \in V$ is at distance $x+y$ of site $(0,0)$. Secondly, we define the *height* $h(s)$ of a site $s=(x,y) \in V$ as the value $-vx+uy$. Sites of height 0 are (iu,iv) with $0 \leq i \leq n$, these sites are on the continuous line linking site $(0,0)$ to site (nu,nv). The sites of extremal height are $(0,nv)$ of height nuv and $(nu,0)$ of height $-nuv$. These two sites are the farthest sites from the continuous line linking $(0,0)$ to (nu,nv). Thus the height is a good value to measure the distance between a site and the objective line. Imagine a process where sites of maximal and minimal height of a configuration are flipped, then this process will quickly converges to a good discrete approximation of our objective line but unfortunately this process requires the knowledge of the whole configuration. Nevertheless the height is still useful, remark that for a configuration c, for $0 \leq i \leq m-1$ if $c_{i+1} = c_i + a$, then $h(c_{i+1}) = h(c_i) - v$ and if $c_{i+1} = c_i + b$, then $h(c_{i+1}) = h(c_i) + u$. Thus, computing the difference of height between two sites c_i and c_j requires only the word associated to the segment $[c_i, \dots, c_j]$ of c, in particular we do not need to know $h(c_i)$ or $h(c_j)$. If a site of a configuration detects locally a great variation of height among its neighbors then it knows that there is an error in the configuration and he can act to try to correct it by flipping.

As told before, we want to introduce a random process which transforms any configuration into a good approximation of the continuous line linking $(0,0)$ to (nu,nv). We call *flat line* the best possible approximation of a discrete line. A configuration c is a flat line of height h if and only if for all $0 \leq i \leq m$, $h \leq h(c_i) \leq h+u+v-1$. Flat line of height h are unique. Note that, we will be interested only in flat line of height h with $-u-v+1 \leq h \leq 0$, since all configurations start in $(0,0)$ of height 0.

Now, we have everything to identify the good sites to flip for reorganizing a configuration into a flat line. Such sites are called *active* and we explain how to identify them. A site c_i is *right decreasing* (resp *left decreasing*) in c if there exists an integer k_0, with $1 \leq k_0 \leq u+v$, such that

1. for each k such that $1 \leq k < k_0$, we have $h(c_i) - u - v + 1 \leq h(c_{i+k}) \leq h(c_i)$ (resp we have $h(c_i) - u - v + 1 \leq h(c_{i-k}) \leq h(c_i)$);
2. $h(c_{i+k_0}) \leq h(c_i) - u - v$ (resp $h(c_{i-k_0}) \leq h(c_i) - u - v$).

Similarly, site c_i is *right increasing* (resp *left increasing*) in c if there exists an integer k_0, with $1 \leq k_0 \leq u+v$, such that

1. for each k such that $1 \leq k < k_0$, we have $h(c_i) \leq h(c_{i+k}) \leq h(c_i) + u + v - 1$ (resp we have $h(c_i) \leq h(c_{i-k}) \leq h(c_i) + u + v - 1$);
2. $h(c_{i+k_0}) \geq h(c_i) + u + v$ (resp $h(c_{i-k_0}) \geq h(c_i) + u + v$).

To interpret these definitions, a site of a configuration will look at the $u+v$ previous sites of the configuration (its left) and the $u+v$ next sites of the configuration (its right), a direction can be decreasing if the site detects a drop of at least $u+v$ or can be increasing if the site detects a rise of height by at least $u+v$. Now, with these information a site can decide if it wants to flip: an increasing (resp. decreasing) site is active if and only if at least one direction is increasing (resp. decreasing) and no direction is decreasing (resp. increasing).

Now, we have to present how the active sites are selected to be flipped. For this, we introduce the following rule δ which takes as input a configuration c of $\mathbb{P}$ and output a random configuration $\delta(c)$ defined as follow: a site i is selected uniformly at random in $\{1, m-1\}$ and if this site is active then it is flipped in $\delta(c)$ and all other sites remain unchanged:

– If c_i is active and increasing, then replace c_i by $c_i - a + b$ in $\delta(c)$.
– Otherwise, if c_i is active and decreasing, then replace c_i by $c_i + a - b$ in $\delta(c)$.

Time is discrete and let c^t design the configuration at time t, c^0 is the *initial configuration*. The configuration at time $t+1$ is a random variable defined by $c^{t+1} = \delta(c^t)$.

Now, we want to prove that a configuration evolving according to our process will evolve quickly to reach a flat line. When the dynamics reaches a flat line no more site are active and nothing happens. We call a configuration with no active site a *stable* configuration, *i.e.* a configuration is stable if and only if $\delta(c) = c$ with probability 1.

Proposition 1. *The flat lines of height h with $-u-v+1 \leq h \leq 0$ are stable configurations for rule δ.*

Proof. Consider a configuration c which is flat line, c contains site $(0,0)$ of height 0 then c is a flat line of height h with $-u-v+1 \leq h \leq 0$. Now, by definition of a flat line, for any site of the configuration, the condition 2 of increasing and decreasing cannot be satisfied. Then, for any site of a flat line, both directions are neither increasing nor decreasing. Thus, there is no active site in a flat line.

In fact, these flat lines are the only stable configurations for rule δ but this result will be indirectly shown when we prove in Theroem 2 that any configuration evolving according to rule δ reaches a flat line. Consider a sequence of configurations $(c^t)_{t\geq 0}$ updated with rule δ. We now aim to give an upper bound on the *convergence time* $T = \min\{t : c^t \text{ is stable}\}$, *i.e.* the first time that our random process hits a stable configuration. The rest of the paper is dedicated to the proof of the following theorem.

Theorem 1. *For any initial configuration c^0, the expected convergence time $\mathbb{E}[T]$ is $O(nuvm^3)$.*

In fact, we conjecture that the convergence time is in fact $O(m^3)$ (see the open question in section 5 for more details).

3 Toolbox

In this short section we present the lemma used to prove the upper bound of Theorem 1. Lemma 1 is a classical result on martingales, its proof can be found in [3]. The way to use this lemma is to affect a value between 0 and k (with $k \in \mathbb{N}$) to each configuration, this value will be called the *energy* $E(c)$ of configuration c. If wisely defined, this energy will behave as a random walk: its expected variation will be less than 0 for any configuration. A non-biased one dimensional random walk on $\{0,\ldots,k\}$ hits the value 0 on $O(k^2)$ time step. Once again if the energy is wisely defined, only flat lines will have an energy of 0 and thus our process hits a flat line in polynomial time. A key part of this lemma is to bound the expected variation of energy, so we introduce the following notation:

$$\Delta E(c^t) = E(c^{t+1}) - E(c^t) \text{ (or } \Delta E(c) = E(\delta(c)) - E(c)).$$

Lemma 1. *Let $k \in \mathbb{Z}^+$ and $\epsilon > 0$. Consider $(c^t)_{t\geq 0}$ a random sequence of configurations, and E an energy function such that $\forall c \in \mathbb{P}$ $E(c) \in \{0,\ldots,k\}$. Assume that if $0 < E(c^t) < k$, then $\mathbb{E}[\Delta E(c^t)|c^t] \leq 0$ and $Prob\{|\mathbb{E}[\Delta E(c^t)]| \geq 1\} \geq \epsilon$ and if $E(c^t) = k$, then $\mathbb{E}[\Delta E(c^t)|c^t] \leq \epsilon$. Let $T = \min\{t : E(c^t) = 0\}$ denote the random variable for the first time t where $E(c^t) = 0$. Then, $\mathbb{E}[T] \leq \frac{E(c^0)(k-E(c^0))}{2\epsilon}$.*

4 Collapsing of the top of the configurations

In this part, we show Theorem 1 using Lemma 1. Considering the whole configuration is yet too difficult, so we will focus on a subset of critical sites to ease the analysis: the sites of maximum height. Consider a configuration c of $\mathbb{P}$, let $\hat{h}(c) = \max_{0\leq i\leq n}(h(c_i))$ be the maximum height of sites of c and let $\Delta(c) = \hat{h}(c) - \min_{0\leq i\leq n}(h(c_i))$ be the difference between the maximum height and the minimum height of c. For a sequence of configurations $(c^t)_{t\geq 0}$ of $\mathbb{P}$, we denote its initial maximum height $\hat{h}(c^0)$ by $\hat{h}$ and we denote by $\hat{\Delta}$ the difference between the maximum height and the minimum height at time 0, *i.e.* $\hat{\Delta} = \Delta(c^0)$.

Lemma 2. *For any configuration c of $\mathbb{P}$, updating c according to rule δ does not increase the maximum height,* i.e. $\hat{h}(\delta(c)) \leq \hat{h}(c)$.

Proof. The only way to increase the height of a site is to flip an active increasing one. Consider such as site c_i, then by definition of active and increasing there is a site of height at least $h(c_i)+u+v$ in configuration c and thus $\hat{h}(c) \geq h(c_i)+u+v$. Moreover, flipping site i increases its height by $u+v$ so we can conclude that the maximum height is non-increasing according to time.

Corollary 1. *Consider a sequence of configurations $(c^t)_{t\geq 0}$ of $\mathbb{P}$, then for all $t \geq 0$, $\hat{h}(c^t) \leq \hat{h}$.*

Remark that all sites of maximal height are decreasing (otherwise their neighbors would be higher). So, if we find a way to flip all this sites then an irreversible update happens: the maximal height of the configuration decreases. By repeating this reasoning, the difference between the maximum height and the minimum height will decrease over time and the dynamics will reach a flat line. So from now on, we focus on the sites of maximal height. First, we show that these sites have a very specific position on the configuration, the distance between two sites of maximal height is a multiple of $u+v$. For all $i \in \{0,\ldots,u+v-1\}$ and for all $h \in \{-nv,\ldots,+nu\}$, we define $V_i^h = \{(x,y) \in V : -vx+uy = h \text{ and } x+y = i \mod u+v\}$ as the set of sites of height h which are at distance $i \mod u+v$ of site $(0,0)$. We define $H^h(c) = \{c_i \in c : h(c_i) = h\}$ as the set of sites of c of height h. For a sequence of configurations $(c^t)_{t\geq 0}$, we denote by $\hat{H}(c^t) = H^{\hat{h}}(c^t)$ the set of sites of c^t which are at the initial maximum height.

Lemma 3. *Consider a sequence of configurations $(c^t)_{t\geq 0}$, then there exists $\ell \in \{0,\ldots,u+v-1\}$ such that for all $t \geq 0$, we have $\hat{H}(c^t) \subset V_\ell^{\hat{h}}$.*

Proof. We prove this result by recurrence over t. Consider the site c_j with $j = min_{0\leq i\leq m}\{i : c_i \in \hat{H}(c^0)\}$, there exists $0 \leq l \leq u+v-1$ such that $c_j \in V_l^{\hat{h}}$. Consider a site c_k of $\hat{H}(c^0)$ and the word w associated to segment $[c_j, c_k]$, when moving by b along this path the height is increased by u and when moving by a the height is decreased by v. Since $-v = u \mod u+v$, we have $h(c_j) = h(c_k) + |w|u \mod u+v$. Since $h(c_j) = h(c_k) = \hat{h}$, then $0 = |w|u \mod u+v$. Remember that $gcd(u, u+v) = 1$, so $|w| = 0 \mod u+v$ and then $c_k \in V_l^{\hat{h}}$. This conclude the proof for $t = 0$. Now, suppose that for $t \geq 0$ we have $\hat{H}(c^t) \subset V_l^{\hat{h}}$. Only one site is updated between t and $t+1$ and three cases occur:

- $\hat{H}(c^t) = \emptyset$ then $\hat{h}(c^t) < \hat{h}$ and by corollary 1, $\hat{H}(c^{t+1}) = \emptyset$;
- $\hat{H}(c^{t+1}) \subset \hat{H}(c^t)$ and then $\hat{H}(c^{t+1}) \subset V_l^{\hat{h}}$;
- there exists $0 \leq j \leq m$ such that $\hat{H}(c^{t+1}) = \hat{H}(c^t) \cup \{c_j\}$ and $\hat{H}(c^t) \neq \emptyset$. By recurrence there exists $0 \leq k \leq m$ such that $c_k \in \hat{H}(c^t) \cap V_l^{\hat{h}}$ and by the same argument as before the path between c_j and c_k has a length $0 \mod u+v$ and $c_j \in V_l^{\hat{h}}$.

Thus for each case, the recurrence is true.

Consider a random sequence of configurations $(c^t)_{t\geq 0}$, from the previous lemma there exists ℓ such for all $t \geq 0, \hat{H}(c^t) \subset V_\ell^{\hat{h}}$, we denote this set as $\hat{V}$. Now we focus on sites of $\hat{V}$. Here, we start the second part of the proof. We have to study the first time when $c^t \cap \hat{V}$ becomes empty. Analyzing in details the evolution of $c^t \cap \hat{V}$ would be too hard. Thus, we introduce a new rule $\hat{\delta}$. This new rule is in fact a cellular automaton called ECA 178 previously studied in [3]. A configuration $\hat{c}$ of this cellular automaton will encode the fact that sites of $\hat{V}$ belong to c or not. The cellular automaton $\hat{\delta}$ will be defined such that it is both easier to analyze it than δ and close enough to the dynamics of δ to be used as a bound in the study of δ. Let $\mathbb{T} = \{0, \dots, |\hat{V}| - 1\}$ be a set of *cells*. A configuration $\hat{c}$ of $\mathbb{T}$ is a function $\mathbb{T} \to \{0, 1\}$ that assigns to each cell $i \in \mathbb{T}$ a *state* $\hat{c}_i \in \{0, 1\}$. For all $i, j \in \mathbb{T}$, cells i and j are *neighbors* if $|i - j| = 1$.

Definition 1. *We say that a configuration $\hat{c}$ of $\mathbb{T}$ and a configuration c of $\mathbb{P}$ are* synchronized*, denoted $c \sim \hat{c}$, if and only if for all $i \in \mathbb{T}$, we have $\hat{c}_i = 1$ if and only if $h(c_{\ell+i(u+v)}) = \hat{h}(c)$.*

We say that a configuration $\hat{c}$ of $\mathbb{T}$ dominates *a configuration c of $\mathbb{P}$, denoted $\hat{c} \succ c$, if and only if for all $i \in \mathbb{T}$, we have $h(c_{\ell+i(u+v)}) = \hat{h}(c)$ implies $\hat{c}_i = 1$.*

Remark that, for each configuration c of $\mathbb{P}$ there exists an unique configuration $\hat{c}$ of $\mathbb{T}$ such that $c \sim \hat{c}$. Now, it is time to introduce the energy of a configuration and since the structure of $\hat{c}$ is simpler than c we will define the energy on configuration $\hat{c}$ and the energy of c will be the energy of $\hat{c}$. Now, for an efficient way to define an energy function see [4] where an efficient formalism is introduced to allow an automatization of the proof. We give here a simplified sketch of the proof for two reasons: we want the paper to be self-contained and the method developed in [4] are for configurations with periodic boundary conditions, so we need to apply some patches to their tools. To define an energy function, we give a potential to each cell which depends on the states of the neighboring cells. The energy of a configuration will be the sum of the potentials of all its cells. We will see later that this way of defining an energy function locally will ease the computation of its expected variation.

Definition 2. *Consider a configuration $\hat{c}$ of $\mathbb{T}$ then the potential p_i of cell i is defined as $p_i = p_i^1 + p_i^l + p_i^r$ where:*

- $p_i^1 = 4$ *if* $\hat{c}_i = 1$ *and* 0 *otherwise;*
- $p_i^l = 1$ *if* $i > 0$ *and* $\hat{c}_i \neq \hat{c}_{i-1}$ *and* 0 *otherwise;*
- $p_i^r = 1$ *if* $i < |\mathbb{T}| - 1$ *and* $\hat{c}_i \neq \hat{c}_{i+1}$ *and* 0 *otherwise.*

Definition 3. *The energy $E(\hat{c})$ of a configuration $\hat{c}$ of $\mathbb{T}$ is defined as:*

$$E(\hat{c}) = M(\hat{c}) + \sum_{0 \leq i \leq |\mathbb{T}|-1} p_i$$

where $M(\hat{c})$ is a parameter whose value is 2 if for all $i \in \mathbb{T}$, $\hat{c}_i = 1$ and 0 otherwise. The energy $E(c)$ of a configuration c of $\mathbb{P}$ is $P(\hat{c})$ where $\hat{c}$ is the configuration of $\mathbb{T}$ such that $\hat{c} \sim c$.

So the potential of a cell is 4 if the cell is in state 1 plus 1 per neighboring cells in a different state than itself. After summing the potential of all cells, the energy of a configuration is four times the number of 1 in the configuration plus 1 per pattern 01 and 10. Note that the only way to decrease the number of 01 in the configuration is to switch an isolated 0 (pattern 101) to 1 or an isolated 1 (pattern 010) to 0. Also note that, the energy of a configuration belongs to $\{0, \dots, 4|\mathbb{T}| + 2\}$ and the only configuration of energy 0 is the configuration where all cells are in state 0 and the only configuration of energy $4|\mathbb{T}| + 2$ is the configuration where all cells are in state 1. The parameter M will later be important in the proofs for a case where cells are in state 1 except one. The next step is to show that increasing the number of one in a configuration also increases its energy.

Lemma 4. *For a configuration c of $\mathbb{P}$, for all configurations $\hat{c}$ of $\mathbb{T}$ such that $\hat{c} \succ c$, we have $E(\hat{c}) \geq E(c)$.*

Proof. First, remark that by switching a cell $0 \leq i \leq |\mathbb{T}| - 1$ of a configuration of $\mathbb{T}$ from state 0 to state 1, the potential of cell i increases by at least 2 and eventually the potential of cell $i-1$ decreases by 1 and the potential of cell $i+1$ decreases by 1. Then switching a cell from 0 to 1 in a configuration of $\mathbb{T}$ is a non-decreasing operation for the energy. Now, consider the configuration $\hat{c}'$ of $\mathbb{T}$ synchronized with c then by definition $\forall i \in \mathbb{T}$, we have $\hat{c}'_i \leq \hat{c}_i$. Then $E(\hat{c}') \leq E(\hat{c})$.

Here it would be tempting to consider a sequence of configurations $(c)_{t \geq 0}$ of $\mathbb{P}$ evolving under rule δ and to consider the sequence of configurations $(\hat{c})_{t \geq 0}$ of $\mathbb{T}$ such that for all $t \geq 0$, we have $c \sim \hat{c}$. This is not the good way to proceed because the second sequence will be too hard to analyze. Instead we introduce a new rule $\hat{\delta}$ on configurations of $\mathbb{T}$ which behaves almost like δ but which is easier to analyze. We say that a cell $i \in \mathbb{T}$ is *active* if and only if there exists $j \in \mathbb{T}$ such that $|i - j| = 1$ and $\hat{c}_i \neq \hat{c}_j$. Consider a configuration c of $\mathbb{P}$, the configuration $\delta(c)$ and a configuration $\hat{c}$ of $\mathbb{T}$, we define by a coupling with δ the following rule $\hat{\delta}$ which associates to each configuration $\hat{c}$ a random configuration $\hat{\delta}(\hat{c})$ of $\mathbb{T}$ as follows:

- if a site $c \in V \setminus \hat{V}$ is updated in c then no cell is updated in $\hat{c}$;
- if there exists i such that site $c_{\ell+i(u+v)} \in \hat{V}$ is updated in c and cell i of $\hat{c}$ is active then $\hat{c}$ changes its states:

$$\hat{\delta}(\hat{c})_i = \begin{cases} 1 - \hat{c}_i & \text{if } i \text{ is active and } i(u+v)+\ell \text{ is updated in } c \\ \hat{c}_i & \text{otherwise} \end{cases}$$

  and for $j \in \mathbb{T} \setminus \{i\}, \hat{\delta}(\hat{c})_j = \hat{c}_j$.

No, we prove that two synchronized configurations won't differ too much after one time step if one evolves with rule δ and the other with rule $\hat{\delta}$.

Lemma 5. *Consider $c \in \mathbb{P}$ and $\hat{c} \in \mathbb{T}$ such that $c \sim \hat{c}$. Then, $\hat{\delta}(\hat{c}) \succ \delta(c)$.*

Proof. We call $\hat{c}'$ the configuration of $\mathbb{T}$, such that $\hat{c}' \sim \delta(c)$. If a site of $V \setminus \hat{V}$ is updated then by definition of the coupling, $\hat{\delta}(\hat{c}) = \hat{c} = \hat{c}'$ and $\hat{\delta}(\hat{c}) \sim \delta(c)$. Now, consider that a site $i(u+v)+\ell$ of $\hat{V}$ is updated. Then by definition of the coupling, $\hat{c}'$ and $\hat{\delta}(\hat{c})$ may differ only on cell i. We have to show that the case $\hat{c}'_i > \hat{\delta}(\hat{c})_i$ may not occur:

- If cell i is inactive in $\hat{c}$ and $\hat{c}_i = 1$, then $\hat{\delta}(\hat{c})_i = 1$ and $\hat{\delta}(\hat{c}) \succ \delta(c)$.

- If cell i is inactive in $\hat{c}$ and $\hat{c}_i = 0$, then since $\hat{c}_i = 0$, we have site $c_{i(u+v)+\ell} \notin \hat{V}$ and $h(c_{i(u+v)+\ell}) \leq \hat{h} - (u+v)$. If $h(c_{i(u+v)+\ell}) < \hat{h} - (u+v)$ then $h(\delta(c)_{i(u+v)+\ell}) \leq \hat{h} - (u+v)$ and $\hat{c}'_i = 0$. If $h(c_{i(u+v)+\ell}) = \hat{h} - (u+v)$, since cell i of $\hat{c}$ is inactive, then for all $j \in \{-1, 1\}$, $\hat{c}_{i+j} = 0$. Since $\hat{c} \sim c$, for all $j \in \{-1, 1\}$, $h(c_{(i+j)(u+v)+\ell}) \leq \hat{h} - (u+v)$. Moreover, only sites of $\hat{V}$ can be of height $\hat{h}$, then for all $k \in \{-(u+v), \ldots, (u+v)\}$, $h(c_{i(u+v)+\ell+k}) < \hat{h}$ and no direction can be increasing for site $c_{i(u+v)+\ell}$ in c since condition 2 is not true. Then, $h(\delta(c)_{i(u+v)+\ell}) \leq \hat{h} - (u+v)$ and $\hat{c}'_i = 0$.
- If cell i is active in $\hat{c}$ and $\hat{c}_i = 1$, then we have $c_{i(u+v)+\ell} \in \hat{V}$ and $h(c_{i(u+v)+\ell}) = \hat{h}$. Since cell i is active in $\hat{c}$, there exists $j \in \{-1, 1\}$ such that $\hat{c}_{i+j} = 0$. Since $\hat{c} \sim c$, there exists $j \in \{-1, 1\}$ such that $h(c_{(i+j)(u+v)+\ell}) \leq \hat{h} - u - v$. Then, site $c_{i(u+v)+\ell}$ is decreasing in one direction since there exists $k \in \{-(u+v), \ldots, (u+v)\}$, such that conditions 1 and 2 are satisfied. Since $h(c_{i(u+v)+\ell}) = \hat{h}$, site $c_{i(u+v)+\ell}$ cannot be increasing in the other direction (condition 2 cannot be satisfied by any site). Thus, site $c_{i(u+v)+\ell}$ is flipped, its height is decreased, $h(\delta(c)_{i(u+v)+\ell}) < \hat{h}$ and $\hat{c}'_i = 0$.
- If cell i is active in $\hat{c}$ and $\hat{c}_i = 0$, then $\hat{\delta}(\hat{c})_i = 1$ and $\hat{\delta}(\hat{c})$ dominates $\delta(c)$.

Thus, in every case $\hat{\delta}(\hat{c})$ dominates $\delta(c)$.

Here it would be tempting to consider a sequence of configurations $(c^t)_{t\geq 0}$ of $\mathbb{P}$ evolving under rule δ and the sequence of configurations $(\hat{c}^t)_{t\geq 0}$ of $\mathbb{T}$ evolving under rule $\hat{\delta}$ and where $c^0 \sim \hat{c}^0$. Again, this is not the right way to proceed because Lemma 5 is not strong enough to deduce a correlation between these two sequences for $t \geq 2$. Lemma 4 and Lemma 5 are used to deduce a bound on the expected variation of energy as follow: consider c of $\mathbb{P}$ updated by δ and $\hat{c}$ of $\mathbb{T}$ updated by $\hat{\delta}$ such that $\hat{c} \sim c$, then:

$$\Delta E[c] \leq \Delta E[\hat{c}]$$

where $\Delta E[c] = E[\delta(c)] - E[c]$ and $\Delta E[\hat{c}] = E[\hat{\delta}(\hat{c})] - E[\hat{c}]$. Now, everything is in place to bound the expected variation of energy of a configuration c evolving under rule δ.

Lemma 6. *Consider a configuration c of $\mathbb{P}$ such that $\hat{h}(c) \geq 1$ and $\Delta(c) \geq u+v$ then:*

- *if $E(c) = 4|\mathbb{T}| + 2$ then $\mathbb{E}[\Delta E(c)] \leq -\frac{1}{m}$;*
- *otherwise, $\mathbb{E}[\Delta E(c)] \leq 0$ and $Prob(|\Delta E(c)| \geq 1) \geq \frac{1}{m}$.*

Proof. Let $\hat{c}$ of $\mathbb{T}$, such that $c \sim \hat{c}$. First consider that c is a configuration of maximum energy, $E(c) = 4|\mathbb{T}| + 2$, *i.e.* for all $i \in \mathbb{T}$, $\hat{c}_i = 1$ then there exists $i, j \in \{0, \ldots, m\}$ such that $h(c_i) \leq \hat{h}(c) - u - v$, $|i - j| < u + v$ and $j \in \hat{V}(c)$. Then site j is of height $\hat{h}(c)$ is active and decreasing. With probability $\frac{1}{m}$, site j is flipped; $h(\delta(c)_j) < \hat{h}(c)$ and $E[\delta(c)] = 4|\hat{V}|$ and $\Delta E[c] = -2$. Then, to conclude for this case we have $\mathbb{E}[\Delta E(c)] \leq -\frac{2}{m}$. For the other cases, the rest of proof relies on the fact that we will not compute $\Delta E[c]$ directly but we will use $\Delta E[\hat{c}] = E(\hat{\delta}(\hat{c})) - E(\hat{c})$ since by Lemma 5, we have $\Delta E[c] \leq \Delta E[\hat{c}]$.

Now, to deal with the global parameter $M(\hat{c})$, we consider the case where there exists i such that $\hat{c}_i = 0$ and for all $j \neq i$, $\hat{c}_j = 1$. Then, the only active cells for rule $\hat{c}$ are cell i and cells j such that $|j - i| = 1$ (the worst case for the following analysis is when there is only one neighbor, $i = 0$ or $i = |\mathbb{T}| - 1$):

- the variation of energy for firing cell i is less than 4 (+4 for the number of 1; +2 for the parameter M and less than -2 for the number of pattern 01 and 10 since the 0 is isolated);
- the variation of energy for firing cell j with $|j - 1| = 1$ is less than -4 (since a 1 is switched to 0).

Thus, the total variation of energy is less than 0 and $Prob(|\Delta(E(c))| \geq 1) \geq \frac{1}{m}$.

Now, we can focus on the general case where there are at least two cells in state 0 and forget about parameter M whose value is always 0. We compute the expected variation of energy by the following formula:

$$\Delta(E(\hat{c})) = \sum_{0 \leq i \leq |\mathbb{T}|-1} Prob(\text{cell } i \text{ is updated}) \Delta(E(\hat{c}))_{\text{cell } i \text{ is updated}}$$

When a cell is switched from 0 to 1 then the energy of the configuration increases by at most 4 and when a cell is switched from 1 to 0 then the energy decreases by 8 if both neighboring cells are in state 0 and by 4 otherwise. Now, by the definition of an active cell, each active cell in state 0 (the only cells which can increase the energy of the configuration by being updated) has an active neighboring cell in state 1. Then, the contribution of these two cells to the total of the expected variation of energy is less than 0. Note that if two active cells in state 1 have the same neighbors in state 0, firing the cell in state 0 decreases the potential by 8 and the contribution of these three cells to the total of the expected variation of energy is less than 0. Thus, the lemma is true for this final case.

And now, only the main theorem is left.

Theorem 2. *Consider a sequence of configuration $(c^t)_{t \geq 0}$, for any initial configuration c^0, the expected convergence time $\mathbb{E}[T]$ is $O(\hat{\Delta} m^3)$.*

Proof. By Lemma 1 and Lemma 6, after $O(m^3)$ the dynamics has reached a configuration c such that either c is a flat line or $\Delta(c) < \hat{\Delta}$. By iterating this reasoning (which is also true for the minimum height), we reach in $O(\hat{\Delta} m^3)$ iterations a configuration where the maximum difference of height is strictly less than $u + v$, *i.e.* a flat line and a stable configuration.

5 Conclusion

In this paper, we have introduced and analyzed a random process which enables a twisted thread to reorganize itself. Nevertheless, there is still a lot of open questions. For the case $u = v = 1$ (gradient $\frac{1}{2}$), our random process is the same one as the one introduced and analyzed in [2] but our analyze gives an upper bound on the convergence time of $O(m^4)$ whereas in [2] they prove an upper bound of $O(m^3)$. We were able to generalize the random process for any rational gradient but not the analysis. In fact, we conjecture that our random process converges in $O(m^3)$, our analysis here considers only the sites of maximal height and forgets about a lot of useful updates which are done at the same time. There are also a lot of negative results to prove: showing than any random process based on local decisions requires $\Omega(m^3)$ steps; showing that it is not possible to approximate lines with irrational gradient; showing that sites need to look at a distance $u + v$ to decide be active or not and showing that there are no deterministic process to achieve this goal. From the theoretical side, we have done a reduction, we proved that our random process encodes cellular automaton ECA 178. We need to develop a better formalism for these kinds of reduction. This work is also part of a study of cooling processes in crystallography [1, 2, 5]. Here we studied a "simple" case. A more complicated case is to study a $2D$ structure instead of a $1D$ thread. A $2D$ tilling was studied in [5] but the next step is to study a $2D$ aperiodic tilings like Penrose tilings.

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