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Bohr-Neugebauer type theorem for some partial neutral functional differential equations ⋆

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Abstract

In this work, we study the existence of almost periodic solutions for some partial neutral functional differential equations. Using the variation of constants formula and the spectral decomposition of the phase space developed in [6], we prove that the existence of an almost periodic solution is equivalent to the existence of a bounded solution on $\mathbb{R}^+$.

Key words:
neutral equation, Hille-Yosida condition, semigroup, variation of constants formula, essential growth, spectral decomposition, almost periodic solution.

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1 Introduction

The purpose of this work is to study the existence of almost periodic solutions of the following class of partial neutral functional differential equations

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t) + f(t), \text{ for } t \geq \sigma, \\ u_\sigma = \varphi \in C := C([-r,0]\,;X), \end{cases} \tag{1}$$

where A is a linear operator on a Banach space X, not necessarily densely defined and satisfies the known Hille-Yosida condition:

$(\mathbf{H_0})$ there exist $\bar{M} \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, +\infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{\bar{M}}{(\lambda - \omega)^n}, \ \ n \in \mathbb{N}, \ \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$, for $\lambda \in \rho(A)$.

C is the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology. $\mathcal{D} : C \to X$ is a bounded linear operator which has the following form

$$\mathcal{D}\varphi := \varphi(0) - \int_{-r}^{0} [d\eta(\theta)]\, \varphi(\theta),\ \varphi \in C,$$

for a mapping $\eta : [-r, 0] \to \mathcal{L}(X)$ of bounded variation and non atomic at zero, which means that there exists a continuous nondecreasing function $\delta : [0, r] \to [0, +\infty)$ such that $\delta(0) = 0$ and

$$\left|\int_{-s}^{0} [d\eta(\theta)]\, \varphi(\theta)\right| \leq \delta(s) \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|\,, \ \ \varphi \in C, \ \ s \in [0, r]\,,$$

where $\mathcal{L}(X)$ denotes the space of bounded linear operators from X to X. For every $t \geq \sigma$, the history function $u_t \in C$ is defined by

$$u_t(\theta) = u(t + \theta), \text{ for } \theta \in [-r, 0]\,.$$

L is a bounded linear operator from C into X and f is a continuous function from $\mathbb{R}$ to X.

In [26] and [27], the authors studied a system of partial neutral functional differential-difference equations defined on the unit circle S, which is a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions. This system is

$$\frac{\partial}{\partial t}\left[u(.,t) - qu(.,t-r)\right] = k\frac{\partial^2}{\partial x^2}\left[u(.,t) - qu(.,t-r)\right] + \zeta(u_t), \text{ for } t \geq 0, \tag{2}$$

where $x \in S$, k is a positive constant, ζ is a continuous function and $0 \leq q < 1$. The phase space is $C([-r,0]\,;H^1(S))$. In [16] and [17], the author studied the qualitative behavior of solutions of Equation (2). He obtained several results about stability, attractiveness and bifurcation of solutions near an equilibrium. In [2], motivated by the above works, the authors gave the basic theory for the following partial neutral functional differential equation

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - Fu(t-r)\right] = A\left[u(t) - Fu(t-r)\right] + P(u_t),\ t \geq 0, \\ u_0 = \varphi \in C, \end{cases}$$

where A is not necessarily densely defined and satisfies the Hille-Yosida condition on a Banach space X, F is a bounded linear operator from X to X and P is a bounded linear operator from C to X. It was proved in particular, that the solutions generate a locally Lipschitz continuous integrated semigroup. In [4] and [5], the authors studied the existence, regularity and stability of solutions for a more general class of nonlinear partial neutral functional differential equations.

The existence of periodic solutions or almost periodic solutions is very important in the qualitative studies of many problems. Among numerous results in this topics, we mention the following result which is classical in the theory of ordinary differential equations. Let us consider the following system of differential equations in finite dimensional space

$$\frac{d}{dt}x(t) = Bx(t) + g(t),\ t \in \mathbb{R}, \tag{3}$$

where B is a constant $n \times n$-matrix and $g : \mathbb{R} \to \mathbb{R}^n$ is a continuous and ω-periodic function. In [20], Massera studied the existence of periodic solutions of Equation (3). He proved the equivalence between the existence of bounded solutions on $\mathbb{R}^+$ and the existence of ω-periodic solutions. As a generalization of this result, Bohr and Neugebauer, see [13], studied the existence of almost periodic solutions of Equation (3) in the case where the function g is almost periodic. More precisely, they proved that the existence of a bounded solution on $\mathbb{R}^+$ implies the existence of an almost periodic solution and every bounded solution on $\mathbb{R}$ is almost periodic. Another direction of generalization of these two classical results is to discuss the existence of periodic or almost periodic solutions for partial functional differential equations (i.e. Equation (1) in the case $\mathcal{D}(\varphi) = \varphi(0)$). Many works are devoted to this subject. For more information, we refer to [8], [9], [12], [14], [18] and [21].

In [6], the authors discussed the fundamental linear theory of Equation (1). In particular, they studied the asymptotic behavior of the solution semigroup of the homogeneous equation

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}u_t = A\mathcal{D}u_t + L(u_t), \text{ for } t \geq 0, \\ u_0 = \varphi \in C. \end{cases} \tag{4}$$

They obtained a new variation of constants formula associated to Equation (1). Moreover, they established the existence of periodic and almost periodic solutions in the case where the semigroup associated to Equation (4) is hyperbolic.

The goal of this work is to prove the existence of almost periodic solutions of Equation (1) without the hyperbolicity condition. More precisely, we will show that the existence of an almost periodic solution of Equation (1) is equivalent to the existence of a bounded solution on $\mathbb{R}^+$. Our approach is based on the variation of constants formula and the spectral decomposition of the phase space developed in [6].

This work is organized as follows: in Section 2, we recall the variation of constants formula obtained in [6]. In section 3, we develop several fundamental results about the spectral decomposition of solutions of Equation (1). As a consequence, we obtain a finite dimensional reduction of Equation (1). In Section 4, we prove the main result of this work which states the equivalence between the existence of bounded solutions on $\mathbb{R}^+$ and the existence of almost periodic solutions of Equation (1). To illustrate our approach, we propose an application for the model (2).

2 Variation of constants formula

Throughout this paper, we suppose that the operator $A : D(A) \subset X \longrightarrow X$ satisfies the Hille-Yosida condition $(\mathbf{H_0})$.

We need the following definition and results which are taken from [4] and [6].

Definition 1 *[6] A continuous function u from $[-r+\sigma, +\infty)$ into X is an integral solution of Equation (1), if*

(i) $\displaystyle\int_\sigma^t \mathcal{D}u_s ds \in D(A)$, *for* $t \geq \sigma$,

(ii) $\displaystyle\mathcal{D}u_t = \mathcal{D}\varphi + A\int_\sigma^t \mathcal{D}u_s ds + \int_\sigma^t [L(u_s) + f(s)]\, ds$, $\quad$ *for* $t \geq \sigma$,

(iii) $u_\sigma = \varphi$.

From the closedness property of the operator A, we can see that if u is an integral solution of Equation (1), then $\mathcal{D}u_t \in \overline{D(A)}$ for all $t \geq \sigma$. In particular, $\mathcal{D}\varphi \in \overline{D(A)}$.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ defined by

$$\begin{cases} D(A_0) = \left\{ x \in D(A) : Ax \in \overline{D(A)} \right\}, \\ A_0 x = Ax, \;\; \text{for } x \in D(A_0). \end{cases}$$

Lemma 2 *[7, Lemma 3.3.12, pp. 140] Assume that* $(\mathbf{H}_0)$ *holds .Then* A_0 *generates a strongly continuous semigroup* $(T_0(t))_{t\geq 0}$ *on* $\overline{D(A)}$.

The integrated form of Equation (1) is given by the next result.

Theorem 3 *[4] Assume that* $(\mathbf{H_0})$ *holds. Then, for all* $\varphi \in C$ *such that* $\mathcal{D}\varphi \in \overline{D(A)}$, *Equation (1) has a unique integral solution* u *on* $[-r+\sigma, +\infty)$. *Moreover,* u *is given by*

$$\mathcal{D}u_t = T_0(t-\sigma)\mathcal{D}\varphi + \lim_{\lambda \to +\infty} \int_\sigma^t T_0(t-s) B_\lambda [L(u_s) + f(s)] ds, \text{ for } t \geq \sigma,$$

where $B_\lambda = \lambda R(\lambda, A)$, *for* $\lambda > \omega$.

In the sequel of this work, $u(., \sigma, \varphi, f)$ denotes the integral solution of Equation (1). The phase space C_0 of Equation (1) is given by

$$C_0 := \left\{ \varphi \in C : \mathcal{D}\varphi \in \overline{D(A)} \right\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t)\varphi = v_t(., \varphi),$$

where $v(., \varphi)$ is the integral solution of the homogeneous equation (4). We have the following result.

Proposition 4 *[6, Proposition 2] Assume that* $(\mathbf{H_0})$ *holds. Then* $(\mathcal{U}(t))_{t\geq 0}$ *is a strongly continuous semigroup on* C_0, *that is:*
(**i**) *for all* $t \geq 0$, $\mathcal{U}(t)$ *is a bounded linear operator on* C_0,
(**ii**) $\mathcal{U}(0) = I$,
(**iii**) $\mathcal{U}(t+s) = \mathcal{U}(t)\mathcal{U}(s)$, *for all* $t, s \geq 0$,
(**iv**) *for all* $\varphi \in C_0$, $\mathcal{U}(t)\varphi$ *is a continuous function of* $t \geq 0$ *with values in* C_0. *Moreover,*
(**v**) $(\mathcal{U}(t))_{t\geq 0}$ *satisfies, for* $\varphi \in C_0$, $t \geq 0$ *and* $\theta \in [-r, 0]$, *the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) & \text{if } \; t+\theta \geq 0, \\ \varphi(t+\theta) \;\; \text{if } \; t+\theta \leq 0. \end{cases}$$

We investigate, in the next result, the infinitesimal generator of $(\mathcal{U}(t))_{t\geq 0}$.

Theorem 5 *[6, Theorem 3] Assume that* $(\mathbf{H_0})$ *holds. Then the operator* $\mathcal{A}_{\mathcal{U}}$ *defined on* C_0 *by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \left\{ \varphi \in C^1([-r,0]\,;X) : \mathcal{D}\varphi \in D(A),\ \mathcal{D}\varphi' \in \overline{D(A)} \text{ and } \mathcal{D}\varphi' = A\mathcal{D}\varphi + L(\varphi) \right\}, \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi', \text{ for } \varphi \in D(\mathcal{A}_{\mathcal{U}}), \end{cases}$$

is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ *on* C_0.

In order to give a variation of constants formula associated to Equation (1), we need to extend the semigroup $(\mathcal{U}(t))_{t\geq 0}$ to the space $C_0 \oplus \langle X_0 \rangle$ where $\langle X_0 \rangle$ is the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in X\},$$

the function $X_0 c$ is given, for $c \in X$, by

$$(X_0 c)(\theta) = \begin{cases} 0 \text{ if } & \theta \in [-r,0), \\ c \text{ if } & \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $\|\phi + X_0 c\| = |\phi| + |c|$, for $(\phi, c) \in C_0 \times X$, is a Banach space. Consider the extension $\widetilde{\mathcal{A}}_{\mathcal{U}}$ of the operator $\mathcal{A}_{\mathcal{U}}$ on $C_0 \oplus \langle X_0 \rangle$ defined by

$$\begin{cases} D\left(\widetilde{\mathcal{A}}_{\mathcal{U}}\right) &= \left\{ \varphi \in C^1([-r,0]\,;X) : \mathcal{D}\varphi \in D(A) \text{ and } \mathcal{D}\varphi' \in \overline{D(A)} \right\}, \\ \widetilde{\mathcal{A}}_{\mathcal{U}}\phi &= \varphi' + X_0 \left(A\mathcal{D}\varphi + L\varphi - \mathcal{D}\varphi' \right). \end{cases}$$

In order to compute the resolvent operator $R(\lambda, \widetilde{\mathcal{A}}_{\mathcal{U}})$, we need to make the following assumption

$(\mathbf{H_1})$ $\mathcal{D}(e^{\lambda \cdot} c) \in D(A)$, for all $c \in D(A)$ and all complex λ, where $e^{\lambda \cdot} c \in C$ is defined by

$$(e^{\lambda \cdot} c)(\theta) = e^{\lambda\theta} c, \text{ for } \theta \in [-r,0].$$

Lemma 6 *[6, Theorem 13] Assume that* $(\mathbf{H}_0)$ *and* $(\mathbf{H}_1)$ *hold. Then* $\widetilde{\mathcal{A}}_{\mathcal{U}}$ *satisfies the Hille-Yosida condition on* $C_0 \oplus \langle X_0 \rangle$ *: there exist* $\widetilde{M} \geq 0$ *and* $\widetilde{\omega} \in \mathbb{R}$ *such that* $(\widetilde{\omega}, +\infty) \subset \rho(\widetilde{\mathcal{A}}_{\mathcal{U}})$ *and*

$$\left| R(\lambda, \widetilde{\mathcal{A}}_{\mathcal{U}})^n \right| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n}, \ n \in \mathbb{N}, \ \lambda > \widetilde{\omega}.$$

Now, we can state the variation of constants formula associated to Equation (1).

Theorem 7 *[6, Theorem 16] Assume that* $(\mathbf{H_0})$ *and* $(\mathbf{H_1})$ *hold. Then, for all* $\varphi \in C_0$*, the integral solution* u *of Equation (1) is given by the following variation of constants formula*

$$u_t = \mathcal{U}(t-\sigma)\varphi + \lim_{\lambda \to +\infty} \int_\sigma^t \mathcal{U}(t-s)\left(\widetilde{B}_\lambda(X_0 f(s))\right) ds, \text{ for } t \geq \sigma,$$

where $\widetilde{B}_\lambda = \lambda R(\lambda, \widetilde{\mathcal{A}}_{\mathcal{U}})$ *for* $\lambda > \widetilde{\omega}$.

3 Spectral decomposition of the phase space

In order to determine the asymptotic behavior of the semigroup $(\mathcal{U}(t))_{t\geq 0}$, we need to introduce some preliminary results. In the beginning, we introduce a definition.

Definition 8 *[15, Definition 3.1, pp. 275] The operator* $\mathcal{D}$ *is said to be stable if there exist positive constants* η *and* μ *such that the solution of the homogenous difference equation*

$$\begin{cases} \mathcal{D}u_t = 0, \ t \geq 0, \\ u_0 = \varphi, \end{cases}$$

where $\varphi \in \{\psi \in C : \ \mathcal{D}\psi = 0\}$*, satisfies*

$$|u_t(.,\varphi)| \leq \mu e^{-\eta t} |\varphi|, \text{ for } t \geq 0.$$

Example 9 *The operator* $\mathcal{D}$ *defined by*

$$\mathcal{D}\varphi = \varphi(0) - q\varphi(-r)$$

is stable if and only if $|q| < 1$.

In the following, we add two supplementary assumptions

$(\mathbf{H_2})$ The semigroup $(T_0(t))_{t\geq 0}$ is compact on $\overline{D(A)}$, for each $t > 0$.

$(\mathbf{H_3})$ The operator $\mathcal{D}$ is stable.

Then, we have the following fundamental result on the semigroup $(\mathcal{U}(t))_{t\geq 0}$.

Theorem 10 *[6, lemma 10] Assume that* $(\mathbf{H_0})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. Then the semigroup* $(\mathcal{U}(t))_{t\geq 0}$ *is decomposed on* C_0 *as follows*

$$\mathcal{U}(t) = \mathcal{U}_1(t) + \mathcal{U}_2(t), \text{ for } t \geq 0,$$

where $(\mathcal{U}_1(t))_{t\geq 0}$ *is an exponentially stable semigroup on* C_0*, which means that there are positive constants* α_0 *and* N_0 *such that*

$$|\mathcal{U}_1(t)\varphi| \leq N_0 e^{-\alpha_0 t} |\varphi|, \text{ for } t \geq 0 \text{ and } \varphi \in C_0,$$

and $\mathcal{U}_2(t)$ *is compact for every* $t > 0$*. More exactly,* $(\mathcal{U}_1(t))_{t\geq 0}$ *is the semigroup associated to the equation*

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}u_t = (A + \delta I)\mathcal{D}u_t, \ t \geq 0, \\ u_0 = \varphi, \end{cases}$$

where δ *is taken such that*

$$\left| e^{\delta t} T_0(t) \right| \leq \gamma e^{-\beta t}, \quad t \geq 0,$$

for some positive constants β *and* γ*.*

Our next goal is to reduce Equation (1) to a finite dimensional space. We introduce the Kuratowski's measure of noncompactness $\alpha(.)$ of bounded sets K in a Banach space Y by

$$\alpha(K) = \inf \{k > 0 : K \text{ has a finite cover of balls of diameter} < k\}.$$

For a bounded linear operator B on Y, $|B|_\alpha$ is defined by

$$|B|_\alpha = \inf \{c > 0 : \ \alpha(B(K)) \leq c\alpha(K), \text{ for any bounded set } K \text{ of } Y\}.$$

The essential growth bound $\omega_{ess}(\mathcal{U})$ of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is defined by

$$\begin{aligned} \omega_{ess}(\mathcal{U}) &= \lim_{t\to+\infty} \frac{1}{t} \log |\mathcal{U}(t)|_\alpha, \\ &= \inf_{t>0} \frac{1}{t} \log |\mathcal{U}(t)|_\alpha. \end{aligned}$$

By Theorem 10, we deduce that $\omega_{ess}(\mathcal{U}) < 0$. Consequently by [11, Theorem 5.3.7, pp. 333], we get the following spectral decomposition.

Theorem 11 *Assume that* $(\mathbf{H_0})$*,* $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. Then* C_0 *is decomposed as follows*

$$C_0 = S \oplus V,$$

where S *is* $\mathcal{U}$*-invariant and there are positive constants* α *and* N *such that*

$$|\mathcal{U}(t)\varphi| \leq N e^{-\alpha t} |\varphi|, \text{ for } t \geq 0 \text{ and } \varphi \in S, \tag{5}$$

V *is a finite dimensional space and the restriction of* $\mathcal{U}$ *to* V *becomes a group.*

Let C_0^* be the dual space of C_0 and $d = \dim(V)$. Take a basis vectors $\Phi = \{\phi_1, ..., \phi_d\}$ of V. Then there exist d-elements $\{\psi_1, ..., \psi_d\}$ in C_0^* such that:

$$\begin{cases} \langle \psi_i, \phi_j \rangle = \delta_{ij}, \\ \langle \psi_i, \phi \rangle = 0, \text{ for all } \phi \in S \text{ and } i \in \{1, ..., d\}, \end{cases} \tag{6}$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases}$$

and $\langle ., . \rangle$ denotes the duality pairing between C_0^* and C_0. Let $\Psi = col\{\psi_1, ..., \psi_d\}$ and $\langle \Psi, \Phi \rangle$ be a $d \times d$ -matrix, with $\langle \psi_i, \phi_j \rangle$ its (i, j)-component. Then $\langle \Psi, \Phi \rangle = I_{d\times d}$. Denote by Π^s and Π^v the projections respectively on S and V, and by $\mathcal{U}^s(t)$ and $\mathcal{U}^v(t)$ the restrictions of $\mathcal{U}(t)$ respectively on S and V, which correspond to the above decomposition of the phase space C_0. Let $\varphi \in C_0$. Then $\varphi = \Pi^s\varphi + \Pi^v\varphi$ with $\Pi^v\varphi = \sum\limits_{i=1}^{d} \alpha_i\phi_i$ and $\alpha_i \in \mathbb{R}$. By (6), we conclude that

$$\alpha_i = \langle \psi_i, \varphi \rangle .$$

Hence

$$\begin{aligned} \Pi^v\varphi &= \sum_{i=1}^{d} \langle \psi_i, \varphi \rangle \phi_i, \\ &= \Phi \langle \Psi, \varphi \rangle . \end{aligned}$$

Since $(\mathcal{U}^v(t))_{t\geq 0}$ is a group on V, then there exists a $d \times d$-matrix G such that

$$\mathcal{U}^v(t)\Phi = \Phi e^{tG}, \text{ for } t \in \mathbb{R}.$$

Moreover, $\sigma(G) = \{\lambda \in \sigma(\mathcal{A}_{\mathcal{U}}) : Re(\lambda) \geq 0\}$.
For any $n \geq n_0 > \widetilde{\omega}$ and $i \in \{1, ..., d\}$, we define the linear mapping $x_{i,n}^*$ by

$$x_{i,n}^*(a) = \left\langle \psi_i, \widetilde{B}_n(X_0 a) \right\rangle, \text{ for } a \in X.$$

Since $\left|\widetilde{B}_n\right| \leq \dfrac{n}{n - \widetilde{\omega}}\widetilde{M}$, for any $n \geq n_0$, then $x_{i,n}^*$ is a bounded linear operator from X to $\mathbb{R}$ such that

$$\left|x_{i,n}^*\right| \leq \frac{n}{n - \widetilde{\omega}}\widetilde{M}\,|\psi_i|\,, \text{ for any } n \geq n_0.$$

Define the d-column vector $x_n^* = col\left(x_{1,n}^*, ..., x_{d,n}^*\right)$. Then

$$\langle x_n^*, a \rangle = \left\langle \Psi, \widetilde{B}_n(X_0 a) \right\rangle, \quad a \in X.$$

This means that

$$\langle x_n^*, a\rangle_i = \left\langle \psi_i, \widetilde{B}_n(X_0 a)\right\rangle \text{ for } i = 1, ..., d \text{ and } a \in X.$$

Moreover,

$$\sup_{n\geq n_0} |x_n^*| \leq 2\widetilde{M} \sup_{1\leq i\leq d} |\psi_i| < \infty.$$

Which implies that $(x_n^*)_{n\geq n_0}$ is a bounded sequence in $\mathcal{L}(X, \mathbb{R}^d)$.

Theorem 12 *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. There exists* $x^* \in \mathcal{L}(X, \mathbb{R}^d)$, *such that* $(x_n^*)_{n\geq n_0}$ *converges weakly to* x^* *in the sense that*

$$\langle x_n^*, x\rangle \underset{n\to\infty}{\to} \langle x^*, x\rangle, \text{ for all } x \in X.$$

For the proof of Theorem 12, we need the following fundamental results.

Theorem 13 *[23, pp. 776] Let* Y *be any separable Banach space and* $(z_n^*)_{n\in\mathbb{N}}$ *be a bounded sequence in* Y^*. *Then, there exists a subsequence* $\left(z_{n_k}^*\right)_{k\in\mathbb{N}}$ *of* $(z_n^*)_{n\in\mathbb{N}}$ *which converges weakly in* Y^* *in the sense that there exists* $z^* \in Y^*$ *such that*

$$\left\langle z_{n_k}^*, x\right\rangle \underset{n\to\infty}{\to} \langle z^*, x\rangle, \text{ for all } x \in Y.$$

Lemma 14 *Assume that* $(\mathbf{H_0})$, $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3})$ *hold. Let* $u(., \sigma, 0, f)$ *be the solution of Equation (1) with* $\varphi = 0$. *Then*

$$\Pi^v u_t(., \sigma, 0, f) = \Phi \lim_{n\to+\infty} \int_\sigma^t e^{(t-\xi)G} \langle x_n^*, f(\xi)\rangle \, d\xi, \ t \geq \sigma.$$

Proof of the lemma.

The solution $u(., \sigma, 0, f)$ of Equation (1) with $\varphi = 0$ is given, for $t \geq \sigma$, by

$$u_t(., \sigma, 0, f) = \lim_{n\to+\infty} \int_\sigma^t \mathcal{U}(t-\xi) \left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi.$$

Then

$$\Pi^v u_t(., \sigma, 0, f) = \lim_{n\to+\infty} \int_\sigma^t \mathcal{U}^v(t-\xi) \, \Pi^v \left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi.$$

Since

$$\Pi^v \left(\widetilde{B}_n(X_0 f(\xi))\right) = \Phi \left\langle \Psi, \widetilde{B}_n(X_0 f(\xi))\right\rangle = \Phi \langle x_n^*, f(\xi)\rangle,$$

it follows that

$$\begin{aligned}\Pi^v u_t(., \sigma, 0, f) &= \Phi \lim_{n\to+\infty} \int_\sigma^t e^{(t-\xi)G} \left\langle \Psi, \widetilde{B}_n X_0 f(\xi)\right\rangle d\xi, \\ &= \Phi \lim_{n\to+\infty} \int_\sigma^t e^{(t-\xi)G} \langle x_n^*, f(\xi)\rangle \, d\xi, \ t \geq \sigma. \qquad \square\end{aligned}$$

Proof of Theorem 12. Let Z_0 be any closed separable subspace of X. Since $(x_n^*)_{n\geq n_0}$ is a bounded sequence, then by Theorem 13 we get that the sequence $(x_n^*)_{n\geq n_0}$ has a subsequence $\left(x_{n_k}^*\right)_{k\in\mathbb{N}}$ which converges weakly to some $x_{Z_0}^*$ in Z_0. We claim that the whole sequence $(x_n^*)_{n\geq n_0}$ converges weakly to $x_{Z_0}^*$ in Z_0. We proceed by contradiction and suppose that there exists a subsequence $\left(x_{n_p}^*\right)_{p\in\mathbb{N}}$ of $(x_n^*)_{n\geq n_0}$ which converges weakly to an element $\widetilde{x}_{Z_0}^*$ with $\widetilde{x}_{Z_0}^* \neq x_{Z_0}^*$. Let $a \in Z_0$. By Lemma 14, we get that

$$\lim_{k\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle x_{n_k}^*, a\right\rangle d\xi = \lim_{p\to+\infty}\int_\sigma^t e^{(t-\xi)G}\left\langle x_{n_p}^*, a\right\rangle d\xi, \text{ for } a \in Z_0.$$

This implies that

$$\int_\sigma^t e^{(t-\xi)G}\left\langle x_{Z_0}^*, a\right\rangle d\xi = \int_\sigma^t e^{(t-\xi)G}\left\langle \widetilde{x}_{Z_0}^*, a\right\rangle d\xi, \text{ for } a \in Z_0.$$

Consequently $x_{Z_0}^* \equiv \widetilde{x}_{Z_0}^*$, which gives a contradiction. We conclude that the sequence $(x_n^*)_{n\geq n_0}$ converges weakly to $x_{Z_0}^*$ in Z_0. Let Z_1 be anther closed separable subspace of X. By using the same argument as above, we get that $(x_n^*)_{n\geq n_0}$ converges weakly to $x_{Z_1}^*$ in Z_1. Since $Z_0 \cap Z_1$ is a closed separable subspace of X, we get that $x_{Z_1}^* \equiv x_{Z_0}^*$ on $Z_0 \cap Z_1$. For any $x \in X$, we define x^* by

$$\langle x^*, x\rangle = \langle x_Z^*, x\rangle,$$

where Z is any closed separable subspace of X such that $x \in Z$. Then x^* is well defined on X and it is a bounded linear mapping on X such that

$$|x^*| \leq \sup_{n\geq n_0}|x_n^*| < \infty,$$

and $(x_n^*)_{n\geq n_0}$ converges weakly to x^* in X. □

As an immediate consequence of the above theorem, we obtain that

Corollary 15 *For any continuous function $h : \mathbb{R} \to X$, we have, for $t, \sigma \in \mathbb{R}$,*

$$\lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v(t-\xi)\,\Pi^v\left(\widetilde{B}_n(X_0 h(\xi))\right)d\xi = \Phi\int_\sigma^t e^{(t-\xi)G}\langle x^*, h(\xi)\rangle\, d\xi.$$

The expression in the above corollary is well defined for all $t, \sigma \in \mathbb{R}$, since $(\mathcal{U}^v(t))_{t\in\mathbb{R}}$ is a group. We are now in the position to state a finite dimensional reduction of Equation (1).

Theorem 16 *Assume that $(\mathbf{H_0}), (\mathbf{H_1}), (\mathbf{H_2})$ and $(\mathbf{H_3})$ hold. Let u be an integral solution of Equation (1) on $\mathbb{R}$. Then, $z(t) = \langle\Psi, u_t\rangle$ is a solution of the* *ordinary differential equation*

$$\frac{d}{dt}z\left(t\right) = Gz\left(t\right) + \left\langle x^*, f\left(t\right)\right\rangle, \ t \in \mathbb{R}. \tag{7}$$

Conversely, if f is a bounded function on $\mathbb{R}$ and z is a solution of Equation (7) on $\mathbb{R}$, then the function u given by

$$u(t) = \left[\Phi z\left(t\right) + \lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\widetilde{B}_n(X_0 f\left(\xi\right))\right)d\xi\right](0)\,, \text{ for } t \in \mathbb{R},$$

is an integral solution of Equation (1) on $\mathbb{R}$.

PROOF. Let u be an integral solution of Equation (1) on $\mathbb{R}$. Then

$$u_t = \Pi^s u_t + \Pi^v u_t, \text{ for all } t \in \mathbb{R},$$

and, for $t, \sigma \in \mathbb{R}$, one has

$$\Pi^v u_t = \mathcal{U}^v\left(t-\sigma\right)\Pi^v u_\sigma + \lim_{n\to+\infty}\int_{\sigma}^{t}\mathcal{U}^v\left(t-\xi\right)\Pi^v\left(\widetilde{B}_n(X_0 f\left(\xi\right))\right)d\xi\ .$$

Since $\Pi^v u_t = \Phi\left\langle\Psi, u_t\right\rangle$ and by Corollary 15, we get that

$$\Phi\left\langle\Psi, u_t\right\rangle = \mathcal{U}^v\left(t-\sigma\right)\Phi\left\langle\Psi, u_\sigma\right\rangle + \Phi\int_{\sigma}^{t} e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi,$$

$$= \Phi e^{(t-\sigma)G}\left\langle\Psi, u_\sigma\right\rangle + \Phi\int_{\sigma}^{t} e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi.$$

Let $z\left(t\right) = \left\langle\Psi, u_t\right\rangle$. Then

$$z(t) = e^{(t-\sigma)G}z(\sigma) + \int_{\sigma}^{t} e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi, \text{ for } t, \sigma \in \mathbb{R}.$$

Consequently, z is a solution of the ordinary differential equation (7) on $\mathbb{R}$. Conversely, assume that f is bounded on $\mathbb{R}$. Then $\int_{-\infty}^{t}\mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\widetilde{B}_n(X_0 f\left(\xi\right))\right)d\xi$ is well defined on $\mathbb{R}$. Let z be a solution of (7) on $\mathbb{R}$ and define v by

$$v(t) = \Phi z\left(t\right) + \lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s\left(t-\xi\right)\Pi^s\left(\widetilde{B}_n(X_0 f\left(\xi\right))\right)d\xi, \text{ for } t \in \mathbb{R}.$$

Since

$$z(t) = e^{(t-\sigma)G}z(\sigma) + \int_{\sigma}^{t} e^{(t-\xi)G}\left\langle x^*, f(\xi)\right\rangle d\xi, \text{ for } t, \sigma \in \mathbb{R},$$

using Corollary 15, the function v_1 given by

$$v_1(t) = \Phi z\left(t\right), \text{ for } t \in \mathbb{R},$$

satisfies the integral equation

$$v_1(t) = \mathcal{U}^v(t-\sigma)\, v_1(\sigma) + \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v(t-\xi)\,\Pi^v\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi, \text{ for } t,\sigma\in\mathbb{R}.$$

Moreover, the function v_2 given by

$$v_2(t) = \lim_{n\to+\infty}\int_{-\infty}^t \mathcal{U}^s(t-\xi)\,\Pi^s\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi, \text{ for } t\in\mathbb{R},$$

satisfies

$$v_2(t) = \mathcal{U}^s(t-\sigma)\, v_2(\sigma) + \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^s(t-\xi)\,\Pi^s\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi, \text{ for } t\geq\sigma.$$

Then, for all $t\geq\sigma$, we have

$$\begin{aligned}\mathcal{U}(t-\sigma)\, v(\sigma) &= \mathcal{U}^v(t-\sigma)\, v_1(\sigma) + \mathcal{U}^s(t-\sigma)\, v_2(\sigma),\\ &= v_1(t) - \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^v(t-\xi)\,\Pi^v\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi + v_2(t) -\\ &\qquad \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}^s(t-\xi)\,\Pi^s\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi,\\ &= v(t) - \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}(t-\xi)\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi.\end{aligned}$$

Therefore

$$v(t) = \mathcal{U}(t-\sigma)\, v(\sigma) + \lim_{n\to+\infty}\int_\sigma^t \mathcal{U}(t-\xi)\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi, \text{ for } t\geq\sigma.$$

By Theorem 7, we obtain that the function u defined by $u(t) = v(t)(0)$ is an integral solution of Equation (1) on $\mathbb{R}$. $\square$

4 Almost periodic solutions for Equation (1)

First of all, we recall some properties about almost periodic functions. Let $\mathcal{BC}(\mathbb{R}, X)$ be the space of all bounded continuous functions from $\mathbb{R}$ to X, provided with the uniform norm topology. For $g\in\mathcal{BC}(\mathbb{R}, X)$ and for every $\tau\in\mathbb{R}$, we define the function g_τ by

$$g_\tau(s) = g(\tau+s), \text{ for all } s\in\mathbb{R}.$$

Definition 17 *[13] A function g is said to be almost periodic if the set*

$$\{g_\tau : \tau \in \mathbb{R}\}$$

is relatively compact in $\mathcal{BC}(\mathbb{R}, X)$.

Consider the ordinary differential Equation (3), where B is a constant $n \times n$-matrix and $g : \mathbb{R} \to \mathbb{R}^n$ is a continuous function.

Theorem 18 *[13, Theorem 5.8, pp. 86] Assume that g is an almost periodic function. Then the following are equivalent:*
(**i**) *existence of a bounded solution on $\mathbb{R}^+$ of Equation (3),*
(**ii**) *existence of an almost periodic solution of Equation (3).*
Moreover, every bounded solution on $\mathbb{R}$ is almost periodic.

For the existence of almost periodic solutions of Equation (1), we assume that

$(\mathbf{H_4})$ f is an almost periodic function.

Theorem 19 *Assume that $(\mathbf{H_0}), (\mathbf{H_1}), (\mathbf{H_2}), (\mathbf{H_3})$ and $(\mathbf{H_4})$ hold. Then the following are equivalent:*
(**i**) *existence of a bounded solution on $\mathbb{R}^+$ of Equation (1),*
(**ii**) *existence of an almost periodic solution of Equation (1).*

PROOF. Let u be a bounded solution of Equation (1) on $\mathbb{R}^+$. By Theorem 16, the function $z(t) = \langle \Psi, u_t \rangle$, for $t \geq 0$, is a solution of the ordinary differential Equation (7) and it is bounded on $\mathbb{R}^+$. Moreover, the function

$$\nu(t) = \langle x^*, f(t) \rangle, \text{ for } t \in \mathbb{R},$$

is almost periodic from $\mathbb{R}$ to $\mathbb{R}^d$. By Theorem 18, we get that the reduced system (7) has an almost periodic solution $\widetilde{z}$. Consequently, $\Phi\widetilde{z}(.)$ is an almost periodic function on $\mathbb{R}$. By Theorem 16, the function $u(t) = v(t)(0)$, where

$$v(t) = \Phi\widetilde{z}(t) + \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\, \Pi^s\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi, \text{ for } t \in \mathbb{R},$$

is an integral solution of Equation (1) on $\mathbb{R}$. To end the proof, we will show that the function

$$y(t) = \lim_{n \to +\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-\xi)\, \Pi^s\left(\widetilde{B}_n(X_0 f(\xi))\right) d\xi, \text{ for } t \in \mathbb{R},$$

is almost periodic. In fact for any sequence of real numbers $\left(\alpha_p'\right)_{p \geq 0}$ there exists a subsequence $(\alpha_p)_{p \geq 0} \subset \left(\alpha_p'\right)_{p \geq 0}$ such that $f(. + \alpha_p)$ converges uniformly on

$\mathbb{R}$ to a function $\widetilde{f}$. We can see also that $y(.+\alpha_p)$ converges uniformly on $\mathbb{R}$ to the function

$$\widetilde{y}(t)=\lim_{n\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-\xi)\,\Pi^s\left(\widetilde{B}_n(X_0\widetilde{f}(\xi))\right)d\xi,\text{ for } t\in\mathbb{R}.$$

Consequently, y is an almost periodic function and v is an almost periodic solution of Equation (1). $\square$

5 Application

In order to apply the abstract result of the previous section, we consider the model proposed in [26]:

$$\begin{cases}\dfrac{\partial}{\partial t}\left[u(t,x)-qu(t-r,x)\right]=\dfrac{\partial^2}{\partial x^2}\left[u(t,x)-qu(t-r,x)\right]+\\[2ex] \qquad\qquad\displaystyle\int_{-r}^{0}G(\theta)u(t+\theta,x)d\theta+h\left(t,x\right),\ \text{for } t\geq\sigma \text{ and } x\in[0,\pi],\\[2ex] u(t,x)-qu(t-r,x)=0,\ \text{for } x=0,\pi \text{ and } t\geq\sigma,\\[2ex] u(\sigma+\theta,x)=\psi(\theta,x),\ \text{for } \theta\in[-r,0] \text{ and } x\in[0,\pi],\end{cases}\tag{8}$$

where $G:[-r,0]\to\mathbb{R}$, $\psi:[-r,0]\times[0,\pi]\to\mathbb{R}$ and $h:\mathbb{R}\times[0,\pi]\to\mathbb{R}$ are continuous functions q is a positive constant in $(0,1)$.

In order to write System (8) in an abstract form, we introduce $X=C\left([0,\pi];\mathbb{R}\right)$ the space of continuous functions from $[0,\pi]$ to $\mathbb{R}$ endowed with the uniform norm topology. Define the operator $A:D(A)\subset X\to X$ by

$$\begin{cases}D(A)=\left\{y\in C^2\left([0,\pi];\mathbb{R}\right):y(0)=y(\pi)=0\right\},\\ Ay=y''.\end{cases}$$

Lemma 20 *[10, Proposition 14.6] The operator A satisfies the Hille-Yosida condition on X:*

$$(0,+\infty)\subset\rho(A)\ \textit{and}\ \left|(\lambda I-A)^{-1}\right|\leq\frac{1}{\lambda},\ \textit{for}\ \lambda>0.$$

This lemma implies that condition $(\mathbf{H}_0)$ is satisfied. On the other hand, we can see that

$$\overline{D(A)} = \{y \in X : y(0) = y(\pi) = 0\}.$$

Let us introduce the bounded linear operator $\mathcal{D} : C := C([-r,0]\,;X) \to X$ by

$$\mathcal{D}\phi := \phi(0) - q\phi(-r).$$

Since $0 < q < 1$, then $\mathcal{D}$ is stable and Condition $(\mathbf{H}_3)$ holds. Moreover, by definitions of the operators A and $\mathcal{D}$, it follows that Condition $(\mathbf{H}_1)$ is satisfied.

Let $L : C \to X$ be the operator defined by

$$L(\phi)(x) = \int_{-r}^{0} G(\theta)\phi(\theta)(x)d\theta, \text{ for } x \in [0,\pi] \text{ and } \phi \in C,$$

$f : \mathbb{R} \longrightarrow X$ be the mapping defined by

$$f(t)(x) = h(t,x), \text{ for } t \in \mathbb{R} \text{ and } x \in [0,\pi],$$

and the initial data $\varphi \in C$ is given by

$$\varphi(\theta)(x) = \psi(\theta,x), \text{ for } \theta \in [-r,0] \text{ and } x \in [0,\pi].$$

L is a bounded linear operator from C to X. By continuity of h, the function f is continuous from $\mathbb{R}$ to X. Let $w(t) = u(t,.)$, for $t \geq \sigma$. Then Equation (8) takes the abstract form

$$\begin{cases} \dfrac{d}{dt}\mathcal{D}w_t = A\mathcal{D}w_t + L(w_t) + f(t), \text{ for } t \geq \sigma, \\ w_\sigma = \varphi \in C. \end{cases} \tag{9}$$

Let A_0 be the part of the operator A in $\overline{D(A)}$. Then A_0 is given by

$$\begin{cases} D(A_0) = \left\{y \in C^2([0,\pi]\,;\mathbb{R}) : y(0) = y(\pi) = y''(0) = y''(\pi) = 0\right\}, \\ A_0 y = y'', \text{ for } y \in D(A_0). \end{cases}$$

It is well known from [11, Example 1.4.34 , pp. 123], that the operator A_0 generates a strongly continuous compact semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$ and

$$|T_0(t)| \leq e^{-t}, \text{ for } t \geq 0.$$

This implies that $(\mathbf{H_2})$ holds. By Theorem 3, for any $\varphi \in C$ such that

$$\mathcal{D}\varphi \in \{y \in X : y(0) = y(\pi) = 0\},$$

there exists a unique integral solution of Equation (9) on $[-r+\sigma, +\infty)$.

In order to study the existence of almost periodic solutions of Equation (9), we suppose that

$(\mathbf{H_5})$ h is almost periodic in t uniformly for $x \in [0,\pi]$, which means that for any $\varepsilon > 0$, there is a positive number $l(\varepsilon)$, such that any interval of length $l(\varepsilon)$ contains a τ for which

$$|h(t+\tau,x) - h(t,x)| < \varepsilon, \text{ for all } (t,x) \in \mathbb{R} \times [0,\pi] .$$

By Assumption $(\mathbf{H_5})$, we deduce that the function $f : \mathbb{R} \rightarrow X$ is almost periodic.

Moreover, we suppose that

$(\mathbf{H_6})$ there exists a constant $\beta \in (0,1)$ such that

$$\int_{-r}^{0} |G(\theta)|\, d\theta \leq (1-q)\,\beta.$$

Proposition 21 *Assume that* $(\mathbf{H_5})$ *and* $(\mathbf{H_6})$ *hold. Then, Equation (9) has a bounded solution on* $\mathbb{R}^+$*, and it has an almost periodic solution.*

Proof. The first step is to prove that Equation (9) has a bounded solution on $\mathbb{R}^+$. Let

$$\rho = \frac{1}{1+q}\left(1 + \frac{|f|}{1-\beta}\right), \tag{10}$$

where $|f| = \sup_{s\in\mathbb{R}} |f(s)|$. Take $\varphi \in C$ such that $|\varphi| < \rho$. Then

$$|\varphi\,(0) - q\varphi(-r)| < (1+q)\,\rho.$$

Let w be the integral solution of Equation (9) with the initial condition φ. We claim that

$$|w\,(t) - qw(t-r)| \leq (1+q)\,\rho, \text{ for all } t \geq 0. \tag{11}$$

We proceed by contradiction. Let t_0 be the first time such that (11) is not true. Then

$$t_0 = \inf\left\{t > 0 : |w\,(t) - qw(t-r)| > (1+q)\,\rho\right\}.$$

By continuity, one has

$$|w\,(t_0) - qw(t_0 - r)| = (1+q)\,\rho,$$

and there exists a positive constant $\varepsilon > 0$ such that

$$|w\,(t) - qw(t-r)| > (1+q)\,\rho, \text{ for } t \in (t_0, t_0+\varepsilon)\,.$$

Using the variation of constants formula, we obtain

$$|w(t_0) - qw(t_0 - r)| \leq e^{-t_0}(1+q)\rho + \int_0^{t_0} e^{-(t_0-s)}\left[\int_{-r}^{0} |G(\theta)|\,|u(s+\theta)|\,d\theta + |f|\right] ds.$$

Since $|w(t) - qw(t-r)| \leq (1+q)\rho$, for $t \leq t_0$, then

$$|w(t)| \leq (1+q)\rho + q\,|w(t-r)|\,, \text{ for } t \in [-r, t_0]\,.$$

Moreover, since $|\varphi| < \rho$, we can see that

$$|w(t)| \leq \frac{1+q}{1-q}\rho, \text{ for } t \in [-r, t_0].$$

Then

$$|w(t_0) - qw(t_0 - r)| \leq e^{-t_0}(1+q)\rho + \left(1 - e^{-t_0}\right)\left[\int_{-r}^{0} |G(\theta)|\,d\theta\ \frac{1+q}{1-q}\rho + |f|\right].$$

Condition ($\mathbf{H_6}$) implies that

$$|w(t_0) - qw(t_0 - r)| \leq e^{-t_0}(1+q)\rho + \left(1 - e^{-t_0}\right)\left((1+q)\beta\rho + |f|\right).$$

Thanks to (10), we obtain

$$|w(t_0) - qw(t_0 - r)| \leq e^{-t_0}(1+q)\rho + \left(1 - e^{-t_0}\right)(1+q)\rho - \left(1 - e^{-t_0}\right)(1-\beta)\,.$$

Consequently, we obtain that

$$|w(t_0) - qw(t_0 - r)| \leq (1+q)\rho - \left(1 - e^{-t_0}\right)(1-\beta) < (1+q)\rho.$$

By continuity, there exists a positive ε_0 such that

$$|w(t) - qw(t-r)| < (1+q)\rho, \text{ for } t \in (t_0, t_0 + \varepsilon_0)\,.$$

Which gives a contradiction. We deduce that

$$|w(t) - qw(t-r)| \leq (1+q)\rho, \text{ for } t \geq 0.$$

We claim that

$$|w(t)| \leq \frac{1+q}{1-q}\rho, \text{ for } t \geq 0.$$

Let $t \in [0, r]$. Then

$$|w(t)| \leq (1+q)\rho + q\rho \leq (1+q)(1+q)\rho,$$

and for $t \in [r, 2r]$

$$|w(t)| \leq (1+q)\left(1 + q + q^2\right)\rho.$$

We proceed by steps and we obtain that for $t \in [(n-1)r, nr]$

$$|w(t)| \leq (1+q)\left(1+q+q^2+\ldots+q^n\right)\rho.$$

Consequently,

$$|w(t)| \leq (1+q)\,\rho \sum_{n\geq 0} q^n = \frac{1+q}{1-q}\rho, \text{ for all } t \geq 0.$$

Then Equation (9) has a bounded integral solution w on $\mathbb{R}^+$. By Theorem 19, we deduce that Equation (9) has an almost periodic solution. □

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