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# CONJUGACIES OF MODEL SETS

JOHANNES KELLENDONK AND LORENZO SADUN

ABSTRACT. Let  $M$  be a model set meeting two simple conditions: (1) the internal space  $H$  is a product of  $\mathbb{R}^n$  and a finite group, and (2) the window  $W$  is a finite union of disjoint polyhedra. Then any point pattern with finite local complexity (FLC) that is topologically conjugate to  $M$  is mutually locally derivable (MLD) to a model set  $M'$  that has the same internal group and window as  $M$ , but has a different projection from  $H \times \mathbb{R}^d$  to  $\mathbb{R}^d$ . In cohomological terms, this means that the group  $H_{an}^1(M, \mathbb{R})$  of asymptotically negligible classes has dimension  $n$ . We also exhibit a counterexample when the second hypothesis is removed, constructing two topologically conjugate FLC Delone sets, one a model set and the other not even a Meyer set.

## 1. INTRODUCTION AND STATEMENT OF RESULTS.

A substantial part of the analysis of Delone sets (or tilings) is based on the study of their associated dynamical systems. This includes characterizing certain classes of Delone sets by ergodic and topological properties of their dynamical systems. For instance, the dynamical system of a repetitive finite local complexity (FLC) Delone set of  $\mathbb{R}^d$  is topologically conjugate to that of a repetitive Meyer set if and only if it has  $d$  independent topological eigenvalues [KS]. Similar characterizations are known for model sets (see [ABKL] for a review).

This suggests a natural definition and a question. Two Delone sets  $\Lambda$  and  $\Lambda'$  are called *topologically conjugate* whenever their associated dynamical systems are topologically conjugate. We call them *pointed topologically conjugate* if the conjugacy maps  $\Lambda$  to  $\Lambda'$ . Which properties of FLC Delone sets are preserved under topological conjugacy or pointed topological conjugacy?

In [KS] we showed that the Meyer property is *not* always preserved under topological conjugacy. In view of this, we call a Meyer set *rigid* if every FLC Delone set topologically conjugate to  $\Lambda$  is a Meyer set. Our aim in this article is to study rigidity for model sets.

Recall that a Meyer set is a Delone set  $\Lambda$  such that the set of difference vectors  $\Lambda - \Lambda$  is uniformly discrete, and that model sets are Meyer sets arising from a particular construction. This construction involves a cut & project scheme, that is, an internal (locally compact abelian) group  $H$ , a lattice  $\Gamma \subset H \times \mathbb{R}^d$  and a strip  $S = W \times \mathbb{R}^d$  where  $W \subset H$ , the so-called window, is a compact subset that is the closure of its interior. The projection set

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for these data is the set of points arising by projecting the points of  $S \cap \Gamma$  onto  $\mathbb{R}^d$  along  $H$ . Generically (but not always) the projection set is repetitive in which case we call any element of its hull, that is, any point set which has the same local patches up to translation, a model set. In particular, all model sets are repetitive according to this definition.

We say that a model set  $M'$  is a *reprojection* of a model set  $M$  if it arises from the same setup, except that the projection of  $S \cap \Gamma$  onto  $\mathbb{R}^d$  is not along  $H$ , but is along  $H' \subset H \times \mathbb{R}^d$  with  $H' = \{(h, g(h)) | h \in H\}$  where  $g : H \rightarrow \mathbb{R}^d$  is a continuous group homomorphism.

We pay particular attention to *polyhedral* model sets, by which we mean model sets satisfying two additional assumptions:

- H1. The internal space  $H$  is the product of the vector space  $\mathbb{R}^n$  with a finite (discrete, pure torsion) group  $C$ .
- H2. The window  $W$  is a finite union of polyhedra.

Our main result states that such model sets are extremely rigid:

**Theorem 1.1.** *If  $M$  is a polyhedral model set and  $\Lambda$  is a Delone set of finite local complexity that is pointed topologically conjugate to  $M$ , then  $\Lambda$  is mutually locally derivable (MLD) to a reprojection of  $M$ .*

Without hypothesis H2, this theorem is false. In Section 8 we exhibit a one dimensional model set  $M$  satisfying H1 but not H2, and a Delone set  $\Lambda$  of finite local complexity that is pointed topologically conjugate to  $M$ , with  $\Lambda$  not being a Meyer set, much less a model set or a reprojection of  $M$ .

A first result in the direction of studying topological conjugacies can be found in [KS]. It says that any pointed topological conjugacy between repetitive FLC Delone sets is the composition of an MLD transformation followed by a *shape conjugacy* (defined below) which can be chosen arbitrarily close to the identity. Given that MLD transformations are well-understood, this reduces the task of understanding topological conjugacies to a study of shape conjugacies. Shape conjugacies modulo MLD transformations are parametrized (at least infinitesimally) by a subgroup of the first tiling cohomology with  $\mathbb{R}^d$ -coefficients  $H^1(\Lambda, \mathbb{R}^d)$ , called the *asymptotically negligible group* in [CS], which we denote by  $H_{an}^1(\Lambda, \mathbb{R}^d)$ .

If  $\Lambda$  is a Meyer set, then within  $H_{an}^1(\Lambda, \mathbb{R}^d)$  there is a subgroup of shape deformations that preserves the Meyer property. We call these *nonslip*, and denote the subgroup  $H_{ns}^1(\Lambda, \mathbb{R}^d)$ . If  $\Lambda$  is a model set, there is a further subgroup, denoted  $H_{repr}^1(\Lambda, \mathbb{R}^d)$ , corresponding to reprojections. One can similarly define  $H_{an}^1(\Lambda, \mathbb{R})$ ,  $H_{ns}^1(\Lambda, \mathbb{R})$ , and  $H_{repr}^1(\Lambda, \mathbb{R})$ . (See Sections 2 and 4, below.) In cohomological terms, Theorem 1.1 can be restated as follows:

**Theorem 1.2.** *If  $M$  is a polyhedral model set then  $H_{repr}^1(M, \mathbb{R}) = H_{ns}^1(M, \mathbb{R}) = H_{an}^1(M, \mathbb{R})$ .*

Another cohomological restatement is as follows: There is a natural map from the cohomology of  $H \times \mathbb{R}^d / \Gamma$  to the cohomology of a model set constructed from the data  $(\Gamma, H, \mathbb{R}^d)$ . Let us denote its image by  $H_{max}^1(M, \mathbb{R})$

**Theorem 1.3.** *If  $M$  is a polyhedral model set then  $H_{an}^1(M, \mathbb{R})$  is  $n$ -dimensional and is contained in  $H_{max}^1(M, \mathbb{R})$ .*

The organization of the paper is as follows. In Section 2 we review the machinery of Delone dynamical systems that is needed in the remainder of the paper. In Section 3 we review the theory of model sets, identifying how different model sets with the same parameter can differ. In Section 4 we introduce the notion of nonslip generators of shape conjugacies and show (Theorem 4.7) that all nonslip generators are, up to local deformation, generators of reprojections. In Section 5 we show (Theorem 5.1) that asymptotically negligible classes are represented by coboundaries of nonslip generators. Taken together, this proves Theorem 1.1. In Section 6 we interpret these results in terms of cohomology and prove Theorems 1.2 and 1.3.

Nonslip generators are introduced as a means of proving Theorem 1.1, but we believe that they have independent interest. In Section 7 we explore the significance of the nonslip property for model sets that do not necessarily satisfy hypotheses H1 and H2, and for more general Meyer sets. We prove

**Theorem 1.4.** *Let  $\Lambda$  be a repetitive Meyer set and  $F$  a generator of a shape deformation. If  $F$  is not nonslip, then the deformed set  $\Lambda^F$  is not Meyer.*

In Section 8 we exhibit a model set that does not satisfy H2 and a generator of shape conjugacies that is not nonslip, and hence a deformation of a model set that is not Meyer. We do not know whether it is ever possible to construct a nonslip class that is not a reprojection.

## 2. PRELIMINARIES ON POINT SETS AND THEIR DYNAMICAL SYSTEMS

In this section we review some of the necessary background on Delone sets and their dynamical systems. Apart from the notion of reprojection for model sets, this is all well-established in the literature.

**2.1. Dynamical system of a Delone set.** A *Delone set* is a set  $\Lambda \subset \mathbb{R}^d$  that is uniformly discrete and relatively dense. That is, there exists an  $r > 0$  such that every ball of radius  $r$  contains at most one point of  $\Lambda$ , and there exists an  $R > 0$  such that every ball of radius  $R$  contains at least one point of  $\Lambda$ .

A *Meyer set* is a Delone set  $\Lambda$  for which  $\Lambda - \Lambda$  (i.e., the set of displacement vectors between points of  $\Lambda$ ) is uniformly discrete. Equivalently,  $\Lambda$  is Meyer if 0 is not a limit point of  $(\Lambda - \Lambda) - (\Lambda - \Lambda)$ .

Let  $B$  be a compact subset of  $\mathbb{R}^d$ . The  $B$ -patch of a point set  $\Lambda \subset \mathbb{R}^d$  is the intersection  $P = \Lambda \cap B$  of  $\Lambda$  with  $B$ . We denote it by  $(P, B)$  or simply by  $P$ . An  $R$ -patch of  $\Lambda$  at  $x$  is the intersection of  $\Lambda$  with  $B = B_R(x)$ , the ball of radius  $R$  at  $x$ .

A Delone set has *finite local complexity*, or FLC, if for each  $R > 0$  the set  $\{B_R(0) \cap (\Lambda - x) \mid x \in \Lambda\}$  is finite, or stated differently, the number of  $R$ -patches occurring at points of  $\Lambda$  and counted up to translation is finite. A Delone set is *repetitive* if for every patch  $P$  of  $\Lambda$ , there exists an  $R$  such that every  $R$ -patch of  $\Lambda$  contains at least one translated copy of  $P$ . *Henceforth all Delone sets in this paper will be assumed to have FLC and to be repetitive.*

Delone sets are associated with dynamical systems as follows. We pick a metric on the space of Delone sets with given inner and outer radii  $r$  and  $R$  such that two Delone sets are close if their restriction to a large ball around the origin are close in the Hausdorff metric. If the Delone sets have FLC, this means that they agree exactly on a large ball, up to a small translation.  $\mathbb{R}^d$  acts on the space of Delone sets by translation. The closure of the orbit of a Delone set  $\Lambda$  is called the *continuous hull* of  $\Lambda$ , and is denoted  $\Omega_\Lambda$ , or just  $\Omega$  when there is no ambiguity about which Delone set is being considered. If  $\Lambda$  is a repetitive FLC Delone set, then  $(\Omega_\Lambda, \mathbb{R}^d)$  is a minimal dynamical system. We will also consider the *canonical transversal*  $\Xi_\Lambda$  (or simply  $\Xi$ ) of  $\Omega_\Lambda$  which is given by the closure of the set  $\{\Lambda - x : x \in \Lambda\}$ .  $\Xi_\Lambda$  consists of all point patterns of  $\Omega_\Lambda$  that contain the origin.

A Delone set  $\Lambda'$  is *locally derived* from  $\Lambda$  if there exists a radius  $R > 0$  such that, whenever  $\Lambda - x_1$  and  $\Lambda - x_2$  agree to radius  $R$  around the origin,  $\Lambda' - x_1$  and  $\Lambda' - x_2$  agree to radius 1 around the origin. If  $\Lambda'$  is locally derived from  $\Lambda$  and  $\Lambda$  is locally derived from  $\Lambda'$ , we say that  $\Lambda$  and  $\Lambda'$  are *mutually locally derivable*, or MLD.

A local derivation of  $\Lambda'$  from  $\Lambda$  extends to a factor map from  $\Omega_\Lambda$  to  $\Omega_{\Lambda'}$ . If  $\Lambda$  and  $\Lambda'$  are MLD, then this factor map is a topological conjugacy called an *MLD map*.

**2.2. Maximal equicontinuous factor.** An important factor of the dynamical system  $(\Omega, \mathbb{R}^d)$  is the largest factor (up to conjugacy) on which the action is equicontinuous. We denote this so-called *maximal equicontinuous factor* by  $\Omega_{max}$  and the factor map by  $\pi_{max}$ .

The equivalence relation

$$\mathcal{R}_{max} = \{(\Lambda_1, \Lambda_2) \in \Omega_\Lambda \times \Omega_\Lambda : \pi_{max}(\Lambda_1) = \pi_{max}(\Lambda_2)\}$$

can be alternatively described using the (regional) proximality relation and its strong version. This description can be used to show the following two results needed later on:

- Any topological conjugacy preserves  $\mathcal{R}_{max}$ .
- If two elements  $\Lambda_1, \Lambda_2$  in the hull of a Meyer set satisfy  $\pi_{max}(\Lambda_1) = \pi_{max}(\Lambda_2)$ , then they share a point.

Proofs can be found in [\[BK\]](#).

### 3. MODEL SETS

A model set (or cut & project set) is a Meyer set that is obtained by a particular construction.

**3.1. Cut and project scheme.** To construct a model set one needs a *cut and project scheme*  $(\Gamma, H, \mathbb{R}^d)$  and a subset  $W \subset H$  called the *window*.

The cut & project scheme consists of the space  $\mathbb{R}^d$  in which the model set lives (the *parallel* or *physical* space), a locally compact abelian group  $H$  (called the *internal group* or *perpendicular space*) and a lattice (a cocompact discrete subgroup)  $\Gamma \subset H \times \mathbb{R}^d$ . The set  $S := W \times \mathbb{R}^d$  is called the *strip*. As usual, we require three further assumptions:

- (1) The projection onto the second factor  $\pi^{\parallel} : H \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  is injective when restricted to the lattice  $\Gamma$ ,
- (2) Projection onto the first factor  $\pi^{\perp} : H \times \mathbb{R}^d \rightarrow H$  maps the lattice  $\Gamma$  densely into  $H$ ,
- (3) If  $W + h = W$  for  $h \in H$  then  $h = 0$ .

We also use the notation  $\Gamma^{\parallel} = \pi^{\parallel}(\Gamma)$ ,  $\Gamma^{\perp} = \pi^{\perp}(\Gamma)$ ,  $x^{\parallel} = \pi^{\parallel}(x)$  and  $x^{\perp} = \pi^{\perp}(x)$ . We write  $\pi_{\Gamma}^{\perp}$  for the restriction of  $\pi^{\perp}$  to  $\Gamma$ . The point set

$$\mathcal{L}_{\xi}(W) := \{\pi^{\parallel}(\gamma) : \gamma \in S \cap (\Gamma + \xi)\}$$

is called the projection set of the cut & project scheme with window  $W$  and parameter  $\xi \in H \times \mathbb{R}^d / \Gamma$ . Note that any element of  $\ker \pi_{\Gamma}^{\perp}$  is a period of the projection set  $\mathcal{L}_{\xi}(W)$ .

We are interested in windows  $W$  that are compact and the closure of their interiors. In this case, the projection set  $\mathcal{L}_{\xi}(W)$  is repetitive if the parameter  $\xi$  is such that  $\pi^{\perp}(\Gamma + \xi) \cap \partial W$  is empty. We call such parameters *non-singular* and denote the set of non-singular parameters by  $NS$ .

**Definition 3.1.** *A model set (with window  $W$ ) is an element in the hull of a projection set  $\mathcal{L}_{\xi}(W)$  whose parameter is non-singular. We always assume that  $W$  is compact and the closure of its interior. We call the model set polyhedral if  $H$  is the product of  $\mathbb{R}^n$  with a finite group and the window is a finite disjoint union of polyhedra.*

Model sets are repetitive Meyer sets. It is well-known that MLD maps send model sets to model sets, and send Meyer sets to Meyer sets.

**Remark 1.** Our use of the term *model set* is slightly different than that of some other authors. First, a model set in our sense need not to be a projection set with closed window, but might be what elsewhere is called an *inter-model set*, containing some but not all points  $x^{\parallel}$  for which  $x^{\perp}$  lies on the boundary of  $W$ . Second, the requirement that the parameter be non-singular automatically makes our model sets repetitive. So model sets in our sense might elsewhere be called repetitive inter-model sets (or complete Meyer sets). We should also mention that the window being the closure of its interior is not required by all authors.

For instance [BG] demand only that the window be relatively compact and have non-empty interior.

**3.2. Reprojection.** The projection  $\pi^\parallel$  is along  $H$  (onto  $\mathbb{R}^d$ ). If we change the direction along which we project, that is, if we tilt the space  $H$ , but otherwise leave the strip  $S$  and the parameter  $\xi$  fixed, then this affects the projection set  $\mathcal{L}_\xi(W) = \{\pi^\parallel(x) : x \in S \cap (\Gamma + \xi)\}$  rather mildly. Let  $\pi' : H \times \mathbb{R}^d \rightarrow \mathbb{R}^d$  be the projection onto  $\mathbb{R}^d$  along another group  $H' \subset H \times \mathbb{R}^d$  transversal to  $\mathbb{R}^d$ . We call

$$\mathcal{L}'_\xi(W) = \{\pi'(x) : x \in S \cap (\Gamma + \xi)\}$$

the *reprojection* of  $\mathcal{L}_\xi(W)$  along  $H'$ . More generally, if  $\Lambda$  is a subset of  $\pi^\parallel(\Gamma + \xi)$  we call

$$\Lambda' = \{\pi' \circ \pi_\xi^{\parallel -1}(\lambda) : \lambda \in \Lambda\}$$

its *reprojection* along  $H'$ . Here  $\pi_\xi^\parallel$  is the restriction of  $\pi^\parallel$  to  $\Gamma + \xi$  which is injective by our assumption. We can think of  $H'$  as being the image of  $H$  under a group isomorphism which corresponds to a shear transformation in the case  $H$  is a vector space: there exists a continuous group homomorphism  $g : H \rightarrow \mathbb{R}^d$  such that  $H' = \{(h, g(h)) : h \in H\}$ .

**3.3. Non-singular parameters and model sets.** A model set is called *non-singular* if it is a projection set  $\mathcal{L}_\xi(W)$  with non-singular parameter  $\xi$ . Since the window is the closure of its interior, this occurs for a dense  $G_\delta$ -set of  $\xi$ . Therefore these model sets are also called *generic*.

For a fixed window  $W$ , the hull of a non-singular model set  $\mathcal{L}_{\xi_0}(W)$  contains all other model sets  $\mathcal{L}_\xi(W)$  with  $\xi \in NS$ . In other words, the hull of a non-singular model set with window  $W$  depends not on the choice of the non-singular parameter  $\xi$  but only on the window. We therefore denote the hull with  $\Omega(W)$ . Hence a model set with window  $W$  is an element of  $\Omega(W)$ .

It is not difficult to see that  $\mathcal{L}_\xi(W) = \mathcal{L}_{\xi'}(W)$  if and only if  $\xi - \xi' \in \Gamma$ . The map  $\mathcal{L}_\xi(W) \mapsto \xi$  (for  $\xi \in NS$ ) is thus surjective. It is called the torus parametrization map.

It turns out that this torus is also the maximal equicontinuous factor,  $\Omega_{max} = (H \times \mathbb{R}^d)/\Gamma$ , the action being given by left translation on the second factor  $\mathbb{R}^d$  [BK]. The factor map  $\pi_{max} : \Omega(W) \rightarrow (H \times \mathbb{R}^d)/\Gamma$  is the continuous extension of the map  $\mathcal{L}_\xi(W) \mapsto \xi$  (for  $\xi \in NS$ ) which associates to a non-singular model set  $\mathcal{L}_\xi(W)$  its parameter  $\xi$ . This map is injective precisely on its pre-image of  $NS$ . The elements of the hull that are mapped by  $\pi_{max}$  to singular points are called *singular model sets*.

**3.4. The star map.** The star map sends points in  $\mathbb{R}^d$  to points in  $H$ . There are several related star maps to be considered:

- The *general star map*  $\sigma$  is a group homomorphism  $\Gamma^\parallel \rightarrow H$ ,  $\sigma(x) = \pi^\perp(\gamma)$  where  $\gamma \in \Gamma$  is the unique lift of  $x \in \Gamma^\parallel$  in  $\Gamma$  under  $\pi^\parallel$ . In particular,  $[(\sigma(x), x)]_\Gamma = [(0, 0)]_\Gamma$ ,

where  $[\cdot]_\Gamma$  denotes an equivalence class mod  $\Gamma$ . We denote the general star map also simply with a star,  $\sigma(x) = x^*$ . The general star map is not continuous if one gives  $\pi^\parallel(\Gamma)$  the relative topology induced by  $\mathbb{R}^d$ .

- The *general star map*  $\sigma_\xi$  with parameter  $\xi$  sends  $\pi^\parallel(\Gamma + \xi)$  to  $H$ , where  $(\sigma_\xi(x), x)$  is the unique lift of  $x \in \pi^\parallel(\Gamma + \xi)$  to  $\Gamma + \xi$ . Like the general star map without parameter (or equivalently, with parameter 0), this has a dense domain in  $\mathbb{R}^d$ , dense range in  $H$ , and is not continuous.
- If  $\Lambda$  is a Delone subset of  $\pi^\parallel(\Gamma + \xi)$  we denote the restriction of  $\sigma_\xi$  to  $\Lambda$  also by  $\sigma_\Lambda$  and call it the *star map of  $\Lambda$* . Note that the support of  $\sigma_\Lambda$  is uniformly discrete. This will allow us later to talk about weak pattern equivariance of  $\sigma_\Lambda$ . If  $\Lambda$  is a model set with window  $W$  then the image of  $\sigma_\Lambda$  is a dense subset of  $W$ .

Note that if  $x_1$  and  $x_2$  are points of  $M$  and  $\pi_{max}(M) = \xi$ , then both  $(\sigma_M(x_1), x_1)$  and  $(\sigma_M(x_2), x_2)$  are in  $\Gamma + \xi$ , so  $x_2 - x_1 \in \Gamma^\parallel$  and

$$\sigma_M(x_2) - \sigma_M(x_1) = \sigma_\xi(x_2) - \sigma_\xi(x_1) = (x_2 - x_1)^*.$$

The factor map  $\pi_{max}$  is related to the star map of a pattern as follows. If  $M$  is a model set and  $\xi = \pi_{max}(M)$  then  $(\sigma_M(x), x) \in \Gamma + \xi$  for all points  $x \in M$ . Thus

$$\pi_{max}(M) = [\sigma_M(x), x]_\Gamma.$$

In particular, if  $0 \in M$ , then  $\pi_{max}(M) = [\sigma_M(0), 0]_\Gamma$ . Let  $\iota_H : H \rightarrow H \times \mathbb{R}^d/\Gamma$  be given by  $\iota_H(h) = [h, 0]_\Gamma$ . By assumption  $\iota_H$  is injective. Furthermore, by equivariance of  $\pi_{max}$  we have  $\pi_{max}(M - x) = [\sigma_M(x), 0]_\Gamma$  for all  $x \in M$ . Thus

$$(1) \quad \sigma_M(x) = \iota_H^{-1} \circ \pi_{max}(M - x)$$

for all  $x \in M$ .

**3.5. Acceptance domains of patches.** Let  $M$  be a model set with window  $W$  let  $B$  be a compact set and let  $P = M \cap B$ . We call  $P$  the “ $B$ -patch of  $M$ ”. More generally we say that a finite set  $P \in \mathbb{R}^d$  is a patch for a model set with window  $W$  if there is  $M \in \Omega(W)$  and a compact set  $B \in \mathbb{R}^d$  such that  $P = M \cap B$ .

We wish to determine a condition for another model set  $M'$  in the hull of  $M$  to have  $M' \cap B = P$ . It will be sufficient for our applications to consider the case that  $P$  contains the origin which we will assume throughout.

Let  $D$  be the preimage  $\sigma^{-1}((W - W) \cap \Gamma^\perp)$ . This gives the set of all possible displacements between points in the same pattern, since if  $x_1, x_2 \in M$ , then  $\sigma_M(x_2) - \sigma_M(x_1) = \sigma(x_2 - x_1) \in W - W$ . Let  $P' = (D \cap B) \setminus P$ .  $P'$  the set of points that can appear in a  $B$ -patch of some model set which contains 0 but are not in the  $B$ -patch  $P$ . Note that  $P'$  is a finite set. Let

$$W_P^o = \bigcap_{x \in P} (\text{Int}(W) - x^*) \cap \bigcap_{x' \in P'} (W^c - x'^*).$$

$W_P := \overline{W_P^o}$  is called the *acceptance domain* of  $P$ . By construction, it is the closure of an open set and hence the closure of its interior.

**Proposition 3.2.** *Let  $P$  be a  $B$ -patch for a model set with window  $W$ . We assume that  $P$  contains the origin. Let  $M' \in \Omega(W)$  be a possibly different model set which contains the origin. If  $M'$  is non-singular then, for all  $x \in M$ ,  $\sigma_{M'}(x) \in \text{Int}(W_P)$  if and only if  $P = (M - x) \cap B$ .*

*Proof.* Let  $\xi = \pi_{\max}(M')$ . Assuming that  $M'$  is non-singular this means that  $M' = \mathcal{L}_\xi(W)$ . The condition  $M' \cap B = P$  has two parts:

- (1) All the points of  $P$  should be in  $M'$ . This is equivalent to having  $\sigma_\xi(x) \in W$  for all  $x \in P$ . But  $\sigma_\xi(x) - \sigma_\xi(0) = x^*$ , so this is in turn equivalent to  $\sigma_{M'}(0) \in W - x^*$ .
- (2) All the points of  $P'$  should not be in  $M'$ . That is, for each  $x' \in P'$ ,  $\sigma_\xi(x') \in W^c$ , so  $\sigma_{M'}(0) \in W^c - x'^*$ , where  $W^c$  denotes the complement of  $W$ .

Thus  $P \in M'$  if and only if

$$(2) \quad \sigma_{M'}(0) \in \bigcap_{x \in P} (W - x^*) \cap \bigcap_{x' \in P'} (W^c - x'^*).$$

Recall that  $\pi_{\max}(M') = [\sigma_{M'}(0)]_\Gamma$ . Therefore and since  $M$  was assumed non-singular,  $\sigma_{M'}(0)$  cannot lie on the boundary of  $W_P^o$ . Thus we can replace the right-hand side with the closed set  $W_P$ .  $\square$

**Lemma 3.3.** *Suppose that  $M$  is a model set containing the origin and satisfying H1 and H2. The acceptance domain of every (non-empty) patch of  $M$  containing the origin can be written as a finite union of closed convex sets that have non-empty interior.*

*Proof.* Since  $W$  is a finite union of connected polyhedra this is also the case for  $W_P$ . We can decompose  $W_P$  into a finite union of connected polyhedra and each of those into a finite union of convex polyhedra. Since  $W_P$  is the closure of its interior the convex polyhedra can be taken to have non-empty interior.  $\square$

We formulate the result of the above lemma as a hypothesis on the window  $W$  that is weaker than H2.

H2'. The acceptance domain of every patch containing the origin can be written as a finite union of closed convex sets that have non-empty interior.

**Lemma 3.4.** *Consider a cut  $\mathcal{E}$  project scheme  $(\Gamma, H, \mathbb{R}^d)$  and a compact subset  $W \subset H$  that is the closure of its interior. For any neighborhood  $U$  of  $0 \in H$  there exists a finite set  $J \subset W \cap \pi^\perp(\Gamma)$  such that  $\emptyset \neq \bigcap_{u \in J} W - u \subset U$ .*

*Proof.* Without loss of generality we can assume that  $U$  is contained in the compact set  $W - W$ , so that  $K = (W - W) \setminus U$  is compact. Since  $W \cap \pi^\perp(\Gamma)$  is countable, we can

find a sequence of nested finite sets  $J_1 \subset J_2 \subset \dots$  with  $\bigcup J_i = W \cap \pi^\perp(\Gamma)$ . If every intersection  $\bigcap_{u \in J_i} W - u$  contains a point  $x_i \in K$  then, by compactness, there is a limit point

$x_\infty \in K$  such that  $x_\infty \in \bigcap_{u \in \pi^\perp(\Gamma)} W - u$ . But Schlottmann proved [Sc][Lemma 4.1] that

$\bigcap_{u \in W \cap \pi^\perp(\Gamma)} W - u = \{0\}$ , which is a contradiction.  $\square$

**Corollary 3.5.** *Let  $M$  be a model set containing the origin. Any open subset of the window of  $M$  contains the acceptance domain for a patch of  $M$  that contains the origin.*

*Proof.* Let  $M$  be a model set with window  $W$  and let  $U \subset W$  be an open subset. Pick  $x \in M$  such that  $\sigma_M(x) \in U$ , and let  $U' = U - \sigma_M(x)$ , which is an open neighborhood of 0.

We apply Lemma 3.4 to obtain a finite set  $J \subset W \cap \pi^\perp(\Gamma)$  such that  $\emptyset \neq \bigcap_{u \in J} W - u \subset U'$ .

For each  $u$  pick  $q \in M$  such that  $\sigma_M(q) - \sigma_M(x) = \sigma(q - x) = u$ . Denoting by  $Q$  the set of such points  $q$  we have  $\emptyset \neq \bigcap_{q \in Q} W - \sigma(q) \subset U$ . Let  $B \subset \mathbb{R}^d$  be any compact neighborhood of the origin containing  $Q$ . The acceptance domain of  $P = B \cap M$  is then a subset of  $U$ .

Furthermore,  $P$  contains the origin.  $\square$

**3.6. Singular model sets.** We wish to describe the singular model sets, that is, the elements in the hull  $\Omega(W)$  that are not themselves projection sets with non-singular parameter.

**Proposition 3.6.** *Let  $M$  be a model set with window  $W$  and  $\xi = \pi_{max}(M)$ . Then*

$$\mathcal{A}_\xi(Int(W)) \subset M \subset \mathcal{A}_\xi(W).$$

*Proof.* A model set  $M$  with window  $W$  is a limit of a sequence  $(\mathcal{A}_\eta - x_n)_n$  where  $\eta$  is non-singular and  $x_n \in \mathbb{R}^d$ . But  $\mathcal{A}_\eta - x_n = \mathcal{A}_{\xi_n}$ , where  $\xi_n = \eta + [0, x_n]_\Gamma$ . By continuity of  $\pi_{max}$  we thus have  $\xi = \lim \xi_n$  and so the sequence  $(\xi_n)_n$  lifts to a sequence  $(\tilde{\xi}_n)_n \subset H \times \mathbb{R}^d$  which converges to a lift  $\tilde{\xi}$  of  $\xi$ . We may quickly restrict to the case that  $\tilde{\xi}^\parallel = 0$ , because otherwise we can replace  $x_n$  by  $x_n - \tilde{\xi}_n^\parallel$  and  $\eta$  by  $\eta - [0, \tilde{\xi}^\parallel]_\Gamma$  to reduce to that situation. Then  $M$  and all  $\mathcal{A}_{\xi_n}$  are subsets of  $\Gamma^\parallel$ . In particular,  $\Gamma^\parallel$  is the common domain of all functions  $\sigma_{\xi_n}$ , and the sequence of functions  $(\sigma_{\xi_n})_n$  converges uniformly to  $\sigma_\xi$ , since  $(\tilde{\xi}_n)_n$  converges to  $\tilde{\xi}$ . Hence if  $\sigma_\xi(x) \in Int(W)$  then an open neighborhood of  $\sigma_\xi(x)$  in  $Int(W)$  contains all  $\sigma_{\xi_n}(x)$  for  $n$  sufficiently large which shows that  $\sigma_{\xi_n}(x) \in W$  for all  $n$  sufficiently large. The latter means that  $x$  belongs to  $\mathcal{A}_{\xi_n}$  for all  $n$  sufficiently large and thus also to  $M$ . This shows that  $\mathcal{A}_\xi(Int(W)) \subset M$ .

To obtain the other inclusion we use the same kind of argument but for the complement  $W^c$  instead of  $Int(W)$ . Indeed, if  $\sigma_\xi(x) \in W^c$  then an open neighborhood of  $\sigma_\xi(x)$  in  $W^c$  contains all  $\sigma_{\xi_n}(x)$  for  $n$  sufficiently large which shows that  $x$  cannot belong to  $M$ .  $\square$

We can now generalize one direction of Proposition 3.2 to singular model sets.

**Corollary 3.7.** *Let  $P$  be a  $B$ -patch for a model set with window  $W$  and  $M' \in \Omega(W)$  a possibly singular model set. We assume that  $P$  and  $M'$  contain the origin. Let  $x \in M'$ . If  $\sigma_{M'}(x) \in \text{Int}(W_P)$  then  $P = (M' - x) \cap B$ . In particular  $\{\sigma_{M'}(x) : P = (M' - x) \cap B\}$  is dense in  $W_P$ .*

*Proof.* Set  $\xi = \pi_{\max}(M')$ . We see from the description of  $M'$  given in Theorem 3.8 that the projection set  $\mathcal{L}_\xi(\text{Int}(W_P))$  is contained in  $M'$ . Hence condition (2) also applies to a singular  $M'$ , as long as  $\sigma_{M'}(0)$  does not lie on the boundary of  $W_p$ . Hence if  $\sigma_{M'}(x) \in \text{Int}(W_P)$  then  $P = (M' - x) \cap B$ . Denseness of  $\{\sigma_{M'}(x) : P = (M' - x) \cap B\}$  follows directly from that fact that  $\{\sigma_{M'}(x) : x \in M'\}$  is dense in  $W$ .  $\square$

We now describe the potential difference between two singular model sets with the same parameter. For that we assume that the window  $W$  is polyhedral and decompose it as a polyhedral complex. Let  $\mathcal{F}(W)$  be the set of open faces of  $W$ . Thus  $W$  is the disjoint union of its interior  $\text{Int}(W)$  with the  $f \in \mathcal{F}(W)$  (where vertices are considered open 0-cells). Let  $V(f)$  be the vector space parallel to  $f$ , that is, the space spanned by  $f - f$ .

Let  $\tilde{H}_\xi(f)$  be the closure of  $V(f) \cap (\Gamma + \xi)^\perp$ .  $\tilde{H}_\xi(f)$  might be empty, but if it is not empty it is of the form  $\tilde{H}_\xi(f) = H_\xi(f) + \Delta_\xi(f)$  where  $H_\xi(f)$  is a real vector space and  $\Delta_\xi(f)$  a discrete subset of  $(\Gamma + \xi)^\perp$ . In that case  $\sigma^{-1}(H_\xi(f) \cap \Gamma^\perp)$  is a sublattice and we let  $E_\xi(f)$  be its real span. We define  $E_\xi(f)$  to be empty if  $\tilde{H}_\xi(f)$  is empty.

Furthermore, given that  $f$  is bounded, there is a finite subset  $\Phi_\xi(f) \subset \Delta_\xi(f)$  such that

$$(3) \quad f \cap (\Gamma + \xi)^\perp \subset \bigcup_{\eta \in \Phi_\xi(f)} \eta + H_\xi(f).$$

**Theorem 3.8.** *Let  $M$  be an arbitrary model set with polyhedral window  $W$  and let  $\xi = \pi_{\max}(M)$ . Then*

$$M \setminus \mathcal{L}_\xi(\text{Int}(W)) \subset \bigcup_{f \in \mathcal{F}(W)} \bigcup_{\eta \in \Phi_\xi(f)} E_\xi(f) + \sigma_\xi^{-1}(\eta).$$

*Proof.* By Proposition 3.6 we have

$$M \setminus \mathcal{L}_\xi(\text{Int}(W)) \subset \bigcup_{f \in \mathcal{F}(W)} \mathcal{L}_\xi(f).$$

If  $\mathcal{L}_\xi(f)$  is not empty then (3) implies that

$$\mathcal{L}_\xi(f) = \bigcup_{\eta \in \Phi_\xi(f)} \eta + \mathcal{L}_\xi(f_\eta)$$

where  $f_\eta = (f - \eta) \cap H_\xi(f)$  is an open subset of  $H_\xi(f)$ . Moreover,

$$\mathcal{L}_\xi(f_\eta) = \{\pi^\parallel(x) : x \in \Gamma, \pi^\perp(x) \in f_\eta\} \subset \sigma^{-1}(H_\xi(f) \cap \Gamma^\perp) \subset E_\xi(f).$$

$\square$

**Lemma 3.9.**  *$E_f$  has codimension at least 1.*

*Proof.* Let  $\Gamma_\xi(f) = \pi_\Gamma^{-1}(H_\xi(f) \cap \Gamma^\perp)$  and  $V$  a bounded open subset of  $H_\xi(f)$ . Then we can rewrite  $\mathcal{L}_\xi(V) = \{\pi^\parallel(x) : x \in \Gamma_\xi(f), \pi^\perp(x) \in V\}$  and so we see that  $\mathcal{L}_\xi(V)$  is also the projection set with window  $V$  and parameter 0 for the cut & project scheme  $(\Gamma_\xi(f), H_\xi(f), E_\xi(f))$ . Since  $V$  is open  $\mathcal{L}_\xi(V)$  is relatively dense in  $E_\xi(f)$  [BG] and thus has strictly positive lower density in  $E_\xi(f)$ .

Since  $\pi^\perp(\Gamma)$  is dense in  $H$  we can find a subset  $V \subset f_\eta$ , open in the topology of  $H_\xi(f)$ , and an infinite subset  $\Psi \subset \pi^\perp(\Gamma)$  such that  $V + \psi \subset W$  for all  $\psi \in \Psi$  and such that the sets  $V + \psi$  have pairwise empty intersection. Hence  $\mathcal{L}_\xi(W)$  contains the disjoint union of all  $\mathcal{L}_\xi(V + \psi)$ ,  $\psi \in \Psi$ . Now the lower density of  $\mathcal{L}_\xi(V + \psi)$  in  $E_f$  is independent of  $\psi$ . Therefore, if  $E_f$  has dimension  $d$  and hence is all of  $\mathbb{R}^d$  then the lower density of  $\mathcal{L}_\xi(W)$  must be infinite, which is a contradiction.  $\square$

**Corollary 3.10.** *Consider two singular elements  $M_1, M_2$  of the hull of a model set. If  $\pi_{max}(M_1) = \pi_{max}(M_2) = \xi$  then the difference set  $M_1 \Delta M_2$  is contained in a finite affine hyperplane arrangement which we denote  $\mathcal{A}(M_1, M_2)$ .*

*Proof.* By Theorem 3.8 the symmetric difference  $M_1 \Delta M_2$  is contained in  $\bigcup_{f \in \mathcal{F}(W)} \bigcup_{\eta \in \Phi_\xi(f)} E_\xi(f) + \sigma_\xi^{-1}(\eta)$ . By Lemma 3.9 the sets  $E_\xi(f) + \sigma_\xi^{-1}(\eta)$  are proper affine hyperplanes. Indeed, if  $\sigma_\xi^{-1}(\eta)$  is not finite that it contains a period which must be contained in  $E_\xi(W)$ . There are only finitely many affine hyperplanes, because  $\mathcal{F}(W)$  and  $\Phi_\xi(f)$  are finite.  $\square$

The affine hyperplanes making up  $\mathcal{A}(M_1, M_2)$  depend on the pair  $M_1, M_2$ . However, their number is uniformly bounded:

**Lemma 3.11.** *Given the hull  $\Omega(W)$  of a model set. There is a finite number  $N$  such that for all  $M_1, M_2 \in \Omega(W)$  with  $\pi_{max}(M_1) = \pi_{max}(M_2)$  the number of hyperplanes in the arrangement  $\mathcal{A}(M_1, M_2)$  is bounded by  $N$ .*

*Proof.* There is a finite number of faces and each face  $f$  gives rise to a set  $\Phi_\xi(f)$  which can vary with  $\xi$ , but the number of its elements is bounded from above since  $f$  is bounded and  $\Delta_f$  discrete.  $\square$

**Proposition 3.12.** *Given the hull  $\Omega(W)$  of a model set. Given  $r > 0$  there exists a  $\rho > 0$  such that for all  $M_1, M_2 \in \Omega(W)$  which satisfy  $\pi_{max}(M_1) = \pi_{max}(M_2)$  and all  $x \in \mathbb{R}^d$  the ball  $B_\rho(x)$  contains at least one point at which  $M_1$  and  $M_2$  agree out to distance  $r$ .*

*Proof.* Let  $\xi = \pi_{max}(M_1) = \pi_{max}(M_2)$ . Pick a patch  $(P, B)$  where  $B$  contains a ball of radius  $r$ . Since the interior of the acceptance domain  $W_P$  is in the interior of  $W$ ,  $\mathcal{L}_\xi(W_P^0)$  is a subset of both  $M_1$  and  $M_2$ .

This is a relatively dense subset of  $M$ , so there exists a  $\rho$  such that for all  $x \in \mathbb{R}^d$  the ball  $B_\rho(x)$  contains at least one point of  $\mathcal{L}_\xi(Int(W_P))$ , and around this point  $M_1$  and  $M_2$  agree out to distance  $r$ .  $\square$

**3.7. Pattern equivariant functions.** Let  $\Lambda \subset \mathbb{R}^d$  be a FLC-Delone set and  $Y$  some set. A function  $f : \mathbb{R}^d \rightarrow Y$  is called *strongly pattern equivariant* if there exists an  $R > 0$  (called the radius) such that, whenever  $x_1, x_2 \in \mathbb{R}^d$  are such that  $\Lambda - x_1$  and  $\Lambda - x_2$  agree exactly on the ball  $B_R(0)$ , then  $f(x_1) = f(x_2)$ . In other words, each function value  $f(x)$  is determined exactly by the pattern of  $\Lambda$  in a ball of radius  $R$  around  $x$ . For most of our purposes we need functions only defined on  $\Lambda$ , or the CW-complex it defines. So we call a function  $\phi : \Lambda \rightarrow Y$  strongly pattern equivariant if there exists an  $R > 0$  such that, whenever  $x_1, x_2 \in \Lambda$  are such that  $\Lambda - x_1$  and  $\Lambda - x_2$  agree exactly on the ball  $B_R(0)$ , then  $\phi(x_1) = \phi(x_2)$ . It can be shown that if  $Y$  is a finite dimensional real vector space then any strongly pattern equivariant function on  $\Lambda$  is the restriction of a smooth strongly pattern equivariant function on  $\mathbb{R}^d$  [K2].

Any locally constant function  $\tilde{f} : \Xi_\Lambda \rightarrow Y$  defines a strongly pattern equivariant function on  $f : \Lambda \rightarrow Y$  via  $f(x) = \tilde{f}(\Lambda - x)$ ; this defines a bijective correspondence.

Now let  $Y$  be a topological space. Any continuous function  $\tilde{f} : \Omega_\Lambda \rightarrow Y$  is uniquely determined by the function  $f : \mathbb{R}^d \rightarrow Y$ ,  $f(x) = \tilde{f}(\Lambda - x)$ . We call a function  $f : \mathbb{R}^d \rightarrow Y$  arising in such a way from a continuous function  $\tilde{f} : \Omega_\Lambda \rightarrow Y$  *weakly pattern equivariant*. Likewise  $\phi : \Lambda \rightarrow Y$  is *weakly pattern equivariant* if it arises in the above way from a corresponding function  $\tilde{\phi} : \Xi_\Lambda \rightarrow Y$ . Equivalently we may say that  $\phi : \Lambda \rightarrow Y$  is weakly pattern equivariant for  $\Lambda$  iff the function  $\{\Lambda - x : x \in \Lambda\} \ni (\Lambda - x) \mapsto \phi(x)$  is uniformly continuous in the topology of  $\Xi_\Lambda$ . It is not difficult to see that if  $Y = \mathbb{R}^k$  is a finite dimensional vector space then  $\phi$  is weakly pattern equivariant if and only if it is the uniform limit of strongly pattern equivariant functions.

The following is a very important example. A direct proof not using the maximal equicontinuous factor map can be obtained from Corollary 3.5, see also [BL].

**Lemma 3.13.** *The star map  $\sigma_M : M \rightarrow H$  of a model set is weakly pattern equivariant.*

*Proof.* Recall that the image of  $\sigma_M$  lies in  $W$ . We can hence rewrite (1) as  $\sigma_M = \iota_H^{-1}|_W \circ \pi_{max}(M - \cdot)$ . Since  $W \subset H$  is compact  $\iota_H^{-1}|_W : [W, 0]_\Gamma \rightarrow H$  is uniformly continuous. Since also  $\pi_{max}$  is uniformly continuous  $\sigma_M$  is weakly pattern equivariant.  $\square$

A point pattern  $\Lambda$  can always be realized as the vertex set of a polygonal tiling of  $\mathbb{R}^d$ . Such a tiling gives a cell decomposition of  $\mathbb{R}^d$ . Functions on  $\Lambda$  with values in an abelian group  $A$  can then be viewed as 0-cochains on this CW complex. Likewise, we can consider 1-cochains, 2-cochains, etc. These cochains are considered strongly pattern equivariant if their values on a cell are determined exactly by the pattern of  $\Lambda$  in some fixed finite radius around that cell, and weakly pattern equivariant if they can be uniformly approximated by strongly pattern equivariant cochains.

The coboundary of a strongly pattern equivariant cochain is strongly pattern equivariant. The cohomology of the complex of strongly pattern equivariant cochains with values in  $A$  is

called the  $A$ -valued cohomology of  $\Lambda$  and denoted  $H^*(\Lambda, A)$ . It is naturally isomorphic to the Čech cohomology  $\check{H}^*(\Omega_\Lambda, A)$  [S1].

There is an equivalent description of the cohomology of  $\Lambda$  provided that  $A = \mathbb{R}$  (or  $\mathbb{R}^k$ ). We can consider de Rham forms on  $\mathbb{R}^d$  which are strongly pattern equivariant for  $\Lambda$ . These form a sub complex of the usual de Rham complex for  $\mathbb{R}^d$  and  $H^*(\Lambda, \mathbb{R})$  can be seen as the cohomology of this sub complex [K1].

#### 4. SHAPE CONJUGACIES AND NONSLIP GENERATORS

A shape conjugation is a particular shape deformation in the sense of [CS, K2], and shape deformations arise as follows: Consider a function  $F : \Lambda \rightarrow \mathbb{R}^d$ , defined on an FLC Delone set  $\Lambda$ . It defines a new set

$$\Lambda^F := \{x + F(x) : x \in \Lambda\}.$$

We assume that the coboundary  $\delta F$  of  $F$  is strongly pattern equivariant. This means that the elements of  $\Lambda^F - \Lambda^F$  can be locally derived from  $\Lambda - \Lambda$  and implies in particular that  $\Lambda^F$  has FLC. We think of  $\Lambda^F$  as a deformation of  $\Lambda$  (as if we had turned on  $F$  slowly) and call the function  $F$  the *generator* of the deformation.

One possible way for  $\delta F$  to be strongly pattern equivariant is for  $F$  already to be strongly pattern equivariant. This is the case precisely if  $\Lambda^F$  can be locally derived from  $\Lambda$ .

A deformation is a *shape semi-conjugacy* if the map  $\Lambda - x \mapsto \Lambda^F - x$  extends from the orbit of  $\Lambda$  in  $\Omega_\Lambda$  to a topological semi-conjugacy  $\mathfrak{s}_F : \Omega_\Lambda \rightarrow \Omega_{\Lambda^F}$ .

Our ultimate aim is to understand the extent to which the dynamical system of a Delone set determines the Delone set. More specifically, given an FLC-Delone set  $\Lambda$ , we ask which FLC-Delone sets are topologically equivalent to it? To investigate this question we recall the following theorem.

**Theorem 4.1** ([KS], Theorem 5.1). *Let  $\Lambda$  and  $\Lambda'$  be FLC Delone sets that are pointed topologically conjugate. For each  $\epsilon > 0$  there exists a FLC Delone set  $\Lambda_\epsilon$  that is MLD with  $\Lambda$  and a function  $F_\epsilon : \Lambda_\epsilon \rightarrow \mathbb{R}^d$  whose coboundary is strongly pattern equivariant such that  $\Lambda' = \Lambda^{F_\epsilon}$  and  $\mathfrak{s}_{F_\epsilon} : \Omega_{\Lambda_\epsilon} \rightarrow \Omega_{\Lambda'}$  is a topological conjugacy; in other words  $F_\epsilon$  is a generator of a shape conjugation mapping  $\Lambda_\epsilon$  to  $\Lambda'$ . Moreover  $\sup_x |F_\epsilon(x)| \leq \epsilon$ .*

**Remark 2.** The term “deformation of a model set” has been used in the literature (see [BL, BD]) also for other kinds of deformations for which, in particular, the deformed set is no longer necessarily FLC. These could be achieved by functions  $F$  whose co-boundaries are not strongly pattern equivariant. Without the FLC requirement there are many more possible deformations and our rigidity results do not apply.

**4.1. Auto-conjugacy.** Theorem 4.1 is about pointed topological conjugacy. To understand what it means for the two patterns to be merely topologically conjugated we need to understand when, given two elements  $\Lambda_1, \Lambda_2$  of the same hull, the map  $\Lambda_1 \rightarrow \Lambda_2$  extends to a topological conjugacy. This is then an auto-conjugacy.

A first observation to make is that an auto-conjugacy  $\varphi$  must preserve the equivalence relation given by  $\pi_{max}$ , i.e. if  $\pi_{max}(\Lambda_1) = \pi_{max}(\Lambda_2)$  then  $\pi_{max}(\varphi(\Lambda_1)) = \pi_{max}(\varphi(\Lambda_2))$ . Furthermore, the map induced by  $\varphi$  on  $\Omega_{max}$  must be the rotation by  $\eta := \pi_{max}(\Lambda_2) - \pi_{max}(\Lambda_1)$ , since this is a homeomorphism (and hence the only one) on  $\Omega_{max}$  mapping  $\pi_{max}(\Lambda_1) - x$  to  $\pi_{max}(\Lambda_2) - x$  for all  $x \in \mathbb{R}^d$ . It follows then that for model sets the rotation by  $\eta$  must leave the set  $NS$  of nonsingular points invariant. But for a generic choice of the window the only translations leaving  $NS$  invariant are the elements of  $(\{0\} \times \mathbb{R}^d + \Gamma)/\Gamma$ . It follows that on the orbit of a non-singular model set  $\varphi$  is given by a global translation. By continuity  $\varphi$  must then be everywhere this global translation. Thus for generic model sets auto-conjugacies are global translations and hence in particular MLD transformations. The more general case in which  $NS$  admits symmetries is presently under investigation.

**4.2. Asymptotically negligible co-chains.** It turns out that the map  $\Lambda - x \mapsto \Lambda^F - x$  extends to a topological semi-conjugacy  $\mathfrak{s}_F : \Omega_\Lambda \rightarrow \Omega_{\Lambda^F}$  if and only if  $F$  is weakly pattern equivariant [CS, K2]. So the generator of a shape semi-conjugacy is a weakly pattern equivariant function  $F : \Lambda \rightarrow \mathbb{R}^d$  whose coboundary is strongly pattern equivariant. If  $F$  were strongly pattern equivariant, then the shape deformation would be a local derivation. If  $F$  is small enough, then this procedure can be inverted [K2], so that  $\mathfrak{s}_F$  is a shape *conjugacy*.

Shape semi-conjugacies (and small shape conjugacies), up to MLD shape conjugacies, are thus parametrized by the sub-group of the first cohomology group consisting of strongly pattern equivariant co-cycles which are coboundaries of weakly pattern equivariant functions. Such co-cycles are called *asymptotically negligible* and we denote the subgroup by  $H_{an}^1(\Lambda, \mathbb{R}^d)$ .

**Proposition 4.2.** *The coboundary of the star map  $\sigma_M$  of a model set is an asymptotically negligible 1-cocycle.*

**Remark 3.** The above proposition can be understood as a generalization of a result from Boulmezaoud's thesis [B].<sup>1</sup>

*Proof.* By Lemma 3.13  $\sigma_M : M \rightarrow H$  is weakly pattern equivariant. That its coboundary is strongly pattern equivariant is a direct consequence of its additivity. In fact, if  $e$  is an edge between points  $x_1, x_2$  in  $M$  then  $\delta\sigma_M(e) = \sigma_M(\partial e) = \sigma_M(x_2) - \sigma_M(x_1) = (x_2 - x_1)^*$ , and so depends only on the displacement between  $x_1$  and  $x_2$ .  $\delta\sigma_M(e)$  is thus strongly pattern equivariant for any radius  $R$  greater than the maximal distance between two neighboring points.  $\square$

<sup>1</sup>Boulmezaoud showed that in the case that the internal group is  $\mathbb{R}^n$  the star map  $\mathbb{R}^d \rightarrow \mathbb{R}^n$ ,  $x \mapsto x^*$  extends to a weakly pattern equivariant smooth function whose differential is strongly pattern equivariant.

**Corollary 4.3.** *Consider a model set  $M$  with internal group  $H$ . Let  $L : H \rightarrow \mathbb{R}^d$  be a continuous group homomorphism. Then  $F_L : M \rightarrow \mathbb{R}^d$ ,  $F_L(x) = L(\sigma_M(x))$  is a generator of a shape conjugation. The model set  $M^{F_L}$  resulting from the shape change is the reprojection along  $H' = \{(h, -L(h)) \in H \times \mathbb{R}^d : h \in H\}$ .*

*Proof.* Continuity of  $L$  implies that  $F_L$  is weakly pattern equivariant. Additivity of  $L$  implies that  $F_L$  has strongly pattern equivariant coboundary. Finally, we have  $M^{F_L} = \{x + L(\sigma_M(x)) : x \in M\} = \{(\pi^\parallel + L \circ \pi^\perp)(\sigma_M(x), x) : x \in M\}$ . Now the elements in the kernel of  $\pi^\parallel + L \circ \pi^\perp$  have the form  $(h, -L(h))$ ,  $h \in H$  and so  $\pi^\parallel + L \circ \pi^\perp$  is the projection onto  $\mathbb{R}^d$  along the subspace  $H' = \{(h, -L(h)) \in H \times \mathbb{R}^d : h \in H\}$ .  $\square$

**4.3. Nonslip generators.** We will introduce a property for generators of shape conjugations which characterizes those we have discussed above in the context of model sets. We state the definition for arbitrary weakly pattern equivariant functions, but the primary application is for vector-valued functions. Recall that  $F$  being weakly pattern equivariant means that there exists a continuous function  $\tilde{F} : \Xi_\Lambda \rightarrow \mathbb{R}^d$  such that  $F(x) = \tilde{F}(\Lambda - x)$ . We denote  $\mathcal{R}_{max}^\Xi = \mathcal{R}_{max} \cap \Xi \times \Xi$ .

**Definition 4.4.** *Let  $\Lambda$  be a Meyer set. A weakly pattern equivariant function  $F : \Lambda \rightarrow Y$  is nonslip if there exists an  $\epsilon > 0$  such that, for all  $(\Lambda_1, \Lambda_2) \in \mathcal{R}_{max}^\Xi$  we have  $\tilde{F}(\Lambda_1) = \tilde{F}(\Lambda_2)$  whenever  $\Lambda_1$  and  $\Lambda_2$  agree out to a radius of  $\epsilon^{-1}$ . We call  $R = \epsilon^{-1}$  the nonslip radius of  $F$ .*

A strongly pattern equivariant function is manifestly nonslip. The star map is not strongly pattern equivariant, but we shall see below that it is nonslip.

**Lemma 4.5.** *The star map of  $M$  is nonslip.*

*Proof.* Let  $M_1, M_2 \in \Omega_M$ . If  $\pi_{max}(M_1) = \pi_{max}(M_2) = \xi$  and  $x \in M_1 \cap M_2$ , then equation (1) implies that  $\sigma_{M_1}(x) = \sigma_{M_2}(x)$ . Since  $\tilde{\sigma}_M(M_i - x) = \sigma_{M_i}(x)$  for  $i = 1, 2$  the star map is non-slip for every positive radius.  $\square$

**Corollary 4.6.** *Every generator of a shape conjugation of the form  $F_L$  is nonslip.*

**4.4. Nonslip = reprojection.** We have just seen that continuous linear functions  $L : H \rightarrow \mathbb{R}^d$  define nonslip generators of shape conjugations, and hence that reprojections are generated by nonslip generators. We next show the converse, that under hypotheses H1 and H2', all nonslip generators are essentially of this form. The following theorem is the main result of this section, and proves half of Theorem 1.1.

**Theorem 4.7.** *Let  $M$  be a model set satisfying hypotheses H1 and H2'. Let  $F : M \rightarrow \mathbb{R}$  be a weakly pattern equivariant function that is nonslip, and whose coboundary  $\delta F$  is strongly pattern equivariant. Then  $F$  can be written as  $F(x) = L(\sigma_M(x)) + \psi(x)$ , where  $L : H \rightarrow \mathbb{R}$  is a continuous linear map and  $\psi : M \rightarrow \mathbb{R}$  is strongly pattern equivariant.*

*Proof.* Note that since all elements of  $C$  have finite order,  $\{0\} \times C$  must lie in the kernel of  $L$  and  $L$  is determined on  $\mathbb{R}^n$  alone.

To view the nonslip property from a different angle we consider an inverse limit construction for the canonical transversal  $\Xi$ . Let  $M_1 \sim_\epsilon M_2$  if  $\pi_{max}(M_1) = \pi_{max}(M_2)$  and the patterns  $M_1, M_2$  agree out to radius  $\epsilon^{-1}$ . This is the intersection of two closed equivalence relations and therefore itself a closed equivalence relation; the quotient space  $\Xi / \sim_\epsilon$  is a compact Hausdorff space. Denote by  $\pi_\epsilon : \Xi \rightarrow \Xi / \sim_\epsilon$  the canonical projection. Then

$$\Xi = \lim_{0 \leftarrow \epsilon} \Xi / \sim_\epsilon$$

and a weakly pattern equivariant  $F$  is nonslip iff  $\tilde{F}$  is the pullback by  $\pi_\epsilon$  of a continuous function on some approximant  $\Xi / \sim_\epsilon$ .

Let  $F : M \rightarrow \mathbb{R}$  be a nonslip, weakly pattern equivariant function, whose coboundary  $\delta F$  is strongly pattern equivariant. Since the star map  $\sigma_M$  is also nonslip, there exists an  $\epsilon > 0$  such that we have the commutative diagram

$$\begin{array}{ccccc} \Xi & \xrightarrow{\pi_\epsilon} & \Xi / \sim_\epsilon & \xrightarrow{\widetilde{\sigma}_M^\epsilon} & H \\ \tilde{F} \downarrow & & \tilde{F}^\epsilon \downarrow & & \\ \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} \end{array}$$

where  $\tilde{F}^\epsilon$  and  $\widetilde{\sigma}_M^\epsilon$  are the pullbacks of the continuous maps  $\tilde{F}$  and  $\widetilde{\sigma}_M$  induced by  $F$  and  $\sigma_M$ . We need to show that there exists a  $0 < \eta \leq \epsilon$ , a strongly pattern equivariant function  $\psi$  and a continuous group homomorphism  $L$  such that the right side of the diagram can be completed to a commutative diagram

$$\begin{array}{ccc} \Xi / \sim_\eta & \xrightarrow{\widetilde{\sigma}_M^\eta} & H \\ \tilde{F}^\eta - \tilde{\psi}^\eta \downarrow & & \downarrow L \\ \mathbb{R} & = & \mathbb{R} \end{array}$$

( $\eta^{-1}$  is hence at least as large as the strongly pattern equivariant radius of  $\psi$  and the nonslip radius of  $F$ ). Given that a global translation of a point set is an MLD transformation which may be absorbed in the definition of  $\psi$  we may assume that  $M$  contains the origin.

By FLC there are a finite number of possible  $\epsilon^{-1}$ -patches at 0 which contain the origin. By H2' the acceptance domain of each  $\epsilon^{-1}$ -patch can be written as a finite union of closed convex sets that have non-empty interior. We call these convex sets *sectors* and let  $I$  index the sectors of all  $\epsilon^{-1}$ -patches. Thus  $\alpha \in I$  denotes both a patch (located for reference at the origin) and a closed convex subset  $W^\alpha \subset W$  of non-empty interior. For each such  $\alpha$ , let  $\Xi^\alpha$  be the corresponding subset of  $\Xi$ , i.e. the set of point patterns that (a) contain the origin, (b) have the correct  $\epsilon^{-1}$  patch around the origin, and (c) are mapped to sector  $W^\alpha$  by  $\widetilde{\sigma}_M$ .

The crucial observation is that on  $\Xi^\alpha$  the equivalence relation  $\sim_\epsilon$  coincides with the relation defined by  $\pi_{max}$ , and hence for  $M_1, M_2 \in \Xi^\alpha$  we have  $M_1 \sim_\epsilon M_2$  iff  $\widetilde{\sigma}_M(M_1) = \widetilde{\sigma}_M(M_2)$ . In other words, the restriction of  $\widetilde{\sigma}_M^\epsilon$  to  $\Xi^\alpha / \sim_\epsilon$  is a homeomorphism between  $\Xi^\alpha / \sim_\epsilon$  and  $W^\alpha$ .

It follows that for each  $\alpha$  there is a unique function  $f^\alpha : W^\alpha \rightarrow \mathbb{R}$  such that

$$(4) \quad \begin{array}{ccccc} \Xi^\alpha & \xrightarrow{\pi_\xi} & \Xi^\alpha / \sim_\epsilon & \xrightarrow{\widetilde{\sigma}_M^\epsilon} & W^\alpha \\ \tilde{F} \downarrow & & \tilde{F}^\epsilon \downarrow & & \downarrow f^\alpha \\ \mathbb{R} & = & \mathbb{R} & = & \mathbb{R} \end{array}$$

commutes.

Let  $y \in D = \sigma^{-1}((W - W) \cap \Gamma^\perp)$  be a possible displacement between two points in the same pattern. Define

$$\Delta_y \tilde{F} : \Xi \cap (\Xi + y) \rightarrow \mathbb{R}$$

by

$$\Delta_y \tilde{F}(M') = \tilde{F}(M' - y) - \tilde{F}(M').$$

We consider also the restriction  $\Delta_y^{\alpha, \alpha'} \tilde{F}$  of  $\Delta_y \tilde{F}$  to  $\Xi^\alpha \cap (\Xi^{\alpha'} + y)$ . Taking into account the fact that  $\widetilde{\sigma}_M(M' - y) - \widetilde{\sigma}_M(M') = \sigma_{M'}(y) - \sigma_{M'}(0) = y^*$ , the preceding paragraph shows that

$$\Delta_y^{\alpha, \alpha'} \tilde{F}(M') = \Delta_{y^*}^{\alpha, \alpha'} f(\widetilde{\sigma}_M(M'))$$

where  $\Delta_v^{\alpha, \alpha'} f(u) = f^{\alpha'}(u + v) - f^\alpha(u)$ .

Since  $F$  has strongly pattern equivariant coboundary, and since strongly pattern equivariant functions are locally constant on  $\Xi$ ,  $\Delta_y \tilde{F}$  is locally constant on  $\Xi \cap (\Xi + y)$ . Hence by (4) the function  $\Delta_{y^*}^{\alpha, \alpha'} f$  is locally constant on  $W^\alpha \cap (W^{\alpha'} - y^*)$ . Since  $W^\alpha$  and  $W^{\alpha'}$  are convex,  $W^\alpha \cap (W^{\alpha'} - y^*)$  is connected or empty. Hence the function  $\Delta_{y^*}^{\alpha, \alpha'} f$  is actually constant on  $W^\alpha \cap (W^{\alpha'} - y^*)$ .

There are finitely many sectors and each sector has non-empty interior. So there is an open neighborhood  $U \subset H$  of 0 such that for all  $\alpha$  and all  $v \in U$  we have  $W^\alpha \cap (W^\alpha - v) \neq \emptyset$ . We claim that for  $v \in U$  the value of  $\Delta_v^{\alpha, \alpha} f(u) = f^\alpha(u + v) - f^\alpha(u)$  is independent of  $\alpha$  as well as independent of  $u \in W^\alpha \cap (W^\alpha - v)$  (we know already that it is independent of  $u$  if  $v = y^*$  with  $y \in D$ ).

The sectors have non-empty interior, so by Corollary 3.5 for each sector  $\alpha$  we can find a patch  $P^\alpha$  of  $M$  whose acceptance domain is contained in the interior of  $W^\alpha$ . By repetitivity, there is a radius such that every ball of that radius contains at least one copy of each patch  $P^\alpha$ . Let  $P$  be a patch of  $M$  of that radius, so that  $P$  contains translates of all the patches  $P^\alpha$ . That is, there are  $x^\alpha \in \mathbb{R}^d$  such that  $P^\alpha + x^\alpha$  is a subpatch of  $P$ . It follows that  $\widetilde{\sigma}_M(M - x^\alpha)$  lies in the interior of  $W^\alpha$ . By Corollary 3.7 the set  $\{\sigma_M(x) : P = (M - x) \cap B\}$  is dense in the acceptance domain  $W_P$ . If  $P = (M - x_1) \cap B = (M - x_2) \cap B$  we call  $x_2 - x_1$  a *return vector* of  $P$ . The possible values of  $y^*$  for return vectors  $y$  of  $P$  are thus dense in  $W_P - W_P$ .

Pick a  $y$  such  $y^*$  is a return vector of  $P$ . We then have

$$\Delta_y \tilde{F}(M - x^\alpha) - \Delta_y \tilde{F}(M - x^{\alpha'}) = \tilde{F}(M - x^\alpha) - \tilde{F}(M - x^{\alpha'}) - (\tilde{F}(M - y - x^\alpha) - \tilde{F}(M - y - x^{\alpha'})).$$

Now  $\tilde{F}(M - x^\alpha) - \tilde{F}(M - x^{\alpha'})$  is obtained by adding the  $\delta F(e_i)$  over the edges  $e_i$  along a path in  $P$  which joins  $x^\alpha$  to  $x^{\alpha'}$ , while  $\tilde{F}(M - y - x^\alpha) - \tilde{F}(M - y - x^{\alpha'})$  is obtained by

summing the values of  $\delta F(e_i)$  over the corresponding path in  $P + y$ . Since  $y$  is a return vector to  $P$  and  $\delta F$  is strongly pattern equivariant, the result is the same. Hence

$$\Delta_{y^*}^{\alpha, \alpha} f(\widetilde{\sigma}_M(M - x^\alpha)) - \Delta_{y^*}^{\alpha', \alpha'} f(\widetilde{\sigma}_M(M - x^{\alpha'})) = \Delta_y \widetilde{F}(M - x^\alpha) - \Delta_y \widetilde{F}(M - x^{\alpha'}) = 0.$$

To summarize, we have established that for all  $y^* \in (W_P - W_P) \cap \Gamma^\perp$  the value of  $\Delta_{y^*}^{\alpha, \alpha} f(u)$  is the same for all  $\alpha$  and all  $u \in W^\alpha \cap (W^\alpha - y^*)$ . Moreover, for fixed  $u$  in the interior of  $W^\alpha \cap (W^\alpha - y^*)$ , the function  $v \mapsto \Delta_v^{\alpha, \alpha} f(u)$  is continuous in a neighborhood of  $y^*$ . It follows that  $\Delta_v^{\alpha, \alpha} f(u)$  is independent of  $\alpha$  and  $u \in W^\alpha \cap (W^\alpha - v)$  for all  $v \in W_P - W_P$ .

Let  $\tilde{U} = (W_P^o - W_P^o) \cap U$ .  $\tilde{U}$  is an open neighborhood of the identity in  $H$ . We define  $L : \tilde{U} \rightarrow \mathbb{R}$  such that  $L(v)$  is the constant value that the function  $\Delta_v^{\alpha, \alpha} f$  takes on  $W^\alpha \cap (W^\alpha - v)$ . We saw that  $L$  is continuous. We claim that  $L$  is additive where sums are defined. Indeed, if  $u_1, u_2, u_1 + u_2 \in U$  then there is  $u \in W^\alpha$  such that also  $u + u_1$  and  $u + u_1 + u_2$  lie in  $W^\alpha$ . It follows that, for all  $u \in \tilde{U}$

$$L(u_1 + u_2) = f^\alpha(u + u_1 + u_2) - f^\alpha(u + u_1) + f^\alpha(u + u_1) - f^\alpha(u) = L(u_2) + L(u_1).$$

A continuous additive function on a neighborhood of the origin is necessarily linear. That is,  $L$  equals its derivative. We may then extend  $L$  to a group homomorphism on the group generated by  $\tilde{U}$  and thus obtain a linear function  $L : H \rightarrow \mathbb{R}$  that is trivial on the torsion factor.

Now let  $\psi(x) = F(x) - L(\sigma_M(x))$  for  $x \in M$ . This then defines a function  $\tilde{\psi} : \Xi \rightarrow \mathbb{R}$ ,  $\tilde{\psi} = \tilde{F} - L \circ \widetilde{\sigma}_M$  which is again continuous (on  $\Xi$ ). If  $\widetilde{\sigma}_M(M - x) = \widetilde{\sigma}_M(M - y) \in W^\alpha$  then

$$\psi(y) - \psi(x) = \Delta_{y-x}^\alpha \tilde{F}(M - x) - L((y - x)^*) = 0$$

by the construction above and so  $\tilde{\psi}$  is a continuous function which is constant on  $\Xi^\alpha$ . Moreover, if  $W^\alpha \cap W^\beta \neq \emptyset$  then continuity implies that  $\tilde{\psi}$  takes the same value on  $\Xi^\alpha$  and  $\Xi^\beta$ . It follows that  $\tilde{\psi}$  is constant on the pre-images under  $\widetilde{\sigma}_M$  of the connected components of the acceptance domains of the  $\epsilon^{-1}$ -patches at 0. However, different components of the same central  $\epsilon^{-1}$  patch are separated by a nonzero distance in  $W$ , and so can be distinguished by the  $R$ -patches at 0 for some (possibly large) fixed  $R > 0$ . Different central patches are distinguished by their patterns out to distance  $\epsilon^{-1}$ . Thus  $\psi(x)$  is in fact strongly pattern equivariant with radius  $\eta^{-1}$  where  $\eta = \min(R^{-1}, \epsilon)$ .  $\square$

**Corollary 4.8.** *If  $F$  is a nonslip generator of a shape conjugation for a model set  $M$  satisfying  $H1$  and  $H2'$  then, up to MLD transformations,  $M^F$  is a reprojection of  $M$ . In particular  $M^F$  is a model set.*

*Proof.* Theorem 4.7 applied to vector valued functions and Corollary 4.3 imply that  $F$  is the generator of a reprojection plus a strongly pattern equivariant function. Hence, up to an MLD transformation,  $M^F$  is a reprojection of  $M$ , and is a model set.  $\square$

## 5. ASYMPTOTICALLY NEGLIGIBLE = NONSLIP

We now turn to the question of when an asymptotically negligible cocycle is nonslip. Here we need the stronger assumption H2.

**Theorem 5.1.** *Let  $M$  be a model set satisfying assumptions H1 and H2, and let  $F : M \rightarrow \mathbb{R}$  be a weakly pattern equivariant function whose coboundary is strongly pattern equivariant. Then  $F$  is nonslip.*

*Proof.* Let  $F : M \rightarrow \mathbb{R}$  be weakly pattern equivariant with strongly pattern equivariant  $\delta F$ . We need to show that there exists  $R > 0$  such that, for any choice of pair  $M_1, M_2$  in the hull of  $M$  and point  $x \in M_1$  such that  $\pi_{max}(M_1) = \pi_{max}(M_2)$  and  $B_R \cap (M_1 - x) = B_R \cap (M_2 - x)$  then  $F_2(x) - F_1(x) = 0$ .

We denote by  $R_0$  the radius of pattern equivariance of  $\delta F$ . Fix  $R > R_0$  and consider *doubly pointed double  $R$ -patches*. These are double- $R$ -patches  $(P, Q; B_R(z))$  of  $(M_1, M_2)$  which are centered in a point  $z \in M_1 \cup M_2$ , i.e.  $P = M_1 \cap B_R(z)$  and  $Q = M_2 \cap B_R(z)$ , together with two points  $x, y \in P \cap Q$  which are at least distance  $R_0$  away from  $z$ .

We denote such an object by  $P^{(2)}(x, y) = (x, y; P, Q; B_R(z))$ . By FLC there are finitely many up to translation. Since  $\delta F$  is strongly pattern equivariant with radius  $R_0$  the expression

$$F(P^{(2)}(x, y)) = \delta F_2(x, y) - \delta F_1(x, y) = F_2(y) - F_1(y) - (F_2(x) - F_1(x))$$

depends only on the translational congruence class of  $P^{(2)}(x, y)$ . Hence the set  $D_R$  of possible values  $F$  can take on doubly pointed double  $R$ -patches is finite.

We now need the following lemma.

**Lemma 5.2.** *There exists  $\tilde{N}$  and  $R_1 > 0$  such that for all pairs  $(M_1, M_2)$  with  $\pi_{max}(M_1) = \pi_{max}(M_2)$  and all  $x \in M_1 \cap M_2$  we have  $F_2(x) - F_1(x) \in \tilde{D}_{R_1} := D_{R_1} + \dots + D_{R_1}$  ( $\tilde{N}$  copies).*

*Proof of the lemma.* Let  $\mathcal{V}$  be the collection of subspaces  $E_f$  of  $\mathbb{R}^d$  associated to the faces of  $W$ . Let  $\nu \in \mathbb{R}^d$  be a vector of length one and pick  $\omega > 0$  such that the cone

$$C := \bigcup_{\lambda \geq 0} B_{\lambda\omega}(\lambda\nu)$$

intersects a vectorspace of  $\mathcal{V}$  only trivially. Therefore  $C$  intersects each hyperplane from  $\mathcal{A}(M_1, M_2)$  at most once and so, by Lemma 3.11, intersects in total at most  $N$  hyperplanes.

Now let  $M_1, M_2 \in \Omega_M$  satisfy  $\pi_{max}(M_1) = \pi_{max}(M_2)$  and consider a point  $x \in M_1 \cap M_2$ . Application of Proposition 3.12 with  $r > 0$  guaranties that there is  $\rho > 0$  so that we may choose a point  $x_n \in B_\rho(x + n\rho\nu)$  at which  $M_1$  and  $M_2$  agree, and this for all  $n \in \mathbb{N}$ . We choose  $x_0 = x$  and thus obtain a sequence  $(x_n)_n \subset M_1 \cap M_2$ . Let  $R_1 = 2\rho + R_0$  and consider the sequence of doubly pointed double  $R_1$ -patches  $P_n^{(2)}(x_n, x_{n+1}) = (x_n, x_{n+1}; P_n, Q_n; B_{2\rho}(x + (n + 1/2)\rho\nu))$ . If  $n \geq 4/\omega$  then  $P_n^{(2)}(x_n, x_{n+1})$  is contained in  $x + C$ . Since  $x + C$  intersects at most  $N$  hyperplanes from  $\mathcal{A}(M_1, M_2)$  at most  $\tilde{N} = N + 4/\omega$  double patches  $(P_n, Q_n)$

are distinct and so for at most  $\tilde{N}$  values of  $n$  we have  $F(P_n^{(2)}) \neq 0$ . Once all hyperplanes are crossed,  $M_1 - x_n$  and  $M_2 - x_n$  agree out to distance approximately  $n\rho\omega$  and hence  $\lim_{n \rightarrow \infty} (F_2(x_n) - F_1(x_n)) = 0$  by the weak pattern equivariance of  $F$ . Thus  $F_2(x) - F_1(x) = \sum_n F(P_n^{(2)}) \in \tilde{D}_{R_1}$  with  $R_1 = 2\rho + R_0$ .  $\square$

We continue the proof of Theorem 5.1. If  $\tilde{D}_{R_1} = \{0\}$ , then  $F$  is nonslip with radius  $R_1$ . Otherwise, let  $c = \min\{|d| : d \in \tilde{D}_{R_1} \setminus \{0\}\}$ . Since  $F$  is weakly pattern equivariant  $\tilde{F}$  is uniformly continuous, so there exists  $R$  such that  $B_R \cap (M_1 - x) = B_R \cap (M_2 - x)$  implies  $|\tilde{F}(M_1 - x) - \tilde{F}(M_2 - x)| < c$ . But if  $B_R \cap (M_1 - x) = B_R \cap (M_2 - x)$  then  $x \in M_1 \cap M_2$ , so the inequality implies  $|\tilde{F}(M_1 - x) - \tilde{F}(M_2 - x)| = 0$ .  $\square$

*Proof of Theorem 1.1.* If  $M'$  is pointed topologically conjugate to  $M$ , then  $M'$  is MLD to a pattern  $M''$  that is shape conjugate to  $M$ . Let  $F$  be the generator of that shape conjugacy. By Theorem 5.1 applied to vector-valued functions,  $F$  is nonslip. By Theorem 4.7 applied to vector-valued functions,  $F$  is then the sum of a linear map  $L : H \rightarrow \mathbb{R}^d$  and a strongly pattern equivariant function  $\psi_\epsilon$ . Since  $L$  induces a reprojection and  $\psi_\epsilon$  induces an MLD transformation,  $M''$  is MLD to a reprojection of  $M$ . Since  $M'$  is MLD to  $M''$ ,  $M'$  is also MLD to a reprojection of  $M$ .  $\square$

## 6. COHOMOLOGICAL INTERPRETATION

We have already described how the coboundary  $\delta F$  of a generator of a shape deformation defines an element in the first cohomology  $H^1(\Lambda, \mathbb{R}^d)$  of  $\Lambda$  with coefficients in  $\mathbb{R}^d$ . Here the cohomology is the cohomology of strongly pattern equivariant cochains on the  $CW$ -complex defined by  $\Lambda$ . Alternatively we can describe  $H^1(\Lambda, \mathbb{R}^d)$  as the cohomology of the complex of strongly pattern equivariant  $\mathbb{R}^d$ -valued de Rham 1-forms on  $\mathbb{R}^d$ . Its elements then arise as differentials  $dF$  of smooth functions  $F : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and the relation between the two description is given by integration: Given a 1-chain defined by  $\Lambda$ , that is, essentially an oriented edge  $e$  in a tiling whose vertices are given by  $\Lambda$ ,  $\delta F(e) = \int_e dF = F(y) - F(x)$ , where  $x, y \in \Lambda$  are the source and the range vertex of the edge.

There are a number of canonical subgroups of  $H^1(\Lambda, \mathbb{R}^d)$ , and our results can be viewed as saying when these subgroups are, and aren't, equal.

One subgroup, denoted  $H_{lin}^1(\Lambda, \mathbb{R}^d)$  is given by generators  $F$  that are the restriction to  $\Lambda$  of linear maps  $L : \mathbb{R}^d \rightarrow \mathbb{R}^d$ . For each such  $F$ , the deformed set is simply the result of applying the linear transformation  $\text{id} + L$  to the points of  $\Lambda$ . If  $L$  is non-zero then the deformation cannot be a local derivation, so  $H_{lin}^1(\Lambda, \mathbb{R}^d)$  is isomorphic to  $\text{Hom}(\mathbb{R}^d, \mathbb{R}^d) \cong \mathbb{R}^{d^2}$  (as a vector space).

For model sets, reprojections give another subgroup,  $H_{repr}^1(\Lambda, \mathbb{R}^d)$ . Elements of  $H_{repr}^1(\Lambda, \mathbb{R}^d)$  correspond to generators of the form  $F_L(x) = L(\sigma_M(x))$ , where  $L : H \rightarrow \mathbb{R}^d$  is a continuous group homomorphism.

The *asymptotically negligible* classes [CS]  $H_{an}^1(\Lambda, \mathbb{R}^d)$  are represented by the coboundaries of shape semi-conjugacies. That is, by strongly pattern equivariant 1-cochains that are the coboundaries of weakly pattern equivariant functions. Recall that a generator of a shape semi-conjugacy is weakly pattern equivariant and hence bounded. (In fact, a generator  $F$  is weakly pattern equivariant iff it is bounded [KS]). Since linear maps are unbounded, it follows that  $H_{lin}^1(\Lambda, \mathbb{R}^d) \cap H_{an}^1(\Lambda, \mathbb{R}^d) = \{0\}$ .

We also considered nonslip generators. Recall that any strongly pattern equivariant generator is nonslip and a nonslip generator is asymptotically negligible. Hence the classes of nonslip generators define a subgroup  $H_{ns}^1(\Lambda, \mathbb{R}^d)$  of  $H_{an}^1(\Lambda, \mathbb{R}^d)$ . Furthermore, in the context of a model set  $M$ ,  $F_L$  is nonslip and so  $H_{repr}^1(M, \mathbb{R}^d)$  is a subgroup of  $H_{ns}^1(M, \mathbb{R}^d)$ . Thus we have a sequence of inclusions

$$H_{repr}^1(M, \mathbb{R}^d) \subset H_{ns}^1(M, \mathbb{R}^d) \subset H_{an}^1(M, \mathbb{R}^d).$$

A natural question is whether these groups coincide.

**6.1. Reinterpretation of Theorems 4.7 and 5.1.** We can also consider functions and cochains with values in  $\mathbb{R}$  rather than with values in  $\mathbb{R}^d$ , with

$$H_{repr}^1(\Lambda, \mathbb{R}^d) = H_{repr}^1(\Lambda, \mathbb{R}) \otimes \mathbb{R}^d, \quad H_{ns}^1(\Lambda, \mathbb{R}^d) = H_{ns}^1(\Lambda, \mathbb{R}) \otimes \mathbb{R}^d, \quad H_{an}^1(\Lambda, \mathbb{R}^d) = H_{an}^1(\Lambda, \mathbb{R}) \otimes \mathbb{R}^d.$$

The following are immediate corollaries of Theorem 4.7 and Theorem 5.1.

**Corollary 6.1.** *If  $M$  is a model set satisfying H1 and H2', then  $H_{repr}^1(M, \mathbb{R}) = H_{ns}^1(M, \mathbb{R})$ .*

*Proof.* By Theorem 4.7, each nonslip generator is the sum of a linear function on  $H$  and a strongly pattern equivariant function, so the 1-cohomology class of its coboundary is in  $H_{repr}^1(M, \mathbb{R})$ . □

**Corollary 6.2.** *If  $M$  is a model set satisfying H1 and H2, then  $H_{ns}^1(M, \mathbb{R}) = H_{an}^1(M, \mathbb{R})$ .*

*Proof.* Theorem 5.1 gives this result on the level of cochains. Every generator of a shape semi-conjugacy is nonslip, period. □

The last two corollaries together give Theorem 1.2.

**6.2. Image of the first cohomology of the maximal equicontinuous torus.** The maximal equicontinuous factor map  $\pi_{max} : \Omega \rightarrow \Omega_{max}$  induces an injective map in cohomology  $\pi_{max}^* : H^1(\Omega_{max}, \mathbb{Z}) \rightarrow H^1(\Omega, \mathbb{Z})$ . We have thus a fourth subgroup which is worth comparing with the other, namely the image under  $\pi_{max}^*$  of  $H^1(\Omega_{max}, \mathbb{R}^d)$  which we denote by  $H_{max}^1(\Lambda, \mathbb{R}^d)$ . To do that we need a better understanding of the image of  $\pi_{max}^*$ .

We recall from [BKS] that  $\Omega_{max}$  can be alternatively described with the help of the topological eigenvalues of the action. Let  $\hat{\mathbb{R}}^d = Hom(\mathbb{R}^d, U(1))$  be the Pontryagin dual of  $\mathbb{R}^d$ , where we require each homomorphism to be continuous on  $\mathbb{R}^d$ . An element  $\chi \in \hat{\mathbb{R}}^d$  is a topological eigenvalue if there exists a non-vanishing continuous function  $f : \Omega \rightarrow \mathbb{C}$  such that  $f(\Lambda - t) = \chi(t)f(\Lambda)$ . Topological eigenvalues form a countable subgroup  $\mathcal{E}$  of  $\hat{\mathbb{R}}^d$  and  $\Omega_{max}$  can be identified with the dual  $\hat{\mathcal{E}} = Hom(\mathcal{E}, U(1))$ , where  $\mathcal{E}$  is given the discrete topology.

For a general topological space  $X$ ,  $H^1(X, \mathbb{Z})$  is isomorphic to  $[X, S^1]$ , the homotopy classes of continuous maps  $X \rightarrow S^1$ . Furthermore, if  $\varphi : X \rightarrow Y$  is a continuous map then  $\varphi^* : H^1(Y, \mathbb{Z}) \rightarrow H^1(X, \mathbb{Z})$  can be identified with the mapping  $[Y, S^1] \ni [f] \mapsto [f \circ \varphi] \in [X, S^1]$ . We apply this to  $\pi_{max} : \Omega \rightarrow \Omega_{max}$ . Since  $\Omega_{max} \cong \hat{\mathcal{E}}$  the elements of  $[\Omega_{max}, S^1]$  are the homotopy classes of characters on  $\hat{\mathcal{E}}$ . Hence  $[\Omega_{max}, S^1] \cong \mathcal{E}$  and the image of  $\chi \in \mathcal{E}$  under  $\pi_{max}^*$  in  $[\Omega, S^1]$  is given by the homotopy class of an eigenfunction  $f_\chi$  of  $\chi$  ( $f_\chi$  is normalized so as to have modulus 1). This describes the image of  $\pi_{max}^*$  (in degree one) in  $[\Omega, S^1]$  [BKS].

To obtain the image of  $\pi_{max}^*$  in pattern equivariant cohomology we consider the restriction of a representative  $f$  of an element of  $[\Omega, S^1]$  to the orbit of  $\Lambda$  and define

$$\check{f} : \mathbb{R}^d \rightarrow S^1, \quad \check{f}(x) := f(\Lambda - x).$$

**Lemma 6.3.** *Any element of  $[\Omega, S^1]$  admits a representative  $f$  such that  $\check{f}(x) := f(\Lambda - x)$  is strongly pattern equivariant.*

*Proof.*  $[\Omega, S^1]$  can be seen as the direct limit of  $[\mathcal{G}_n, S^1]$  where  $\mathcal{G}_n$  is the  $n$ th approximant in the Gähler complex [S2]. Each element thus comes from some  $[\mathcal{G}_n, S^1]$  and the latter elements produce strongly pattern equivariant functions when considered on the orbit.  $\square$

Since  $\mathbb{R}^d$  is simply connected we can lift  $\check{f}$  to a continuous function  $\tau : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\check{f}(x) = \exp 2\pi i \tau(x)$ . We define  $F$  to be the restriction of  $\tau$  to  $\Lambda$ . Then  $\delta F$  is strongly pattern equivariant and so we have a map  $[\Omega, S^1] \rightarrow H^1(\Lambda, \mathbb{Z})$ :  $[f] \mapsto [\delta F]$ .

**Lemma 6.4.** *With the above notation  $[f] \mapsto [\delta F]$  is a group homomorphism whose image corresponds to the image of  $[\Omega, S^1]$  in  $H^1(\Omega, \mathbb{R})$  under the identification  $H^1(\Omega, \mathbb{Z}) \cong H^1(\Lambda, \mathbb{Z}) \subset H^1(\Lambda, \mathbb{R})$ .*

*Proof.* Using Gähler's approximation this statement boils down to consider the map between  $[\mathcal{G}_n, S^1]$  and  $H^1(\mathcal{G}_n, \mathbb{R})$  where the latter can be considered as de Rham cohomology on a branched manifold. If  $[f] \in [\mathcal{G}_n, S^1]$  then (assuming without restriction of generality that  $f$  is smooth)  $\frac{1}{2\pi i} f^{-1} df$  represents the element in  $H^1(\mathcal{G}_n, \mathbb{R})$  under the map  $[\mathcal{G}_n, S^1] \rightarrow H^1(\mathcal{G}_n, \mathbb{Z}) \subset H^1(\mathcal{G}_n, \mathbb{R})$ . This is well-known (see also [KP]). Now in order to obtain the map on cellular cohomology of  $\mathcal{G}_n$  one just needs to integrate over 1-chains. The result is a 1-cochain which, when interpreted as strongly pattern equivariant 1-cochain on  $\Lambda$  coincides precisely with  $\delta F$ .  $\square$

If we combine the two arguments we see that the image of  $H^1(\Omega_{max}, \mathbb{Z}) \cong \mathcal{E}$  in pattern equivariant cohomology can be described as follows.

**Corollary 6.5.** *Upon the above identification of  $H^1(\Omega_{max}, \mathbb{Z})$  with  $\mathcal{E}$  and the identification of  $H^1(\Omega, \mathbb{Z})$  with  $H^1(\Lambda, \mathbb{Z})$  the map  $\pi_{max}^*$  becomes the map*

$$\mathcal{E} \ni \chi \mapsto [\delta \tilde{\beta}|_{\Lambda}] \in H^1(\Lambda, \mathbb{Z})$$

where  $\tilde{\beta}|_{\Lambda}$  is the restriction to  $\Lambda$  of a continuous function  $\tilde{\beta} : \mathbb{R}^d \rightarrow \mathbb{R}$  such that  $\exp(2\pi i \tilde{\beta})$  is strongly pattern equivariant and homotopic to  $\chi$ .

Each eigenvalue  $\chi$  can be lifted, that is there exists a  $\beta \in \mathbb{R}^{d^*}$  such that  $\chi(x) = \exp 2\pi i \beta(x)$ . Therefore  $\tilde{\beta} - \beta$  must be bounded and hence weakly pattern equivariant. It follows that  $[\delta \tilde{\beta}] - [\delta \beta] \in H_{an}^1(\Lambda, \mathbb{R})$  and, since  $\beta$  is linear, we see that

$$H_{max}^1(\Lambda, \mathbb{R}^d) \subset H_{lin}^1(\Lambda, \mathbb{R}^d) + H_{an}^1(\Lambda, \mathbb{R}^d).$$

Stated differently, the image of  $H_{max}^1(\Lambda, \mathbb{R}^d)$  in the quotient group<sup>2</sup>

$$H_m^1(\Lambda, \mathbb{R}^d) := H^1(\Lambda, \mathbb{R}^d) / H_{an}^1(\Lambda, \mathbb{R}^d)$$

can be identified with a subspace of the vector space of linear deformations. Indeed let  $\psi : H^1(\Omega_{max}, \mathbb{R}^d) \rightarrow H_m^1(\Lambda, \mathbb{R}^d)$  be the composition of  $\pi_{max}^*$  with the canonical projection. It is induced by  $[\exp 2\pi i \beta] \mapsto [\delta \beta]$ .

**Lemma 6.6.** *Let  $\Lambda$  be a repetitive FLC Delone set. We have*

$$\psi(H^1(\Omega_{max}, \mathbb{R}^d)) = (H_{lin}^1(\Lambda, \mathbb{R}^d) + H_{an}^1(\Lambda, \mathbb{R}^d)) / H_{an}^1(\Lambda, \mathbb{R}^d) \cong \text{Hom}(\mathbb{R}^d, \mathbb{R}^d)$$

whenever  $\Lambda$  is topologically conjugate to a Meyer set.

*Proof.* The above shows that  $\psi(H^1(\Omega_{max}, \mathbb{R}^d)) \subset (H_{lin}^1(\Lambda, \mathbb{R}^d) + H_{an}^1(\Lambda, \mathbb{R}^d)) / H_{an}^1(\Lambda, \mathbb{R}^d)$  and the r.h.s. is clearly isomorphic to the vector space of all linear deformations. Thus equality holds precisely if the real span of  $\{\beta \in \mathbb{R}^{d^*} : \exp 2\pi i \beta \in \mathcal{E}\}$  has dimension  $d$ . By the results of [KS] this is equivalent to saying that  $\Lambda$  is topologically conjugate to a Meyer set.  $\square$

**6.3. Case of model sets.** In the case of model sets which satisfy H1 we can say more, because we have a more explicit model for the maximal equicontinuous factor, namely the torus parametrization. Indeed, if  $H = \mathbb{R}^n \times C$  then, by cocompactness of  $\Gamma$ ,  $\Omega_{max} = H \times \mathbb{R}^d / \Gamma$  is an  $n+d$ -torus and so we can identify  $H^1(\Omega_{max}, \mathbb{Z}) \cong \mathcal{E} \cong \hat{\Omega}_{max}$  with the so-called reciprocal lattice  $\Gamma^{rec}$  which is given by those continuous group homomorphisms  $\alpha : H \times \mathbb{R}^d \rightarrow \mathbb{R}$  which satisfy  $\alpha(\gamma) \in \mathbb{Z}$  for all  $\gamma \in \Gamma$ . Again,  $\alpha$  must be trivial on the torsion component  $C$  and restricts to a linear map  $\mathbb{R}^n \times \mathbb{R}^d \rightarrow \mathbb{R}$ .

<sup>2</sup>This quotient group is the mixed group of [K2] studied in [B] for projection method tilings.

**Proposition 6.7.** *Let  $M$  a model set satisfying H1. Upon the identification of  $H^1(\Omega_{max}, \mathbb{Z})$  with  $\Gamma^{rec}$  and the identification of  $H^1(\Omega, \mathbb{Z})$  with  $H^1(M, \mathbb{Z})$  the map  $\pi_{max}^*$  becomes the map*

$$\Gamma^{rec} \ni \alpha \mapsto [\delta\alpha(\sigma_M(\cdot), \cdot)] \in H^1(M, \mathbb{Z}).$$

*Proof.* The eigenvalue  $\chi \in \mathcal{E}$  corresponding to  $\alpha \in \Gamma^{rec}$  is  $\chi(x) = \exp 2\pi i\alpha(0, x)$ ,  $x \in \mathbb{R}^d$ . Recall that  $\sigma_M : M \rightarrow H$  is weakly pattern equivariant and has strongly pattern equivariant coboundary. We may therefore extend it to a weakly pattern equivariant function on all of  $\mathbb{R}^d$  whose differential is strongly pattern equivariant [K2]. We denote the extension also by  $\sigma_M$ . As  $\{\sigma_M(x) : x \in M\}$  lies in a compact subset of  $H$ ,  $\chi_t(x) = \exp 2\pi i\alpha(t\sigma_M(x), x)$  is a homotopy between  $\chi$  and  $\chi_1$ . Furthermore, if  $x, y \in M$  then  $(\sigma_M(y), y) - (\sigma_M(x), x) = ((y-x)^*, y-x) \in \Gamma$  and so  $M \ni x \mapsto \exp 2\pi i\alpha(\sigma_M(x), x)$  is constant and hence a strongly pattern equivariant function on  $M$ . It follows that  $\chi_1$  is a strongly pattern equivariant function on  $\mathbb{R}^d$  and thus we may apply Corollary 6.5 to obtain the statement.  $\square$

**Proposition 6.8.** *Let  $M$  be a model set satisfying H1, that is,  $H = \mathbb{R}^n \times C$  with some finite group  $C$ . Then  $H_{repr}^1(M, \mathbb{R})$  has dimension  $n$  and*

$$H_{max}^1(M, \mathbb{R}^d) = H_{repr}^1(M, \mathbb{R}^d) \oplus H_{lin}^1(M, \mathbb{R}^d).$$

*Proof.* We have  $H_{max}^1(M, \mathbb{R}) \cong \Gamma^{rec} \otimes_{\mathbb{Z}} \mathbb{R} \cong (\mathbb{R}^n \times \mathbb{R}^d)^* \cong \mathbb{R}^{n*} \oplus \mathbb{R}^{d*}$ . Thus an element corresponding to  $\alpha \in \Gamma^{rec} \otimes_{\mathbb{Z}} \mathbb{R}$  can be split into  $(\alpha^\perp, \alpha^\parallel) \in \mathbb{R}^{n*} \oplus \mathbb{R}^{d*}$ . Under this splitting the coboundary  $\delta\alpha(\sigma_M(\cdot), \cdot)$  becomes  $(\delta\alpha^\perp \circ \sigma_M, \delta\alpha^\parallel)$ . With  $\mathbb{R}^d$  coefficients this induces exactly the splitting  $H_{max}^1(M, \mathbb{R}^d) = H_{repr}^1(M, \mathbb{R}^d) \oplus H_{lin}^1(M, \mathbb{R}^d)$ . This also shows that the dimension of  $H_{repr}^1(M, \mathbb{R})$  is  $n$ .  $\square$

With this proposition at hand we see that Theorem 1.2 is equivalent to Theorem 1.3. Indeed, it shows that for polyhedral model sets the statement  $\dim H_{an}^1(M, \mathbb{R}) = n$  is equivalent to  $H_{an}^1(M, \mathbb{R}) \subset H_{max}^1(M, \mathbb{R})$ .

## 7. NONSLIP SETS AND THE MEYER PROPERTY

In this section we explore a little further the concept of nonslip generators. Our hope is that this may turn out useful later for the study of shape conjugations of Delone sets which are not model. We consider an analogous property of sets, and show that a generator of a shape conjugation is nonslip if and only if its associated shape conjugacy preserves that property.

**Definition 7.1.** *A Delone set  $\Lambda$  is nonslip if for all  $R > 0$  one can find an  $\epsilon > 0$  such that for all  $\Lambda_1, \Lambda_2 \in \Omega_\Lambda$  we have that if  $\pi_{max}(\Lambda_1) = \pi_{max}(\Lambda_2)$  and  $d(\Lambda_1, \Lambda_2) \leq \epsilon$ , then both sets agree on  $B_R(0)$ .*

**Lemma 7.2.** *Every Meyer set is nonslip.*

*Proof.* Recall that for Meyer sets  $\pi_{max}(\Lambda_1) = \pi_{max}(\Lambda_2)$  implies that  $0 \in \Lambda_1 - \Lambda_2$ . But  $\Lambda_1 - \Lambda_2$  is uniformly discrete by the Meyer property. Hence if  $\Lambda_1$  and  $\Lambda_2$  are close enough they have to coincide on a ball of radius equal to the inverse of their distance.  $\square$

Recall that a generator of shape conjugacy of an FLC Delone set  $\Lambda \subset \mathbb{R}^d$  is a function  $F : \Lambda \rightarrow \mathbb{R}^d$  such that  $\Lambda \mapsto \Lambda^F = \{x + F(x) : x \in \Lambda\}$  extends to an  $\mathbb{R}^d$ -equivariant homeomorphism  $\mathfrak{s}_F : \Omega_\Lambda \rightarrow \Omega_{\Lambda^F}$ . We saw that in this case  $F$  extends to a continuous map  $\tilde{F} : \Omega_\Lambda \rightarrow \mathbb{R}^d$  and so we may define  $F_{\Lambda'} : \Lambda' \rightarrow \mathbb{R}^d$  by  $F_{\Lambda'}(x) = \tilde{F}(\Lambda' - x)$ . It is not difficult to see that  $F_{\Lambda'}$  is weakly pattern equivariant for  $\Lambda'$  and that  $\mathfrak{s}_F(\Lambda') = \Lambda'^{F_{\Lambda'}} = \{x + F_{\Lambda'}(x) : x \in \Lambda'\}$ . Indeed  $\mathfrak{s}_F(\Lambda') = \lim \mathfrak{s}_F(\Lambda) - x_n$  for some sequence  $(\Lambda - x_n)_n$  converging to  $\Lambda'$ , and  $\mathfrak{s}_F(\Lambda) - x_n = \{x + \tilde{F}(\Lambda - x) : x \in \Lambda\} - x_n = \{y + \tilde{F}(\Lambda - x_n - y) : y \in \Lambda - x_n\}$ . Now since  $\tilde{F}$  is bounded and continuous we conclude that  $\lim \mathfrak{s}_F(\Lambda) - x_n = \{y - \tilde{F}(\Lambda' - y) : y \in \Lambda'\}$ .

Nonslip generators of shape conjugacies and nonslip sets are closely related.

For a more general (not necessarily Meyer) Delone set we generalize the concept of a nonslip weakly pattern equivariant function as follows:

**Definition 7.3.** *Let  $\Lambda$  be an FLC Delone set. A weakly pattern equivariant function  $F : \Lambda \rightarrow Y$  is nonslip if there exists an  $\epsilon > 0$  such that for all  $\Lambda_1, \Lambda_2 \in \mathcal{R}_{max}^\Xi$  we have  $\tilde{F}(\Lambda_1) = \tilde{F}(\Lambda_2)$  whenever  $d(\Lambda_1, \Lambda_2) \leq \epsilon$ .*

Note that if  $\Lambda$  is nonslip, then the above definition reduces to the definition we previously gave for Meyer sets. This follows as in the proof of the last lemma from the fact that  $\pi_{max}(\Lambda_1) = \pi_{max}(\Lambda_2)$  implies that  $\Lambda_1$  and  $\Lambda_2$  agree on balls once they are close.

**Proposition 7.4.** *Let  $\Lambda$  be nonslip.  $\Lambda^F$  is nonslip iff  $F$  is nonslip.*

*Proof.* “ $\Rightarrow$ ” We suppose that  $\Lambda^F$  is nonslip. Hence, given  $R$  there exists  $\delta$  such that for all  $(\Lambda_1, \Lambda_2) \in \mathcal{R}_{max}$  we have  $d(\mathfrak{s}_F(\Lambda_1), \mathfrak{s}_F(\Lambda_2)) < \delta$  implies  $B_R[\mathfrak{s}_F(\Lambda_1)] = B_R[\mathfrak{s}_F(\Lambda_2)]$ . Moreover,  $\tilde{F}$  is uniformly continuous so there exists  $\epsilon_1$  such that for all  $\Lambda_1, \Lambda_2 \in \Xi$  we have  $d(\Lambda_1, \Lambda_2) < \epsilon_1$  implies  $\|\tilde{F}(\Lambda_1) - \tilde{F}(\Lambda_2)\| < R^{-1}$ . Furthermore,  $\mathfrak{s}_F$  is uniformly continuous so there exists  $\epsilon_2$  such that for all  $\Lambda_1, \Lambda_2 \in \Omega$  we have  $d(\Lambda_1, \Lambda_2) < \epsilon_2$  implies  $d(\mathfrak{s}_F(\Lambda_1), \mathfrak{s}_F(\Lambda_2)) < \delta$ . Finally,  $\Lambda$  is nonslip so there exists  $\epsilon_3$  such that for all  $(\Lambda_1, \Lambda_2) \in \mathcal{R}_{max}$  we have  $d(\Lambda_1, \Lambda_2) < \epsilon_3$  implies  $B_1[\Lambda_1] = B_1[\Lambda_2]$ . Let  $\epsilon = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \}$  (which depends on  $R$ ). Then for all  $(\Lambda_1, \Lambda_2) \in \mathcal{R}_{max}^\Xi$  we have  $d(\Lambda_1, \Lambda_2) < \epsilon$  implies that  $0 \in \Lambda_1 \cap \Lambda_2$ ,  $B_R[\mathfrak{s}_F(\Lambda_1)] = B_R[\mathfrak{s}_F(\Lambda_2)]$  and  $\|F_{\Lambda_1}(0) - F_{\Lambda_2}(0)\| < R^{-1}$ . Since  $\mathfrak{s}_F(\Lambda_i) = \{x + F_{\Lambda_i}(x) : x \in \Lambda_i\}$  we see that, if  $R$  is large enough, this implies  $F_{\Lambda_1}(0) = F_{\Lambda_2}(0)$ .

“ $\Leftarrow$ ” We suppose that  $F$  is nonslip, hence there exists  $\delta$  such that for all  $(\Lambda_1, \Lambda_2) \in \mathcal{R}_{max}^\Xi$  and all  $x \in \Lambda_1 \cap \Lambda_2$  we have that  $d(\Lambda_1 - x, \Lambda_2 - x) < \delta$  implies  $F_{\Lambda_1}(x) = F_{\Lambda_2}(x)$ . By definition of the metric there exists  $\epsilon_1$  such that if  $d(\Lambda_1, \Lambda_2) < \epsilon_1$  then  $d(\Lambda_1 - x, \Lambda_2 - x) < \delta$  for all  $x$  of size smaller or equal to the radius of relative denseness. Let  $R > 0$  and  $\|F\| = \sup_{\Lambda' \in \Omega} \|\tilde{F}(\Lambda')\|$  which

is finite, by the continuity of  $\tilde{F}$ . Since  $\Lambda$  is nonslip there exists  $\epsilon_2$  such that for all  $(\Lambda_1, \Lambda_2) \in \mathcal{R}_{max}$  we have  $d(\Lambda_1, \Lambda_2) < \epsilon_2$  implies  $B_{R+\|F\|}[\Lambda_1] = B_{R+\|F\|}[\Lambda_2]$ . Let  $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ . Then, if  $(\Lambda_1, \Lambda_2) \in \mathcal{R}_{max}$  and  $d(\Lambda_1, \Lambda_2) < \epsilon$  we have  $B_{R+\|F\|}[\Lambda_1] = B_{R+\|F\|}[\Lambda_2]$  and  $F_{\Lambda_1}(x) = F_{\Lambda_2}(x)$  for all  $x \in \Lambda_1 \cap \Lambda_2$  of size smaller or equal to the radius of relative denseness. It follows that  $B_R[\mathfrak{s}_F(\Lambda_1)] = B_R[\mathfrak{s}_F(\Lambda_2)]$ . Thus  $\Lambda^F$  is nonslip.  $\square$

The following corollary is just a special case:

**Corollary 7.5.** *Let  $\Lambda$  be a Meyer set and  $F$  be a generator of a shape conjugacy. If  $\Lambda^F$  is a Meyer set then  $F$  must be nonslip.*

The contrapositive says that if  $F$  is not nonslip, then  $\Lambda^F$  is not Meyer. That is Theorem 1.4.

Applying these observations to model sets we obtain:

**Corollary 7.6.** *Given a model set  $M$  which satisfies H1. If  $H_{ns}^1(M, \mathbb{R}) \subset H_{max}^1(M, \mathbb{R})$  then any shape conjugation of  $M$  which is a Meyer set is a reprojection of  $M$ .*

*Proof.* Let  $F$  be the generator of a shape conjugation of  $M$  such that  $M^F$  is a Meyer set. By Corollary 7.5,  $F$  must be nonslip. By Proposition 6.8,  $H_{ns}^1(M, \mathbb{R}) \subset H_{max}^1(M, \mathbb{R})$  is equivalent to the equality  $H_{ns}^1(M, \mathbb{R}) = H_{repr}^1(M, \mathbb{R})$ . Hence  $M^F$  is a reprojection.  $\square$

## 8. A MODEL SET THAT IS NOT RIGID

Theorem 1.1 states that most common examples of model sets, constructed by direct application of the cut & project method with polyhedral windows, are rigid. In this example we exhibit a model set with a Euclidean internal space (so satisfying H1) that is not. The example is constructed by a substitution; and model sets arising from substitutions may have very complicated (fractal) windows.

Consider the 1-dimensional substitution  $\sigma$  on four letters:

$$\begin{aligned}\sigma(a_1) &= a_1 b_1 a_2 \\ \sigma(b_1) &= a_1 b_2 \\ \sigma(a_2) &= a_1 b_2 a_2 \\ \sigma(b_2) &= a_2 b_1.\end{aligned}$$

Its substitution matrix is

$$A_\sigma = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

and has eigenvalues  $\phi^2$ ,  $-\phi$ ,  $\phi^{-1}$  and  $\phi^{-2}$  where  $\phi = (1 + \sqrt{5})/2$  is the golden mean. We choose tile length proportional to the right Perron Frobenius eigenvalue, namely  $b_1$  and  $b_2$

tiles to have unit length, and  $a_1$  and  $a_2$  tiles to have length equal to  $\phi = (1 + \sqrt{5})/2$ . This is then a geometric primitive aperiodic unimodular Pisot substitution (which is not irreducible). By the results of [BSW] the dynamical spectrum is pure point, indeed the balanced pair  $(a_1b_1, b_1a_1)$  terminates with coincidence (see [BSW] for explanations on this notion). A lot is known about such substitution tilings. The following can be found more or less implicit in, for instance, [BM, BBK, Si].

- Since the spectrum is pure point, the tilings are MLD to (possibly colored) regular model sets. Moreover, the set  $M$  of left boundary points of  $a_1$ -tiles in a tiling is MLD to the tiling and hence also a regular model set.
- Since the substitution is unimodular, the maximal equicontinuous factor  $\Omega_{max}$  of the associated dynamical system is a torus of dimension  $J$  where  $J$  is the algebraic degree of the Perron Frobenius eigenvalue. This eigenvalue is here  $\phi^2$  and hence  $J = 2$ . This, in turn implies that  $M$  has a cut & project scheme in which the internal space  $H$  is  $\mathbb{R}$ .

One readily computes using the technique of [AP] that  $H^1(\Omega_M, \mathbb{R}) = \mathbb{R}^4$ , and that substitution acts on  $H^1(\Omega_M, \mathbb{R})$  by the transpose  $A_\sigma^T$  of the substitution matrix. For substitution tilings,  $H_{an}^1(\Omega_M, \mathbb{R})$  is the span of all of the generalized eigenspaces of this action with eigenvalues strictly inside the unit circle [CS]. In our case this means that  $H_{an}^1(\Omega_M, \mathbb{R}) = \mathbb{R}^2$  is the span of the  $\phi^{-1}$  and  $\phi^{-2}$  eigenvectors of  $A_\sigma^T$ .

The generator of a shape conjugacy corresponding to the  $\phi^{-2}$  eigenvector induces a reprojection and hence is nonslip. It corresponds to a shape conjugacy in which all of the  $a$  tiles are lengthened and the  $b$  tiles are shortened (or vice-versa), while maintaining  $|a_1| = |a_2|$  and  $|b_1| = |b_2|$  and preserving the quantity  $|a_1|\phi + |b_1| = \phi^2 + 1$ .

The generator  $F$  of a shape conjugacy corresponding to the  $\phi^{-1}$  eigenvector is not nonslip, and results in a Delone set  $M^F$  that is not Meyer. To see this, let  $A_i^n = \sigma^n(a_i)$  and  $B_i^n = \sigma^n(b_i)$  be  $n$ -th order supertiles, and let  $|A_i|$  and  $|B_i|$  be the Euclidean lengths of these supertiles after deformation. This time  $|A_1| \neq |A_2|$ , in fact,  $|A_1^n| - |A_2^n|$  is proportional to  $\phi^{-n}$ . We have  $|A_1^n| \in M^F - M^F$ , since  $A_1^n B_1^n$  appears in our tiling and  $A_1^n$  and  $B_1^n$  start with  $a_1$ . Likewise,  $|A_2^n| \in M^F - M^F$ , since  $A_2^n B_1^n$  appears in the tiling and also  $A_2^n$  starts with  $a_1$ . However, since  $|A_1^n| - |A_2^n|$  is proportional to  $\phi^{-n}$  the set of differences  $M^F - M^F$  cannot be uniformly discrete, so  $M^F$  is not Meyer.

The key feature of this example is that the substitution matrix is reducible. The maximal equicontinuous factor is determined by the dynamical spectrum, which for self-similar tile lengths is determined by the Perron-Frobenius eigenvalue  $\lambda_{PF}$ . If  $\lambda_{PF}$  is a Pisot number, then a basis for the nonslip generators of shape conjugacy is given by the eigenvectors with eigenvalues algebraically conjugate to  $\lambda_{PF}$ . However, a basis for  $H_{an}^1$  is given by *all* the eigenvectors with eigenvalue strictly smaller than 1. The eigenvectors whose small eigenvalues

are not conjugate to  $\lambda_{PF}$  correspond to weakly pattern equivariant functions that are not nonslip.

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