Random Fields of Bounded Variation and Computation of their Variation Intensity

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Abstract

The main purpose of this paper is to define and characterize random fields of bounded variation, that is random fields with sample paths in the space of functions of bounded variation, and to study their mean total variation. Simple formulas are obtained for the mean total directional variation of random fields, based on known formulas for the directional variation of deterministic functions. It is also shown that the mean variation of random fields with stationary increments is proportional to the Lebesgue measure, and an expression of the constant of proportionality, called the *variation intensity*, is established. This expression shows in particular that the variation intensity only depends on the family of two-dimensional distributions of the stationary increment random field. When restricting to random sets, the obtained results gives generalizations of well-known formulas from stochastic geometry and mathematical morphology. The interest of these general results is illustrated by computing the variation intensities of several classical stationary random field and random sets model, namely Gaussian random fields and excursion sets, Poisson shot noises, Boolean models, dead leaves models, and random tessellations.

Keywords: Functions of bounded variation; Directional variation; Variation intensity; Specific perimeter; Stationary increment random fields; Germ-grain models;

MSC2010 subject classification: Primary 60G60; Secondary 60G17; 60G51; 60D05

1 Introduction

This paper focuses on the definition and characterizations of random fields (r.f.) of bounded variation, that is, random fields having their paths in the space of functions of bounded variation, as well as to provide explicit formulas for the computation of their mean total variation. Its motivations stem from the modeling of textures by random fields in image processing where functions of bounded variation are an important model.

The functional space $BV(U)$ of functions of bounded variation defined over some open set $U \subset \mathbb{R}^d$ is a subspace of $L^1(U)$ for which the functions are weakly differentiable in the sense that their distributional derivative Df can be represented by a finite Radon measure over U [AFP00, EG92]. The total variation of $f \in L^1(U)$ is then defined by $|Df|(U)$, the total variation of the vector-valued measure Df, if $f \in BV(U)$, and by $+\infty$ otherwise.

Ever since the seminal paper of Rudin, Osher and Fatemi [ROF92], the total variation has been widely used for various tasks in image processing such as denoising with the TV-L^2 and TV-L^1 models [ROF92, Cha04, Nik04, AVCM04, CE05, DAG09], zooming [GM98] or

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also deconvolution [CW98]. Concerning textures, the total variation generally appears in the problem of decomposing on image in a cartoon part (or geometric part) and a textural part following the framework proposed by Meyer [Mey01]. For this problem it is considered that the total variation of a textural part should be high or even infinite, whereas the total variation of the cartoon part should be low, in comparison with another norm adapted to oscillatory images [Mey01, VO03, AABFC05, AC05, BLMV10]. Another work which relates total variation and textures is the texture synthesis method proposed by Fadili and Peyré [FP10].

Even though the total variation of textures is generally considered to be high or even infinite, to the best of our knowledge little is known on the total variation of classical texture models such as Gaussian random fields, shot noises, or the other germ-grain models (Boolean model, dead leaves models). However, for the later germ-grain models, the random field models corresponds to random geometric images with few contours, and intuitively their total variation should depend on the geometry of the grains. Hence it is natural to ask the following question: What is the mean total variation of classical germ-grain models ?

Even though the above question is intuitively simple, provide a solution to the problem raised by this question necessitates the development of several theoretical results presented in this paper. To the best of our knowledge random fields (r.f.) of bounded variation over $\mathbb{R}^d$ have never been studied for $d \geq 2$, with the exception of the short paper [Ibr95] and the very recent paper [BD13] that studies the excursion sets of shot noise random fields that have bounded variation.

Let us know detail the main results of the paper.

For a r.f. f, our general strategy is first to deal with directional variations $|D_u f|$, $u \in S^{d-1}$ and then integrate over all directions $u \in S^{d-1}$ to obtain results on the variation $|Df|$. The advantage of dealing with directional variations is that it gives simple expressions only involving difference quotients, and it also provides additional information on the anisotropy of the function or the r.f. f.

The first section of this chapter is devoted to deterministic functions of bounded directional variation. Relations between the directional variation $|D_u f|$ of a function f and the integral of its difference quotients are emphasized, yielding to the fundamental relation

$$|D_u f|(U) = \lim_{r \to 0} \int_{U \ominus [0,ru]} \frac{|f(x+ru) - f(x)|}{|r|} dx,$$

for any open set $U \subset \mathbb{R}^d$. Since the stated results are not found in the reference textbooks on the subject (e.g. [AFP00, EG92]), they are presented with a complete proof.

After this preliminary study of deterministic functions of bounded directional variation, random fields of (locally) bounded (directional) variation are defined and characterized. In particular, one defines the *directional variation intensity measure* $\Theta_{V_u}(f, \cdot)$ of a r.f. f as the expectation of the directional variation of f, and it is shown that

$$\Theta_{V_u}(f, U) = \lim_{r \to 0} \int_{U \ominus [0,ru]} \frac{\mathbb{E}(|f(x+ru) - f(x)|)}{|r|} dx.$$

A particular interest is then given to r.f. f with stationary increments having locally bounded (directional) variation. If f is such a r.f., it is proved that the mean directional variation of f on every domain U is proportional to the Lebesgue measure of U. The constant of proportionality is called the *directional variation intensity* of the stationary r.f. f and is denoted by $\theta_{V_u}(f)$. Along with the definition of $\theta_{V_u}(f)$, a practical formula is derived:

$$\theta_{V_u}(f) = \lim_{r \to 0} \frac{\mathbb{E}(|f(ru) - f(0)|)}{|r|}. \quad (1)$$

In particular, the directional variation intensity $\theta_{V_u}(f)$ only depends on the family of two-dimensional distributions of the stationary increment r.f. f.

As mentioned above, we are aware of only one general result on the variation of r.f. which is due to Ibragimov [Ibr95]. Ibragimov theorem shows that if a r.f. is *Lipschitz in mean*, that is, if there exists $K > 0$ such that for all $x, h \in \mathbb{R}^d$,

$$\mathbb{E}(|f(x+h) - f(x)|) \leq K|h|,$$

then f has a.s. locally bounded variation. After recalling this theorem, we establish an equivalent of Ibragimov theorem for r.f. of bounded directional variation. Besides, we demonstrate afterwards that the converse of Ibragimov theorem holds when the r.f. has stationary increments: a locally integrable stationary increment r.f. has bounded variation with a finite variation intensity if and only if it is Lipschitz in mean.

Section 5 focuses on random sets. Let us recall that, by definition, the variation of an indicator function is its perimeter [AFP00, EG92]. Hence as a particular case of our results on the mean variation of r.f., one obtains formulas for the mean perimeter of random sets. In particular the specific directional variation of a stationary random set $X \subset \mathbb{R}^d$ is given by

$$\theta_{V_u}(X) = 2 \lim_{r \to 0} \frac{\nu_X(ru)}{|r|}, \tag{2}$$

where $\nu_X(y) = \mathbb{P}(y \in X,\ 0 \notin X)$ is the variogram of X. Integrating over all directions, one gets a similar formula for the specific variation $\theta_{V_u}(X)$ of the set X, also called specific perimeter. These formulas extend known results from the theory of random sets which are due to Matheron [Mat67, Mat75, Lan02] and that relates the mean perimeter of random sets to the derivatives in zero of the mean geometric covariogram (or variogram for stationary random sets). These results were first established by the author in [Gal11], but the proposed proofs were then unnecessary complicated since the general results on the variation of random fields were not at hand. In addition, mean coarea formulas for mean total variation are established. In particular it is shown that the variation intensity of a stationary r.f. f is equal to the integral over t of the specific variation of its excursion sets $\{f > t\} := \{y \in \mathbb{R}^d,\ f(y) > t\}$, that is

$$\theta_{V_u}(f) = \int_{-\infty}^{+\infty} \theta_{V_u}(\{f > t\})dt \quad \text{and} \quad \theta_V(f) = \int_{-\infty}^{+\infty} \theta_V(\{f > t\})dt.$$

Finally, Section 6 illustrates the interest of the general formulas (1) and (2) by computing the variation intensity of several classical stationary random field models. We first study the variation of stationary Gaussian r.f and the perimeter of their excursion sets. It is shown that a Gaussian r.f. f_G has finite variation intensity if and only if the one-dimensional restrictions of its covariance are twice differentiable at 0. According to [Sch10], this condition also implies that the sample paths of f_G are a.s. in the Sobolev space $W^{1,1}_{\text{loc}}(\mathbb{R}^d)$, and in particular the variation $|Df_G|$ is a.s. absolutely continuous with respect to the Lebesgue measure. In short, functions of bounded variation are not an interesting model for stationary Gaussian random fields. We also show that the same condition of differentiability of the covariance characterizes the finiteness of the specific perimeter of Gaussian excursion sets. The second part of Section 6 provides expressions of the directional and non directional variation intensities of several germ-grain r.f.namely: Poisson shot noise of random sets, Boolean models, colored dead leaves r.f., transparent dead leaves process, and colored tessellations. A germ-grain model defines a random field by combining colored random sets according to an interaction principle (addition, supremum, occultation, transparency,...). These constructions result in r.f. models that present numerous discontinuities along the geometric contours of the grains. For these germ-grain models, the obtained formulas explicitly clarifies the somewhat intuitive relation between the geometry of the grains X and the total variation of the germ-grain r.f. Moreover, they show that for all the considered germ-grain models, there are only two geometric features of influence on variation intensity of the germ-grain r.f.: the mean perimeter and the mean Lebesgue measure of the grains.

2 Functions of Bounded Directional Variation and Difference Quotients

This section gathers several necessary results from the theory of functions of bounded variation, with a particular interest in functions of bounded directional variation. For a general treatment of the subject we refer to the textbook of Ambrosio, Fusco and Pallara [AFP00]. In a first part, some basic definitions are recalled. In a second part, relations between the directional variation of a function and the integral of its difference quotients are emphasized. Although the stated results are well-known in the calculus of variations community[1], they are not, to the best of our knowledge, presented in any reference textbooks on the subject (e.g. [AFP00, EG92]). This is why each result of this second part is presented with a complete proof.

Notation For any open subset $U \subset \mathbb{R}^d$, $\mathcal{B}(U)$ denotes the set of Borel subsets of U, and we write $V \subset\subset U$ if $V \subset U$ is open and relatively compact in U. $L^1(U)$ (resp. $L^1_{\text{loc}}(U)$) denotes the space of functions integrable (resp. locally integrable) over U. $\mathcal{C}^0(U,\mathbb{R})$, $\mathcal{C}^1(U,\mathbb{R})$, and $\mathcal{C}^\infty(U,\mathbb{R})$ denotes the space of functions from U to $\mathbb{R}$ that are respectively continuous, continuously differentiable, and infinitely differentiable. The vector field spaces $\mathcal{C}^0(U,\mathbb{R}^d)$, $\mathcal{C}^1(U,\mathbb{R}^d)$, and $\mathcal{C}^\infty(U,\mathbb{R}^d)$ are defined similarly. The spaces $\mathcal{C}^0_c(U,\mathbb{R})$, $\mathcal{C}^\infty_c(U,\mathbb{R}^d)$, etc. denotes the spaces of functions or vector fields that have a compact support. S^{d-1} denotes the unit Euclidean sphere in $\mathbb{R}^d$. $\mathcal{H}^{d-1}$ denotes the Hausdorff measure of index $d-1$.

2.1 Functions of Bounded Variation

Definition 2.1 (Functions of bounded variation)**.** *Let U be an open set of $\mathbb{R}^d$. We say that $f \in L^1(U)$ is a* function of bounded variation *in U if the distributional derivative of f is representable by a finite Radon measure, i.e. if there exists a $\mathbb{R}^d$-valued Radon measure, noted $Df = (D_1 f, \dots, D_d f)$, such that $|Df|(U) < +\infty$ and for all $\varphi = (\varphi_1, \dots, \varphi_d) \in \mathcal{C}^\infty_c(U,\mathbb{R}^d)$*

$$\int_U f(x) \operatorname{div} \varphi(x) dx = -\sum_{i=1}^{d} \int_U \varphi_i(x) D_i f(dx).$$

The vector space of all functions of bounded variation in U is denoted by $BV(U)$.

A function $f \in L^1_{\text{loc}}(U)$ has locally bounded variation *in U if $f \in BV(V)$ for all open set $V \subset\subset U$. The space of functions of locally bounded variation in U is denoted by $BV_{\text{loc}}(U)$.*

If $\varphi \in \mathcal{C}^1(U,\mathbb{R})$ and $u \in S^{d-1}$, we write

$$\frac{\partial \varphi}{\partial u}(x) = \langle \nabla \varphi(x), u \rangle, \quad x \in \mathbb{R}^d,$$

for the directional derivative of φ in the direction u.

[1] Luigi Ambrosio, personal communication.

Definition 2.2 (Functions of bounded directional variation)**.** *Let U be an open set of $\mathbb{R}^d$ and let $u \in S^{d-1}$. $f \in L^1(U)$ is a* function of bounded directional variation *in U in the direction u if the directional distributional derivative of f in the direction u is representable by a finite Radon measure, i.e. if there exists a signed Radon measure, noted $D_u f$, such that $|D_u f|(U) < +\infty$ and for $\varphi \in \mathcal{C}_c^\infty(U, \mathbb{R})$*

$$\int_U f(x) \frac{\partial \varphi}{\partial u}(x) dx = -\int_U \varphi(x) D_u f(dx).$$

The corresponding space is denoted by $BV_u(U)$.

The space $BV_{u,\mathrm{loc}}(U)$ is defined as the subspace of functions $f \in L^1_{\mathrm{loc}}(U)$ such that $f \in BV_u(V)$ for all open set $V \subset\subset U$.

If $f \in BV(U)$ then $|Df|(U)$ is called the *variation* of f in U. Similarly, if $f \in BV_u(U)$, $|D_u f|(U)$ is called the *directional variation* of f in the direction u in U.

Proposition 2.3 (Variation and directional variations)**.** *Let U be an open set of $\mathbb{R}^d$ and let $f \in L^1(U)$. Then, the three following assertions are equivalent:*

(i) $f \in BV(U)$.

(ii) $f \in BV_u(U)$ for all $u \in S^{d-1}$.

(iii) For all vector e_i of the canonical basis, $f \in BV_{e_i}(U)$.

In addition, if $f \in BV(U)$ we have for all measurable set $A \in \mathcal{B}(U)$,

$$D_u f(A) = \langle Df(A), u \rangle = \sum_{i=1}^d u_i D_i f(A), \quad u = (u_1, \dots, u_d) \in S^{d-1}, \tag{3}$$

and

$$|Df|(A) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} |D_u f|(A) \mathcal{H}^{d-1}(du), \tag{4}$$

where ω_{d-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{d-1}$.

Remark. In consequence of Equation (3), the maps $u \mapsto D_u f(a)$ and $u \mapsto |D_u f|(A)$ are continuous on S^{d-1} for all fixed $A \in \mathcal{B}(U)$ as soon as $f \in BV(U)$.

Proof. This proposition is mostly from [Chl97]. The proof is given for the convenience of the reader. Clearly, Assertion (ii) implies Assertion (iii). Let us show that (i) implies (ii). Let $f \in BV(U)$, let $Df = (D_1 f, \dots, D_d f)$ be the Radon measure representing its distributional derivative, and let $u \in S^{d-1}$. Then $\langle Df, u \rangle := \sum_{i=1}^d u_i D_i f$ is a signed Radon measure which represents the directional derivative of f in the direction u, and by the Cauchy-Schwarz inequality

$$|\langle Df, u \rangle| (U) \leq \sum_{i=1}^d |u_i||D_i f|(U) \leq |u||Df|(U) = |Df|(U) < +\infty.$$

Hence $f \in BV_u(U)$. Let us now show that (iii) implies (i). For all vector e_i of the canonical basis, $f \in BV_{e_i}(\mathbb{R}^d)$ and there exists a signed Radon measure $D_{e_i} f$ which represents the distributional partial derivatives of f. But then one easily checks that $(D_{e_1} f, \dots, D_{e_d} f)$ is a $\mathbb{R}^d$-valued finite Radon measure which represents the distributional derivative of f, and thus $f \in BV(U)$. As for the two announced equalities, note that we have already shown that $D_u f = \langle Df, u \rangle$. To finish let us show Formula (4). Let $f \in BV(U)$ and let $A \in \mathcal{B}(U)$. By the polar decomposition theorem [AFP00, Corollary 1.29] there exists a unique $|Df|$-integrable function $\sigma : U \to S^{d-1}$ such that $Df = \sigma|Df|$. With this notation, observe that for all $u \in S^{d-1}$

$$D_u f(A) = \langle Df, u\rangle(A) = \int_A \langle \sigma(x), u\rangle |Df|(dx).$$

Hence, by [AFP00, Proposition 1.23]

$$|D_u f|(A) = \int_A |\langle \sigma(x), u\rangle| \, |Df|(dx).$$

For all $\nu \in S^{d-1}$ we have the following well-known identity

$$\int_{S^{d-1}} |\langle \nu, u\rangle| \, \mathcal{H}^{d-1}(du) = 2\omega_{d-1}.$$

Hence by Fubini theorem

$$\int_{S^{d-1}} |D_u f|(A)\mathcal{H}^{d-1}(du) = \int_A \left(\int_{S^{d-1}} |\langle \sigma(x), u\rangle| \, \mathcal{H}^{d-1}(du) \right) |Df|(dx) = 2\omega_{d-1}|Df|(A).$$

□

2.2 Directional Variation and Difference Quotients

Within this section, given a function $f \in L^1(U)$, $U \subset \mathbb{R}^d$ open, and a direction $u \in S^{d-1}$, we will consider difference quotients of the form $x \mapsto \frac{f(x+ru)-f(x)}{r}$ for some $r \neq 0$. Such functions are not defined on the whole domain U but on $U \cap (-ru + U)$. Since we will typically attempt at letting r tends to zero, we will restrict further this domain of definition to point x such that the whole segment $[x, x+ru]$ is included in U. This set is denoted by $U \ominus [0, ru] = \{x \in U,\ [x, x+ru] \subset U\}$.

Lemma 2.4. *Let U be an open subset of $\mathbb{R}^d$ and $u \in S^{d-1}$. Then for all functions $f \in BV_u(U)$ and $r \in \mathbb{R}$,*

$$\int_{U \ominus [0,ru]} |f(x+ru) - f(x)| dx \leq |r||D_u f|(U).$$

Proof. The proof is based on the one of Lemma 3.24 p. 133 of [AFP00] but is slightly more precise. Let us first suppose that $f \in \mathcal{C}^1(U)$. Then, since

$$f(x+ru) - f(x) = \int_0^1 \frac{\partial f}{\partial u}(x + rtu) r dt,$$

one has

$$\begin{aligned}\int_{U \ominus [0,r]} |f(x+ru) - f(x)| dx &= \int_{U \ominus [0,ru]} \left| \int_0^1 \frac{\partial f}{\partial u}(x+rtu) r dt \right| dx \\ &\leq |r| \int_0^1 \int_{U \ominus [0,ru]} \left| \frac{\partial f}{\partial u}(x+rtu) \right| dx dt \\ &\leq |r| \int_0^1 |D_u f| \left(rtu + U \ominus [0, ru] \right) dt.\end{aligned}$$

Now for all $t \in [0,1]$, $rtu + U \ominus [0,ru] \subset U$ and thus for all $t \in [0,1]$, $|D_u f| \left(rtu + U \ominus [0,ru] \right) \leq |D_u f| \left(U \right)$. Hence,

$$\int_{U \ominus [0,ru]} |f(x+ru) - f(x)| dx \leq |r||D_u f| \left(U \right).$$

This inequality extends to all functions in $BV_u(U)$ thanks to the density of smooth functions in $BV_u(U)$: for all $f \in BV_u(U)$ there exists a sequence $(f_n)_n$, $f_n \in \mathcal{C}^1(U)$, such that (f_n) tends to f in $L^1(U)$ and the total directional variation $|D_u f_n|(U) = \int_U \left|\frac{\partial f}{\partial u}\right|$ tends to $|D_u f|(U)$ (using the directional equivalent of [AFP00, Theorem 3.9 p. 122]). $\square$

Let us recall the definition of weak* for signed Radon measure [AFP00, p. 26-27]. Let $\mathcal{C}_0^0(U,\mathbb{R})$ denote the closure of the space of continuous and compactly supported functions $\mathcal{C}_c^0(U,\mathbb{R})$ for the sup norm. Then one says that a sequence $(\mu_n)_{n\in\mathbb{N}}$ of signed Radon measure on U weakly* converges to the signed Radon measure μ is for every $\varphi \in \mathcal{C}_0^0(U,\mathbb{R})$,

$$\lim_{n\to+\infty} \int_U \varphi(x)\mu_n(dx) = \int_U \varphi(x)\mu(dx).$$

The next theorem provides a characterization of function of finite directional variation in terms of difference quotient.

Theorem 2.5. *Let U be an open subset of $\mathbb{R}^d$, $u \in S^{d-1}$ and $f \in L^1(U)$. Then, the three following assertions are equivalent:*

(i) $f \in BV_u(U)$.

(ii) The family of signed measures

$$\mu_r : A \mapsto \int_{U\ominus[0,ru]} \frac{f(x+ru)-f(x)}{r} \mathbb{1}_A(x)dx, \quad A \in \mathcal{B}(U), \ r \neq 0,$$

weakly converges to some signed Radon measure μ.*

(iii) $\liminf_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx < +\infty.$

If any of these assertions holds, then $\mu = D_u f$ and

$$|D_u f|(U) = \lim_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx. \tag{5}$$

Besides, Formula (5) is also valid in the degenerate case: f is not in $BV_u(U)$ if and only if

$$\lim_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx = +\infty.$$

Proof. Let us begin with a preliminary observation: For all $\varphi \in \mathcal{C}_c^1(U,\mathbb{R})$,

$$\lim_{r\to 0} \int_U \varphi(x)\mu_r(dx) = -\int_U \frac{\partial\varphi}{\partial u}(x) f(x)dx. \tag{6}$$

Indeed, for $\varphi \in \mathcal{C}_c^1(U,\mathbb{R})$ and $r \neq 0$,

$$\begin{aligned}
\int_U \varphi(x)\mu_r(dx) &= \int_{U\ominus[0,ru]} \varphi(x)\frac{f(x+ru)-f(x)}{r} dx \\
&= \frac{1}{r}\int_{U\ominus[-ru,0]} \varphi(x-ru)f(x)dx - \frac{1}{r}\int_{U\ominus[0,ru]} \varphi(x)f(x)dx \\
&= \int_U \frac{\varphi(x-ru)\mathbb{1}_{U\ominus[-ru,0]}(x) - \varphi(x)\mathbb{1}_{U\ominus[0,ru]}(x)}{r} f(x)dx.
\end{aligned}$$

Hence, since for all $x \in U$,

$$\lim_{r \to 0} \frac{\varphi(x - ru)\mathbb{1}_{U \ominus [-ru,0]}(x) - \varphi(x)\mathbb{1}_{U \ominus [0,ru]}(x)}{r} = -\frac{\partial \varphi}{\partial u}(x)$$

and for all r such that $|r|$ is small enough such that $\mathrm{supp}(\varphi) \subset U \ominus [0, ru]$,

$$\frac{\varphi(x - ru)\mathbb{1}_{U \ominus [-ru,0]}(x) - \varphi(x)\mathbb{1}_{U \ominus [0,ru]}(x)}{r} f(x) = \frac{\varphi(x - ru) - \varphi(x)}{r} f(x) \le \left\| \frac{\partial \varphi}{\partial u} \right\|_\infty f(x) \in L^1(U),$$

one obtains (6) by dominated convergence.

We will prove $(i) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$.

$(i) \Rightarrow (iii)$ is immediate thanks to Lemma 2.4.

Let us show $(iii) \Rightarrow (ii)$. Remark that for all $r \neq 0$,

$$\int_{U \ominus [0,ru]} \frac{|f(x + ru) - f(x)|}{|r|} dx = |\mu_r|(U).$$

Since by hypothesis $\liminf_{r \to 0} |\mu_r|(U) < +\infty$, there exists a sequence (r_n) converging to 0 such that

$$\lim_{n \to +\infty} \int_{U \ominus [0,ru]} \frac{|f(x + r_n u) - f(x)|}{|r_n|} dx = \liminf_{r \to 0} \int_U \frac{|f(x + ru) - f(x)|}{|r|} dx.$$

Since a convergent sequence is bounded, there exists $C > 0$ such that for all $n \in \mathbb{N}$, $|\mu_{r_n}(U)| \le C$. By weak* compactness [AFP00, Theorem 1.59 p. 26], there exists a subsequence $(r_{n_k})_{k \in \mathbb{N}}$ such that $(\mu_{r_{n_k}})$ weakly* converges to some signed Radon measure μ. Hence, by definition of the weak* convergence, for all $\psi \in \mathcal{C}_c^0(U, \mathbb{R})$,

$$\int_U \psi(x)\mu(dx) = \lim_{k \to +\infty} \int_U \psi(x)\mu_{r_{n_k}}(dx).$$

But, according to (6), for all $\varphi \in \mathcal{C}_c^1(U, \mathbb{R})$,

$$\int_U \varphi(x)\mu(dx) = -\int_U \frac{\partial \varphi}{\partial u}(x) f(x) dx = \lim_{r \to 0} \int_U \varphi(x)\mu_r(dx).$$

By the density of $\mathcal{C}_c^1(U, \mathbb{R})$ in $\mathcal{C}_c^0(U, \mathbb{R})$ for the uniform convergence, the above equality $\int_U \varphi\mu = \lim \int_U \varphi\mu_r$ extends to all $\varphi \in \mathcal{C}_c^0(U, \mathbb{R})$, which shows that the whole family (μ_r) weakly* converges to μ.

Let us now show that $(ii) \Rightarrow (i)$. Since (μ_r) weakly* converges to μ and according to (6), for all $\varphi \in \mathcal{C}_c^1(U, \mathbb{R})$,

$$\int_U \varphi(x)\mu(dx) = \lim_{r \to 0} \int_U \varphi(x)\mu_r(dx) = -\int_U \frac{\partial \varphi}{\partial u}(x) f(x) dx$$

that is $f \in BV_u(U)$ and $D_u f = \mu$.

It remains to show Equation (5). Thanks to the inequality of Lemma 2.4, it is enough to show that

$$|D_u f|(U) \le \liminf_{r \to 0} \int_{U \ominus [0,ru]} \frac{|f(x + ru) - f(x)|}{|r|} dx.$$

But this inequality translates in $|\mu|(U) \le \liminf |\mu_r|(U)$ which is immediate since the total variation application $\nu \mapsto |\nu|(U)$ is lower-semicontinuous with respect to the weak* convergence [AFP00, Theorem 1.59 p. 26]. □

3 Random Fields of Bounded Variation

3.1 Integrable Random Fields

Following the generally approved definition, a *random field* (r.f.) of $\mathbb{R}^d$ is a a family of random variables $\xi_x : (\Omega, \mathcal{A}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ indexed by $x \in \mathbb{R}^d$ (see e.g. [Doo53, p. 46] or [GS74, p. 41]). However, the study of the variation of a r.f. requires more constraint, namely that the r.f. sample paths are integrable functions.

Definition 3.1 (Integrable random field)**.** *Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space and U an open subset of $\mathbb{R}^d$. An* integrable random field *f defined on U is a random variable of $L^1(U)$, that is a measurable map*

$$f : (\Omega, \mathcal{A}) \to (L^1(U), \mathcal{B}(L^1(U))),$$

where $L^1(U)$ is endowed with the Borel σ-algebra of the $L^1(U)$-convergence. Similarly a locally integrable random field *f defined on U is a random variable of $L^1_{\text{loc}}(U)$.*

We will speak of a.s. integrable random field to allow the function f to be outside $L^1(U)$ with probability 0. Let us observe that if f is an integrable random field, then the integrals

$$\omega \mapsto \int_{\mathbb{R}^d} f(\omega, x)\varphi(x)dx, \quad \varphi \in \mathcal{C}_c^0\left(\mathbb{R}^d, \mathbb{R}\right),$$

are well-defined random variables and thus $f\mathcal{L}^d$ defines a unique signed Radon measures. Let us also remark that if $f : \left(\Omega \times \mathbb{R}^d, \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^d)\right) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is a jointly measurable random field such that its sample paths are almost surely integrable, then f defines an integrable random field in the sense of the above definition

3.2 Definition of Random Fields of Bounded Variation

Definition 3.2 (Random field of bounded variation)**.** *Let $U \subset \mathbb{R}^d$ be an open set. An a.s. integrable random field $f \in L^1(U)$ is a* random field of bounded variation *in U if there exists some random $\mathbb{R}^d$-valued Radon measure $Df = (D_1 f, \dots, D_d f)$ such that $|Df|(U)$ is a.s. finite and for all $\varphi = (\varphi_1, \dots, \varphi_d) \in \mathcal{C}_c^\infty\left(U, \mathbb{R}^d\right)$,*

$$\int_U f(x) \operatorname{div} \varphi(x) dx = -\sum_{i=1}^d \int_U \varphi_i(x) D_i f(dx) \ \text{a.s.}$$

An a.s. locally integrable r.f. $f \in L^1_{\text{loc}}(U)$ is a r.f. of locally bounded variation *in U if for all $V \subset\subset U$ the restriction of f to V is a r.f. of bounded variation in V.*

Definition 3.3 (**Variation intensity measure**)**.** *Let $U \subset \mathbb{R}^d$ be an open set and f be a r.f. of locally bounded variation in U. Then the intensity measure of the variation $|Df|$, that is the measure $A \mapsto \mathbb{E}(|Df|(A))$, is called the* variation intensity measure *of f and is denoted $\Theta_V(f, \cdot)$.*

The variation intensity measure is an important characteristic of a r.f. f of bounded variation: for all measurable set A, $\Theta_V(f, A)$ is the mean variation of f in A. The mean variation $\Theta_V(f, U)$ on the whole domain U will be called the *mean total variation* of f.

Definition 3.4 (Random field of bounded directional variation)**.** *Let $U \subset \mathbb{R}^d$ be an open set and let $u \in S^{d-1}$. An a.s. integrable random field $f \in L^1(U)$ is a* r.f. of bounded directional variation *in U if there exists some random signed Radon measure $D_u f$ such that $|D_u f|(U)$ is a.s. finite and for all $\varphi \in \mathcal{C}_c^\infty(U,\mathbb{R})$,*

$$\int_{\mathbb{R}^d} f(x)\frac{\partial \varphi}{\partial u}(x)dx = -\int_{\mathbb{R}^d} \varphi(x) D_u f(dx).$$

An a.s. locally integrable r.f. $f \in L^1_{\text{loc}}(U)$ is a r.f. of locally bounded directional variation *in U if for all $V \subset\subset U$ the restriction of f to V is a r.f. of bounded variation in V.*

Definition 3.5 (**Directional variation intensity measure**)**.** *Let $U \subset \mathbb{R}^d$ be an open set, let $u \in S^{d-1}$, and let f be a r.f. of locally bounded directional variation in U. Then the intensity measure of the variation $|D_u f|$, that is the measure $A \mapsto \mathbb{E}(|D_u f|(A))$, is called the* directional variation intensity measure *of f in the direction u and is denoted $\Theta_{V_u}(f,\cdot)$.*

$\Theta_{V_u}(f,U)$ will be called the *mean total directional variation* of f in the direction u. One easily establishes the analog of Proposition 2.3 for the case of r.f. of bounded variation. In particular we have the following formula.

Proposition 3.6 (Integral geometric formula for directional variation intensity measures)**.** *Let $U \subset \mathbb{R}^d$ be an open set, and let f be a r.f. of bounded variation in U. Then f is a r.f. of bounded directional variation in U for all directions $u \in S^{d-1}$, and for all $A \in \mathcal{B}(U)$,*

$$\Theta_V(f,A) = \frac{1}{2\omega_{d-1}}\int_{S^{d-1}} \Theta_{V_u}(f,A)du.$$

Proof. As for Proposition 2.3, if Df is a random $\mathbb{R}^d$-valued Radon measure representing the distributional derivative of f then $\langle Df, u\rangle$ is a random signed Radon measure which represents the directional distributional derivative of f. Hence f is a r.f. of bounded directional variation in U for all directions $u \in S^{d-1}$. The integral geometric formula is obtained in applying Fubini theorem to Formula (4). □

We conclude this section in showing that the proposed definition of r.f. of bounded variation is equivalent with a more instinctive one: a r.f. has bounded variation if its sample paths have a.s. bounded variation.

Proposition 3.7 (Sample paths of r.f. of bounded variation)**.** *Let $f : \Omega \times U \to \mathbb{R}$ be a jointly measurable random field such that $f(\omega,\cdot) \in L^1(U)$ a.s. and let $u \in S^{d-1}$. Then f is a r.f. of bounded directional variation in the direction u (in the sense of Definition 3.4) if and only if its sample paths $x \mapsto f(\omega,x)$ are in $BV_u(U)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Similarly, f is a r.f. of bounded directional variation in $\mathbb{R}^d$ if and only if its sample paths $x \mapsto f(\omega,x)$ are in $BV(U)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

Proof. Let us first show the equivalence for r.f. of bounded directional variation. The direct sense is immediate from the definition: since $D_u f$ is a.s. a Radon measure, the sample paths are a.s. in $BV_u(U)$. Conversely, note $\Omega' \subset \Omega$ the set of $\omega \in \Omega$ for which $x \mapsto f(\omega,x)$ are in $BV_u(U)$. Then, for all $\omega \in \Omega'$, there exists a signed Radon measure $\mu(\omega,\cdot)$ such that for all $\varphi \in \mathcal{C}_c^\infty(U,\mathbb{R})$,

$$\int_U f(\omega,x)\frac{\partial\varphi}{\partial u}(x)dx = -\int_U \varphi(x)\mu(\omega,dx).$$

The only difficulty is to ensure that $\omega \mapsto \mu(\omega,\cdot)$ is a well-defined random signed Radon measure, that is a measurable map. Let (r_n) be a sequence converging to 0. According to Theorem 2.5, for all $\omega \in \Omega'$, $\mu(\omega,\cdot)$ is the weak* limit of the sequence of signed Radon measures

$$\mu_{r_n}(\omega,A) = \int_{U\ominus[0,ru]} \frac{f(\omega,x+r_n u) - f(\omega,x)}{r_n}\mathbb{1}_A(x)dx, \quad A \in \mathcal{B}(U).$$

Hence μ is measurable since it is the a.s. limit of the weakly* convergent sequence of random signed Radon measures $(\mu_n)_{n\in\mathbb{N}}$. The corresponding equivalence for r.f. of bounded variation is straightforward using the fact that $f(\omega,\cdot) \in BV(U)$ if and only if $BV_{e_i}(U)$ for all e_i in the canonical basis (see Proposition 2.3). $\square$

Remark (Notation)**.** Proposition 3.7 extends directly to random fields of locally bounded variation and locally bounded directional variation. In view of Proposition 3.7 we will use the notation $f \in BV(U)$ a.s., $f \in BV_{u,\mathrm{loc}}(U)$ a.s., etc. to express that a r.f. f has bounded variation, locally bounded directional variation, etc.

3.3 Characterization in Terms of Difference Quotient

Let us transcript Theorem 2.5 in the case of random fields to establish an integral expression of the total variation intensity.

Proposition 3.8 (Characterization of r.f. of bounded directional variation)**.** *Let $f \in L^1(U)$ a.s. and $u \in S^{d-1}$. Then the three following assertions are equivalent:*

(i) $f \in BV_u(U)$ *a.s.*

(ii) $\liminf_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx < +\infty$ *a.s.*

(iii) $\lim_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx$ *exists and is finite a.s.*

If any of these conditions holds, then the total variation intensity is

$$\Theta_{V_u}(f,U) = \lim_{r\to 0} \int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(|f(x+ru)-f(x)|\right)}{|r|} dx. \tag{7}$$

Proof. The characterization is a consequence of Theorem 2.5. Let us now turn to the expression of the total variation intensity. According to Theorem 2.5,

$$\lim_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx = |D_u f|(U) \text{ a.s.},$$

and by Lemma 2.4 for all $r \neq 0$,

$$\int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx \leq |D_u f|(U) \text{ a.s..}$$

Hence if the r.v. $|D_u f|(U)$ is integrable, Formula (7) is obtained by dominated convergence and Fubini theorem. If $\mathbb{E}\left(|D_u f|(\mathbb{R}^d)\right) = \Theta_{V_u}\left(f,\mathbb{R}^d\right) = +\infty$, then Formula (7) is still valid since by Fatou lemma

$$\begin{aligned} +\infty = \Theta_{V_u}(f,U) &= \mathbb{E}\left(\liminf_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx\right) \\ &\leq \liminf_{r\to 0} \mathbb{E}\left(\int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx\right). \end{aligned}$$

$\square$

Remark (Degenerate case)**.** Formula (7) can be extended to the degenerate case where the limit on the right-hand side is infinite. However let us precise that this degenerate case is more subtle than in the deterministic case (see Theorem 2.5). Indeed there are two different cases for which the limit of Formula (7) is infinite: either the r.f. f is not of bounded variation or f is of bounded variation but its total variation $|D_u f|(U)$ has infinite expectation. In both cases, it is coherent to say that the mean total variation of the process is infinite. This convention will be used in the remaining of the paper.

3.4 A Sufficient Condition for Locally Bounded Directional Variation

As mentioned in the introduction, we are aware of only one result dealing with the variation of random fields defined over $\mathbb{R}^d$ for $d \geq 2$. This result, which is reproduced below, is due to Ibragimov [Ibr95] and it gives a sufficient condition for a r.f. to be of bounded variation. We will later demonstrate in Section 4 that this sufficient condition is also necessary for r.f. with stationary increments (see Theorem 4.4).

Theorem 3.9 (Ibragimov theorem [Ibr95])**.** *Let $U \subset\subset \mathbb{R}^d$ and let $f : \Omega \times U \to \mathbb{R}$ be a measurable and separable random field. Suppose that there exists $x_0 \in U$ such that $\mathbb{E}(f(x_0)) < +\infty$ and that there exists $K > 0$ such that for any x, $y \in U$,*

$$\mathbb{E}(|f(x) - f(y)|) \leq K|x - y|.$$

Then the realizations of f are a.s. in $BV(U)$, and there is a constant $C > 0$ such that for all $A \in \mathcal{B}(U)$

$$\Theta_V(f, A) \leq C\mathcal{L}^d(A).$$

In the remaining of this section we will establish results similar to Ibragimov theorem for both r.f. of bounded variation and r.f. of bounded directional variation.

Proposition 3.10 (A sufficient condition for locally bounded directional variation)**.** *Let $f : \Omega \times U \to \mathbb{R}$ be a jointly measurable random field such that $f(\omega, \cdot) \in L^1_{\text{loc}}(U)$ a.s. and let $u \in S^{d-1}$. Suppose that there exists a constant $K > 0$ such that for all $r \in \mathbb{R}$ and $x \in U \ominus [0, ru]$,*

$$\mathbb{E}(|f(x + ru) - f(x)|) \leq K|r|. \tag{8}$$

Then f is a r.f. of locally bounded directional variation in the direction u and for all $W \subset\subset U$,

$$\Theta_{V_u}(f, W) \leq K\mathcal{L}^d(W).$$

Proof. Let $W \subset\subset U$. By Fatou lemma and Fubini theorem

$$\begin{aligned}\mathbb{E}\left(\liminf_{r\to 0} \int_{W\ominus[0,ru]} \frac{|f(x + ru) - f(x)|}{|r|} dx\right) &\leq \liminf_{r\to 0} \mathbb{E}\left(\int_{W\ominus[0,ru]} \frac{|f(x + ru) - f(x)|}{|r|} dx\right) \\ &\leq K \liminf_{r\to 0} \mathcal{L}^d(W \ominus [0, ru]) \\ &\leq K\mathcal{L}^d(W) < +\infty.\end{aligned}$$

In particular, $\liminf \int_{W\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|} dx < +\infty$ a.s. and thus $f \in BV_u(W)$ a.s. by Proposition 3.8. This is valid for all $W \subset\subset U$, and thus $f \in BV_{u,\text{loc}}(U)$ a.s. The inequality $\Theta_{V_u}(f, W) \leq K\mathcal{L}^d(W)$ is immediate using Formula (7). $\square$

In the following, Property (8) will be referred to as *Lipschitzness in mean in the direction* u. Let us precise that the upper bound given in Proposition 3.10 is optimal. Indeed, as it will be shown later (see Theorem 4.4), any r.f. f with stationary increments and having locally bounded directional variation satisfies $\Theta_{V_u}(f, W) = K\mathcal{L}^d(W)$ for some constant K.

The previous proposition can be adapted to deal with r.f. of bounded variation. It gives a slightly improved version of Ibragimov theorem since the obtained upper bound on the variation intensity is optimal.

Proposition 3.11 (A sufficient condition for locally bounded variation)**.** *Let* $f : \Omega \times U \to \mathbb{R}$ *be a jointly measurable random field such that* $f(\omega, \cdot) \in L^1_{\text{loc}}(U)$ *a.s. Suppose that there exists a constant* $K > 0$ *such that for all* $x, y \in U$*,*

$$\mathbb{E}\left(|f(x) - f(y)|\right) \leq K|x - y|.$$

Then f *is a r.f. of locally bounded variation, and for all* $W \subset\subset U$*,*

$$\Theta_V(f, W) \leq \frac{d\omega_d}{2\omega_{d-1}} K\mathcal{L}^d(W).$$

Proof. By Proposition 3.10, f has a.s. bounded directional variation in the d directions of the canonical basis, and thus $f \in BV(U)$ a.s. (see Proposition 2.3 or [AFP00, Section 3.11]). In addition, Proposition 3.10 shows that for all $W \subset\subset U$ and for all $u \in S^{d-1}$, $\Theta_{V_u}(f, W) \leq K\mathcal{L}^d(W)$. Hence, by Proposition 3.6, for all $U \subset\subset \mathbb{R}^d$,

$$\Theta_V(f, W) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \Theta_{V_u}(f, W)\mathcal{H}^{d-1}(du) \leq \frac{d\omega_d}{2\omega_{d-1}} K\mathcal{L}^d(W),$$

where we used that $\mathcal{H}^{d-1}\left(S^{d-1}\right) = d\omega_d$. □

Similarly to the case of Proposition 3.10, the upper bound of the variation intensity of Proposition 3.11 is shown to be reached for isotropic r.f. with stationary increments.

4 Variation Intensity of Random Fields with Stationary Increments

Let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a jointly measurable random field. f is said to be have stationary increments or to be a stationary increment random field, if for all $y \in \mathbb{R}^d$, both random fields $x \mapsto f(x + y) - f(y)$ and $x \mapsto f(x) - f(0)$ have the same finite-dimensional distributions.

The next theorem, which constitutes the main result of the paper, defines and gives an expression of the directional variation intensity $\theta_{V_u}(f)$ of a stationary increment r.f. f.

Theorem 4.1 (**Definition and computation of the directional variation intensity of a stationary increment r.f.**)**.** *Let* $u \in S^{d-1}$ *and let* $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ *be a jointly measurable stationary increment r.f. such that* $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ *a.s. The following assertions are equivalent:*

(i) $f \in BV_{u,\text{loc}}(\mathbb{R}^d)$ *a.s. and its directional variation intensity measure* $\Theta_{V_u}(f, \cdot)$ *is locally finite.*

(ii) $f \in BV_{u,\text{loc}}(\mathbb{R}^d)$ *a.s. and its directional variation intensity measure* $\Theta_{V_u}(f, \cdot)$ *is proportional to the Lebesgue measure: there exists a constant* $\theta_{V_u}(f) \geq 0$ *such that for all* $A \in \mathcal{B}(\mathbb{R}^d)$*,* $\Theta_{V_u}(f, A) = \theta_{V_u}(f)\mathcal{L}^d(A)$*.*

(iii) $\liminf_{r\to 0} \dfrac{\mathbb{E}(|f(ru)-f(0)|)}{|r|} < +\infty$.

(iv) $\lim_{r\to 0} \dfrac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}$ *exists and is finite.*

If any of the above assertions holds, the constant of proportionality $\theta_{V_u}(f)$ is given by

$$\theta_{V_u}(f) = \liminf_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|} = \lim_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}.$$

The constant $\theta_{V_u}(f)$ is called the directional variation intensity *of f in the direction u. It is the mean amount of directional variation of f per unit volume.*

Proof. We will show the following chain of implications: (ii) $\Leftrightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i). First remark that (iv) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (i) are trivial.

Remark that since f has stationary increments, thanks to Fubini theorem for all $U \subset\subset \mathbb{R}^d$ et $r \neq 0$, one always has

$$\mathbb{E}\left(\int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|}dx\right) = \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}\mathcal{L}^d(U\ominus[0,ru]). \tag{9}$$

Let us show (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (ii). Suppose that $f \in BV_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. and that $\Theta_{V_u}(f,\cdot)$ is locally finite. Let $U \subset\subset \mathbb{R}^d$. According to Lemma 2.4,

$$\int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|}dx \leq |D_u f|(U).$$

By hypothesis, $|D_u f|(U)$ is an L^1-r.v. Hence, one can apply the reverse Fatou lemma,

$$\begin{aligned}
\limsup_{r\to 0} \frac{\mathbb{E}\left(|f(ru)-f(0)|\right)}{|r|}\mathcal{L}^d(U) &= \limsup_{r\to 0} \frac{\mathbb{E}\left(|f(ru)-f(0)|\right)}{|r|}\mathcal{L}^d(U\ominus[0,ru]) \\
&= \limsup_{r\to 0} \mathbb{E}\left(\int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|}dx\right) \\
&\leq \mathbb{E}\left(\limsup_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|}dx\right) \\
&\leq \mathbb{E}\left(|D_u f|(U)\right) \\
&= \Theta_{V_u}(f,U) < +\infty.
\end{aligned}$$

Besides, by Fatou lemma,

$$\begin{aligned}
\Theta_{V_u}(f,U) &= \mathbb{E}\left(|D_u f|(U)\right) \\
&= \mathbb{E}\left(\liminf_{r\to 0} \int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|}dx\right) \\
&\leq \liminf_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}\mathcal{L}^d(U\ominus[0,ru]) \\
&\leq \liminf_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}\mathcal{L}^d(U).
\end{aligned}$$

Hence we have shown that for all $U \subset\subset \mathbb{R}^d$,

$$\limsup_{r\to 0} \frac{\mathbb{E}\left(|f(ru)-f(0)|\right)}{|r|}\mathcal{L}^d(U) \le \Theta_{V_u}\left(f,U\right) \le \liminf_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}\mathcal{L}^d(U).$$

This shows that $\frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}$ has a finite limit, say $\theta_{V_u}(f)$, and that for all bounded open sets U, $\Theta_{V_u}\left(f,U\right) = \theta_{V_u}(f)\mathcal{L}^d(U)$. This equality extends to all Borel sets $A \in \mathcal{B}(\mathbb{R}^d)$ and thus Θ_{V_u} is proportional to the Lebesgue measure.

It remains to show (iii) $\Rightarrow$ (i). Thanks to Fatou lemma and Equation (9),

$$\begin{aligned}\mathbb{E}\left(\liminf_{r\to 0}\int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|}dx\right) &\le \liminf_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}\mathcal{L}^d(U\ominus[0,ru]) \\ &\le \mathcal{L}^d(U)\liminf_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|} < +\infty.\end{aligned}$$

In particular, $\liminf_{r\to 0}\int_{U\ominus[0,ru]} \frac{|f(x+ru)-f(x)|}{|r|}dx$ is almost surely finite, and thus by Proposition 3.8 $f \in BV_u(U)$ a.s. This is valid for all $U \subset\subset \mathbb{R}^d$, and thus $f \in BV_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. Besides, by Formula (5), the above equation reads $\Theta_{V_u}\left(f,U\right) \le \liminf_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}\mathcal{L}^d(U) < +\infty$, and thus $\Theta_{V_u}\left(f,\cdot\right)$ is locally finite. □

Remark (Degenerate Case)**.** As for Proposition 3.8, one extends the definition of $\theta_{V_u}(f)$ to the degenerate case where $\lim \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|} = +\infty$.

Integrating over all the directions, one obtains the equivalent of Theorem 4.1 for non directional variation.

Corollary 4.2 (**Definition and computation of the variation intensity of a stationary increment r.f.**)**.** *$f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a jointly measurable stationary increment r.f. such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. Then $f \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. with a locally finite variation intensity measure $\Theta_V\left(f,\cdot\right)$ if and only if for all $u \in S^{d-1}$ the limit*

$$\lim_{r\to 0} \frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}$$

exists and is finite. In this case the variation intensity measure $\Theta_V\left(f,\cdot\right)$ is proportional to the Lebesgue measure, and the constant of proportionality $\theta_V(f)$ is given by

$$\begin{aligned}\theta_V(f) &= \frac{1}{2\omega_{d-1}}\int_{S^{d-1}} \theta_{V_u}(f)\mathcal{H}^{d-1}(du) \\ &= \frac{1}{2\omega_{d-1}}\int_{S^{d-1}} \lim_{r\to 0} \frac{\mathbb{E}\left(|f(ru)-f(0)|\right)}{|r|}\mathcal{H}^{d-1}(du) \\ &= \lim_{r\to 0} \frac{1}{2\omega_{d-1}}\int_{S^{d-1}} \frac{\mathbb{E}\left(|f(ru)-f(0)|\right)}{|r|}\mathcal{H}^{d-1}(du).\end{aligned} \tag{10}$$

The constant $\theta_V(f)$ is called the variation intensity *of f.*

Proof. The results are straightforward consequences of Theorem 4.1 and the integral geometric formula of Proposition 3.6 which becomes Formula (10) in this context. The fact that the limit and the integral commute follows from the bounded convergence theorem using the bound

$$\frac{\mathbb{E}\left(|f(ru)-f(0)|\right)}{|r|} \le \theta_{V_u}(f)\frac{\mathcal{L}^d\left(B(0,1)\right)}{\mathcal{L}^d(B(0,1)\ominus[0,ru])} \le \theta_{V_u}(f)\frac{1}{(1-R)^d}, \quad r \in [-R,R]\setminus\{0\},\ R<1,$$

given by Lemma 2.4 and Theorem 4.1. □

Let us remark that the directional variation intensities $\theta_{V_u}(f)$, $u \in S^{d-1}$, as well as the variation intensity $\theta_V(f)$ only depend on the two-dimensional distributions of the stationary increment r.f. f. In addition, if the random field f has almost surely $\mathbb{C}^1$ sample paths, then one simply has $\theta_V(f) = \mathbb{E}(\|\nabla f(0)\|)$ and $\theta_{V_u}(f) = \mathbb{E}(|\langle \nabla f(0), u\rangle|)$, $u \in S^{d-1}$.

To finish this section, it is shown that the directional Lipschitzness in mean introduced in Section 3.4 is a necessary and sufficient condition for stationary increment r.f. having finite directional variation intensity.

Lemma 4.3 (Directional Lipschitzness in mean of stationary increment r.f.). *Let $u \in S^{d-1}$ and $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a jointly measurable stationary increment r.f. such that $f \in BV_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. Then for all $x \in \mathbb{R}^d$ and all $r \in \mathbb{R}$,*

$$\mathbb{E}\left(|f(x+ru) - f(x)|\right) \leq \theta_{V_u}(f)|r|.$$

Proof. First, since f has stationary increments, for all $x \in \mathbb{R}^d$ and $r \in \mathbb{R}$, $\mathbb{E}\left(|f(x+ru) - f(x)|\right) = \mathbb{E}\left(|f(ru) - f(0)|\right)$. Let $r \neq 0$ and $\rho > 0$. By Lemma 2.4,

$$\int_{B(0,\rho)\ominus[0,ru]} |f(x+ru) - f(x)|\, dx \leq |r|\, |D_u f|\, (B(0,\rho)) \text{ a.s.}$$

Hence, by Fubini theorem

$$\mathbb{E}\left(|f(ru) - f(0)|\right) \mathcal{L}^d(B(0,\rho) \ominus [0,ru]) \leq |r|\theta_{V_u}(f)\mathcal{L}^d(B(0,\rho)),$$

and thus,

$$\mathbb{E}\left(|f(ru) - f(0)|\right) \leq |r|\theta_{V_u}(f)\frac{\mathcal{L}^d(B(0,\rho))}{\mathcal{L}^d(B(0,\rho) \ominus [0,ru])}.$$

Letting $\rho \to +\infty$ ones obtains the announced inequality. $\square$

Combining the results of Proposition 3.10 and Lemma 4.3 we obtain that a stationary increment r.f. has finite directional variation intensity if and only if it is directionally Lipschitz in mean.

Theorem 4.4 (Characterization of stationary increment r.f. with locally bounded directional variation via directional Lipschitzness in mean). *Let $u \in S^{d-1}$ and let $f : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a jointly measurable stationary increment r.f. such that $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. The three following assertions are equivalent:*

(i) f has locally bounded variation in the direction u and its directional variation intensity $\theta_{V_u}(f)$ is finite.

(ii) There exists a constant $K > 0$ such that

$$\mathbb{E}\left(|f(x+ru) - f(x)|\right) \leq K|r|, \quad x \in \mathbb{R}^d,\ r \in \mathbb{R}.$$

(iii) There exists a constant $K > 0$ and a real $R > 0$ such that

$$\mathbb{E}\left(|f(ru) - f(0)|\right) \leq K|r|, \quad r \in [0,R).$$

Besides the directional variation intensity $\theta_{V_u}(f)$ is the least constant K such that (ii) and (iii) hold.

Proof. (i) $\Rightarrow$ (ii) has already been shown with Lemma 4.3: if f has locally bounded variation in the direction u with a finite directional variation intensity $\theta_{V_u}(f)$ then (ii) holds with $K = \theta_{V_u}(f)$. Clearly (ii) implies (iii). To finish, if (iii) holds, then

$$\liminf_{r\to 0} \frac{\mathbb{E}(|f(ru) - f(0)|)}{|r|} \leq K < +\infty,$$

and by Theorem 4.1 f has locally bounded variation and $\theta_{V_u}(f) \leq K$. $\square$

As already mentioned in Section 3.4, Theorem 4.4 shows that the upper bound of Proposition 3.10 (resp. Proposition 3.11) becomes an equality for any stationary increment r.f. of bounded directional variation having a finite directional variation intensity (resp. any r.f. with stationary and isotropic increments that has a finite variation intensity).

5 The Case of Random Sets

In this section we apply the general results of the previous sections to the special case of random indicator functions, which leads to extension of classical results regarding the mean perimeter of random sets. Propositions 5.3 and 5.5 were first established by the author in [Gal11], but the proposed proofs were then unnecessary complicated since the general results of the previous section were not at hand. As discussed in [Gal11], these results generalized known formulas first established by Matheron and widely stated in the mathematical morphology literature [HMS67, Mat75, Ser82, Lan02].

5.1 Random Measurable Sets

The classical framework for the study of random sets is the one of random closed sets (RACS) [Mol05]. The framework of random integrable functions gives another framework for random sets, namely the one of random measurable sets (RAMS). We refer to [GLR14] for a discussion regarding the links between RAMS and RACS. To be more precise, a *random measurable set (RAMS)* X is a measurable map

$$\begin{array}{rcl} X : (\Omega, \mathcal{A}) & \to & \left(\mathcal{B}(\mathbb{R}^d), \mathcal{B}(\mathcal{B}(\mathbb{R}^d))\right), \\ \omega & \mapsto & X(\omega) \end{array}$$

where $\mathcal{B}(\mathcal{B}(\mathbb{R}^d))$ denotes the Borel σ-algebra induced by the local convergence in measure (The local convergence in measure of a sequence $(A)_n$ towards A simply corresponds to the convergence in $L^1_{\text{loc}}(\mathbb{R}^d)$ of the indicator functions $\mathbb{1}_{A_n}$ towards $\mathbb{1}_A$).

Let us recall that the (variational) perimeter $\text{Per}(A)$ of a measurable set A is defined as

$$\text{Per}(A) = \begin{cases} |D\mathbb{1}_A|(\mathbb{R}^d) & \text{if } \mathbb{1}_A \in BV(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

Similarly the directional variation $V_u(A)$ in the direction $u \in S^{d-1}$ of A is

$$V_u(A) = \begin{cases} |D_u\mathbb{1}_A|(\mathbb{R}^d) & \text{if } \mathbb{1}_A \in BV_u(\mathbb{R}^d), \\ +\infty & \text{otherwise.} \end{cases}$$

5.2 Variation of Random Sets and Mean Covariogram

In this section we will consider RAMS X for which $\mathbb{E}(\mathcal{L}^d(X)) < +\infty$.

Definition 5.1 (Mean covariogram of a random measurable set). *Let X be a RAMS of $\mathbb{R}^d$ having finite mean Lebesgue measure, that is $\mathbb{E}(\mathcal{L}^d(X)) < +\infty$. The* mean covariogram γ_X *of X is the function $\gamma_X : \mathbb{R}^d \to [0, \infty[$ defined by*

$$\gamma_X(y) = \mathbb{E}\left(\mathcal{L}^d\left(X \cap (y + X)\right)\right) = \int_{\Omega\times\mathbb{R}^d} \mathbb{1}_{X(\omega)}(x)\mathbb{1}_{X(\omega)}(x+y)\mathbb{P}(d\omega)dx.$$

Lemma 5.2 (Matheron [Mat86]). *Let X be a RAMS of $\mathbb{R}^d$ having finite mean Lebesgue measure and let γ_X be its mean covariogram. Then for all $y \in \mathbb{R}^d$*

$$\gamma_X(0) - \gamma_X(y) = \frac{1}{2}\mathbb{E}\left(\int_{\mathbb{R}^d} |\mathbb{1}_X(x+y) - \mathbb{1}_X(x)|\, dx\right).$$

Proof. One simply has to take expectation in the following identity:

$$\int_{\mathbb{R}^d} |\mathbb{1}_X(x+y) - \mathbb{1}_X(x)|\, dx = \int_{\mathbb{R}^d} (\mathbb{1}_X(x+y) - \mathbb{1}_X(x))^2\, dx = 2\left(\mathcal{L}^d(X) - \mathcal{L}^d(X \cap (y+X))\right).$$

□

Proposition 5.3 (Mean covariogram and mean variation). *Let X be a random set of $\mathbb{R}^d$ satisfying $\mathbb{E}\left(\mathcal{L}^d(X)\right) < +\infty$ and let γ_X be its mean covariogram. Then for all $u \in S^{d-1}$,*

$$\lim_{r\to 0} \frac{\gamma_X(0) - \gamma_X(ru)}{|r|} = \frac{1}{2}\mathbb{E}\left(V_u(X)\right),$$

and, noting $(\gamma_X^u)'(0) = \lim_{r\to 0^+} \dfrac{\gamma_X(ru) - \gamma_X(0)}{r}$,

$$-\frac{1}{\omega_{d-1}}\int_{S^{d-1}} (\gamma_X^u)'(0)\mathcal{H}^{d-1}(du) = \mathbb{E}\left(\mathrm{Per}(X)\right).$$

Proof. By Lemma 5.2,

$$\frac{\gamma_X(0) - \gamma_X(ru)}{|r|} = \frac{1}{2}\mathbb{E}\left(\int_{\mathbb{R}^d} \frac{|\mathbb{1}_X(x+ru) - \mathbb{1}(x)|}{|r|}\right),$$

hence the first formula is just a transcription of the identity of Proposition 3.8

$$\Theta_{V_u}(f, U) = \lim_{r\to 0}\int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(|f(x+ru) - f(x)|\right)}{|r|}dx,$$

with $f = \mathbb{1}_X$ and $U = \mathbb{R}^d$. The second formula is obtained by integration over all the directions $u \in S^{d-1}$ and by applying the dominated convergence theorem, since by Lemma 2.4,

$$\left|\frac{\gamma_X(ru) - \gamma_X(0)}{r}\right| = \frac{1}{2}\mathbb{E}\left(\int_{\mathbb{R}^d} \frac{|\mathbb{1}_X(x+ru) - \mathbb{1}(x)|}{|r|}\right) \le \frac{1}{2}\mathbb{E}\left(V_u(X)\right).$$

□

5.3 Specific Variation and Variogram of Stationary Random Sets

In this section, we call *stationary* any random set $X : \Omega \to \mathcal{B}(\mathbb{R}^d)$ such that the map $(\omega, x) \mapsto \mathbb{1}_{X(\omega)}(x)$ is a jointly measurable and stationary random field.

A stationary random set is said to be of *locally bounded variation* if the stationary random field $\mathbb{1}_X : (\omega, x) \mapsto \mathbb{1}_{X(\omega)}(x)$ is of locally bounded variation, and one defines similarly random sets of *locally bounded directional variation.* One writes $\theta_V(X) := \theta_V(\mathbb{1}_X)$ which is referred to as the *variation intensity* or the *specific variation* of the stationary random set X. Similarly, $\theta_{V_u}(X) := \theta_{V_u}(\mathbb{1}_X)$ is called the *directional variation intensity* or the *specific directional variation* in the direction u.

Definition 5.4 (Variogram of a stationary random set)**.** *Let X be a stationary random set. The variogram ν_X of X is the function $\nu_X : \mathbb{R}^d \to \mathbb{R}$ defined by*

$$\nu_X(y) = \mathbb{P}\left(y \in X,\ 0 \notin X\right).$$

Clearly $\nu_X(0) = 0$ and the variogram ν_X is even: indeed, by stationarity

$$\nu_X(-y) = \mathbb{P}\left(0 \notin X\right) - \mathbb{P}\left(-y \notin X,\ 0 \notin X\right) = \mathbb{P}\left(0 \notin X\right) - \mathbb{P}\left(0 \notin X,\ y \notin X\right) = \nu_X(y).$$

As the next proposition shows, the variogram $\nu_X(y)$ plays the same role for stationary random sets as the difference $\gamma_X(0) - \gamma_X(y)$ for random sets of finite mean Lebesgue measure.

Proposition 5.5 (Specific variations and variogram)**.** *Let X be a stationary random set and let ν_X be its variogram. Then for all $u \in S^{d-1}$ the limit*

$$\left(\nu_X^u\right)'(0) := \lim_{r\to 0} \frac{1}{|r|}\nu_X(ru) \in [0, +\infty]$$

exists, and the specific directional variation $\theta_{V_u}(X)$ is given by

$$\theta_{V_u}(X) = 2\left(\nu_X^u\right)'(0) = 2\lim_{r\to 0}\frac{1}{|r|}\mathbb{P}\left(ru \in X, 0 \notin X\right).$$

In other words, the specific directional variation is twice the directional derivative of the variogram at the origin. Integrating over all directions, one obtains the specific variation of X:

$$\theta_V(X) = \frac{1}{\omega_{d-1}}\int_{S^{d-1}} \left(\nu_X^u\right)'(0)\mathcal{H}^{d-1}(du). \tag{11}$$

Proof. Since X is a random set $|\mathbb{1}_X(ru) - \mathbb{1}_X(0)| \in \{0, 1\}$, hence for all $r \in \mathbb{R}$,

$$\mathbb{E}(|\mathbb{1}_X(ru) - \mathbb{1}_X(0)|) = \mathbb{P}(ru \in X,\ 0 \notin X) + \mathbb{P}(ru \notin X,\ 0 \in X) = 2\nu_X(ru).$$

By Theorem 4.1, the limit $\lim_{r\to 0} \frac{1}{|r|}\nu_X(ru)$ exists and

$$\theta_{V_u}(X) = 2\lim_{r\to 0}\frac{1}{|r|}\nu_X(ru),$$

and the formula of $\theta_V(X)$ is given by Corollary 4.2. $\square$

5.4 Random Excursion Sets and Coarea Formula for Mean Variations

First let us recall the coarea formula for deterministic functions. In what follows, U is an open subset of $\mathbb{R}^d$. Recall that for any measurable set A, one defines the *perimeter* of A in U as the variation of the indicator function $\mathbb{1}_A$ in U, and one writes $\mathrm{Per}(A, U) := |D\mathbb{1}_A|(U)$. Similarly, one defines $V_u(A, U) := |D_u\mathbb{1}_A|(U)$ the directional variation of A in U. If $f : U \to \mathbb{R}$ is a measurable function and $t \in \mathbb{R}$, $\{f > t\}$ denotes the set $\{x \in U,\ f(x) > t\}$ and is called the *(upper) level set* of level t.

Proposition 5.6 (Coarea formula)**.** *Let $f \in L^1(U)$. Then $f \in BV(U)$ if and only if the sets $\{f > t\}$ are of finite perimeter for $\mathcal{L}^1$-a.a. $t \in \mathbb{R}$ and the function $t \mapsto \mathrm{Per}(\{f > t\}, U)$ is in $L^1(\mathbb{R})$, and in this case*

$$|Df|(U) = \int_{-\infty}^{+\infty} \mathrm{Per}(\{f > t\}, U)\, dt.$$

We refer to [AFP00, p. 145] for the proof of the coarea formula. Let us mention that the coarea formula remains valid if the upper level sets are replaced by other level sets: $\{f \geq t\}$, $\{f < t\}$ or $\{f \leq t\}$. Besides, a coarea formula also holds for directional variation:

$$|D_u f|(U) = \int_{-\infty}^{+\infty} D_u(\{f > t\}, U)\, dt,\ u \in S^{d-1}.$$

Using the coarea formula (see Proposition 5.6), one obtains a relation between the variation intensity of f and the variation intensity of its level sets. Let us recall that by definition,

$$\Theta_{V_u}(f, U) = \mathbb{E}\left(|D_u f|(U)\right).$$

Proposition 5.7. *(Coarea formula for total variation intensity) Let U be an open subset of $\mathbb{R}^d$, let f be a r.f. a.s. in $L^1(U)$, and let $u \in S^{d-1}$. Then $f \in BV_u(U)$ a.s. with finite mean total directional variation $\Theta_{V_u}(f, U)$ if and only if for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ its level sets $\{f > t\}$ have a.s. finite directional variation in U in the direction u and $t \mapsto \Theta_{V_u}(\{f > t\}, U)$ is in $L^1(\mathbb{R})$, and in this case*

$$\Theta_{V_u}(f, U) = \int_{-\infty}^{+\infty} \Theta_{V_u}(\{f > t\}, U)\, dt.$$

Similarly, $f \in BV(U)$ a.s. with finite mean total variation if and only if for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ its level sets $\{f > t\}$ have a.s. finite variation and $t \mapsto \Theta_V(\{f > t\}, U)$ is in $L^1(\mathbb{R})$, and in this case

$$\Theta_V(f, U) = \int_{-\infty}^{+\infty} \Theta_V(\{f > t\}, U)\, dt.$$

Proof. The proof consists in apply Fubini and Lebesgue theorem. First, let us justify that the function

$$\begin{array}{rcl} g : \Omega \times U \times \mathbb{R} & \to & \{0, 1\} \\ (\omega, x, t) & \mapsto & \mathbb{1}_{\{f > t\}}(\omega, x) \end{array}$$

is measurable. Let $(t_n)_{n \in \mathbb{N}}$ be a dense sequence of $\mathbb{R}$, then one easily checks that

$$g^{-1}(1) = \bigcup_{n \in \mathbb{N}} \{(\omega, x),\ f(\omega, x) > t_n\} \times (t_n, +\infty),$$

which is in the product σ-algebra $\mathcal{A} \otimes \mathcal{B}(U) \otimes \mathcal{B}(\mathbb{R})$ since f is jointly measurable. Second, we have the following elementary identity

$$|f(\omega, x + ru) - f(\omega, x)| = \int_{-\infty}^{+\infty} \left|\mathbb{1}_{\{f > t\}}(\omega, x + ru) - \mathbb{1}_{\{f > t\}}(\omega, x)\right| dt.$$

Hence, by Fubini theorem

$$\int_{U\ominus[0,ru]} \mathbb{E}\left(|f(x+ru)-f(x)|\right) dx = \int_{-\infty}^{+\infty}\int_{U\ominus[0,ru]} \mathbb{E}\left(\left|\mathbb{1}_{\{f>t\}}(x+ru)-\mathbb{1}_{\{f>t\}}(x)\right|\right) dxdt.$$

Let us now suppose that for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ the level sets $\{f > t\}$ have a.s. finite directional variation in the direction u and that $t \mapsto \Theta_{V_u}(\{f > t\}, U)$ is in $L^1(\mathbb{R})$. Then, by Proposition 3.8,

$$\lim_{r\to 0}\int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(\left|\mathbb{1}_{\{f>t\}}(x+ru)-\mathbb{1}_{\{f>t\}}(x)\right|\right)}{|r|} dx = \Theta_{V_u}(\{f>t\},U) \quad \text{for } \mathcal{L}^1\text{-a.e. } t,$$

and by Lemma 2.4,

$$\int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(\left|\mathbb{1}_{\{f>t\}}(x+ru)-\mathbb{1}_{\{f>t\}}(x)\right|\right)}{|r|} dx \leq \Theta_{V_u}(\{f>t\},U) \in L^1(\mathbb{R}).$$

Hence Lebesgue theorem applies and

$$\lim_{r\to 0}\int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(|f(x+ru)-f(x)|\right)}{|r|} dx = \int_{-\infty}^{+\infty}\Theta_{V_u}(\{f>t\},U)\, dt < +\infty.$$

By Proposition 3.8 one deduces that f has a.s. bounded directional variation and that

$$\Theta_{V_u}(f,U) = \int_{-\infty}^{+\infty}\Theta_{V_u}(\{f>t\},U)\, dt.$$

Let us now prove the converse implication. Suppose that $f \in BV_u(U)$ a.s. with $\Theta_{V_u}(f,U) < +\infty$. Then, by Fatou lemma and Fubini theorem

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\liminf_{r\to 0}\int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(\left|\mathbb{1}_{\{f>t\}}(x+ru)-\mathbb{1}_{\{f>t\}}(x)\right|\right)}{|r|} dxdt \\
&\leq \liminf_{r\to 0}\int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(|f(x+ru)-f(x)|\right)}{|r|} dx \\
&= \Theta_{V_u}(f,U) < +\infty.
\end{aligned}$$

In particular, for L^1-a.e. $t \in \mathbb{R}$,

$$\liminf_{r\to 0}\int_{U\ominus[0,ru]} \frac{\left|\mathbb{1}_{\{f>t\}}(x+ru)-\mathbb{1}_{\{f>t\}}(x)\right|}{|r|} dx < +\infty \text{ a.s.}$$

Hence, for L^1-a.e. $t \in \mathbb{R}$, Proposition 3.8 ensures that $\{f > t\}$ has a.s. locally bounded variation in U in the direction u, and that

$$\Theta_{V_u}(\{f>t\},U) = \lim_{r\to 0}\int_{U\ominus[0,ru]} \frac{\mathbb{E}\left(\left|\mathbb{1}_{\{f>t\}}(x+ru)-\mathbb{1}_{\{f>t\}}(x)\right|\right)}{|r|} dx.$$

Besides, the above inequality shows that $\int_{-\infty}^{+\infty}\Theta_{V_u}(\{f>t\},U) \leq \Theta_{V_u}(f,U) < +\infty$, that is $t \mapsto \Theta_{V_u}(\{f>t\},U) \in L^1(\mathbb{R})$. To finish, the case of non directional variation easily follows from the integral geometric formula of Proposition 3.6 and Fubini theorem. □

Since the level sets $\{f > t\}$, $t \in \mathbb{R}$, of a stationary r.f. f are stationary random sets, when dealing with stationary r.f., Proposition 5.7 gives a similar results for variation intensities.

Proposition 5.8. *(Coarea formula for variation intensity of stationary r.f.) Let f be an integrable r.f. and let $u \in S^{d-1}$. Then $f \in BV_{u,\mathrm{loc}}(\mathbb{R}^d)$ a.s. with finite directional variation intensity $\theta_{V_u}(f)$ if and only if for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ its level sets $\{f > t\}$ have a.s. locally finite directional variation in the direction u and $t \mapsto \theta_{V_u}(\{f > t\})$ is in $L^1(\mathbb{R})$, and in this case*

$$\theta_{V_u}(f) = \int_{-\infty}^{+\infty} \theta_{V_u}(\{f > t\}) dt.$$

Similarly, $f \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. with finite mean total directional variation $\theta_V(f)$ if and only if for $\mathcal{L}^1$-a.e. $t \in \mathbb{R}$ its level sets $\{f > t\}$ have a.s. locally finite variation and $t \mapsto \theta_V(\{f > t\})$ is in $L^1(\mathbb{R})$, and in this case

$$\theta_V(f) = \int_{-\infty}^{+\infty} \theta_V(\{f > t\})\, dt.$$

Proof. Apply proposition 5.7 and use the fact that $\Theta_{V_u}(f, U) = \theta_{V_u}(f)\, \mathcal{L}^d(U)$ and $\Theta_{V_u}(\{f > t\}, U) = \theta_{V_u}(\{f > t\})\mathcal{L}^d(U)$. □

6 Illustration: Variation Intensities of Some Classical Random Field Models

Section 6 illustrates the different results of Section 4 and 5 by computing the variation intensities of various r.f. and random set models, namely Gaussian random fields, Gaussian excursion sets, Poisson shot noise of random sets, Boolean models, colored dead leaves r.f., transparent dead leaves process, and colored tessellations. Most of the obtained formulas remains valid in the degenerate cases, and thus provides a characterization for the finiteness of the variation intensities of the considered r.f. or random set models.

6.1 Gaussian Random Fields and Gaussian Excursion Sets

A jointly measurable r.f. f_{G} is a *stationary Gaussian r.f.* with mean $\mu \in \mathbb{R}$ and covariance function $C : \mathbb{R}^d \to \mathbb{R}$ if for all $p \in \mathbb{N}$, $x_1, \dots, x_p \in \mathbb{R}^d$, and $w_1, \dots, w_p \in \mathbb{R}$ the r.v. $\sum_{i=1}^{p} w_i f_{\mathrm{G}}(x_i)$ is normal with mean $\sum_{i=1}^{p} w_i \mu$ and variance $\sum_{i,j=1}^{p} w_i w_j C(x_j - x_i)$. Let us mention that the measurability assumption is not restrictive since in what follows we will only consider stationary Gaussian r.f. whose covariance function is regular at the origin, and as soon as the covariance function C is continuous in 0, the r.f. is continuous in probability and thus it has a measurable version [Doo53, p. 61] [GS74, p. 171].

The computation of the directional and non directional variation intensities of f_{G} follows from the following lemma.

Lemma 6.1. *Let f_{G} be a stationary Gaussian r.f. with mean μ and covariance function C. Then for all $u \in S^{d-1}$ and $r \in \mathbb{R}$,*

$$\mathbb{E}\left(|f_{\mathrm{G}}(ru) - f_{\mathrm{G}}(0)|\right) = \frac{2}{\sqrt{\pi}} \sqrt{C(0) - C(ru)}.$$

Proof. We recall that if a r.v. Y has distribution $\mathcal{N}\left(0, \sigma^2\right)$ then $\mathbb{E}(|Y|) = \sqrt{\frac{2}{\pi}}\sigma$. Let f_{G} be a stationary r.f. with mean μ and covariance function C. Then for all $u \in S^{d-1}$ and $r \in \mathbb{R}$,

$f_{\mathrm{G}}(ru) - f_{\mathrm{G}}(0)$ follows a normal distribution with mean 0 and variance $2(C(0) - C(ru))$. Hence

$$\mathbb{E}\left(|f_{\mathrm{G}}(ru) - f_{\mathrm{G}}(0)|\right) = \sqrt{\frac{2}{\pi}}\sqrt{2(C(0) - C(ru))} = \frac{2}{\sqrt{\pi}}\sqrt{C(0) - C(ru)}.$$

□

Proposition 6.2 (Variation intensity of stationary Gaussian r.f.). *Let f_{G} be a stationary Gaussian r.f. with mean μ and covariance C, and let $u \in S^{d-1}$. Then f_{G} has finite directional variation intensity $\theta_{V_u}(f_{\mathrm{G}})$ in the direction u if and only if the one-dimensional restriction of the covariance $C_u : r \mapsto C(ru)$ is twice differentiable at 0, and in this case, noting $C_u''(0)$ the second derivative in 0 of C_u,*

$$\theta_{V_u}(f_{\mathrm{G}}) = \sqrt{\frac{-2C_u''(0)}{\pi}}.$$

Consequently, f_{G} has finite variation intensity $\theta_V(f_{\mathrm{G}})$ if and only if for all $u \in S^{d-1}$ the one-dimensional restrictions of the covariance $C_u : r \mapsto C(ru)$ are twice differentiable at 0, and in this case,

$$\theta_V(f_{\mathrm{G}}) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \sqrt{\frac{-2C_u''(0)}{\pi}} \mathcal{H}^{d-1}(du).$$

Proof. This is a straightforward application of Theorem 4.1. By Lemma 6.1,

$$\lim_{r\to 0} \frac{\mathbb{E}\left(|f_{\mathrm{G}}(ru) - f_{\mathrm{G}}(0)|\right)}{|r|}$$

exists if and only if $\frac{C(0)-C(ru)}{r^2}$ admits a limit in 0, that is if and only if $C_u : r \mapsto C(ru)$ is twice differentiable at 0 with $C_u'(0) = 0$. But since C_u is even, if it is differentiable at 0 then necessarily $C_u'(0) = 0$. As for the expression of $\theta_{V_u}(f_{\mathrm{G}})$, note that

$$\lim_{r\to 0} \frac{C(0) - C(ru)}{r^2} = -\frac{C_u''(0)}{2},$$

hence by Theorem 4.1 and Lemma 6.1,

$$\theta_{V_u}(f_{\mathrm{G}}) = \lim_{r\to 0} \frac{\mathbb{E}\left(|f_{\mathrm{G}}(ru) - f_{\mathrm{G}}(0)|\right)}{|r|} = \sqrt{\frac{-2C_u''(0)}{\pi}}.$$

The case of non directional variation follows from Corollary 4.2. □

It is worth noticing that the necessary and sufficient condition for a stationary Gaussian r.f. to be of locally bounded variation implies a stronger regularity on the sample paths than just being of locally bounded variation. First, the differentiability at the origin of the covariance implies that there exists $\rho > 0$ and $K > 0$ such that for all $x \in B(0, \rho)$

$$C(0) - C(x) \leq \frac{K}{|x|}.$$

In particular, there exists $\alpha > 0$ and $K' > 0$ such that for all $x \in B(0, \rho)$

$$C(0) - C(x) \leq \frac{K'}{|\log(|x|)|^{(1+\alpha)}},$$

and thus, according to Adler and Taylor [AT07, Theorem 1.4.1 p. 20], f has a.s. continuous sample paths. In addition, according to [Sch10], if the covariance function C is twice differentiable at the origin in every direction, then the sample paths of a stationary Gaussian r.f. f_{G}

are a.s. in the Sobolev space $W^{1,2}_{\mathrm{loc}}(\mathbb{R}^d) \subset W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$. Hence by Proposition 6.2, $\theta_V(f_{\mathrm{G}}) < +\infty$ implies that $f_{\mathrm{G}} \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ a.s., and consequently its variation intensity measure is absolutely continuous with respect to the Lebesgue measure. Hence, if $f_{\mathrm{G}} \in BV_{\mathrm{loc}}(\mathbb{R}^d)$ a.s. and a finite variation intensity, then it is a "smooth" function among the functions of locally bounded variation since its variation measure Df_{G} has neither a jump part nor a Cantor part (see [AFP00, Section 3.9] for more details on the decomposition of the variation measure Df of functions of bounded variation). This shows that the space of functions of locally bounded variation is not of interest for stationary Gaussian random fields.

We conclude this section in computing the specific perimeter of the excursion sets of stationary Gaussian r.f. For all $t \in \mathbb{R}$, we consider the random set $\{f_{\mathrm{G}} > t\}$ and we denote by ν_t its variogram, that is the function defined for all $y \in \mathbb{R}^d$ by

$$\nu_t(y) = \mathbb{P}\left(f_{\mathrm{G}}(y) > t,\ f_{\mathrm{G}}(0) \leq t\right).$$

As shown in [Lan02, Proposition 16.1.1],

$$\nu_t(y) = \frac{1}{\pi} \int_0^{\arcsin\left(\sqrt{\frac{C(0)-C(y)}{2C(0)}}\right)} \exp\left(-\frac{t^2}{2C(0)}\left(1+\tan^2 s\right)\right) ds. \tag{12}$$

From this expression Lantuéjoul asserts that the excursion sets of f_{G} have finite "specific perimeter" if and only if $C(0) - C(y)$ is proportional to $|y|^2$ [Lan02, p. 207]. Our next proposition completes this observation in computing the expression of the specific variations of the Gaussian excursion sets $\{f_{\mathrm{G}} > t\}$.

Proposition 6.3 (Specific variation of Gaussian excursion sets)**.** *Let f_{G} be a stationary Gaussian r.f. with mean μ and covariance C, let $t \in \mathbb{R}$, and let $u \in S^{d-1}$. Then $\{f_{\mathrm{G}} > t\}$ has finite specific directional variation in the direction u if and only if the one-dimensional restriction C_u of the covariance is twice differentiable at* 0*, and in this case,*

$$\theta_{V_u}(\{f_{\mathrm{G}} > t\}) = \sqrt{\frac{-2C_u''(0)}{\pi}} \frac{1}{\sqrt{2\pi C(0)}} \exp\left(-\frac{t^2}{2C(0)}\right) = \theta_{V_u}(f_{\mathrm{G}}) p_{f_{\mathrm{G}}}(t),$$

where $p_{f_{\mathrm{G}}}(t) = \frac{1}{\sqrt{2\pi C(0)}} \exp\left(-\frac{t^2}{2C(0)}\right)$ is the first-order density of the r.f. f_{G}. Consequently, $\{f_{\mathrm{G}} > t\}$ has finite specific variation in the direction u if and only if for all $u \in S^{d-1}$ the one-dimensional restrictions $C_u : r \mapsto C(ru)$ are twice differentiable at 0*, and in this case,*

$$\theta_V(\{f_{\mathrm{G}} > t\}) = \frac{1}{2\omega_{d-1}} \int_{S^{d-1}} \sqrt{\frac{-2C_u''(0)}{\pi}} \mathcal{H}^{d-1}(du) \frac{1}{\sqrt{2\pi C(0)}} \exp\left(-\frac{t^2}{2C(0)}\right) = \theta_V(f_{\mathrm{G}}) p_{f_{\mathrm{G}}}(t).$$

Proof. By Proposition 5.5,

$$\theta_{V_u}(\{f_{\mathrm{G}} > t\}) = 2 \lim_{r \to 0} \frac{\nu_t(ru)}{|r|}.$$

The function

$$h : x \mapsto \frac{1}{\pi} \int_0^{\arcsin(x)} \exp\left(-\frac{t^2}{2C(0)}\left(1+\tan^2 s\right)\right) ds$$

is $\mathcal{C}^1$ at $x = 0$ and we have

$$h'(0) = \arcsin'(0) \frac{1}{\pi} \exp\left(-\frac{t^2}{2C(0)}\left(1+\tan^2(\arcsin(0))\right)\right) = \frac{1}{\pi} \exp\left(-\frac{t^2}{2C(0)}\right).$$

Hence, thanks to Formula (12) one deduces that

$$\theta_{V_u}(\{f_{\mathrm{G}} > t\}) = 2 \lim_{r \to 0} \frac{\nu_t(ru)}{|r|} = \frac{2}{\pi} \exp\left(-\frac{t^2}{2C(0)}\right) \lim_{r \to 0} \sqrt{\frac{C(0) - C(ru)}{2C(0)r^2}}.$$

As shown in the proof of Proposition 6.2, the limit on the right-hand side is finite if and only if C_u is twice differentiable in 0, and in this case

$$\lim_{r \to 0} \sqrt{\frac{C(0) - C(ru)}{2r^2}} = \sqrt{\frac{-C_u''(0)}{4C(0)}}.$$

The result follows. □

Remark. For all $t \in \mathbb{R}$ we have

$$\theta_{V_u}(\{f_{\mathrm{G}} > t\}) = \theta_{V_u}(f_{\mathrm{G}}) p_{f_{\mathrm{G}}}(t) \quad \text{and} \quad \theta_V(\{f_{\mathrm{G}} > t\}) = \theta_V(f_{\mathrm{G}}) p_{f_{\mathrm{G}}}(t),$$

and one can check that the coarea formula of Proposition 5.8 holds. These formulas show that the regularity of a stationary Gaussian r.f. corresponds to the one of its level sets: a stationary Gaussian r.f. has finite variation intensity if and only if all its level sets have finite variation intensities if and only if at least one of its level sets has finite variation intensity. Let us also note that the formula $\theta_V(\{f_{\mathrm{G}} > t\}) = \theta_V(f_{\mathrm{G}}) p_{f_{\mathrm{G}}}(t)$ is in accordance with the "Rice formula for the expectation of the geometric measure of level set" derived by Azaïs and Wschebor [AW09, Theorem 6.8]. Indeed, under some smoothness assumption on the paths of f_{G}, thanks to the stationarity of f_{G} one has [AW09, Theorem 6.8]

$$\mathbb{E}(\mathcal{H}^{d-1}(\{y,\ f_{\mathrm{G}}(y) = t\} \cap (0,1)^d)) = \mathbb{E}(\|\nabla f(0)\| | f(0) = t) p_{f_{\mathrm{G}}}(t).$$

But since for a stationary Gaussian r.f., $\nabla f(0)$ and $f(0)$ are independent, one gets

$$\begin{aligned}
\mathbb{E}(\mathcal{H}^{d-1}(\{y,\ f_{\mathrm{G}}(y) = t\} \cap (0,1)^d)) &= \mathbb{E}(\|\nabla f(0)\|) p_{f_{\mathrm{G}}}(t) \\
&= \theta_V(f_{\mathrm{G}}) p_{f_{\mathrm{G}}}(t) \\
&= \theta_V(\{f_{\mathrm{G}} > t\}) \\
&= \mathbb{E}(\mathrm{Per}(\{f_{\mathrm{G}} > t\}, (0,1)^d)).
\end{aligned}$$

6.2 Germ-grain Models

In this section we compute the variation intensities of several germ-grain models under very broad assumptions for the grain distribution. A germ-grain model defines a r.f. by combining a collection of colored random sets given as a marked Poisson process according to an interaction principle (addition, supremum, occultation, transparency,...). The random sets are called the *grains* while the random sets location are called the *germs* [SKM95, Lan02]. Although most of the models evoked here could be defined using RAMS to define the grains (see also the recent report of Rataj regarding random sets of finite perimeter [Rat14]), we prefer to use the classical framework of RACS to simplify the presentation. Each germ-grain model will be illustrated with numerical simulations involving particular grain distribution that are shown in Fig. 1.

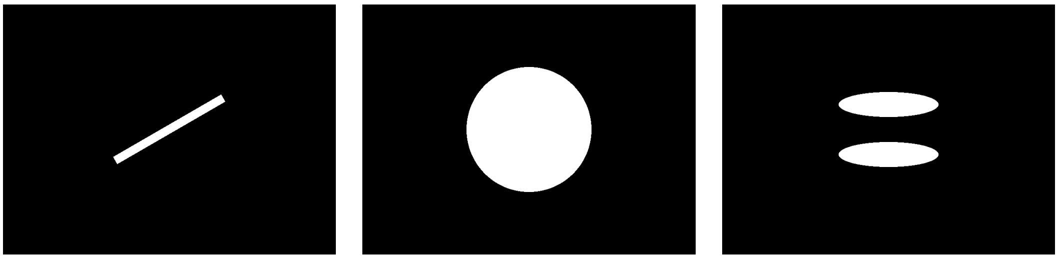

Figure 1: Representation of the three grain distributions used for the simulations of shot noise models, Boolean models, colored dead leaves models, and transparent dead leaves models. (see Fig. 2, Fig. 3, Fig. 4, Fig. 5, and Fig. 6). Left: a rectangle randomly oriented according to a uniform distribution over $[0, \pi]$; Middle: a disc having a radius uniformly distributed over an interval $[0, r_{\max}]$ (the displayed disc has size $r_{\max}$); Right: a fixed non convex set constituted of two disjoint ellipses.

6.2.1 Poisson Shot Noise

Let us first consider the Poisson shot noise model that is obtained in summing random functions placed on the point of a Poisson process. More formally, the *Poisson shot noise* associated with the independently marked Poisson process $\Pi = \{(x_j, \kappa_j)\} \subset \mathbb{R}^d \times K$ and the *impulse function* $h : \mathbb{R}^d \times K \to \mathbb{R}$ is the random field f_{SN} defined by

$$f_{\mathrm{SN}}(x) = \sum_{(x_j,\kappa_j)\in\Pi} h(x - x_j, \kappa_j),$$

where $\Pi = \{(x_j, \kappa_j)\} \subset \mathbb{R}^d \times K$ is an independently marked Poisson process having intensity measure $\lambda\mathcal{L}^d \otimes P_\kappa$, $\lambda > 0$ and P_κ is the probability distribution of the marks (see e.g. [Lan02]). The impulse function $h : \mathbb{R}^d \times K \to \mathbb{R}$ is supposed to be $\mathcal{L}^d \otimes P_\kappa$-integrable, which ensures that f_{SN} has finite expectation by Campbell theorem (see e.g. [Kin93]).

We first show that if the impulse function h has a finite mean total variation then the shot noise f_{SN} has locally bounded variation and its variation intensity is finite.

Proposition 6.4 (Bounds on the variation intensities of Poisson shot noises)**.** *Let $u \in S^{d-1}$ and suppose that $h(\cdot, \kappa) \in BV_u(\mathbb{R}^d)$ P_κ-a.s. with $\mathbb{E}\left(|D_u h(\cdot,\kappa)|(\mathbb{R}^d)\right) < +\infty$. Then the shot noise f_{SN} has locally bounded directional variation in the direction u and*

$$\theta_{V_u}(f_{\mathrm{SN}}) \le \lambda\mathbb{E}\left(|D_u h(\cdot,\kappa)|(\mathbb{R}^d)\right).$$

Consequently, if $h(\cdot,\kappa) \in BV(\mathbb{R}^d)$ P_κ-a.s. and $\mathbb{E}\left(|Dh(\cdot,\kappa)|(\mathbb{R}^d)\right) < +\infty$ then f_{SN} has locally bounded variation and

$$\theta_V(f_{\mathrm{SN}}) \le \lambda\mathbb{E}\left(|Dh(\cdot,\kappa)|(\mathbb{R}^d)\right).$$

Proof. Let $u \in S^{d-1}$ and $r \in \mathbb{R}$. Then

$$|f_{\mathrm{SN}}(ru) - f_{\mathrm{SN}}(0)| \le \sum_{(x_j,\kappa_j)\in\Pi} |h(ru - x_j, \kappa_j) - h(-x_j, \kappa_j)|.$$

By Campbell theorem (see e.g. [Kin93]),

$$\begin{aligned}\mathbb{E}\left(|f_{\mathrm{SN}}(ru) - f_{\mathrm{SN}}(0)|\right) &\le \lambda \int_{\mathbb{R}^d\times K} |h(ru - x,\kappa) - h(-x,\kappa)|\, dx P_\kappa(d\kappa)\\ &\le \lambda\mathbb{E}\left(\int_{\mathbb{R}^d} |h(ru - x,\kappa) - h(-x,\kappa)|\, dx\right).\end{aligned}$$

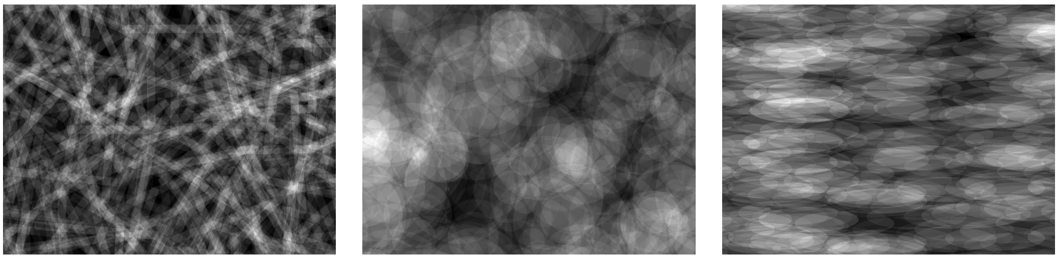

Figure 2: Realizations of three different Poisson shot noises with uniformly distributed gray-levels. The grain distributions are the ones presented in Fig. 1.

By Lemma 2.4 with $U = \mathbb{R}^d$, P_κ-a.s.

$$\int_{\mathbb{R}^d} |h(ru - x, \kappa) - h(-x, \kappa)|\, dx \leq |D_u h(\cdot, \kappa)|(\mathbb{R}^d)|r|.$$

Hence

$$\mathbb{E}\left(|f_{\mathrm{SN}}(ru) - f_{\mathrm{SN}}(0)|\right) \leq \lambda \mathbb{E}\left(|D_u h(\cdot, \kappa)|(\mathbb{R}^d)\right)|r|,$$

that is f_{SN} is directionally Lipschitz in mean. By Theorem 4.4, one concludes that f_{SN} is a.s. in $BV_{u,\mathrm{loc}}(\mathbb{R}^d)$ and $\theta_{V_u}(f_{\mathrm{SN}}) \leq \lambda \mathbb{E}\left(|D_u h(\cdot, \kappa)|(\mathbb{R}^d)\right)$. The upper bound on $\theta_V(f_{\mathrm{SN}})$ is obtained in integrating over all directions $u \in S^{d-1}$. $\square$

Proposition 6.4 simply establishes that the mean variation of the sum is lower than the sum of the mean variations. It only gives an upper bound on the variation intensity of the Poisson shot noise.

Biermé and Desolneux [BD13] recently studied the mean perimeter of excursion sets of Poisson shot noise models within the framework of functions of bounded variation with more specific techniques. Under the hypothesis that the functions $h(\cdot, \kappa)$ have no Cantor part, they obtained an expression for the mean variation intensity of Poisson shot noises [BD13, Theorem 3] which is the sum of two terms: one term for the approximate differential of f_{SN} [AFP00] and one from the jump part of f_{SN}. This second term is explicit and results from the observation that the jump part of f_{SN} is the sum of the jump parts of the derivatives $Dh(\cdot - x_j, \kappa_j)$. Consequently, when the functions h are indicator functions of sets of finite perimeter, the upper bound of Proposition 6.4 is reached, and thus Proposition 6.4 cannot be improved in general.

The following proposition sums up this observation. Figure 2 shows three Poisson shot noise realizations that are the sum of indicator functions.

Proposition 6.5 (Variation of shot noises of random indicator functions)**.** *Consider a shot noise of the form*

$$f_{\mathrm{SN}}(x) = \sum_{(x_j, a_j, X_j) \in \Pi} a_j \mathbb{1}(x \in x_j + X_j),$$

where the Poisson process Π_λ has intensity measure $\lambda \mathcal{L}^d \otimes P_X \otimes P_a$, $\lambda \geq 0$, P_X a probability distribution over the set $\mathcal{F}$ of closed subsets of $\mathbb{R}^d$, and $P_a \in L^1(\Omega)$. Suppose that the RACS $X \sim P_X$ has finite mean Lebesgue measure and finite perimeter. Then f_{SN} has a.s. bounded variation and

$$\theta_V(f) = \lambda \mathbb{E}(|a|)\mathbb{E}(\mathrm{Per}(X)) \quad \text{and} \quad \theta_{V_u}(f) = \lambda \mathbb{E}(|a|)\mathbb{E}(V_u(X)), \quad u \in S^{d-1}.$$

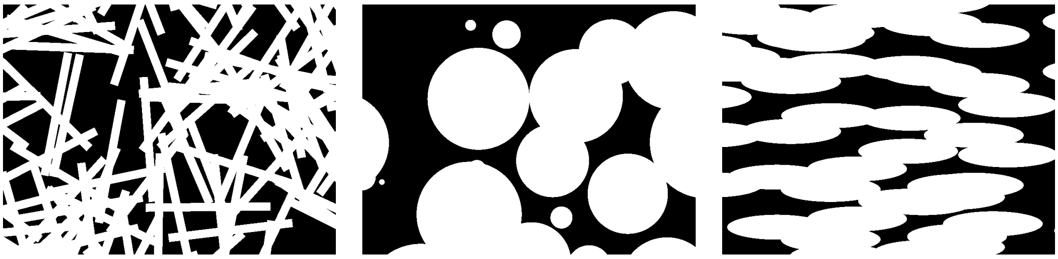

Figure 3: Realizations of three different Boolean random sets for which the grain distributions are the ones presented in Fig. 1.

Proof. The formula $\theta_V(f) = \lambda\mathbb{E}(|a|)\mathbb{E}(\mathrm{Per}(X))$ is given by [BD13, Theorem 3]. By Proposition 6.4, this implies that $\theta_{V_u}(f) = \lambda\mathbb{E}(|a|)\mathbb{E}(V_u(X))$ for $\mathcal{H}^{d-1}$-almost all $u \in S^{d-1}$. We conclude that it is true for all u using a continuity argument (see the remark following Proposition 2.3). $\square$

6.2.2 Boolean Models

We now turn to the computation of the variation intensities of Boolean models. Recall that the homogeneous *Boolean random set* with intensity λ and grain distribution P_X is the stationary random closed sets (RACS) Z_{B} defined by

$$Z_{\mathrm{B}} = \bigcup_{j\in\mathbb{N}} x_j + X_j,$$

where $\{(x_j, X_j)\}$ is an independently marked stationary Poisson process in the space $\mathbb{R}^d \times \mathcal{F}$ having intensity measure $\lambda\mathcal{L}^d \otimes P_X$, $\lambda \geq 0$ (see e.g. [SKM95, Lan02, SW08]). Three examples of Boolean models constructed with the grain distributions described by Fig. 1 are represented in Fig. 3.

The avoiding functional of the Boolean model Z_{B} is well-known: for any compact $K \subset \mathbb{R}^d$ we have

$$\mathbb{P}\left(Z_{\mathrm{B}} \cap K = \emptyset\right) = \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d\left(X \oplus \check{K}\right)\right)\right), \tag{13}$$

where X denotes a RACS with distribution P_X (see e.g. [SKM95, p. 65] or [Lan02, p. 164]). Starting from the general Formula (13), which determines the distribution of Z_{B}, one easily derives the expression of the variogram $\nu_{Z_{\mathrm{B}}}$ of Z_{B}. Indeed, specified for $K = \{0\}$, Formula (13) becomes

$$q := \mathbb{P}\left(0 \notin Z_{\mathrm{B}}\right) = \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right)\right),$$

For $K = \{0, -ru\}$, $r \in \mathbb{R}$ and $u \in S^{d-1}$, remark that we have

$$\mathcal{L}^d\left(X \oplus \check{K}\right) = \mathcal{L}^d\left(X \cup ru + X\right) = 2\mathcal{L}^d(X) - \mathcal{L}^d\left(X \cap ru + X\right).$$

Hence in this case $\mathbb{E}\left(\mathcal{L}^d\left(X \oplus \check{K}\right)\right) = 2\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right) - \gamma_X(ru)$. As a result the variogram $\nu_{Z_{\mathrm{B}}}$ is equal to [SKM95, p. 68], [Lan02, p. 165]

$$\begin{aligned}\nu_{Z_{\mathrm{B}}}(ru) = \mathbb{P}\left(-ru \in Z_{\mathrm{B}} \text{ and } 0 \notin Z_{\mathrm{B}}\right) &= \mathbb{P}\left(0 \notin Z_{\mathrm{B}}\right) - \mathbb{P}\left(Z_{\mathrm{B}} \cap \{0, -ru\} = \emptyset\right) \\ &= q - \exp\left(-\lambda\left(2\mathbb{E}\left(\mathcal{L}^d\left(X\right)\right) - \gamma_X(ru)\right)\right) \\ &= q - q\exp\left(-\lambda\left(\gamma_X(0) - \gamma_X(ru)\right)\right).\end{aligned}$$

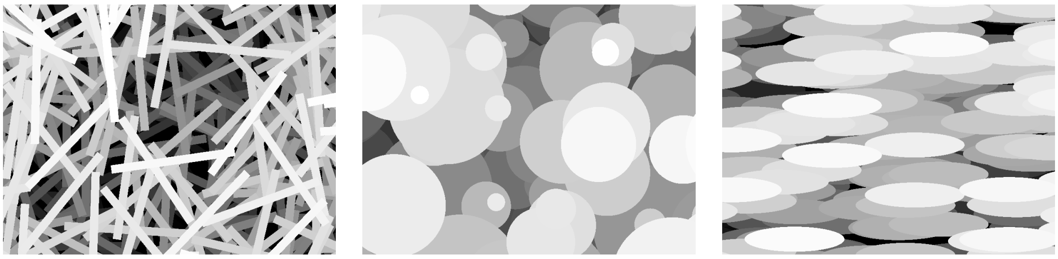

Figure 4: Realizations of three different Boolean random fields with uniformly distributed gray-levels. The grain distributions are the ones presented in Fig. 1.

Thanks to Proposition 5.5, one easily computes the specific variation intensities of Z_{B} from the expression of its variogram.

Proposition 6.6 (Specific variations of Boolean random sets)**.** *Let Z_{B} be the Boolean random set with Poisson intensity λ and grain distribution P_X, and let X be a RACS with distribution P_X. Then for all $u \in S^{d-1}$,*

$$\theta_{V_u}(Z_{\mathrm{B}}) = \lambda \mathbb{E}\left(V_u(X)\right) \exp\left(-\lambda \mathbb{E}\left(\mathcal{L}^d(X)\right)\right)$$

and

$$\theta_V(Z_{\mathrm{B}}) = \lambda \mathbb{E}\left(\mathrm{Per}(X)\right) \exp\left(-\lambda \mathbb{E}\left(\mathcal{L}^d(X)\right)\right). \tag{14}$$

Proof. By Proposition 5.5 and Proposition 5.3,

$$\theta_{V_u}(Z_{\mathrm{B}}) = 2\left(\nu_{Z_{\mathrm{B}}}^u\right)'(0) = 2q\lambda\left(\gamma_X^u\right)'(0) = q\lambda\mathbb{E}\left(V_u(X)\right),$$

and the result follows in replacing q by its expression. $\theta_V(Z_{\mathrm{B}})$ is obtained in integrating over all directions u. □

Let us emphasize that Equation (14), proved by the author in [Gal11], is valid for *any* grain distribution P_X and that it generalizes known results for Boolean models with convex grains [SW08, p. 386]. Similar generalizations involving intensity of surface measures deriving from Steiner formula have recently been established [HLW04, Vil10], under some technical hypotheses on the RACS X. Our result is similar but not identical since the outer Minkowski content of a set differs from its (variational) perimeter [Vil09].

Several random field models can be considered as generalizations of Boolean random sets [Ser88]. Here we consider a simple example called random Boolean islands. Let $\Phi = \{(x_j, X_j, a_j)\}$ be an independently marked Poisson process taking values in $\mathbb{R}^d \times \mathcal{F} \times [0, +\infty)$ and having intensity measure $\lambda\mathcal{L}^d \otimes P_X \otimes P_a$, $\lambda \geq 0$. We define the *Boolean random field* f_{B} associated to this process by

$$f_{\mathrm{B}}(y) = \sup\left(\{0\} \cup \{a_j,\ y \in x_j + X_j\}\right).$$

Three examples of Boolean random fields are represented in Fig. 4. Remark that with this model, the colored random sets are superimposed according to a hierarchy: the lighter sets are placed above the darker ones.

Note that if $a_j = 1$ a.s., then f_{B} is the indicator function of the Boolean random set Z_{B}. More generally, the upper level sets of f_{B} are Boolean random sets: indeed, for all $t \geq 0$,

$$\begin{aligned}\{y,\ f_{\mathrm{B}}(y) > t\} &= \{y,\ \exists (x_j, X_j, a_j) \in \Phi,\ y \in x_j + X_j \text{ and } a_j > t\} \\ &= \bigcup_{\Phi \cap \mathbb{R}^d \times \mathcal{F} \times (t, +\infty)} x_j + X_j,\end{aligned}$$

that is to say $\{y,\ f_{\mathrm{B}}(y) > t\}$ is the Boolean model associated with the Poisson process $\sum_{\Phi} \mathbb{1}(a_j > t)\,\delta_{x_j,X_j}$. Relying on this observation, one deduces an expression of the variation intensities of Boolean random fields using the coarea formula for variation intensities (see Proposition 5.8).

Proposition 6.7 (Variation intensities of Boolean random fields)**.** *Let f_{B} be the Boolean random field with Poisson intensity λ, grain distribution P_X, and gray-level distribution P_a. Let X denote a RACS with distribution P_X and a a r.v. with distribution P_a. Then for all $u \in S^{d-1}$,*

$$\theta_{V_u}(f_{\mathrm{B}}) = \lambda\mathbb{E}\left(V_u(A)\right) \int_0^{+\infty} P_a(\{a > t\}) \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d(X)\right) P_a(\{a > t\})\right) dt$$

and

$$\theta_{V}(f_{\mathrm{B}}) = \lambda\mathbb{E}\left(\mathrm{Per}(A)\right) \int_0^{+\infty} P_a(\{a > t\}) \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d(X)\right) P_a(\{a > t\})\right) dt.$$

Proof. $\{y,\ f_{\mathrm{B}}(y) > t\}$ is the Boolean model associated with the Poisson process $\sum_{\Phi} \mathbb{1}(a_j > t)\,\delta_{x_j,X_j}$. This Poisson process has grain distribution P_X and intensity $\lambda P_a(\{a > t\})$. Hence by Proposition 6.6

$$\theta_{V_u}(\{f_{\mathrm{B}} > t\}) = \lambda\mathbb{E}\left(V_u(A)\right) P_a(\{a > t\}) \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d(X)\right) P_a(\{a > t\})\right).$$

By the coarea formula for variation intensity (see Proposition 5.8)

$$\theta_{V_u}(f_{\mathrm{B}}) = \lambda\mathbb{E}\left(V_u(A)\right) \int_0^{+\infty} P_a(\{a > t\}) \exp\left(-\lambda\mathbb{E}\left(\mathcal{L}^d(X)\right) P_a(\{a > t\})\right) dt.$$

□

6.2.3 Colored Dead Leaves Model

The dead leaves model [CT94, Jeu97, BGR06], also initially introduced by Matheron [Mat68], is a germ-grain model where the interaction rule is occultation, that is where the grains $x_j + X_j$ hide each other. For this germ-grain model the grains are chronologically ordered by a time $t_j \in (-\infty, 0)$, called *falling time*. More precisely the leaves are the points of the Poisson process

$$\Phi = \{(t_j, x_j, X_j, a_j)\} \subset (-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R}$$

with intensity measure $\mathcal{L}^1 \otimes \mathcal{L}_d \otimes P_X \otimes P_a$. For each leaf (t_j, x_j, X_j, a_j), the random set $x_j + X_j$ associated with the random color a_j is partially or totally hidden by its subsequent leaves, that is the leaves which fall after $t = t_j$. In the end, at time $t = 0$ the only remaining part of $x_j + X_j$ is the *visible part* V_j, that is the set[2]

$$V_j = (x_j + X_j) \setminus \left(\bigcup_{(t_k, x_k, X_k, a_k) \in \Phi,\ t_k > t_j} x_k + X_k \right).$$

Let us precise that as soon as $\mathbb{E}(\mathcal{L}^d(X)) > 0$, then all the Euclidean space $\mathbb{R}^d$ is covered by the random sets $x_j + X_j$, and consequently each point $y \in \mathbb{R}^d$ belongs to a unique visible part.

The *colored dead leaves r.f.* f_{CDL} is the r.f. defined in assigning to each $y \in \mathbb{R}^d$ the color a_j of the unique visible part V_j such that $y \in V_j$. More formally, f_{CDL} is defined by

$$f_{\mathrm{CDL}}(y) = \sum_{(t_j, x_j, X_j, a_j) \in \Phi} a_j \mathbb{1}(y \in V_j),$$

[2] Our definition of the *visible parts* V_j is slightly different from the one of [BGR06]. This is because we do not enforce the visible parts to be closed sets.

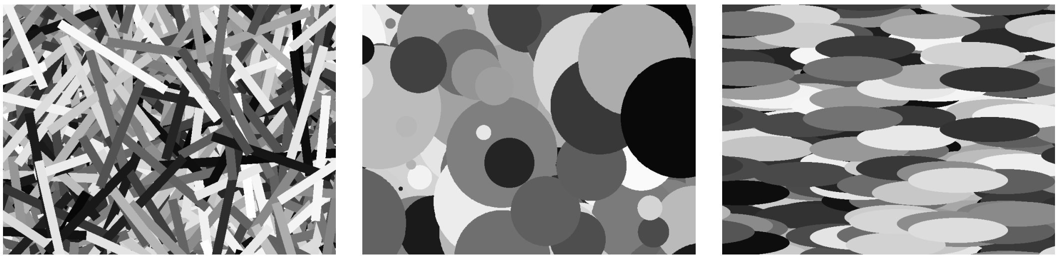

Figure 5: Realizations of three different colored dead leaves r.f. with uniformly distributed gray-levels. The grain distributions are the ones presented in Fig. 1.

but note that for each point y the above sum has only one non null term.

As for the two previous models, three examples of colored dead leaves r.f. having the grain distributions described by Fig. 1 are shown in Fig. 5. Even though occultation between objects is also observable with Boolean r.f., remark that colored dead leaves r.f. vary from this first model. Indeed here the ordering of the objects is not related to their gray-level, and the whole domain is by construction fully covered by objects.

The next proposition gives the variation intensities of this r.f. model. Relying on Theorem 4.1, it consists in computing $\mathbb{E}\left(|f_{\text{CDL}}(ru) - f_{\text{CDL}}(0)|\right)$. To do so one should consider the set of leaves that cover (or hit) the two points 0 and ru (or both).

Let us consider the restriction $\Phi^{\{0,ru\}}$ of the leaves of Φ which hit the set $\{0, ru\}$, that is

$$\Phi^{\{0,ru\}} = \{(t_j, x_j, X_j, a_j) \in \Phi,\ x_j + X_j \cap \{0, ru\} \neq \emptyset\}\,.$$

According to [GG12, Proposition 5], $\Phi^{\{0,ru\}}$ is an independently marked Poisson process with ground process $\left\{t,\ (t,x,X,a) \in \Phi^{\{0,ru\}}\right\}$ of intensity $2\gamma_X(0) - \gamma_X(ru)$ (where γ_X denotes the mean geometric covariogram of X) and marks (x_j, X_j, a_j). The marks a_j are i.i.d. with distribution P_a, and are independent of (x_j, X_j). As for the distributions of the marks (x_j, X_j) we are only interested in the following probabilities.

$$\mathbb{P}\left(\{0, ru\} \subset x_j + X_j\right) = \frac{\mathcal{L}^d \otimes P_X\left(\{(x, X),\ \{0, ru\} \subset x + X\}\right)}{\mathcal{L}^d \otimes P_X\left(\{(x, X),\ \{0, ru\} \cap x + X \neq \emptyset\}\right)} = \frac{\gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)}.$$

and, by symmetry and complementarity,

$$\mathbb{P}\left(0 \in x_j + X_j \text{ and } ru \notin x_j + X_j\right) = \mathbb{P}\left(ru \in x_j + X_j \text{ and } 0 \notin x_j + X_j\right) = \frac{\gamma_X(0) - \gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)}.$$

In addition, if one denotes by (t_0, x_0, X_0, a_0) the last leaf of $\Phi^{\{0,ru\}}$, that is, the leaf such that

$$t_0 = \sup\left\{t_j,\ (t_j, x_j, X_j, a_j) \in \Phi^{\{y,z\}}\right\},$$

then (x_0, X_0, a_0) has the same distribution as any mark (x_j, X_j, a_j) of $\Phi^{\{y,z\}}$, the shifted Poisson process

$$\Phi_{t_0} = \{(t - t_0, x, X, a),\ (t, x, X, a) \in \Phi \text{ and } t < t_0\}$$

has the same distribution as Φ, and (x_0, X_0, a_0) and Φ_{t_0} are independent.

Proposition 6.8 (Variation intensities of the colored dead leaves r.f.)**.** *Suppose that* $0 < \mathbb{E}(\mathcal{L}^d(X)) < +\infty$ *and that* $a \in L^1(\Omega)$*. Let* a_1 *and* a_2 *be two independent r.v. with distribution* P_a*. Then for all* $u \in S^{d-1}$,

$$\theta_{V_u}(f_{\text{CDL}}) = \mathbb{E}(|a_1 - a_2|)\frac{\mathbb{E}\left(V_u(X)\right)}{\mathbb{E}\left(\mathcal{L}^d(X)\right)},$$

and

$$\theta_V(f_{\text{CDL}}) = \mathbb{E}(|a_1 - a_2|)\frac{\mathbb{E}(\text{Per}(X))}{\mathbb{E}(\mathcal{L}^d(X))}.$$

Proof. Let us first compute the expectation $\mathbb{E}(|f_{\text{CDL}}(ru) - f_{\text{CDL}}(0)|)$ for $r \in \mathbb{R}$ and $u \in S^{d-1}$. If the points ru and 0 are in the same visible part V_j, then $f_{\text{CDL}}(ru) = f_{\text{CDL}}(0)$. Otherwise, if ru and 0 are in different visible parts, then both $f_{\text{CDL}}(ru)$ and $f_{\text{CDL}}(0)$ have distribution P_a and they are independent. Hence,

$$\mathbb{E}(|f_{\text{CDL}}(ru) - f_{\text{CDL}}(0)|) = \mathbb{E}(|a_1 - a_2|)\,\mathbb{P}(\{ru \text{ and } 0 \text{ belong to different visible parts}\}).$$

Now, 0 and ru belong to different visible parts if the last leaf covering either ru or 0 does not cover both points. Noting $x_0 + X_0$ the last leaf of $\Phi^{\{0,ru\}}$, we have[3]

$$\begin{aligned}
&\mathbb{P}(\{ru \text{ and } 0 \text{ belong to different visible parts}\})\\
&= \mathbb{P}(0 \in x_0 + X_0 \text{ and } ru \notin x_0 + X_0) + \mathbb{P}(ru \in x_0 + X_0 \text{ and } 0 \notin x_0 + X_0)\\
&= 2\frac{\gamma_X(0) - \gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)}.
\end{aligned}$$

Hence, by Theorem 4.1 and Proposition 5.3,

$$\begin{aligned}
\theta_{V_u}(f_{\text{CDL}}) &= \lim_{r\to 0}\frac{\mathbb{E}(|f_{\text{CDL}}(ru) - f_{\text{CDL}}(0)|)}{|r|}\\
&= \mathbb{E}(|a_1 - a_2|)\lim_{r\to 0} 2\frac{\gamma_X(0) - \gamma_X(ru)}{|r|}\frac{1}{2\gamma_X(0) - \gamma_X(ru)}\\
&= \mathbb{E}(|a_1 - a_2|)\frac{\mathbb{E}(V_u(X))}{\mathbb{E}(\mathcal{L}^d(X))}.
\end{aligned}$$

Integrating over all directions gives the expression of $\theta_V(f_{\text{CDL}})$. □

Remark. The expression of the variation intensity

$$\theta_V(f_{\text{CDL}}) = \mathbb{E}(|a_1 - a_2|)\frac{\mathbb{E}(\text{Per}(X))}{\mathbb{E}(\mathcal{L}^d(X))}.$$

is in accordance with our expectation: Indeed, $\mathbb{E}(|a_1 - a_2|)$ is the mean contrast between two distinct visible parts, whereas the ratio $\dfrac{\mathbb{E}(V_u(X))}{\mathbb{E}(\mathcal{L}^d(X))}$ is known to be the mean length of cell boundary per unit area when the RACS are random polygons [CT94].

6.2.4 Transparent Dead Leaves Random Fields

As for the colored dead leaves model, the transparent dead leaves (TDL) r.f. [GG12] is obtained from the Poisson process of leaves $\Phi = \{(t_j, x_j, X_j, a_j)\}$ taking values in the state space $(-\infty, 0) \times \mathbb{R}^d \times \mathcal{F} \times \mathbb{R}$ and having intensity measure $\mathcal{L}^1 \otimes \mathcal{L}^d \otimes P_X \otimes P_a$.

Informally, the TDL r.f. results from the sequential superposition of transparent leaves of support $x_j + X_j$ and gray-level a_j. The superposition of a transparent leaf $x + X$ with gray level a on an image (a function $f : \mathbb{R}^d \to \mathbb{R}$) results in a new image $\tilde{f}$ defined for each $y \in \mathbb{R}^d$ by

$$\tilde{f}(y) = \begin{cases} \alpha a + (1-\alpha)f(y) & \text{if } y \in x + X,\\ f(y) & \text{otherwise,}\end{cases} \tag{15}$$

[3] Up to our slightly different definition of the visible parts, this probability is also given by the general Formula (12) of [BGR06].

Figure 6: Realizations of three different transparent dead leaves r.f. with uniformly distributed gray-levels and transparency coefficient $\alpha = 0.4$. The grain distributions are the ones presented in Fig. 1.

where $\alpha \in (0, 1]$ is a *transparency coefficient*. Following this construction, given the marked point process Φ, the TDL r.f. f_{TDL} of transparency coefficient $\alpha \in (0, 1]$ is defined by

$$f_{\mathrm{TDL}}(y) = \sum_{(t_j,x_j,X_j,a_j)\in\Phi} \mathbb{1}\left(y \in x_j + X_j\right) \alpha a_j \left(1-\alpha\right)^{\left(\sum_{k\in\mathbb{N}} \mathbb{1}(t_k\in(t_j,0) \text{ and } y\in x_k+X_k)\right)}.$$

A noticeable property of the TDL r.f. is that its set of jumps is dense in $\mathbb{R}^d$ (the r.f. is nowhere continuous as can be seen for the realizations of Figure 6) while having locally finite variations, as asserted by the following proposition that gives the TDL variation intensity.

Proposition 6.9 (Variation intensities of the transparent dead leaves r.f.)**.** *Suppose that* $0 < \mathbb{E}\left(\mathcal{L}^d(X)\right) < +\infty$ *and* $a \in L^1$*. Then for all* $u \in S^{d-1}$*,*

$$\theta_{V_u}(f_{\mathrm{TDL}}) = C_\alpha \frac{\mathbb{E}\left(V_u(X)\right)}{\mathbb{E}\left(\mathcal{L}^d(X)\right)},$$

where C_α *is the mean contrast between an independent leaf color and the TDL r.f., that is*

$$C_\alpha = \mathbb{E}(|a - f_{\mathrm{TDL}}(0)|) = \mathbb{E}\left(\left|a - \sum_{k=0}^{+\infty} \alpha a_k \beta^k\right|\right),$$

where the r.v. a *and* $(a_k)_{k\in\mathbb{N}}$ *are i.i.d. with distribution* P_a*. Consequently,*

$$\theta_V(f_{\mathrm{TDL}}) = C_\alpha \frac{\mathbb{E}\left(\mathrm{Per}(X)\right)}{\mathbb{E}\left(\mathcal{L}^d(X)\right)}.$$

Proof. To abbreviate notation, we note $f = f_{\mathrm{TDL}}$ within this proof. The proof relies on the same technique that enables to compute the TDL covariance [GG12, Proposition 6] as well as to compute the variation intensity of the color dead leaves r.f. in the proof of Proposition 6.8, that is, conditioning with respect to the last leaf covering 0, ru or both. First we give a lower and an upper bound of the expectation

$$\mathbb{E}\left(|f(ru) - f(0)|\right).$$

One decomposes this expectation in conditioning with respect of the coverage of the last leaf $x_0 + X_0$ hitting 0 or ru,

$$\begin{aligned}\mathbb{E}\left(|f(ru) - f(0)|\right) = {} & \mathbb{E}\left(|f(ru) - f(0)|\,|\{0, ru\} \subset x_0 + X_0\right) \frac{\gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)} \\ & + \mathbb{E}\left(|f(ru) - f(0)|\,|0 \in x_0 + X_0 \text{ and } ru \notin x_0 + X_0\right) \frac{\gamma_X(0) - \gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)} \\ & + \mathbb{E}\left(|f(ru) - f(0)|\,|ru \in x_0 + X_0 \text{ and } 0 \notin x_0 + X_0\right) \frac{\gamma_X(0) - \gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)}.\end{aligned}$$

By symmetry the two last terms of the above sum are equal. Let us note $\beta = 1 - \alpha$ and f_{t_0} the TDL r.f. associated with the shifted Poisson process

$$\Phi_{t_0} = \{(t - t_0, x, X, a),\ (t, x, X, a) \in \Phi \text{ and } t < t_0\}.$$

Since Φ_{t_0} and Φ have the same distribution, f_{t_0} and f also have the same distribution. On the event $\{\{0, ru\} \subset x_0 + X_0\}$ we have

$$f(0) = \alpha a_0 + \beta f_{t_0}(0) \quad \text{and} \quad f(ru) = \alpha a_0 + \beta f_{t_0}(ru),$$

so that

$$\mathbb{E}\left(|f(ru) - f(0)|\,|\{0, ru\} \subset x_0 + X_0\right) = \beta\mathbb{E}\left(|f_{t_0}(ru) - f_{t_0}(0)|\right) = \beta\mathbb{E}\left(|f(ru) - f(0)|\right).$$

On the event $\{0 \in x_0 + X_0 \text{ and } ru \notin x_0 + X_0\}$ we have

$$f(0) = \alpha a_0 + \beta f_{t_0}(0) \quad \text{and} \quad f(ru) = f_{t_0}(ru).$$

Hence

$$\begin{aligned} |f(ru) - f(0)| &= |f_{t_0}(ru) - \alpha a_0 - \beta f_{t_0}(0)| \\ &= |\alpha\left(f_{t_0}(ru) - a_0\right) + \beta\left(f_{t_0}(ru) - f_{t_0}(0)\right)|. \end{aligned}$$

Using the triangular inequality we have

$$\alpha\,|f_{t_0}(ru) - a_0| - \beta\,|f_{t_0}(ru) - f_{t_0}(0)| \leq |f(ru) - f(0)| \leq \alpha\,|f_{t_0}(ru) - a_0| + \beta\,|f_{t_0}(ru) - f_{t_0}(0)|.$$

Now let us note $C_\alpha = \mathbb{E}\left(|f(0) - a|\right)$. We have, in taking expectation in the previous inequalities

$$\mathbb{E}\left(|f(ru) - f(0)|\,|0 \in x_0 + X_0 \text{ and } ru \notin x_0 + X_0\right) \leq \alpha C_\alpha + \beta\mathbb{E}\left(|f(ru) - f(0)|\right)$$

and

$$\mathbb{E}\left(|f(ru) - f(0)|\,|0 \in x_0 + X_0 \text{ and } ru \notin x_0 + X_0\right) \geq \alpha C_\alpha - \beta\mathbb{E}\left(|f(ru) - f(0)|\right).$$

Let us now establish the upper bound of $\mathbb{E}\left(|f(ru) - f(0)|\right)$. From the initial decomposition of $\mathbb{E}\left(|f(ru) - f(0)|\right)$ and the previous inequality,

$$\begin{aligned} &\mathbb{E}\left(|f(ru) - f(0)|\right) \\ &\leq \beta\mathbb{E}\left(|f(ru) - f(0)|\right)\frac{\gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)} + 2\left(\alpha C_\alpha + \beta\mathbb{E}\left(|f(ru) - f(0)|\right)\right)\frac{\gamma_X(0) - \gamma_X(ru)}{2\gamma_X(0) - \gamma_X(ru)}. \end{aligned}$$

Rearranging the terms we get

$$\mathbb{E}\left(|f(ru) - f(0)|\right) \leq C_\alpha \frac{2\left(\gamma_X(0) - \gamma_X(ru)\right)}{2\gamma_X(0) - \gamma_X(ru)}.$$

Similarly we obtain the following lower bound

$$\mathbb{E}\left(|f(ru) - f(0)|\right) \geq C_\alpha \frac{2\alpha\left(\gamma_X(0) - \gamma_X(ru)\right)}{(4 - 2\alpha)\gamma_X(0) - (4 - 3\alpha)\gamma_X(ru)}.$$

By Theorem 4.1, the directional variation intensity of the TDL r.f. is equal to

$$\theta_{V_u}(f) = \lim_{r \to 0} \frac{\mathbb{E}\left(|f(ru) - f(0)|\right)}{|r|}.$$

On the other hand, by Proposition 5.3,

$$\lim_{r\to 0} \frac{\gamma_X(0)-\gamma_X(ru)}{|r|} = \frac{1}{2} E\left(V_u(X)\right).$$

Using this last property, observe that the two bounds of $\frac{\mathbb{E}(|f(ru)-f(0)|)}{|r|}$ both tends to $C_\alpha \frac{\mathbb{E}(V_u(X))}{\gamma_X(0)}$ as r tends to 0. Hence

$$\theta_{V_u}(f) = C_\alpha \frac{\mathbb{E}\left(V_u(X)\right)}{\mathbb{E}\left(\mathcal{L}^d(X)\right)}.$$

Integrating this equality over all directions, one gets the variation intensity $\theta_V(f)$. □

Remark (Occlusion case)**.** Observe that when $\alpha = 1$, that is when the transparent leaves are opaque, the formulas of the variation intensities of the TDL r.f. given by Proposition 6.9 boils down to the formulas of the variation intensities of the colored dead leaves r.f. given by Proposition 6.8.

6.2.5 Colored Tessellations

A colored tessellation is the random field obtained in assigning a random color to each subset of a random partition of the plane. The interaction principle which is at work for colored tessellations is arguably juxtaposition.

A (random) tessellation is a random partition $\bigcup_j C_j = \mathbb{R}^d$ of the Euclidean space $\mathbb{R}^d$, the sets C_j being called *cells* of the tessellation. Even though random tessellations have been widely studied, there lacks a general acknowledged definition. This is principally because most studied tessellation models only involve convex cells [SKM95, SW08, Cal10]. Nevertheless, tessellations can be constituted of non convex (and even non connected) cells, such as the tessellation corresponding to the dead leaves model defined in [BGR06]. Following [Sto86, BGR06], we consider a quite general definition: A *(random) tessellation* is a point process $T = \sum_j \delta_{C_j}$ taking values in the set $\mathcal{K}'$ of non empty compact sets and which satisfies the following additional properties:

- For all compact set K, the number of sets C_i intersecting K is finite.
- For all $j \neq k$, $\text{int}C_j \cap \text{int}C_k \neq \emptyset$.
- $\bigcup_j C_k = \mathbb{R}^d$.
- For all j, $\mathcal{L}^d\left(\partial C_j\right) = 0$.

With these conditions, a.e. point $x \in \mathbb{R}^d$ belongs to a unique cell C_j. We will only consider *stationary* tessellations, that is tessellations such that for all $x \in \mathbb{R}^d$, $\sum_j \delta_{x+C_j} \overset{\text{d}}{=} \sum_j \delta_{C_j}$. Thanks to the stationarity, for these tessellations every point $x \in \mathbb{R}^d$ a.s. belongs to a unique cell C_j.

Given a stationary tessellation $T = \sum_j \delta_{C_j}$ one defines a stationary random field f_T by associating a random intensity $a_j \in \mathbb{R}$ to each cell C_j. The real r.v. a_j are i.i.d. with common distribution P_a. More formally, the *colored tessellation* T_c associated to the tessellation T and with color distribution P_a is the independently marked point process $T_c = \sum_j \delta_{(C_j,a_j)}$, where the marks a_j have common distribution P_a. If μ denotes the intensity measure of the point process $\sum_j \delta_{C_j}$, then $T_c = \sum_j \delta_{(C_j,a_j)}$ has intensity measure $\mu \otimes P_a$. Its associated random field

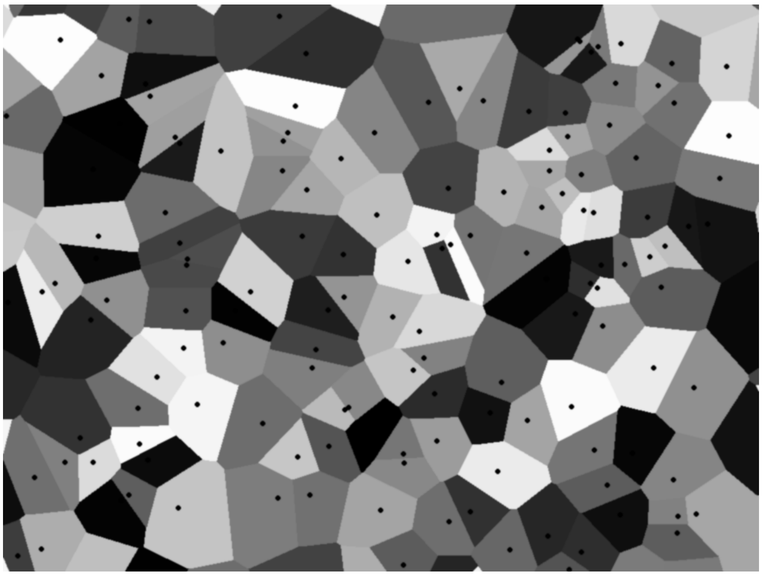

Figure 7: A realization of a colored Poisson-Voronoi tessellation with uniformly distributed gray levels. The points of the underlying Poisson point process are displayed in black.

f_T is defined as follows: $f_T(x) = a_j$ where a_j is the color of the a.s. unique cell C_j containing x. Note that f_T can also be defined as a sum over the marked point process

$$f_T(x) = \sum_j a_j \mathbb{1}\,(x \in C_j)\,.$$

An example of a colored Poisson-Voronoi tessellation is reproduced in Fig. 7. Given a Poisson point process $\Pi = \{x_j\}$, the cells $\{C_j := C(x_j)\}$ of this tessellation are defined by

$$C(x_j) = \left\{y \in \mathbb{R}^d,\ |y - x_j| \leq |y - x_k| \text{ for all } x_k \in \Pi\right\}.$$

We refer to [SKM95, SW08, Cal10] for further properties and references on Poisson-Voronoi tessellations.

In this section we compute the variation intensities of colored tessellations. Not surprisingly, the variation intensity is proportional to the ratio mean perimeter over mean area of the typical cell.

Before establishing the expression of variation intensities of f_T we need to introduce the fundamental notion of typical cell. First, one interprets a stationary tessellation as a point process in $\mathbb{R}^d$ marked with random sets by introducing a centroid map. Recall that $\mathcal{K}'$ denotes the set of non empty compact sets of $\mathbb{R}^d$. A *centroid map* is a measurable application $z : \mathcal{K}' \mapsto \mathbb{R}^d$ such that $z(x+C) = x + z(C)$. Second, given a centroid map z, any stationary tessellation $T = \sum_j \delta_{C_j}$ is decomposed into the stationary marked point process $\sum_j \delta_{(z(C_j), C_j - z(C_j))}$. According to [SW08, Section 4.1], one deduces that for any stationary tessellation T there exists a constant $\lambda > 0$ and a distribution Q over $\mathcal{K}'_0 = \{K \in \mathcal{K}',\ z(K) = 0\}$ such that for all measurable function $f : \mathcal{K}' \mapsto \mathbb{R}+$,

$$\mathbb{E}\left(\sum_j f(C_j)\right) = \lambda \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} f(x+K)dxQ(dK). \tag{16}$$

λ is the intensity of the point process of cell centroids $\sum_j \delta_{z(C_j)}$, and, by definition, Q is the distribution of the *typical cell* of T denoted by $\mathcal{C} \sim Q$. Applying Equation (16) with $f : K \mapsto \mathcal{L}^d\left(K \cap [0,1]^d\right)$ shows that λ is equal to $\dfrac{1}{\mathbb{E}\left(\mathcal{L}^d(\mathcal{C})\right)}$.

The key result to compute the variation intensity of randomly colored stationary tessellations is the following proposition.

Proposition 6.10 (Stationary tessellations and mean covariogram)**.** *Let $T = \sum_j \delta_{C_j}$ be a stationary tessellation and let $\gamma_{\mathcal{C}} : h \mapsto \mathbb{E}\left(\mathcal{C} \cap (h + \mathcal{C})\right)$ be the mean covariogram of its typical cell $\mathcal{C}$. Then for all $h \in \mathbb{R}^d$,*

$$\mathbb{P}\left(\{0 \text{ and } h \text{ belong to the same cell}\}\right) = \frac{\gamma_{\mathcal{C}}(h)}{\gamma_{\mathcal{C}}(0)}.$$

Remark. Proposition 6.10 is stated without proof in [Lan02]. The proof reproduced below is due to Pierre Calka[4].

Proof of Proposition 6.10. Denote $\rho(h) = \mathbb{P}\left(\{0 \text{ and } h \text{ belong to the same cell}\}\right)$. First, by stationarity $\rho(h) = \mathbb{P}\left(\{-h \text{ and } 0 \text{ belong to the same cell}\}\right)$. Second, remark that for all $C \in \mathcal{K}$, $\{0, -h\} \subset C \iff 0 \in C \cap (h + C)$. Hence applying Formula (16) with $K \mapsto \mathbb{1}(0 \in K \cap (h + K))$,

$$\begin{aligned}
\rho(h) &= \mathbb{E}\left(\sum_j \mathbb{1}\left(0 \in C_j \cap (h + C_j)\right)\right) \\
&= \frac{1}{\mathbb{E}\left(\mathcal{L}^d(\mathcal{C})\right)} \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbb{1}\left(0 \in (x + K) \cap (h + x + K)\right) dx Q(dK) \\
&= \frac{1}{\gamma_{\mathcal{C}}(0)} \int_{\mathcal{K}_0} \int_{\mathbb{R}^d} \mathbb{1}\left(-x \in K \cap (h + K)\right) dx Q(dK) \\
&= \frac{1}{\gamma_{\mathcal{C}}(0)} \int_{\mathcal{K}_0} \mathcal{L}^d(K \cap (h + K)) Q(dK) \\
&= \frac{\gamma_{\mathcal{C}}(h)}{\gamma_{\mathcal{C}}(0)}.
\end{aligned}$$

□

Proposition 6.11 (Variation intensities of colored tessellations)**.** *Let $T_c = \sum_j \delta_{(C_j, a_j)}$ be a randomly colored stationary tessellation, let f_T be its associated stationary random field, and denote by $\mathcal{C}$ the typical cell of T. Let a_1 and a_2 be i.d.d. r.v. with distribution P_a. For all $u \in S^{d-1}$,*

$$\theta_{V_u}\left(f_T\right) = \mathbb{E}\left(|a_1 - a_2|\right) \frac{1}{2} \frac{\mathbb{E}\left(V_u(\mathcal{C})\right)}{\mathbb{E}\left(\mathcal{L}^d(\mathcal{C})\right)},$$

and

$$\theta_V\left(f_T\right) = \mathbb{E}\left(|a_1 - a_2|\right) \frac{1}{2} \frac{\mathbb{E}\left(\mathrm{Per}(\mathcal{C})\right)}{\mathbb{E}\left(\mathcal{L}^d(\mathcal{C})\right)}.$$

Proof. Let us compute $\mathbb{E}\left(|f_T(ru) - f_T(0)|\right)$. We have

$$|f_T(ru) - f_T(0)| = \begin{cases} |a_k - a_j| & \text{if } 0 \in C_j \text{ and } ru \in C_k \text{ with } j \neq k, \\ 0 & \text{if } 0 \text{ and } ru \text{ belong to the same cell.} \end{cases}$$

[4] Personal communication.

By Proposition 6.10, and since for $j \neq k$, a_j and a_k are independent,

$$\begin{aligned}\mathbb{E}\left(|f_T(ru) - f_T(0)|\right) &= \mathbb{E}\left(|a_1 - a_2|\right)\mathbb{P}\left(\{0 \text{ and } ru \text{ are in different cells}\}\right)\\ &= \mathbb{E}\left(|a_1 - a_2|\right)\frac{\gamma_{\mathcal{C}}(0) - \gamma_{\mathcal{C}}(h)}{\gamma_{\mathcal{C}}(0)}.\end{aligned}$$

By Proposition 5.3,

$$\lim_{r \to 0} \frac{\gamma_{\mathcal{C}}(0) - \gamma_{\mathcal{C}}(h)}{|r|} = \frac{1}{2}\mathbb{E}\left(V_u(\mathcal{C})\right).$$

Hence, by Theorem 4.1,

$$\theta_{V_u}\left(f_T\right) = \lim_{r \to 0} \frac{\mathbb{E}\left(|f_T(ru) - f_T(0)|\right)}{|r|} = \mathbb{E}\left(|a_1 - a_2|\right)\frac{1}{2}\frac{\mathbb{E}\left(V_u(\mathcal{C})\right)}{\mathbb{E}\left(\mathcal{L}^d(\mathcal{C})\right)}.$$

Integrating over all directions one obtains the expression of the variation intensity $\theta_V\left(f_T\right)$. □

Observe that the formula

$$\theta_V\left(f_T\right) = \mathbb{E}\left(|a_1 - a_2|\right)\frac{1}{2}\frac{\mathbb{E}\left(\mathrm{Per}(\mathcal{C})\right)}{\mathbb{E}\left(\mathcal{L}^d(\mathcal{C})\right)}.$$

of Proposition 6.11 is in accordance with our expectation: Indeed, $\mathbb{E}\left(|a_1 - a_2|\right)$ is the mean contrast between two adjacent cells whereas $\frac{1}{2}\frac{\mathbb{E}(\mathrm{Per}(\mathcal{C}))}{\mathbb{E}\left(\mathcal{L}^d(\mathcal{C})\right)}$ is known to be the mean length of tessellations boundary per unit area [SW08, Section 10.1].

7 Conclusion

In this chapter, general definitions and results related to random fields of (locally) bounded (directional) variation were presented. Our main result is Theorem 4.1 which shows that the directional variation intensity measure of any stationary increment r.f. f is equal to a constant, called the directional variation intensity $\theta_{V_u}(f)$, times the Lebesgue measure, and that this constant $\theta_{V_u}(f)$, that represents the mean directional variation of f per unit volume, is given by

$$\theta_{V_u}(f) = \lim_{r \to 0} \frac{\mathbb{E}(|f(ru) - f(0)|)}{|r|} \in [0, +\infty].$$

When restricting to characteristic functions of sets, several formulas relating the directional variation of random sets to the directional derivatives at the origin of their covariogram or their variogram have been established. It has been shown that they rigorously generalize classical results only established in restricted cases. In particular the notion of specific variation $\theta_V(X)$ of a stationary random set X has been introduced as the variation intensity of the indicator stationary r.f. $\mathbb{1}_X$. The main advantage of the specific variation in comparison with the specific area measure which stems from Steiner formula is that it is defined for any stationary random measurable set, and in particular for any stationary random closed set, without any assumption on the regularity of the set boundary. Besides the directional and non directional specific variations are easily computed once the variogram of the stationary random set is known. Hence the specific variations are an interesting alternative to the usual specific area measure when dealing with d-dimensional random sets.

The relevance of these different formulas has been illustrated with the computation of several variation intensities of classical stationary r.f. and random sets. For germ-grain models, these results formulate explicitly the intuitive relation between the geometry of the grains and the total variation. It is worthy to note that for all the studied models, there are only two geometric features of influence on the total variation of the germ-grain r.f.: the mean perimeter and the mean Lebesgue measure of the grains. Besides, for the colored dead leaves r.f., the transparent dead leaves r.f. and the colored tessellations, the mean variation is proportional to the ratio $\frac{\mathbb{E}(\mathrm{Per}(X))}{\mathbb{E}(\mathcal{L}^d(X))}$. It is well-known that this ratio perimeter/area is related to the notion of scale (see e.g. [LAG09, DAG09] and the references therein). Regarding image modeling, our formulas for the mean variation of these three germ-grain models are in accordance with the above statement: for a high perimeter/area ratio of the grains, the total variation of the concerned germ-grain r.f. is high and thus the r.f. corresponds to a texture, whereas for a low perimeter/area ratio of the grains the r.f. might not be perceived as a texture but rather as a piecewise constant image containing large objects.

This works opens up for several future developments. As a starting point of our study of r.f. of bounded variation, the emphasis was on the mean variation of r.f. It might also be of interest to study the stochastic counterparts of the well-known geometric structures carried by any functions of bounded variation [AFP00]. For example one might study the random set induced by the jumps of a r.f. of bounded variation.

As a another example of further developments, one may consider the sample path properties of random fields and in particular be able to extend to r.f. over $\mathbb{R}^d$ some results of Vervaat [Ver85] where the variation of the sample paths of self-similar one-dimensional stochastic processes with stationary increments is studied.

The good properties of the specific variation $\theta_V(X)$ of a stationary random set X opens new perspectives. As expressed in [Vil10], a problem of interest in stochastic geometry is to define local mean surface densities for inhomogeneous (i.e. non stationary) random sets. One may propose some notion of directional variation densities as an alternative to the local surface densities introduced by Matheron [Mat75, p. 50]

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