# Center manifold and stability in critical cases for some partial functional differential equations

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**Abstract.** In this work, we prove the existence of a center manifold for some partial functional differential equations, whose linear part is not necessarily densely defined but satisfies the Hille-Yosida condition. The attractiveness of the center manifold is also shown when the unstable space is reduced to zero. We prove that the flow on the center manifold is completely determined by an ordinary differential equation in a finite dimensional space. In some critical cases, when the exponential stability is not possible, we prove that the uniform asymptotic stability of the equilibrium is completely determined by the uniform asymptotic stability of the reduced system on the center manifold.
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## 1. **Introduction**

The aim of this paper is to study the existence of a center manifold and stability in some critical cases for the following partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + g(u_t), \ t \geq 0 \\ u_0 = \varphi \in C := C\left([-r,0]\,; E\right), \end{cases} \tag{1.1}$$

where $A$ is not necessarily densely defined linear operator on a Banach space $E$ and $C$ is the space of continuous functions from $[-r,0]$ to $E$ endowed with the uniform norm topology. For every $t \geq 0$ and for a continuous $u : [-r,+\infty) \longrightarrow E$, the function $u_t \in C$ is defined by

$$u_t(\theta) = u(t+\theta) \quad \text{for } \theta \in [-r,0]\,.$$

$L$ is a bounded linear operator from $C$ into $E$ and $g$ is a Lipschitz continuous function from $C$ to $E$ with $g(0) = 0$.

In this work, we assume that $A$ is a Hille-Yosida operator: there exist $\omega \in \mathbb{R}$ and $M_0 \geq 1$ such that $(\omega,\infty) \subset \rho(A)$ and

$$\left|(\lambda I - A)^{-n}\right| \leq \frac{M_0}{(\lambda-\omega)^n} \quad \text{for } \lambda \geq \omega \ \text{ and } \ n \in \mathbb{N},$$

where $\rho(A)$ is the resolvent set of $A$. In [21], the authors proved the existence, regularity and stability of solutions of (1.1) when $A$ generates a strongly continuous semigroup, which is equivalent by Hille-Yosida Theorem to that $A$ is a Hille-Yosida operator and $\overline{D(A)} = E$. In [3], the authors used the integrated semigroup approach to prove the existence and regularity of solutions of (1.1) when $A$ is only a Hille-Yosida operator. Moreover, it was shown that the phase space of equation (1.1) is given by

$$Y := \left\{\varphi \in C : \varphi(0) \in \overline{D(A)}\right\}.$$

Assume that the function $g$ is differentiable at 0 with $g'(0) = 0$. Then the linearized equation of (1.1) around the equilibrium zero is given by

$$\begin{cases} \dfrac{d}{dt}v(t) = Av(t) + L(v_t), \ t \geq 0 \\ v_0 = \varphi \in C. \end{cases} \tag{1.2}$$

If all characteristic values (see section 2) of equation (1.2) have negative real part, then the zero equilibrium of (1.1) is uniformly asymptotically stable. However, if there exists at least one characteristic value with a positive real part, then the zero solution of (1.1) is unstable. In the critical case, when exponential stability is not possible and there exists a characteristic value with zero real part, the situation is more complicated since either stability or instability may hold. The subject of the center manifold is to study the stability in this critical case. For differential equations, the center manifold theory has been extensively studied, we refer to [6], [7], [8], [12], [13], [14], [15], [16], [19], [20] and [24]. In [17] and [22], the authors proved the existence of a center manifold when $\overline{D(A)} = E$. They established the attractiveness of this manifold when the unstable space is reduced to zero. In [11], the authors proved the existence of a center manifold for a given map. Their approach was applied to show the existence of a center manifold for partial functional differential equations in Banach spaces in the case when the linear part generates a compact strongly continuous semigroup. Recently, in [18], the authors studied the existence of invariant manifolds for an evolutionary process in Banach spaces and in particularly for some partial functional differential equations. For more details about the center manifold theory and its applications in the context of partial functional differential equations, we refer to the monograph [27]. Here we consider equation (1.1) when the domain $D(A)$ is not necessarily dense in $E$. The nondensity occurs, in many situations, from restrictions made on the space where the equations are considered (for example, periodic continuous solutions, H"older continuous functions) or from boundary conditions ( the space $C^1$ with null value on the boundary is not dense in the space of continuous functions). For more details, we refer to [1], [2], [3], [4] and [5].

The organization of this work is as follows: in section 2, we recall some results of integral solutions and the semigroup solution and we describe the variation of constants formula for the associated non-homogeneous problem of (1.2). We also give some results on the spectral analysis of the linear equation (1.2). In section 3, we prove the existence of a global center manifold. In section 4, we prove that this center manifold is exponentially attractive when the unstable space is reduced to zero. In section 5, we prove that the flow on the center manifold is governed by an ordinary differential equation in a finite dimensional space. In section 6, we prove a result on the stability of the equilibrium in the critical case. We also establish a new reduction principal for equation (1.1). In section 7, we study the existence of a local center manifold when $g$ is only defined and $C^1$-function in a neighborhood of zero. In the last section, we propose a result on the stability when zero is a simple characteristic value and no characteristic value lies on the imaginary axis.

## 2. **Spectral analysis and variation of constants formula**

In the following we assume

$(\mathbf{H}_1)$ $A$ is a Hille-Yosida operator.

**Definition 2.1.** A continuous function $u : [-r, +\infty) \to E$ is called an integral solution of equation (1.1) if

*i)* $\displaystyle\int_0^t u(s)ds \in D(A)$ for $t \geq 0$,

*ii)* $u(t) = \varphi(0) + A\left(\displaystyle\int_0^t u(s)ds\right) + L\left(\displaystyle\int_0^t u_s ds\right) + \displaystyle\int_0^t g(u_s)ds$ for $t \geq 0$,

*iii)* $u_0 = \varphi$.

We will call, without causing any confusion, the integral solution the function $u_t$, for $t \geq 0$.

Let $A_0$ be the part of the operator $A$ in $\overline{D(A)}$ which is defined by

$$\begin{cases} D(A_0) = \left\{x \in D(A) : Ax \in \overline{D(A)}\right\} \\ A_0 x = Ax \quad \text{for } x \in D(A_0). \end{cases}$$

The following result is well known (see [3]).

**Lemma 2.2.** *$A_0$ generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$.*

For the existence and uniqueness of an integral solution of (1.1), we need the following condition.

$(\mathbf{H}_2)$ $g : C \longrightarrow E$ is Lipschitz continuous.

The following result can be found in [3].

**Proposition 2.3.** *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. Then for* $\varphi \in Y$, *equation (1.1) has a unique global integral solution on* $[-r, \infty)$ *which is given by the following formula*

$$u(t) = \begin{cases} T_0(t)\varphi(0) + \lim_{\lambda\to\infty} \int_0^t T_0(t-s)B_\lambda\left(L(u_s) + g(u_s)\right) ds, & t \geq 0 \\ \varphi(t), \ t \in [-r, 0], \end{cases} \tag{2.1}$$

*where* $B_\lambda = \lambda(\lambda I - A)^{-1}$ *for* $\lambda \geq \omega$.

Assume that $g$ is differentiable at zero with $g'(0) = 0$. Then the linearized equation of (1.1) at zero is given by equation (1.2). Define the operator $U(t)$ on $Y$ by

$$U(t)\varphi = v_t(., \varphi),$$

where $v$ is the unique integral solution of equation (1.2) corresponding to the initial value $\varphi$. Then $(U(t))_{t\geq 0}$ is a strongly continuous semigroup on $Y$. One has the following linearized principle.

**Theorem 2.4.** *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_2)$ *hold. If the zero equilibrium of* $(U(t))_{t\geq 0}$ *is exponentially stable, in the sense that there exist* $N_0 \geq 1$ *and* $\epsilon \geq 0$ *such that*

$$|U(t)| \leq N_0 e^{-\epsilon t} \text{ for } t \geq 0,$$

*then the zero equilibrium of equation (1.1) is locally exponentially stable, in the sense that there exist* $\delta \geq 0$, $\mu \geq 0$ *and* $k \geq 1$ *such that*

$$|x_t(., \varphi)| \leq k e^{-\mu t} |\varphi| \text{ for } \varphi \in Y \text{ with } |\varphi| \leq \delta \text{ and } t \geq 0,$$

*where* $x_t(., \varphi)$ *is the integral solution of equation (1.1) corresponding to initial value* $\varphi$. *Moreover, if* $Y$ *can be decomposed as* $Y = Y_1 \oplus Y_2$ *where* $Y_i$ *are* $U$*-invariant subspaces of* $Y$, $Y_1$ *is a finite-dimensional space and with* $\omega_0 = \lim_{h\to\infty} \frac{1}{h} \log |U(h)|Y_2|$ *we have*

$$\inf\left\{|\lambda| : \lambda \in \sigma\left(U(t)|Y_1\right)\right\} \geq e^{\omega_0 t} \text{ for } t \geq 0,$$

*where* $\sigma\left(U(t)|Y_1\right)$ *is the spectrum of* $U(t)|Y_1$, *then the zero equilibrium of equation (1.1) is unstable, in the sense that there exist* $\varepsilon \geq 0$ *and sequences* $(\varphi_n)_n$ *converging to* $0$ *and* $(t_n)_n$ *of positive real numbers such that* $|x_{t_n}(., \varphi_n)| \geq \varepsilon$.

The above theorem is a consequence of the following result. For more details on the proof, we refer to [3].

**Theorem 2.5.** [9] *Let* $(V(t))_{t\geq 0}$ *be a nonlinear strongly continuous semigroup on a subset* $\Omega$ *of a Banach space* $Z$ *and assume that* $x_0 \in \Omega$ *is an equilibrium of* $(V(t))_{t\geq 0}$ *such that* $V(t)$ *is differentiable at* $x_0$, *with* $W(t)$ *the derivative at* $x_0$ *of* $V(t)$ *for each* $t \geq 0$. *Then,* $(W(t))_{t\geq 0}$ *is a strongly continuous semigroup of bounded linear operators on* $Z$. *If the zero equilibrium of* $(W(t))_{t\geq 0}$ *is exponentially stable, then* $x_0$ *is locally exponentially stable equilibrium of* $(V(t))_{t\geq 0}$. *Moreover, if* $Z$ *can be decomposed as* $Z = Z_1 \oplus Z_2$ *where* $Z_i$ *are* $W$*-invariant subspaces of* $Z$, $Z_1$ *is a finite-dimensional space and with* $\omega_1 = \lim_{h\to\infty} \frac{1}{h} \log |W(h)|Z_2|$ *we have*

$$\inf\left\{|\lambda| : \lambda \in \sigma\left(W(t)|Z_1\right)\right\} \geq e^{\omega_1 t} \text{ for } t \geq 0,$$

*then the equilibrium* $x_0$ *is unstable in the sense that there exist* $\varepsilon \geq 0$ *and sequences* $(y_n)_n$ *converging to* $x_0$ *and* $(t_n)_n$ *of positive real numbers such that* $|V(t_n)y_n - x_0| \geq \varepsilon$.

Some informations of the infinitesimal generator of $(U(t))_{t\geq 0}$ can be found in [5]. For example, we know that

**Theorem 2.6.** *The infinitesimal generator $A_U$ of $(U(t))_{t\geq 0}$ on $Y$ is given by*

$$\begin{cases} D(A_U) = \left\{ \begin{array}{l} \varphi \in C^1([-r,0];E) : \varphi(0) \in D(A),\ \varphi'(0) \in \overline{D(A)} \text{ and} \\ \varphi'(0) = A\varphi(0) + L(\varphi) \end{array} \right\} \\ A_U\varphi = \varphi' \quad \text{for } \varphi \in D(A_U). \end{cases}$$

We now make the next assumption about the operator $A$.

$(\mathbf{H}_3)$ The semigroup $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$ for $t \geq 0$.

**Theorem 2.7.** *Assume that $(\mathbf{H}_3)$ holds. Then, $U(t)$ is a compact operator on $Y$ for $t \geq r$.*

**Proof.** Let $t \geq r$ and $\widetilde{D}$ be a bounded subset of $Y$. We use Ascoli-Arzela's theorem to show that $\left\{U(t)\varphi : \varphi \in \widetilde{D}\right\}$ is relatively compact in $Y$. Let $\varphi \in \widetilde{D}$, $\theta \in [-r,0]$ and $\varepsilon \geq 0$ such that $t+\theta-\varepsilon \geq 0$. Then

$$(U(t)\varphi)(\theta) = T_0(t+\theta)\varphi(0) + \lim_{\lambda\to+\infty} \int_0^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds.$$

Note that

$$\begin{aligned} &\int_0^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds \\ &= \int_0^{t+\theta-\varepsilon} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds + \int_{t+\theta-\varepsilon}^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds \end{aligned}$$

and

$$\lim_{\lambda\to+\infty}\int_0^{t+\theta-\varepsilon} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds = T_0(\varepsilon)\lim_{\lambda\to+\infty}\int_0^{t+\theta-\varepsilon} T_0(t+\theta-\varepsilon-s)B_\lambda L(U(s)\varphi)ds.$$

The assumption $(\mathbf{H}_3)$ implies that

$$T_0(\varepsilon)\left\{\lim_{\lambda\to+\infty}\int_0^{t+\theta-\varepsilon} T_0(t+\theta-\varepsilon-s)B_\lambda L(U(s)\varphi)ds : \varphi \in \widetilde{D}\right\}$$

is relatively compact in $E$. As the semigroup $(U(t))_{t\geq 0}$ is exponentially bounded, then there exists a positive constant $b_1$ such that

$$\left|\lim_{\lambda\to+\infty}\int_{t+\theta-\varepsilon}^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds\right| \leq b_1\varepsilon \ \text{ for } \varphi \in \widetilde{D}.$$

Consequently, the set

$$\left\{\lim_{\lambda\to+\infty}\int_{t+\theta-\varepsilon}^{t+\theta} T_0(t+\theta-s)B_\lambda L(U(s)\varphi)ds : \varphi \in \widetilde{D}\right\}$$

is totally bounded in $E$. We deduce that $\left\{(U(t)\varphi)(\theta) : \varphi \in \widetilde{D}\right\}$ is relatively compact in $E$, for each $\theta \in [-r,0]$. For the completeness of the proof, we need to show the equicontinuity property. Let $\theta$, $\theta_0 \in [-r,0]$ such that $\theta \geq \theta_0$. Then

$$
\begin{aligned}
(U(t)\varphi)(\theta) - (U(t)\varphi)(\theta_0) = {} & (T_0(t+\theta) - T_0(t+\theta_0))\,\varphi(0) \\
& + \lim_{\lambda\to+\infty} \int_0^{t+\theta} T_0(t+\theta-s) B_\lambda L(U(s)\varphi) ds \\
& - \lim_{\lambda\to+\infty} \int_0^{t+\theta_0} T_0(t+\theta_0-s) B_\lambda L(U(s)\varphi) ds.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\int_0^{t+\theta} T_0(t+\theta-s) B_\lambda L(U(s)\varphi) ds = {} & \int_0^{t+\theta_0} T_0(t+\theta-s) B_\lambda L(U(s)\varphi) ds \\
& + \int_{t+\theta_0}^{t+\theta} T_0(t+\theta-s) B_\lambda L(U(s)\varphi) ds.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
|(U(t)\varphi)(\theta) - (U(t)\varphi)(\theta_0)| \le {} & |T_0(t+\theta) - T_0(t+\theta_0)|\,|\varphi(0)| \\
& + \left| \lim_{\lambda\to+\infty} \int_0^{t+\theta_0} (T_0(t+\theta-s) - T_0(t+\theta_0-s))\, B_\lambda L(U(s)\varphi) ds \right| \\
& + \left| \lim_{\lambda\to+\infty} \int_{t+\theta_0}^{t+\theta} T_0(t+\theta-s) B_\lambda L(U(s)\varphi) ds \right|.
\end{aligned}
$$

Assumption ($\mathbf{H}_3$) implies that the semigroup $(T_0(t))_{t\ge0}$ is uniformly continuous for $t \ge 0$. Then

$$
\lim_{\theta\to\theta_0} |T_0(t+\theta) - T_0(t+\theta_0)| = 0.
$$

The semigroup $(U(t))_{t\ge0}$ is exponentially bounded. Consequently, there exists a positive constant $b_2$ such that

$$
\left| \lim_{\lambda\to+\infty} \int_{t+\theta_0}^{t+\theta} T_0(t+\theta-s) B_\lambda L(U(s)\varphi) ds \right| \le b_2\,(\theta-\theta_0)
$$

and

$$
\begin{aligned}
& \lim_{\lambda\to+\infty} \int_0^{t+\theta_0} (T_0(t+\theta-s) - T_0(t+\theta_0-s))\, B_\lambda L(U(s)\varphi) ds \\
& \qquad = (T_0(\theta-\theta_0) - I) \lim_{\lambda\to+\infty} \int_0^{t+\theta_0} T_0(t+\theta_0-s) B_\lambda L(U(s)\varphi) ds.
\end{aligned}
$$

We have proved that there exists a compact set $\widetilde{K}_0$ in $E$ such that

$$
\lim_{\lambda\to+\infty} \int_0^{t+\theta_0} T_0(t+\theta_0-s) B_\lambda L(U(s)\varphi) ds \in \widetilde{K}_0 \ \text{ for } \varphi \in \widetilde{D}.
$$

Using Banach-Steinhaus's theorem, we obtain

$$
\lim_{\theta\to\theta_0} (T_0(\theta-\theta_0) - I)x = 0 \text{ uniformly in } x \in \widetilde{K}_0.
$$

This implies that

$$
\lim_{\theta\to\theta_0^+} (U(t)\varphi)(\theta) - (U(t)\varphi)(\theta_0) = 0 \text{ uniformly in } \varphi \in \widetilde{D}.
$$

We can prove in similar way that

$$\lim_{\theta\to\theta_0^-} \left(U(t)\varphi\right)(\theta) - \left(U(t)\varphi\right)(\theta_0) = 0 \text{ uniformly in } \varphi \in \widetilde{D}.$$

By Ascoli-Arzela's theorem, we conclude that $\overline{\left\{U(t)\varphi : \varphi \in \widetilde{D}\right\}}$ is compact for $t \geq r$.

Now, we consider the spectral properties of the infinitesimal generator $A_U$. We denote by $E$, without causing confusion, the complexication of $E$. For each complex number $\lambda$, we define the linear operator $\Delta(\lambda) : D(A) \to E$ by

$$\Delta(\lambda) = \lambda I - A - L(e^{\lambda \cdot} I), \tag{2.2}$$

where $e^{\lambda \cdot} I : E \to C$ is defined by

$$\left(e^{\lambda \cdot} x\right)(\theta) = e^{\lambda\theta} x,\ x \in E \quad \text{and} \quad \theta \in [-r, 0]\,.$$

**Definition 2.8.** We say that $\lambda$ is a characteristic value of equation (1.2) if there exists $x \in D(A)\backslash\{0\}$ solving the characteristic equation $\Delta(\lambda)x = 0$.

Since the operator $U(t)$ is compact for $t \geq r$, the spectrum $\sigma(A_U)$ of $A_U$ is the point spectrum $\sigma_p(A_U)$. More precisely, we have

**Theorem 2.9.** *The spectrum* $\sigma(A_U) = \sigma_p(A_U) = \{\lambda \in \mathbb{C} : \ker \Delta(\lambda) \neq \{0\}\}$ *.*

**Proof.** Let $\lambda \in \sigma_p(A_U)$. Then there exists $\varphi \in D(A_U)\backslash\{0\}$ such that $A_U\varphi = \lambda\varphi$, which is equivalent to

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0), \text{ for } \theta \in [-r, 0] \text{ and } \varphi'(0) = A\varphi(0) - L(\varphi) \text{ with } \varphi(0) \neq 0.$$

Consequently $\Delta(\lambda)\varphi(0) = 0$. Conversely, let $\lambda \in \mathbb{C}$ such that $\ker \Delta(\lambda) \neq \{0\}$ . Then there exists $x \in D(A)\backslash\{0\}$ such that $\Delta(\lambda)x = 0$. If we define the function $\varphi$ by

$$\varphi(\theta) = e^{\lambda\theta} x \text{ for } \theta \in [-r, 0]\,,$$

then $\varphi \in D(A_U)$ and $A_U\varphi = \lambda\varphi$, which implies that $\lambda \in \sigma_p(A_U)$.

The growth bound $\omega_0(U)$ of the semigroup $(U(t))_{t\geq 0}$ is defined by

$$\omega_0(U) = \inf\left\{\kappa \geq 0 : \sup_{t\geq 0} e^{-\kappa t} |U(t)| < \infty\right\}.$$

The spectral bound $s(A_U)$ of $A_U$ is defined by

$$s(A_U) = \sup\left\{\mathrm{Re}(\lambda) : \lambda \in \sigma_p(A_U)\right\}.$$

Since $U(t)$ is compact for $t \geq r$, then it is well known that $\omega_0(U) = s(A_U)$. Consequently, the asymptotic behavior of the solutions of the linear equation (1.2) is completely obtained by $s(A_U)$. More precisely, we have the following result.

**Corollary 2.10.** *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold. Then, the following properties hold,*
*i) if* $s(A_U) < 0$, *then* $(U(t))_{t\geq 0}$ *is exponentially stable and zero is locally exponentially stable for equation (1.1);*
*ii) if* $s(A_U) = 0$, *then there exists* $\varphi \in Y$ *such that* $|U(t)\varphi| = |\varphi|$ *for* $t \geq 0$ *and either stability or instability may hold;*
*iii) if* $s(A_U) \geq 0$, *then there exists* $\varphi \in Y$ *such that* $|U(t)\varphi| \to \infty$ *as* $t \to \infty$ *and zero is unstable for equation (1.1).*

As a consequence of the compactness of the semigroup $U(t)$ for $t \geq r$ and by Theorem 2.11, p.100, in [10], we get the following general spectral decomposition of the phase space $Y$.

**Theorem 2.11.** *There exist linear subspaces of $Y$ denoted by $Y_-$, $Y_0$ and $Y_+$ respectively with*

$$Y = Y_- \oplus Y_0 \oplus Y_+$$

*such that*
*i)* $A_U(Y_-) \subset Y_-$, $A_U(Y_0) \subset Y_0$, *and* $A_U(Y_+) \subset Y_+$;
*ii)* $Y_0$ *and* $Y_+$ *are finite dimensional;*
*iii)* $\sigma(A_U|Y_0) = \{\lambda \in \sigma(A_U) : \operatorname{Re}\lambda = 0\}$, $\sigma(A_U|Y_+) = \{\lambda \in \sigma(A_U) : \operatorname{Re}\lambda \geq 0\}$;
*iv)* $U(t)Y_- \subset Y_-$ *for* $t \geq 0$, $U(t)$ *can be extended for* $t \leq 0$ *when restricted to* $Y_0 \cup Y_+$ *and* $U(t)Y_0 \subset Y_0, U(t)Y_+ \subset Y_+$ *for* $t \in \mathbb{R}$;
*v) for any* $0 < \gamma < \inf\{|\operatorname{Re}\lambda| : \lambda \in \sigma(A_U) \text{ and } \operatorname{Re}\lambda \neq 0\}$, *there exists* $K \geq 0$ *such that*

$$\begin{aligned} &|U(t)P_-\varphi| \leq Ke^{-\gamma t}\,|P_-\varphi| \quad \text{for } t \geq 0, \\ &|U(t)P_0\varphi| \leq Ke^{\frac{\gamma}{3}|t|}\,|P_0\varphi| \quad \text{for } t \in \mathbb{R}, \\ &|U(t)P_+\varphi| \leq Ke^{\gamma t}\,|P_+\varphi| \quad \text{for } t \leq 0, \end{aligned}$$

*where* $P_-, P_0$ *and* $P_+$ *are projections of* $Y$ *into* $Y_-, Y_0$ *and* $Y_+$ *respectively.*
$Y_-, Y_0$ *and* $Y_+$ *are called stable, center and unstable subspaces of the semigroup* $(U(t))_{t\geq 0}$.

The following result deals with the variation of constants formula for equation (1.1) which are taken from [5]. Let $\langle X_0 \rangle$ be defined by

$$\langle X_0 \rangle = \{X_0 c : c \in E\},$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0), \\ c & \text{if } \theta = 0. \end{cases}$$

We introduce the space $Y \oplus \langle X_0 \rangle$, endowed with the following norm

$$|\varphi + X_0 c| = |\varphi| + |c|.$$

The following result is taken from [5].

**Theorem 2.12.** *The continuous extension* $\widetilde{A_U}$ *of the operator* $A_U$ *defined on* $Y \oplus \langle X_0 \rangle$ *by:*

$$\begin{cases} D(\widetilde{A_U}) = \left\{\varphi \in C^1([-r,0]; E) : \varphi(0) \in D(A) \text{ and } \varphi'(0) \in \overline{D(A)}\right\} \\ \widetilde{A_U}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi'(0)), \end{cases}$$

*is a Hille-Yosida operator on* $Y \oplus \langle X_0 \rangle$*: there exist* $\widetilde{\omega} \in \mathbb{R}$ *and* $\widetilde{M_0} \geq 1$ *such that* $(\widetilde{\omega}, \infty) \subset \rho(\widetilde{A_U})$ *and*

$$\left|(\lambda I - \widetilde{A_U})^{-n}\right| \leq \frac{\widetilde{M_0}}{(\lambda - \widetilde{\omega})^n} \quad \text{for } \lambda \geq \widetilde{\omega} \text{ and } n \in \mathbb{N},$$

*with* $\rho(\widetilde{A_U})$ *the resolvent set of* $\widetilde{A_U}$.
*Moreover, the integral solution* $u$ *of equation (1.1) is given for* $\varphi \in Y$, *by the following variation of constants formula*

$$u_t = U(t)\varphi + \lim_{\lambda \to \infty} \int_0^t U(t-s)\widetilde{B_\lambda}\,(X_0 g(u_s))\,ds \quad \text{for } t \geq 0, \tag{2.3}$$

*where* $\widetilde{B_\lambda} = \lambda(\lambda I - \widetilde{A_U})^{-1}$ *for* $\lambda \geq \widetilde{\omega}$.

**Remarks.** i) Without loss of generality, we assume that $\widetilde{M_0} = 1$. Otherwise, we can renorm the space $Y \oplus \langle X_0 \rangle$ in order to get an equivalent norm for which $\widetilde{M_0} = 1$.
ii) For any locally integrable function $\varrho : \mathbb{R} \to E$, one can see that the following limit exists:

$$\lim_{\lambda\to\infty} \int_s^t U(t-\tau)\widetilde{B_\lambda} X_0 \varrho(\tau) d\tau \quad \text{for } t \geq s.$$

## 3. Global existence of the center manifold

**Theorem 3.1.** *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_3)$ *hold. Then, there exists* $\varepsilon \geq 0$ *such that if*

$$Lip(g) = \sup_{\varphi_1 \neq \varphi_2} \frac{|g(\varphi_1) - g(\varphi_2)|}{|\varphi_1 - \varphi_2|} < \varepsilon,$$

*then, there exists a bounded Lipschitz map* $h_g : Y_0 \to Y_- \oplus Y_+$ *such that* $h_g(0) = 0$ *and the Lipschitz manifold*

$$M_g := \{\varphi + h_g(\varphi) : \varphi \in Y_0\}$$

*is globally invariant under the flow of equation (1.1) on* $Y$.

**Proof.** Let $\mathcal{B} = \mathcal{B}(Y_0, Y_- \oplus Y_+)$ denote the Banach space of bounded maps $h : Y_0 \to Y_- \oplus Y_+$ endowed with the uniform norm topology. We define

$$\mathcal{F} = \{h \in \mathcal{B} : h \text{ is Lipschitz, } h(0) = 0 \text{ and } Lip(h) \leq 1\}.$$

Let $h \in \mathcal{F}$ and $\varphi \in Y_0$. Using the strict contraction principle, one can prove the existence of $v_t^\varphi$ solution of the following equation

$$v_t^\varphi = U(t)\varphi + \lim_{\lambda\to\infty} \int_0^t U(t-\tau)\left(\widetilde{B_\lambda} X_0 g(v_\tau^\varphi + h(v_\tau^\varphi))\right)^0 d\tau,\ t \in \mathbb{R}. \tag{3.1}$$

We now introduce the mapping $T_g : \mathcal{F} \to \mathcal{B}$ by

$$\begin{aligned} T_g(h)\varphi = {} & \lim_{\lambda\to\infty} \int_{-\infty}^0 U(-\tau)\left(\widetilde{B_\lambda} X_0 g(v_\tau^\varphi + h(v_\tau^\varphi))\right)^- d\tau \\ & + \lim_{\lambda\to\infty} \int_{+\infty}^0 U(-\tau)\left(\widetilde{B_\lambda} X_0 g(v_\tau^\varphi + h(v_\tau^\varphi))\right)^+ d\tau. \end{aligned}$$

The first step is to prove that $T_g$ maps $\mathcal{F}$ into itself. Let $\varphi_1, \varphi_2 \in Y_0$ and $t \in \mathbb{R}$. Suppose that $Lip(g) < \varepsilon$. Then

$$|v_t^{\varphi_1} - v_t^{\varphi_2}| \leq K e^{\frac{\gamma}{3}|t|} |\varphi_1 - \varphi_2| + 2K|P_0|\varepsilon \left| \int_0^t e^{\frac{\gamma}{3}|t-\tau|} |v_\tau^{\varphi_1} - v_\tau^{\varphi_2}| d\tau \right|.$$

By Gronwall's lemma, we get that

$$e^{-\frac{\gamma}{3}|t|} |v_t^{\varphi_1} - v_t^{\varphi_2}| \leq K|\varphi_1 - \varphi_2| e^{2K|P_0|\varepsilon|t|}$$

and

$$|v_t^{\varphi_1} - v_t^{\varphi_2}| \leq K|\varphi_1 - \varphi_2| e^{\left[\frac{\gamma}{3} + 2K|P_0|\varepsilon\right]|t|} \text{ for } t \in \mathbb{R}.$$

If we choose $\varepsilon$ such that

$$2K|P_0|\varepsilon < \frac{\gamma}{6}, \tag{3.2}$$

then

$$|v_t^{\varphi_1} - v_t^{\varphi_2}| \leq K\,|\varphi_1 - \varphi_2|\, e^{\frac{\gamma}{2}|t|} \text{ for } t \in \mathbb{R}. \tag{3.3}$$

Moreover,

$$\begin{aligned} |T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq & \int_{-\infty}^{0} |U(-\tau)P_-|\, |g(v_\tau^{\varphi_1} + h(v_\tau^{\varphi_1})) - g(v_\tau^{\varphi_2} + h(v_\tau^{\varphi_2}))|\, d\tau \\ & + \int_0^{+\infty} |U(-\tau)P_+|\, |g(v_\tau^{\varphi_1} + h(v_\tau^{\varphi_1})) - g(v_\tau^{\varphi_2} + h(v_\tau^{\varphi_2}))|\, d\tau. \end{aligned}$$

Consequently,

$$|T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq 2\int_{-\infty}^{0} Ke^{\gamma\tau}\,|P_-|\,\varepsilon\,|v_\tau^{\varphi_1} - v_\tau^{\varphi_2}|\,d\tau + 2\int_0^{+\infty} Ke^{-\gamma\tau}\,|P_+|\,\varepsilon\,|v_\tau^{\varphi_1} - v_\tau^{\varphi_2}|\,d\tau.$$

Using inequality (3.3), we obtain

$$|T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq 2K^2\,|P_-|\,\varepsilon\,|\varphi_1 - \varphi_2| \int_{-\infty}^{0} e^{\frac{\gamma}{2}\tau} d\tau + 2K^2\,|P_+|\,\varepsilon\,|\varphi_1 - \varphi_2| \int_0^{+\infty} e^{-\frac{\gamma}{2}\tau} d\tau.$$

It follows that

$$|T_g(h)\varphi_1 - T_g(h)\varphi_2| \leq \frac{4\varepsilon}{\gamma} K^2\,(|P_-| + |P_+|)\,|\varphi_1 - \varphi_2|\,.$$

If we choose $\varepsilon$ such that

$$\frac{4\varepsilon}{\gamma} K^2\,(|P_-| + |P_+|) < 1,$$

then $T_g$ maps $\mathcal{F}$ into itself.
The next step is to show that $T_g$ is a strict contraction on $\mathcal{F}$. Let $h_1, h_2 \in \mathcal{F}$. For $\varphi \in Y_0$ and for $i = 1, 2$, let $v_t^i$ denote the solution of the following equation

$$v_t^i = U(t)\varphi + \lim_{\lambda\to\infty} \int_0^t U(t-\tau)\left(\widetilde{B_\lambda} X_0 g(v_\tau^i + h_i(v_\tau^i))\right)^0 d\tau \text{ for } t \in \mathbb{R}.$$

Then,

$$\left|v_t^1 - v_t^2\right| \leq \varepsilon K\,|P_0| \left| \int_0^t e^{\frac{\gamma}{3}|t-\tau|} \left( \left|v_\tau^1 - v_\tau^2\right| + \left|h_1(v_\tau^1) - h_1(v_\tau^2)\right| + \left|h_1(v_\tau^2) - h_2(v_\tau^2)\right| \right) d\tau \right|,$$

and

$$\left|v_t^1 - v_t^2\right| \leq 2\varepsilon K\,|P_0| \left| \int_0^t e^{\frac{\gamma}{3}|t-\tau|} \left|v_\tau^1 - v_\tau^2\right| d\tau \right| + \varepsilon K\,|P_0|\,|h_1 - h_2| \left| \int_0^t e^{\frac{\gamma}{3}|t-\tau|} d\tau \right|.$$

By Gronwall's lemma, we obtain that

$$\left|v_t^1 - v_t^2\right| \leq \frac{3\varepsilon K}{\gamma}\,|P_0|\,|h_1 - h_2|\, e^{\left[\frac{\gamma}{3} + 2K|P_0|\varepsilon\right]|t|} \quad \text{for } t \in \mathbb{R}.$$

By (3.2), we obtain

$$\left|v_t^1 - v_t^2\right| \leq \frac{3\varepsilon K}{\gamma}\,|P_0|\,|h_1 - h_2|\, e^{\frac{\gamma}{2}|t|} \quad \text{for all } t \in \mathbb{R}.$$

For $i = 1, 2$, we have

$$T_g(h_i)\varphi = \lim_{\lambda \to \infty} \int_{-\infty}^{0} U(-\tau) \left( \widetilde{B_\lambda} X_0 g(v_\tau^i + h_i(v_\tau^i)) \right)^- d\tau$$
$$+ \lim_{\lambda \to \infty} \int_{+\infty}^{0} U(-\tau) \left( \widetilde{B_\lambda} X_0 g(v_\tau^i + h_i(v_\tau^i)) \right)^+ d\tau.$$

It follows that

$$|T_g(h_1)\varphi - T_g(h_2)\varphi| \le 2K \left|P_-\right| \frac{6\varepsilon^2 K}{\gamma^2} \left|P_0\right| \left|h_1 - h_2\right| + \frac{K \left|P_-\right| \varepsilon}{\gamma} \left|h_1 - h_2\right|$$
$$+ 2K \left|P_+\right| \frac{6\varepsilon^2 K}{\gamma^2} \left|P_0\right| \left|h_1 - h_2\right| + \frac{K \left|P_+\right| \varepsilon}{\gamma} \left|h_1 - h_2\right|.$$

Consequently,

$$|T_g(h_1) - T_g(h_2)| \le (|P_-| + |P_+|) \frac{K\varepsilon}{\gamma} \left[ |P_0| \frac{12\varepsilon K}{\gamma} + 1 \right] |h_1 - h_2|.$$

We choose $\varepsilon$ such that

$$(|P_-| + |P_+|) \frac{K\varepsilon}{\gamma} \left[ |P_0| \frac{12\varepsilon K}{\gamma} + 1 \right] < 1.$$

Then $T_g$ is a strict contraction on $\mathcal{F}$, and consequently it has a unique fixed point $h_g$ in $\mathcal{F} : T_g(h_g) = h_g$.
Finally, we show that

$$M_g := \{\varphi + h_g(\varphi) : \varphi \in Y_0\}$$

is globally invariant under the flow on $Y$. Let $\varphi \in Y_0$ and $v$ be the solution of equation (3.1). We claim that $t \to v_t^\varphi + h_g(v_t^\varphi)$ is an integral solution of equation (1.1) with initial value $\varphi + h_g(\varphi)$. In fact, we have $T_g(h_g)(v_t^\varphi) = h_g(v_t^\varphi)$, $t \in \mathbb{R}$. Moreover, for $t \in \mathbb{R}$, one has

$$T_g(h)(v_t^\varphi) = \lim_{\lambda \to \infty} \int_{-\infty}^{0} U(-\tau) \left( \widetilde{B_\lambda} X_0 g(v_{t+\tau}^\varphi + h_g(v_{t+\tau}^\varphi)) \right)^- d\tau$$
$$+ \lim_{\lambda \to \infty} \int_{+\infty}^{0} U(-\tau) \left( \widetilde{B_\lambda} X_0 g(v_{t+\tau}^\varphi + h_g(v_{t+\tau}^\varphi)) \right)^+ d\tau,$$

which implies that

$$h_g(v_t^\varphi) = \lim_{\lambda \to \infty} \int_{-\infty}^{t} U(t-\tau) \left( \widetilde{B_\lambda} X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi)) \right)^- d\tau$$
$$+ \lim_{\lambda \to \infty} \int_{+\infty}^{t} U(t-\tau) \left( \widetilde{B_\lambda} X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi)) \right)^+ d\tau.$$

Then, for $t \in \mathbb{R}$, we have

$$\begin{aligned} v_t^\varphi + h_g(v_t^\varphi) = U(t)\varphi &+ \lim_{\lambda\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^0 d\tau \\ &+ \lim_{\lambda\to\infty}\int_{-\infty}^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau \\ &+ \lim_{\lambda\to\infty}\int_{+\infty}^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau. \end{aligned}$$

For any $t \geq a$, we have

$$\begin{aligned} &\lim_{\lambda\to\infty}\int_{-\infty}^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau \\ &\quad = \lim_{\lambda\to\infty}\int_{-\infty}^a U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau \\ &\quad\quad + \lim_{\lambda\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau, \end{aligned}$$

and

$$\begin{aligned} &\lim_{\lambda\to\infty}\int_{-\infty}^a U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau \\ &\quad = U(t-a)\left(\lim_{\lambda\to\infty}\int_{-\infty}^a U(a-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau\right). \end{aligned}$$

By the same argument as above, we obtain

$$\begin{aligned} &\lim_{\lambda\to\infty}\int_{+\infty}^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau \\ &\quad = \lim_{\lambda\to\infty}\int_{+\infty}^a U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau \\ &\quad\quad + \lim_{\lambda\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau, \end{aligned}$$

and

$$\begin{aligned} &\lim_{\lambda\to\infty}\int_{+\infty}^a U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau \\ &\quad = U(t-a)\left(\lim_{\lambda\to\infty}\int_{+\infty}^a U(a-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau\right). \end{aligned}$$

Note that

$$\begin{aligned} h_g(v_a^\varphi) = &\lim_{\lambda\to\infty}\int_{+\infty}^a U(a-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau \\ &+ \lim_{\lambda\to\infty}\int_{-\infty}^a U(a-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau, \end{aligned}$$

and in particular

$$v_t^\varphi = U(t-a)v_a^\varphi + \lim_{\lambda\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^0 d\tau.$$

Consequently, for any $t \geq a$, we obtain

$$v_t^\varphi + h_g(v_t^\varphi) = U(t-a)\left(v_a^\varphi + h_g(v_a^\varphi)\right) + \lim_{\lambda\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^0 d\tau$$

$$+ \lim_{\lambda\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^+ d\tau$$

$$+ \lim_{\lambda\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^- d\tau,$$

which implies that for any $t \geq a$,

$$v_t^\varphi + h_g(v_t^\varphi) = U(t-a)\left(v_a^\varphi + h_g(v_a^\varphi)\right) + \lim_{\lambda\to\infty}\int_a^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right) d\tau.$$

Finally, we conclude that $v_{\mathbf{t}}^\varphi + h_g(v_{\mathbf{t}}^\varphi)$ is an integral solution of equation (1.1) on $\mathbb{R}$ with initial value $\varphi + h_g(\varphi)$.

**Theorem 3.2.** *Let $v^\varphi$ be the solution of equation (3.1) on $\mathbb{R}$. Then, for $t \in \mathbb{R}$*

$$|v_t^\varphi| \leq K\,|\varphi|\, e^{\frac{\gamma}{2}|t|} \text{ and } |v_t^\varphi + h_g(v_t^\varphi)| \leq 2K\,|\varphi|\, e^{\frac{\gamma}{2}|t|}.$$

*Conversely, if we choose $\varepsilon$ such that*

$$\frac{2K\varepsilon}{\gamma}\left(|P_-| + |P_+| + 3\,|P_0|\right) < 1,$$

*then for any integral solution $u$ of equation (1.1) on $\mathbb{R}$ with $u_t = O(e^{\frac{\gamma}{2}|t|})$, we have $u_t \in M_g$ for all $t \in \mathbb{R}$.*

**Proof.** Let $v^\varphi$ be the solution of equation (3.1). Then, using the estimate (3.3), we obtain that

$$|v_t^\varphi| \leq K\,|\varphi|\, e^{\frac{\gamma}{2}|t|} \quad \text{for } t \in \mathbb{R}$$

and from the fact that $Lip(h) \leq 1$ and $h_g(0) = 0$, we obtain

$$|v_t^\varphi + h_g(v_t^\varphi)| \leq 2K\,|\varphi|\, e^{\frac{\gamma}{2}|t|} \quad \text{for } t \in \mathbb{R}.$$

Let $u$ be an integral solution of equation (1.1) such that $u_t = O(e^{\frac{\gamma}{2}|t|})$. Then there exists a positive constant $k_0$ such that $|u_t| \leq k_0 e^{\frac{\gamma}{2}|t|}$ for all $t \in \mathbb{R}$. Note that

$$u_t = U(t-s)u_s + \lim_{\lambda\to\infty}\int_s^t U(t-\tau)\widetilde{B_\lambda}X_0 g(u_\tau)d\tau \text{ for } t \geq s.$$

On the other hand,

$$u_t^+ = U(t-s)u_s^+ + \lim_{\lambda\to\infty}\int_s^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(u_\tau)\right)^+ d\tau \text{ for } s \geq t.$$

Moreover, for $s \geq t$ and $s \geq 0$, we have

$$\left|U(t-s)u_s^+\right| \leq Ke^{\gamma(t-s)}\,|P_+u_s| \leq k_0K\,|P_+|\,e^{\gamma(t-s)}e^{\frac{\gamma}{2}|s|} = k_0K\,|P_+|\,e^{\gamma t}e^{-\frac{\gamma}{2}s}.$$

Therefore,

$$\lim_{s\to\infty}\left|U(t-s)u_s^+\right| = 0.$$

It follows that

$$u_t^+ = \lim_{\lambda\to\infty}\int_{+\infty}^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(u_\tau)\right)^+ d\tau.$$

Similarly, we can prove that

$$u_t^- = \lim_{\lambda\to\infty}\int_{-\infty}^{t} U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(u_\tau)\right)^- d\tau.$$

We conclude that

$$u_t = u_t^+ + u_t^- + U(t)u_0^0 + \lim_{\lambda\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(u_\tau)\right)^0 d\tau \quad \text{for } t\in\mathbb{R}.$$

Let $\phi \in Y_0$ such that $\phi = u_0$. By Theorem 3.1, there exists an integral solution $w$ of equation (1.1) on $\mathbb{R}$ with initial value $\phi + h_g(\phi)$ such that $w_t \in M_g$ for all $t\in\mathbb{R}$ and

$$\begin{aligned} w_t = U(t)\phi &+ \lim_{\lambda\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(w_\tau)\right)^0 d\tau \\ &+ \lim_{\lambda\to\infty}\int_{-\infty}^{t} U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(w_\tau)\right)^+ d\tau \\ &+ \lim_{\lambda\to\infty}\int_{+\infty}^{t} U(t-\tau)\left(\widetilde{B_\lambda}X_0 g(w_\tau)\right)^- d\tau. \end{aligned}$$

Then, for all $t\in\mathbb{R}$, we have

$$\begin{aligned} |u_t - w_t| \le &\left|\lim_{\lambda\to\infty}\int_0^t U(t-\tau)\left(\widetilde{B_\lambda}X_0\left(g(u_\tau)-g(w_\tau)\right)\right)^0 d\tau\right| \\ &+\left|\lim_{\lambda\to\infty}\int_{-\infty}^{t} U(t-\tau)\left(\widetilde{B_\lambda}X_0\left(g(u_\tau)-g(w_\tau)\right)\right)^+ d\tau\right| \\ &+\left|\lim_{\lambda\to\infty}\int_{+\infty}^{t} U(t-\tau)\left(\widetilde{B_\lambda}X_0\left(g(u_\tau)-g(w_\tau)\right)\right)^- d\tau\right|. \end{aligned}$$

This implies that

$$\begin{aligned} |u_t - w_t| \le K\varepsilon\Bigg(&|P_0|\left|\int_0^t e^{\frac{\gamma}{3}|t-\tau|}\,|u_\tau - w_\tau|\,d\tau\right| \\ &+ |P_-|\int_{-\infty}^{t} e^{-\gamma(t-\tau)}\,|u_\tau - w_\tau|\,d\tau + |P_+|\int_t^{\infty} e^{\gamma(t-\tau)}\,|u_\tau - w_\tau|\,d\tau\Bigg). \end{aligned}$$

Let $N(t) = e^{-\frac{\gamma}{2}|t|}\,|u_t - w_t|$ for all $t\in\mathbb{R}$. Then, $\widetilde{N} = \sup_{t\in\mathbb{R}} N(t) < \infty$. On the other hand, we have

$$N(t) \le K\varepsilon\widetilde{N}\left(|P_0|\left|\int_0^t e^{-\frac{\gamma}{6}|t-\tau|}d\tau\right| + |P_-|\int_{-\infty}^{t} e^{-\frac{\gamma}{2}(t-\tau)}d\tau + |P_+|\int_t^{\infty} e^{\frac{\gamma}{2}(t-\tau)}d\tau\right).$$

Finally we arrive at

$$\widetilde{N} \le \frac{2K\varepsilon}{\gamma}\left(3\,|P_0| + |P_-| + |P_+|\right)\widetilde{N}.$$

Consequently, $\widetilde{N} = 0$ and $u_t = w_t$ for $t\in\mathbb{R}$.

## 4. **Attractiveness of the center manifold**

In this section, we assume that there exists no characteristic value with a positive real part and hence the unstable space $Y_+$ is reduced to zero. We establish the following result on the attractiveness of the center manifold.

**Theorem 4.1.** *There exist $\varepsilon \geq 0$, $K_1 \geq 0$ and $\alpha \in \left(\frac{\gamma}{3}, \gamma\right)$ such that if $Lip(g) < \varepsilon$, then any integral solution $u_t(\varphi)$ of equation (1.1) on $\mathbb{R}^+$ satisfies*

$$\left|u_t^-(\varphi) - h_g(u_t^0(\varphi))\right| \leq K_1 e^{-\alpha t} \left|\varphi^- - h_g(\varphi^0)\right| \text{ for } t \geq 0. \tag{4.1}$$

**Proof.** The proof of this theorem is based on the following technically lemma.

**Lemma 4.2.** *There exist $\varepsilon \geq 0$, $K_0 \geq 0$ and $\alpha \in \left(\frac{\gamma}{3}, \gamma\right)$ such that if $Lip(g) < \varepsilon$, then there is a continuous bounded mapping $p : \mathbb{R}^+ \times Y_0 \times Y_- \to Y_-$ such that any integral solution $u_t(\varphi)$ of equation (1.1) satisfies*

$$u_t^-(\varphi) = p(t, u_t^0(\varphi), \varphi^-) \quad \text{for } t \geq 0. \tag{4.2}$$

*Moreover,*

$$|p(t, \phi_1, \psi_1) - p(t, \phi_2, \psi_2)| \leq K_0 \left(|\phi_1 - \phi_2| + e^{-\alpha t} |\psi_1 - \psi_2|\right) \tag{4.3}$$

*for all $\phi_1, \phi_2 \in Y_0, \psi_1, \psi_2 \in Y_-$ and $t \geq 0$.*

**Idea of the proof of Lemma 4.2.** The proof is similar to the one given in [22]. Let $\varphi \in Y_0$ and $\psi \in Y_-$. For $t \geq 0$ and $0 \leq \tau \leq t$, we consider the system

$$q(\tau, t, \varphi, \psi) = U(\tau - t)\varphi - \lim_{\lambda \to \infty} \int_\tau^t U(\tau - s) \left(\widetilde{B_\lambda} X_0 g\left(q(s, t, \varphi, \psi) + p(s, q(s, t, \varphi, \psi), \psi)\right)\right)^0 ds,$$

and

$$p(t, \varphi, \psi) = U(t)\psi + \lim_{\lambda \to \infty} \int_0^t U(t - s) \left(\widetilde{B_\lambda} X_0 g(q(s, t, \varphi, \psi) + p(s, q(s, t, \varphi, \psi), \psi))\right)^- ds.$$

Using the contraction principle, we can prove the existence of $q$ and $p$. The expression (4.2) and the estimate (4.3) are obtained in a completely similar fashion to that in [22].

**Proof of Theorem 4.1.** Let $M_g$ be the center manifold of equation (1.1). Then any integral solution lying in $M_g$ must satisfy (4.2). Let $u_t = u_t(\varphi^- + \varphi^0)$ be an integral solution of equation (1.1) on $\mathbb{R}^+$ with initial value $\varphi^- + \varphi^0$. Let $\tau \geq 0$. Then, $u_\tau^0 + h_g(u_\tau^0) \in M_g$ and the corresponding integral solution exists on $\mathbb{R}$ and lies on $M_g$. This solution can be considered as an integral solution of equation (1.1) starting from $\psi^- + \psi^0$ at 0. Let $v_t = v_t(\psi^- + \psi^0)$ be the integral solution corresponding to $\psi^- + \psi^0$. Using Lemma 4.2, we conclude that

$$u_\tau^- - h_g(u_\tau^0) = p(\tau, u_\tau^0, \varphi^-) - p(\tau, u_\tau^0, \psi^-),$$

which implies that

$$\left|u_\tau^- - h_g(u_\tau^0)\right| \leq K_0 e^{-\alpha\tau} \left|\varphi^- - \psi^-\right|. \tag{4.4}$$

Since $Lip(h) \leq 1$, we have

$$\left|u_\tau^- - h_g(u_\tau^0)\right| \leq K_0 e^{-\alpha\tau} \left(\left|\varphi^- - h_g(\varphi^0)\right| + \left|\varphi^0 - \psi^0\right|\right).$$

The initial values $\varphi^0$ and $\psi^0$ correspond to the solutions of the following equations for $0 \leq s \leq \tau$

$$v_s = U(s-\tau)v_\tau + \lim_{\lambda\to\infty}\int_\tau^s U(s-\sigma)\left(\widetilde{B_\lambda}X_0 g(v_\sigma + p(\sigma, v_\sigma, \varphi^-))\right)^0 d\sigma,$$
$$v_s^* = U(s-\tau)v_\tau^* + \lim_{\lambda\to\infty}\int_\tau^s U(s-\sigma)\left(\widetilde{B_\lambda}X_0 g(v_\sigma^* + p(\sigma, v_\sigma^*, \psi^-))\right)^0 d\sigma.$$

Note that $v_\tau = v_\tau^*$. It follows, for $0 \leq s \leq \tau$, that

$$|v_s - v_s^*| \leq K(1+K_0)\varepsilon\,|P_0|\int_s^\tau e^{\frac{\gamma}{3}(\sigma-s)}\,|v_\sigma - v_\sigma^*|\,d\sigma + KK_0\varepsilon\,|P_0|\int_s^\tau e^{\frac{\gamma}{3}(\sigma-s)}e^{-\alpha\sigma}d\sigma\,\left|\varphi^- - \psi^-\right|.$$

Then by Gronwall's lemma, we deduce that there exists a positive constant $\nu$ which depends only on constants $\gamma, K,\ K_0$ and $\varepsilon$ such that, for $0 \leq s \leq \tau$, we have

$$|v_s - v_s^*| \leq \nu\left|\varphi^- - \psi^-\right|.$$

If we assume that $Lip(g)$ is small enough such that $\nu < 1$, then

$$\left|\varphi^0 - \psi^0\right| \leq \nu\left|\varphi^- - \psi^-\right| \leq \nu\left(\left|\varphi^- - h_g(\varphi^0)\right| + \left|h_g(\varphi^0) - h_g(\psi^0)\right|\right),$$

which gives that

$$\left|\varphi^0 - \psi^0\right| \leq \frac{\nu}{1-\nu}\left|\varphi^- - h_g(\varphi^0)\right|.$$

We conclude that

$$\left|u_t^- - h_g(u_t^0)\right| \leq \frac{1}{1-\nu}K_0 e^{-\alpha t}\left|\varphi^- - h_g(\varphi^0)\right| \text{ for } t \geq 0.$$

As an immediate consequence, we obtain the following result on the attractiveness of the center manifold.

**Corollary 4.3.** *Assume that $Lip(g)$ is small enough and the unstable space $Y_+$ is reduced to zero. Then the center manifold $M_g$ is exponentially attractive.*

We also obtain.

**Proposition 4.4.** *Assume that $Lip(g)$ is small enough and the unstable space $Y_+$ is reduced to zero. Let $w$ be an integral solution of equation (1.1) that is bounded on $\mathbb{R}$. Then $w_t \in M_g$ for all $t \in \mathbb{R}$.*

**Proof.** Let $w$ be a bounded integral solution of equation (1.1). Since, the equation (1.1) is autonomous, then for $\sigma \leq 0$, $w_{t'+\sigma}$ is also an integral solution of equation (1.1) for $t' \geq 0$ with initial value $w_\sigma$ at 0. It follows by the estimation (4.1) that

$$\left|w_{t'+\sigma}^-(\varphi) - h_g(w_{t'+\sigma}^0(\varphi))\right| \leq K_1 e^{-\alpha t'}\left|w_\sigma^- - h_g(w_\sigma^0)\right| \text{ for } t' \geq 0.$$

Let $t \geq \sigma$. Then

$$\left|w_t^- - h_g(w_t^0)\right| \leq K_1 e^{-\alpha(t-\sigma)}\left|w_\sigma^- - h_g(w_\sigma^0)\right| \text{ for } t \geq \sigma. \tag{4.5}$$

Since $w$ is bounded on $\mathbb{R}$, letting $\sigma \to -\infty$, we obtain that $w_t^- = h_g(w_t^0)$ for all $t \in \mathbb{R}$.

## 5. **Flow on the center manifold**

In this section, we establish that the flow on the center manifold is governed by an ordinary differential equation in a finite dimensional space. In the sequel, we assume that the function $g$ satisfies the conditions of Theorem 4.1. We also assume that the unstable space $Y_+$ is reduced to zero. Let $d$ be the dimension of the center space $Y_0$ and $\Phi = (\phi_1, ...., \phi_d)$ be a basis of $Y_0$. Then there exists $d$-elements $\Phi = (\phi_1^*, ...., \phi_d^*)$ in $Y^*$, the dual space of $Y$, such that

$$\langle \phi_i^*, \phi_j \rangle := \phi_i^*(\phi_j) = \delta_{ij}, \ \ 1 \leq i, j \leq d,$$

and $\phi_i^* = 0$ on $Y_-$. Denote by $\Psi$ the transpose of $(\phi_1^*, ...., \phi_d^*)$. Then the projection operator $P_0$ is given by

$$P_0 \phi = \Phi \left\langle \Psi, \phi \right\rangle.$$

Since $\left(U^0(t)\right)_{t\geq 0}$ is a strongly continuous group on the finite dimensional space $Y_0$, by Theorem 2.15, p. 102 in [10], we get that there exists a $d \times d$ matrix $G$ such that

$$U^0(t)\Phi = \Phi e^{Gt} \text{ for } t \geq 0.$$

Let $n \in \mathbb{N}$, $n \geq n_0 \geq \widetilde{\omega}$ and $i \in \{1, ...., d\}$. We define the function $x_{ni}^*$ on $E$ by

$$x_{ni}^*(y) = \left\langle \phi_i^*, \widetilde{B_n}\left(X_0 y\right) \right\rangle.$$

Then $x_{ni}^*$ is a bounded linear operator on $E$. Let $x_n^*$ be the transpose of $(x_{n1}^*, ..., x_{nd}^*)$, then

$$\langle x_n^*, y \rangle = \left\langle \Psi, \widetilde{B_n}\left(X_0 y\right) \right\rangle.$$

Consequently,

$$\sup_{n \geq n_0} |x_n^*| < \infty,$$

which implies that $(x_n^*)_{n\geq n_0}$ is a bounded sequence in $\mathcal{L}(E, \mathbb{R}^d)$. Then, we get the following important result.

**Theorem 5.1.** *There exists $x^* \in \mathcal{L}(E, \mathbb{R}^d)$ such that $(x_n^*)_{n\geq n_0}$ converges weakly to $x^*$ in the sense that*

$$\langle x_n^*, y \rangle \to \langle x^*, y \rangle \quad \textit{as } n \to \infty \textit{ for } y \in E.$$

For the proof, we need the following fundamental theorem [25, pp. 776]

**Theorem 5.2.** *Let $X$ be a separable Banach space and $(z_n^*)_{n\geq 0}$ be a bounded sequence in $X^*$. Then there exists a subsequence $\left(z_{n_k}^*\right)_{k\geq 0}$ of $(z_n^*)_{n\geq 0}$ which converges weakly in $X^*$ in the sense that there exists $z^* \in X^*$ such that*

$$\left\langle z_{n_k}^*, y \right\rangle \to \langle z^*, y \rangle \ \textit{ as } k \to \infty \ \textit{ for } x \in X.$$

**Proof of Theorem 5.1.** Let $Z_0$ be a closed separable subspace of $E$. Since $(x_n^*)_{n\geq n_0}$ is a bounded sequence, by Theorem 5.2 there is a subsequence $\left(x_{n_k}^*\right)_{k\in\mathbb{N}}$ which converges weakly to some $x_{Z_0}^*$ in $Z_0$. We claim that all the sequence $(x_n^*)_{n\geq n_0}$ converges weakly to $x_{Z_0}^*$ in $Z_0$. This can be done by way of contradiction. Namely, suppose that there exists a subsequence $\left(x^*_{n_p}\right)_{p\in\mathbb{N}}$ of $(x^*_n)_{n\geq n_0}$ which converges weakly to some $\widetilde{x}^*_{Z_0}$ with $\widetilde{x}^*_{Z_0} \neq x^*_{Z_0}$. Let $\widetilde{u}_t(.,\sigma,\varphi,f)$ denote the integral solution of the following equation

$$\begin{cases} \dfrac{d}{dt}\widetilde{u}(t) = A\widetilde{u}(t) + L(\widetilde{u}_t) + f(t), \ t \geq \sigma \\ \widetilde{u}_\sigma = \varphi \in C, \end{cases}$$

where $f$ is a continuous function from $\mathbb{R}$ to $E$. Then by using the variation of constants formula and the spectral decomposition of solutions, we obtain

$$P_0\widetilde{u}_t(.,\sigma,0,f) = \lim_{n\to+\infty}\int_\sigma^t U(t-\xi)\left(\widetilde{B}_n X_0 f(\xi)\right)^0 d\xi,$$

and

$$P_0\left(\widetilde{B}_n X_0 f(\xi)\right) = \Phi\left\langle \Psi, \widetilde{B}_n X_0 f(\xi)\right\rangle = \Phi\left\langle x^*_n, f(\xi)\right\rangle.$$

It follows that

$$\begin{aligned} P_0\widetilde{u}_t(.,\sigma,0,f) &= \lim_{n\to+\infty}\Phi\int_\sigma^t e^{(t-\xi)G}\left\langle \Psi, \widetilde{B}_n X_0 f(\xi)\right\rangle d\xi, \\ &= \lim_{n\to+\infty}\Phi\int_\sigma^t e^{(t-\xi)G}\left\langle x^*_n, f(\xi)\right\rangle d\xi. \end{aligned}$$

For a fixed $a \in Z_0$, set $f = a$ to be a constant function. Then

$$\lim_{k\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle x^*_{n_k}, a\right\rangle d\xi = \lim_{p\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle x^*_{n_p}, a\right\rangle d\xi \text{ for } a \in Z_0,$$

which implies that

$$\int_\sigma^t e^{(t-\xi)G}\left\langle x^*_{Z_0}, a\right\rangle d\xi = \int_\sigma^t e^{(t-\xi)G}\left\langle \widetilde{x}^*_{Z_0}, a\right\rangle d\xi \text{ for } a \in Z_0.$$

Consequently $x^*_{Z_0} = \widetilde{x}^*_{Z_0}$, which yields a contradiction.
We now conclude that the whole sequence $(x^*_n)_{n\geq n_0}$ converges weakly to $x^*_{Z_0}$ in $Z_0$. Let $Z_1$ be another closed separable subspace of $X$. By using the same argument as above, we obtain that $(x^*_n)_{n\geq n_0}$ converges weakly to $x^*_{Z_1}$ in $Z_1$. Since $Z_0 \cap Z_1$ is a closed separable subspace of $E$, we get that $x^*_{Z_1} = x^*_{Z_0}$ in $Z_0 \cap Z_1$. For any $y \in E$, we define $x^*$ by

$$\langle x^*, y\rangle = \langle x^*_Z, y\rangle,$$

where $Z$ is an arbitrary given closed separable subspace of $E$ such that $y \in Z$. Then $x^*$ is well defined on $E$ and $x^*$ is a bounded linear operator from $E$ to $\mathbb{R}^d$ such that

$$|x^*| \leq \sup_{n\geq n_0}|x^*_n| < \infty,$$

and $(x^*_n)_{n\geq n_0}$ converges weakly to $x^*$ in $E$.

As a consequence, we conclude that

**Corollary 5.3.** *For any continuous function $f : \mathbb{R} \to E$, we have*

$$\lim_{n\to+\infty}\int_\sigma^t U(t-\xi)\left(\widetilde{B}_n X_0 f(\xi)\right)^0 d\xi = \Phi\int_\sigma^t e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi \ \ \text{for } t,\sigma \in \mathbb{R}.$$

Let $\varphi \in Y_0$ such that $\varphi + h_g(\varphi) \in M_g$. From the properties of the center manifold, we know that the integral solution starting from $\varphi + h(\varphi)$ is given by $v_t^\varphi + h_g(v_t^\varphi)$, where $v_t^\varphi$ is the solution of

$$v_t^\varphi = U(t)\varphi + \lim_{\lambda\to\infty} \int_0^t U(t-\tau)\left(\widetilde{B_\lambda} X_0 g(v_\tau^\varphi + h_g(v_\tau^\varphi))\right)^0 d\tau \text{ for } t \in \mathbb{R}.$$

Let $z(t)$ be the component of $v_t^\varphi$. Then $\Phi z(t) = v_t^\varphi$ for $t \in \mathbb{R}$. By Theorem 5.1 and Corollary 5.3, we have

$$\Phi z(t) = v_t^\varphi = \Phi e^{Gt} z(0) + \Phi \int_0^t e^{(t-\tau)G} \langle x^*, g(v_\tau^\varphi + h_g(v_\tau^\varphi))\rangle\, d\tau \text{ for } t \in \mathbb{R}.$$

We conclude that $z$ satisfies

$$z(t) = e^{Gt} z(0) + \lim_{n\to\infty} \int_0^t e^{G(t-\tau)} \langle x_n^*, g(v_\tau^\varphi + h_g(v_\tau^\varphi))\rangle\, d\tau \text{ for } t \in \mathbb{R}.$$

Finally we arrive at the following ordinary differential equation, which determines the flow on the center manifold

$$z'(t) = Gz(t) + \langle x^*, g(\Phi z(t) + h_g(\Phi z(t)))\rangle \text{ for } t \in \mathbb{R}. \tag{5.1}$$

## 6. **Stability in critical cases**

In this section, we suppose that $Lip(g) < \varepsilon$, where $\varepsilon$ is given by Theorem 4.1. Here we study the critical case where the unstable space $Y_+$ is reduced to zero and the exponential stability is not possible, which implies that there exists at least one characteristic value with a real part equals zero.

**Theorem 6.1.** *Assume that $Lip(g)$ is small enough. Then there exists a positive constant $K_2$ such that for each $\varphi \in Y$, there exists $\widetilde{z}_0 \in \mathbb{R}^d$ such that*

$$\left|u_t^0 - \Phi\widetilde{z}(t)\right| + \left|u_t^- - h_g(\Phi\widetilde{z}(t))\right| \le K_2 e^{-\alpha t}\left|\varphi^- - h_g(\varphi^0)\right| \quad \text{for } t \ge 0, \tag{6.1}$$

*where $\widetilde{z}$ is the solution of the reduced system (5.1) with initial value $\widetilde{z}_0$ and $u$ is the integral solution of equation (1.1) with initial value $\varphi$.*

**Proof.** Let $\varphi \in Y$ and $u$ be the corresponding integral solution of equation (1.1). Then

$$u_s^0 = U(s-t)u_t^0 + \lim_{\lambda\to\infty} \int_t^s U(s-\tau)\left(\widetilde{B_\lambda} X_0 g(u_\tau^- + u_\tau^0)\right)^0 d\tau \text{ for } 0 \le s \le t.$$

Also let $s \to w_{s,t}$ be the solution of the following equation

$$w_{s,t} = U(s-t)u_t^0 + \lim_{\lambda\to\infty} \int_t^s U(s-\tau)\left(\widetilde{B_\lambda} X_0 g(w_{\tau,t} + h_g(w_{\tau,t}))\right)^0 d\tau \text{ for } 0 \le s \le t.$$

Then, for $0 \le s \le t$, we have

$$\left|u_s^0 - w_{s,t}\right| \le 2K\varepsilon\,|P_0| \int_s^t e^{\frac{\gamma}{3}(\tau-s)}\left|u_\tau^0 - w_{\tau,t}\right| d\tau + K\varepsilon\,|P_0| \int_s^t e^{\frac{\gamma}{3}(\tau-s)}\left|u_\tau^- - h_g(u_\tau^0)\right| d\tau.$$

It follows that

$$e^{\frac{\gamma}{3}s}\left|u_s^0 - w_{s,t}\right| \le 2K\varepsilon\,|P_0| \int_s^t e^{\frac{\gamma}{3}\tau}\left|u_\tau^0 - w_{\tau,t}\right| d\tau + K\varepsilon\,|P_0| \int_s^t e^{\frac{\gamma}{3}\tau}\left|u_\tau^- - h_g(u_\tau^0)\right| d\tau.$$

Using Gronwall's lemma, we obtain that

$$\left|u_s^0 - w_{s,t}\right| \leq 2K\varepsilon \left|P_0\right| \int_s^t e^{\mu(\tau-s)} \left|u_\tau^- - h_g(u_\tau^0)\right| d\tau \text{ for } 0 \leq s \leq t,$$

where $\mu = 2K\varepsilon \left|P_0\right| + \frac{\gamma}{3}$. By Theorem 4.1, we obtain

$$\left|u_s^0 - w_{s,t}\right| \leq \left(2KK_1\varepsilon \left|P_0\right| \int_s^t e^{(\mu-\alpha)(\tau-s)} d\tau\right) \left|u_s^- - h_g(u_s^0)\right| \text{ for } 0 \leq s \leq t.$$

Let us recall that $\alpha \in \left(\frac{\gamma}{3}, \gamma\right)$. Then we can choose $\varepsilon$ small enough such that $\mu - \alpha < 0$. Consequently,

$$\left|u_s^0 - w_{s,t}\right| \leq \frac{2KK_1\varepsilon \left|P_0\right|}{\alpha - \mu} \left|u_s^- - h_g(u_s^0)\right| \text{ for } 0 \leq s \leq t, \tag{6.2}$$

and for $s = 0$, we have

$$\left|u_0^0 - w_{0,t}\right| \leq \frac{2KK_1\varepsilon \left|P_0\right|}{\alpha - \mu} \left|u_0^- - h_g(u_0^0)\right| \text{ for } t \geq 0.$$

We deduce that $\{w_{0,t} : t \geq 0\}$ is bounded in $Y_0$ and then there exists a sequence $t_n \to \infty$ as $n \to \infty$ and $\phi \in Y_0$ such that

$$w_{0,t_n} \to \phi \text{ as } n \to \infty.$$

Let $\widetilde{w}(.,\phi)$ be the solution of the following equation

$$\widetilde{w}_t(.,\phi) = U(t)\phi + \lim_{\lambda\to\infty} \int_0^t U(t-\tau) \left(\widetilde{B_\lambda} X_0 g(\widetilde{w}_\tau(.,\phi) + h_g(\widetilde{w}_\tau(.,\phi)))\right)^0 d\tau \text{ for } t \geq 0.$$

By the continuous dependence on the initial data, we obtain, for all $s \geq 0$

$$\begin{aligned}\widetilde{w}_s(0,\phi) &= \lim_{n\to\infty} \widetilde{w}_s(0, w_{0,t_n}) = \lim_{n\to\infty} \widetilde{w}_s(0, w_{-t_n}(0, u_{t_n})),\\ &= \lim_{n\to\infty} \widetilde{w}_{s-t_n}(0, u_{t_n}) = \lim_{n\to\infty} w_{s,t_n}.\end{aligned}$$

By (4.4) and (6.2), there exists a positive constant $K_2$ such that

$$\left|u_s^0 - \widetilde{w}_s(0,\phi)\right| \leq K_2 e^{-\alpha s} \left|u_0^- - h_g(u_0^0)\right| \text{ for all } s \geq 0.$$

If we put

$$\Phi\widetilde{z}(t) = \widetilde{w}_t(0,\phi) \text{ for } t \in \mathbb{R},$$

then

$$\widetilde{z}(t) = e^{Gt}\widetilde{z}(0) + \int_0^t e^{(t-\tau)G} \langle x^*, g(\Phi\widetilde{z}(\tau) + h_g(\Phi\widetilde{z}(\tau)))\rangle d\tau \text{ for } t \in \mathbb{R}.$$

Finally, the estimation (6.1) follows from (4.4).

Now, we can state the following result on the stability by using the reduction to the center manifold.

**Theorem 6.2.** *If the zero solution of equation (5.1) is uniformly asymptotically stable (unstable), then the zero solution of equation (1.1) is uniformly asymptotically stable (unstable).*

**Proof.** Assume that 0 is uniformly asymptotically stable for equation (5.1). For $\varsigma \geq 0$, let

$$B_\varsigma = \left\{\varphi^- + \varphi^0 \in Y_- \oplus Y_0 : \left|\varphi^-\right| + \left|\varphi^0\right| < \varsigma\right\},$$

and $M_g \cap B_\rho$ for some $\rho \geq 0$, be the region of attraction of 0 for equation (5.1). First, we prove that 0 is stable for equation (1.1). Let $\varepsilon \geq 0$. Then there exists $\delta < \rho$ such that $|z(t)| < \varepsilon$ for $t \geq 0$, provided that $|z(0)| < \delta$, where $z$ is a solution of (5.1). As 0 is assumed to be uniformly asymptotically stable for equation (5.1), there exists $t_0 = t_0(\delta)$ such that $|z(t)| < \frac{\delta}{2}$, for $t \geq t_0$. Without loss of generality, we can choose $\delta$ and $t_0$ so that $\max(K_1, K_2)e^{-\alpha t_0} < \frac{1}{2}$.

By the continuous dependence on the initial value for equation (1.1), there exists $\delta_1 < \frac{\delta}{2}$ such that if

$$\varphi^- + \varphi^0 \in V_{\delta_1} := \left\{\psi^- + \psi^0 \in Y_- \oplus Y_0 : \left|\psi^0\right| < \frac{\delta_1}{2},\ \left|\psi^- - h_g(\psi^0)\right| < \frac{\delta_1}{2}\right\},$$

then the corresponding integral solution $u_t = u_t(\varphi^- + \varphi^0)$ of equation (1.1) satisfies

$$u_t \in B_\varepsilon \text{ for } t \in [0, t_0].$$

Moreover,

$$\left|u_{t_0}^- - h_g(u_{t_0}^0)\right| < \frac{\delta_1}{2}.$$

Furthermore, by Theorem 6.1, there exists $z_0 \in \mathbb{R}^d$ such that

$$\left|u_t^0 - \Phi\widetilde{z}(t)\right| \leq K_2 e^{-\alpha t}\left|\varphi^- - h_g(\varphi^0)\right| \text{ for } t \geq 0, \tag{6.3}$$

where $\widetilde{z}$ is a solution of the reduced system (5.1) with initial value $\widetilde{z}_0$ such that $|\widetilde{z}_0| < \delta$. It follows that

$$\left|u_{t_0}^0\right| < \delta.$$

Consequently, $u_{t_0} \in B_\varepsilon$ and $u_t$ must be in $B_\varepsilon$ for all $t \geq 0$. This completes the proof of the stability.

Now we deal with the local attractiveness of the zero solution. For a given integral solution $u(., \varphi)$ of equation (1.1) which is assumed to be bounded for $t \geq 0$, it is well known that the $\omega$-limit set $\omega(\varphi)$ is nonempty, compact, invariant and connected since the map $\varphi \to u_t(\cdot, \varphi)$ is compact for $t \geq r$.

For the attractiveness of 0, let $V_\delta$ be chosen as above and $\varphi \in V_\delta$. Then the integral solution $u$ of equation (1.1) starting from $\varphi$ lies in $B_\varepsilon$. The $\omega$-limit set $\omega(\varphi)$ of $u$ is nonempty and invariant and must be in $M_g \cap B_\varepsilon$. Since the equilibrium 0 of (1.1) is uniformly asymptotically stable, we deduce by Theorem 11.4, p. 111 [23] and by the LaSalle invariance principle that the only invariant set in $M_g \cap B_\varepsilon$ must be zero. Consequently, the $\omega$-limit set $\omega(\varphi)$ is zero and

$$u_t(., \varphi) \to 0 \text{ as } t \to 0.$$

Assume now that the zero solution of the reduced system (5.1) is unstable. Then there exist $\sigma_1 \geq 0$, a sequence $(t_n)_n$ of positive real numbers and a sequence $(z_n)_n$ converging to 0 such that $|z(t_n, z_n)| \geq \sigma_1$, where $z(., z_n)$ is a solution of (5.1). On the other hand $\Phi z(., z_n) + h_g(\Phi z(., z_n))$ is an integral solution of equation (1.1) and

$$\left|\Phi z(t_n, z_n) + h_g(\Phi z(t_n, z_n))\right| \geq (1 - Lip(h_g))\left|\Phi z(t_n, z_n)\right|.$$

Moreover, $Lip(h_g)$ can be chosen such that $1 - Lip(h_g) \geq 0$. It follows that

$$|\Phi z(t_n, z_n) + h_g(\Phi z(t_n, z_n))| \geq (1 - Lip(h_g))\, \sigma_2,$$

for some $\sigma_2 \geq 0$. Consequently, the zero solution of equation (1.1) is unstable.

## 7. **Local existence of the center manifold**

In this section we prove the existence of the local center manifold when $g$ is only defined in a neighborhood of zero. We assume that

($\mathbf{H}_4$) There exists $\rho_1 \geq 0$ such that $g : B(0, \rho_1) \to E$ is $C^1$-function, $g(0) = 0$ and $g'(0) = 0$, where $B(0, \rho_1) = \{\varphi \in C : |\varphi| < \rho_1\}$.

For $\rho < \rho_1$, we define the cut-off function $g_\rho : C \to E$ by

$$g_\rho(\varphi) = \begin{cases} g(\varphi) \text{ if } |\varphi| \leq \rho, \\ g(\frac{\rho}{|\varphi|}\varphi) \text{ if } |\varphi| \geq \rho. \end{cases}$$

We consider the following partial functional differential equation

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + g_\rho(u_t) \text{ for } t \geq 0 \\ u_0 = \varphi \in C. \end{cases} \tag{7.1}$$

**Theorem 7.1.** *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_3$) *and* ($\mathbf{H}_4$) *hold. Then there exist* $0 < \rho < \rho_1$ *and Lipschitz continuous mapping* $h_{g_\rho} : Y_0 \to Y_- \oplus Y_+$ *such that* $h_{g_\rho}(0) = 0$ *and the local Lipschitz manifold*

$$M_{g_\rho} = \left\{\varphi + h_{g_\rho}(\varphi) : \varphi \in Y_0\right\}$$

*is globally invariant under the flow associated to equation (7.1).*

**Proof.** Using the same arguments as in [26], Proposition 3.10, p.95, one can show that $g_\rho$ is Lipschitz continuous with

$$Lip(g_\rho) \leq 2 \sup_{|\varphi|<\rho} |g'(\varphi)|.$$

It follows that $Lip(g_\rho)$ goes to zero when $\rho$ goes to zero. According to Theorem 3.1, we deduce that there exist $\rho \geq 0$ and mapping $h_{g_\rho} : Y_0 \to Y_- \oplus Y_+$ with $h_{g_\rho}(0) = 0$ such that the Lipschitz manifold

$$M_{g_\rho} = \left\{\varphi + h_{g_\rho}(\varphi) : \varphi \in Y_0\right\}$$

is globally invariant under the flow of equation (7.1).

**Definition 7.2.** The local center manifold associated to equation (1.1) is defined by

$$\widetilde{M}_{g_\rho} = \left\{\varphi + h_{g_\rho}(\varphi) : \varphi \in Y_0\right\} \cap B(0, \rho).$$

We deduce that the local center manifold contains all bounded integral solutions of equation (1.1) which are bounded by $\rho$. Moreover, the following interesting results hold.

**Theorem 7.3.** ***(Attractiveness)****. Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_3$), ($\mathbf{H}_4$) *hold and the unstable space* $Y_+$ *is reduced to zero. Then there exist* $0 < \rho < \rho_1$, $K_3 \geq 0$ *and* $\alpha \in \left(\frac{\gamma}{3}, \gamma\right)$ *such that any integral solution* $u_t(\varphi)$ *of equation (1.1) with initial data* $\varphi \in B(0, \rho_1)$*, exists on* $\mathbb{R}^+$ *and satisfies*

$$\left|u_t^-(\varphi) - h_{g_\rho}(u_t^0(\varphi))\right| \leq K_3 e^{-\alpha t} \left|\varphi^- - h_{g_\rho}(\varphi^0)\right| \quad \text{for } t \geq 0. \tag{7.2}$$

**Theorem 7.4.** *(**Reduction principle and stability**). Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *hold and the unstable space* $Y_+$ *is reduced to zero. If the zero solution of equation (5.1) is uniformly asymptotically stable (unstable), then the zero solution of equation (1.1) is uniformly asymptotically stable (unstable).*

## 8. **Application**

In this section, we assume that

$(\mathbf{H}_5)$ $\sigma(A_U) \cap \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) \geq 0\} = \{0\}$ and 0 is a simple eigenvalue of $A_U$.

In this case, the reduced system (5.1) on the center manifold becomes

$$z'(t) = \upsilon(z(t)),$$

where $\upsilon$ is the scalar function given by

$$\upsilon(z) = \left\langle x^*, g(\Phi z + h_{g_\rho}(\Phi z))\right\rangle \text{ for } z \in \mathbb{R}.$$

**Theorem 8.1.** *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$, $(\mathbf{H}_5)$ *hold and* $\upsilon$ *satisfies*

$$\upsilon(z) = a_m z^m + a_{m+1} z^{m+1} + .... \tag{8.1}$$

*If* $m$ *is odd and* $a_m < 0$*, then the zero solution of equation (1.1) is uniformly asymptotically stable. If* $a_m > 0$ *then the zero solution of equation (1.1) is unstable.*

**Proof.** The proof is based on Theorem 6.2 and on the following known stability result.

**Theorem 8.2.** [6] *Consider the scalar differential equation*

$$z'(t) = a_m z^m + a_{m+1} z^{m+1} + .... \tag{8.2}$$

*If* $m$ *is odd and* $a_m < 0$*, then the zero solution of equation (8.2) is uniformly asymptotically stable. If* $a_m > 0$*, then the zero solution of equation (8.2) is unstable.*

**Concluding remark.** Assumption (8.1) is natural and it is a consequence of the smoothness of the center manifold, which states that if $g$ is a $C^k$-function, for $k \geq 1$, then $h_{g_\rho}$ is also a $C^k$-function. Consequently if $g$ is a $C^\infty$-function, then the center manifold $h_{g_\rho}$ is also a $C^\infty$-function. Assumption (8.1) can be obtained by using the approximation of the center manifold $h_{g_\rho}$. The proof of the smoothness result is omitted here and it can be done in similar way as in [11].

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## References

[1] M. Adimy and K. Ezzinbi, Semi groupes intégrés et équations différentielles à retard en dimension infinie, C. R. Acad. Sci. Paris, t. 323, série I, 481-486, (1996).
[2] M.Adimy and K.Ezzinbi, A class of linear partial neutral functional differential equations with nondense domain, Journal of Differential Equations, 147, 285-332, (1998).
[3] M. Adimy and K. Ezzinbi, Local existence and linearized stability for partial functional differential equations, Dynamic Systems and Applications, Vol. 7, 389-403, (1998).
[4] M. Adimy and K. Ezzinbi, Existence and stability of solutions for a class of partial neutral functional differential equations, Hiroshima Mathematical Journal, Vol. 34, No. 3, 251-294, (2004).

[5] M. Adimy, K. Ezzinbi and M. Laklach, Spectral decompostion for some partial neutral functional differential equations, Canadian Applied Mathematics Quarterly, Vol. 9, No. 4, 1-34, (2001).
[6] J. Carr, Applications of Center Manifold Theory, Applied Mathematical Sciences, Springer-Verlag, Vol. 35, (1981).
[7] S. N. Chow and K. Lu, Invariant manifolds for flows in Banach spaces, Journal of Differential equations, 74, 285-317, (1988).
[8] G. Da Prato and A. Lunardi, Stability, instability and center manifold theorem for fully nonlinear autonomous parabolic equations in Banach spaces, Archive for Rational Mechanics and Analysis, Vol. 101, 115-141, (1988).
[9] W. Desch and W. Schappacher, Linearized stability for nonlinear semigroups, in " Differential equations in Banach spaces", (A. Favini and E. Obrecht, Eds), Lecture Notes in Mathematics, Springer-Verlag, Vol. 1223, 61-73, (1986).
[10] O. Diekmann, S. A. van Gils, S. M. Verduyn Lunel and H-O Walther, Delay Equations, Functional Complex and Nonlinear Analysis, Applied Mathematical Sciences, Springer-Verlag, (1995).
[11] T. Faria, W. Huang and J. Wu, Smoothness of center manifolds for maps and formal adjoints for semilinear FDEs in general Banach spaces, SIAM, Journal of Mathematical Analysis, 34, 173-203, (2002).
[12] J. K. Hale, Critical cases for neutral functional differential equations, Journal of Differential Equations, 10, 59-82, (1971).
[13] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, (1977).
[14] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Mathematics, Springer-Verlag, Vol. 840, (1981).
[15] A. Keller, The stable, center-stable, center, center-unstable and unstable manifolds, Journal of Differential Equations, 3, 546-570, (1967).
[16] A. Keller, Stability of the center-stable manifold, Journal of Mathematical Analysis and Applications, 18, 336-344, (1967).
[17] X. Lin, J. W. H. So and J. Wu, Centre manifolds for partial differential equations with delays, Proceedings of the Royal Society of Edinburgh, 122, 237-254, (1992).
[18] N. V. Minh and J. Wu, Invariant manifolds of partial functional differential equations, Journal of Differential Equations, 198, 381-421, (2004).
[19] K. Palmer, On the stability of the center manifold, Journal of Applied Mathematics and Physics, (ZAMP), Vol. 38, 273-278, (1987).
[20] S. N. Shimanov, On the stability in the critical case of a zero root for systems with time lag, Prikl. Mat. Mekh. 24, 447-457, (1960).
[21] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, Transactions of the American Mathematical Society, 200, 395-418 (1974).
[22] J. W. H. So, Y. Yang and J. Wu, Center manifolds for functional partial differential equations: Smoothness and attractivity, Mathematica Japonica, 48, 67-81, (1998).
[23] T. Yoshizawa, Stability Theory and the Existence of Periodic Solutions and Almost Periodic Solutions, Applied Mathematical Sciences, Springer-Verlag, Vol. 14, (1975).
[24] A. Vanderbauwhede and S. A. Van Gils, Center manifolds and contractions on a scale of Banach spaces, Journal of Functional Analysis, 72, 209-224, (1987).
[25] E. Zeidler, Nonlinear Functional Analysis and its Applications, Tome I, Fixed Point theorems, Springer-Verlag, (1993).
[26] G. F. Webb, Theory of Nonlinear Age-dependent Population Dynamics, Marcel Dekker, (1985).
[27] J. Wu, Theory and Applications of Partial Functional Differential Equations, Applied Mathematical Sciences, Springer-Verlag, Vol. 119, (1996).