Structural Interactions and Absorption of Structural Rules in BI Sequent Calculus

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Abstract. Development of a contraction-free `BI` sequent calculus, be it in the sense of `G3i` or `G4i`, has not been successful in literature. We address the open problem by presenting such a sequent system. In fact our calculus involves no structural rules.

1 Introduction

Propositional `BI` [10] is a conservative extension of propositional intuitionistic logic `IL` and propositional multiplicative fragment of intuitionistic linear logic `MILL` (*Cf.* [6] for linear logic). It is conservative in the sense that all the theorems of `IL` and `MILL` are a theorem of `BI`. But the extension is not the least conservative. That is, there are expressions of `BI` that are not expressible in `IL` or `MILL` [10]. They shape logical characteristics unique to `BI`, which must be studied. Structural interactions in sequent calculus (interactions between logical rules and structural rules) is one of them for which the details matter. Earlier works $[2, 3, 5, 7, 9]$ on `BI` appear to suggest that the study is non-trivial, however. In this work we solve an open problem of absorption of structural rules, which is of theoretical interest having a foundational implication to automated reasoning. Techniques considered here should be of interest to proof-theoretical studies of other non-classical logics.

1.1 Logic `BI`

`BI` has a proof-theoretical origin. A proof system was defined [10], followed by semantics $[5, 11]$. To speak of the language of `BI` first, if we denote propositional variables by $\mathcal{P}$, signatures of `IL` by $\{\top_0, \bot_0, \wedge_2, \vee_2, \supset_2\}$ and those of `MILL` by $\{{}^*\top_0, *_2, -\!*_2\}$[1] where ${}^*\top$ is the multiplicative top element **1**, $*$ is linear 'times' $\otimes$ and $-\!*$ is linear implication $\multimap$ [6], then it comprises all the expressions that are constructable from $(\mathcal{P}, \{\top_0, \bot_0, {}^*\top_0, \wedge_2, \vee_2, \supset_2, *_2, -\!*_2\})$. Let us suppose two arbitrary expressions (formulas) F and G in the language. Then like in `IL`, we can construct $F \wedge G$, $F \vee G$, $F \supset G$; and, like in `MILL`, we can construct $F * G$, $F -\!* G$. The two types are actively distinguished in `BI` proof systems by two distinct structural connectives. The below examples are given in [10].

$$\frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G} \supset R \quad \frac{\Gamma, F \vdash G}{\Gamma \vdash F -\!* G} -\!*R$$

[1] The sub-scripts denote the arity.

Γ denotes a structure.[2] Note the use of two structural connectives ";" and "," for a structural distinction. If there were only ",", both $\supset R$ and $-\!\!*R$ could apply on $\Gamma, F \vdash G$. The contextual differentiation is a simple way to isolate the two implications. Following the convention of linear logic, the IL structures that ";" form are termed additive; and the MILL structures multiplicative, similarly. One axiom: $F = F \wedge \top = F *^{*}\top$, connects the two types. But ";" and "," do not distribute over one another. So in general a BI structure is a nesting of additive structures $\Gamma_1; \Gamma_2$ and multiplicative structures Γ_1, Γ_2. In the first BI sequent calculus LBI [11], we have the following structural rules as expected:

$$\frac{\Gamma(\Gamma_1; \Gamma_1) \vdash F}{\Gamma(\Gamma_1) \vdash F}\text{ Contraction} \quad \frac{\Gamma(\Gamma_1) \vdash F}{\Gamma(\Gamma_1; \Gamma_1) \vdash F}\text{ Weakening}$$

where $\Gamma(...)$ abstracts any other structures surrounding the focused ones in the sequents. We will formally define the notation later.

1.2 Research problems and contributions

The formulation of BI is intuitive, as we just saw. But that BI is not the least conservative extension of IL and MILL means that IL and MILL interact in parts of BI. Structurally we have an interesting phenomenon. When we consider instances of the contraction rule as were stated earlier, we find that there are several of them, including ones below.

$$\frac{\Gamma((F; F), G) \vdash H}{\Gamma(F, G) \vdash H}\ Ctr_1 \quad \frac{\Gamma(F, (G; G)) \vdash H}{\Gamma(F, G) \vdash H}\ Ctr_2 \quad \frac{\Gamma((F, G); (F, G)) \vdash H}{\Gamma(F, G) \vdash H}\ Ctr_3$$

The first two are simply G1i [12] contractions. The last is not, since what is duplicating bottom-up is a structure. And it poses some proof-theoretical problem: if it is not admissible[3] in LBI, we cannot impose any general restriction on the size of what may duplicate bottom-up, and contraction analysis becomes non-trivial. As we are to state in due course, indeed structural contraction is not admissible in LBI. For a successful contraction absorption, we need to identify what in LBI require the general contraction.

Two issues stand in the way of a successful LBI contraction analysis, however. The first is the structural equivalences $\Gamma, \emptyset_m = \Gamma = \Gamma; \emptyset_a$ (where $\emptyset_a$ denotes the additive nullary structural connective corresponding to $\top$ and $\emptyset_m$ the multiplicative nullary structural connective corresponding to $^{*}\top$) which are by nature bidirectional:

$$\frac{\Gamma \vdash F}{\Gamma; \emptyset_a \vdash F} \quad \frac{\Gamma; \emptyset_a \vdash F}{\Gamma \vdash F} \quad \frac{\Gamma \vdash F}{\Gamma, \emptyset_m \vdash F} \quad \frac{\Gamma, \emptyset_m \vdash F}{\Gamma \vdash F}$$

Apart from being an obvious source of non-termination, it obscures the core mechanism of structural interactions by seemingly implying a free transformation of an additive structure into a multiplicative one and vice versa. The second is the difficulty of isolating the effect of contraction from that of weakening, as a work by Donnelly *et*

[2] Those proof-theoretical terms are assumed familiar. They are found for example in [12]. But formal definitions that we will need for technical discussions will be found in the next section.

[3] An inference rule in sequent calculus is admissible when any sequent which is derivable in the calculus is derivable without the particular rule.

al [3] experienced (where contraction is absorbed into weakening as well as into logical rules). It is also not so straightforward to know whether, first of all, either weakening or contraction is immune to the effect of the structural equivalences. As the result of the technical complications, contraction-free `BI` sequent calculi, be the contraction-freeness in the sense of `G3i` or of `G4i` [4, 12], have remained in obscurity.

The current status of the knowledge of structural interactions within `BI` proof systems is not very satisfactory. From the perspective of theorem proving for example, the presence of the bidirectional rules and contraction as explicit structural rules in `LBI` means that it is difficult to actually prove that an invalid `BI` proposition is underivable within the calculus. This is because `LBI` by itself does not provide termination conditions apart when a (backward) derivation actually terminates: the only case in which no more backward derivation on a `LBI` sequent is possible is when the sequent is empty; the only case in which it is empty is when it is the premise of an axiom.

We solve the open problem of contraction absorption, but even better, of absorbing all the structural rules. We also eliminate nullary structural connectives. The objective of this work is to solve the mentioned long unsolved open problem in proof theory. We do not even require an explicit semantics introduction. Therefore technical dependency on earlier works is pretty small. Only the knowledge of `LBI` [11] is required.

1.3 Structure of the remaining sections

In Section 2 we present technical preliminaries of `BI` proof theory. In Section 3 we introduce our `BI` calculus `LBIZ` with no structural rules. In Section 4 we show its main properties including admissibility of structural rules and equivalence to `LBI`. We also show `Cut` admissibility in [`LBIZ` + `Cut`]. Section 5 concludes.

2 BI Proof Theory - Preliminaries

We assume the availability of the following meta-logical notations. “If and only if” is abbreviated by “iff”.

Definition 1 (Meta-connectives). *We denote logical conjunction (“and”) by $\wedge^\dagger$, logical disjunction (“or”) by $\vee^\dagger$, material implication (“implies”) by $\rightarrow^\dagger$, and equivalence by $\leftrightarrow^\dagger$. These follow the semantics of standard classical logic’s.*

We denote propositional variables by $\mathcal{P}$ and refer to an element of $\mathcal{P}$ by p or q with or without a sub-script. A `BI` formula $F(, G, H)$ with or without a sub-script is constructed from the following grammar: $F := p \mid \top \mid \bot \mid {}^*\top \mid F \wedge F \mid F \vee F \mid F \supset F \mid F * F \mid F -\!\!* F$. The set of `BI` formulas is denoted by $\mathfrak{F}$.

Definition 2 (`BI` structures). `BI` *structure $\Gamma(, Re)$ with or without a sub-/super-script, commonly referred to as a bunch [10], is defined by: $\Gamma := F \mid \Gamma; \Gamma \mid \Gamma, \Gamma$. We denote by $\mathfrak{S}$ the set of* `BI` *structures.*

For binding order, $[\wedge, \vee, *] \gg [\supset, -\!\!*] \gg [;\ ,] \gg [\forall\quad \exists] \gg [\neg^\dagger] \gg [\wedge^\dagger, \vee^\dagger] \gg [\rightarrow^\dagger, \leftrightarrow^\dagger]$ in a decreasing precedence. Connectives in the same group have the same precedence.

Both of the structural connectives “;” and “,” are defined to be fully associative

$$\frac{}{F \vdash F}\ \mathrm{id} \qquad \frac{\Gamma_1 \vdash G \qquad \Gamma(G) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ \mathrm{Cut} \qquad \frac{}{\Gamma(\bot) \vdash H}\ \bot L$$

$$\frac{}{\Gamma \vdash \top}\ \top R \qquad \frac{}{{}^{*}\top \vdash {}^{*}\top}\ {}^{*}\top R \qquad \frac{\Gamma(F;G) \vdash H}{\Gamma(F \wedge G) \vdash H}\ \wedge L$$

$$\frac{\Gamma(F) \vdash H \qquad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\ \vee L \qquad \frac{\Gamma_1 \vdash F \qquad \Gamma(\Gamma_1; G) \vdash H}{\Gamma(\Gamma_1; F \supset G) \vdash H}\ \supset L$$

$$\frac{\Gamma(F,G) \vdash H}{\Gamma(F * G) \vdash H}\ *L \qquad \frac{\Gamma_1 \vdash F \qquad \Gamma(G) \vdash H}{\Gamma(\Gamma_1, F \mathrel{-\!\!*} G) \vdash H}\ \mathrel{-\!\!*} L \qquad \frac{\Gamma \vdash F \qquad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\ \wedge R$$

$$\frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ \vee R \qquad \frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G}\ \supset R \qquad \frac{\Gamma_1 \vdash F \qquad \Gamma_2 \vdash G}{\Gamma_1, \Gamma_2 \vdash F * G}\ *R$$

$$\frac{\Gamma, F \vdash G}{\Gamma \vdash F \mathrel{-\!\!*} G}\ \mathrel{-\!\!*} R \qquad \frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1; \Gamma_2) \vdash H}\ \mathrm{Wk\ L} \qquad \frac{\Gamma(\Gamma_1; \Gamma_1) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ \mathrm{Ctr\ L}$$

$$\frac{\Gamma(\Gamma_1; \top) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ EqAnt_1 \qquad \frac{\Gamma(\Gamma_1, {}^{*}\top) \vdash H}{\Gamma(\Gamma_1) \vdash H}\ EqAnt_2$$

Fig. 1: `LBI`: a `BI` sequent calculus. Inference rules with a double-dotted line are bidirectional. $i \in \{1, 2\}$. Structural connectives are fully associative and commutative.

and commutative, and we assume as such everywhere we talk about `BI` structures. On the other hand, we do not assume distributivity of ";" over ',' or vice versa. A context "$\Gamma(-)$" (with a hole "$-$") takes the form of a tree because of the nesting of additive/multiplicative structures.

Definition 3 (Context). *A context $\Gamma(-)$ is finitely constructed from the following grammar:*

$\Gamma(-) := - \mid -; \Gamma \mid \Gamma; - \mid -, \Gamma \mid \Gamma, - \mid \Gamma(-); \Gamma \mid \Gamma; \Gamma(-) \mid \Gamma(-), \Gamma \mid \Gamma, \Gamma(-)$.

Given any context $\Gamma_1(-)$ and any $\Gamma_2 \in \mathfrak{S}$, we assume that $\Gamma_1(\Gamma_2)$ is some `BI` *structure Γ_3 such that $\Gamma_3 = \Gamma_1(\Gamma_2)$.*

Definition 4 (Sequents). *The set of* `BI` *sequents $\mathfrak{D}$ is defined by:*

$\mathfrak{D} := \{\Gamma \vdash F \mid \Gamma \in \mathfrak{S} \wedge^{\dagger} F \in \mathfrak{F}\}$.

The left hand side of $\vdash$ is termed antecedent, and the right hand side of $\vdash$ consequent.

A variant of the first `BI` sequent calculus `LBI` is found in Figure 1. Notice how already we do not consier the nullary structural connectives. All the additive inference rules share contexts, *e.g.* in $\vee L$ the same context in the conclusion propagates onto both premises. Multiplicative inference rules are context-free [12] or resource sensitive. A good example to illustrate this is $*R$: both Γ_1 and Γ_2 in the conclusion sequent are viewed as resources for the inference rule, and are split into the premises of the rule. Note again our assumption of the full commutativity of "," here. `Cut` is admissible in `LBI`.

Lemma 1 (Cut admissibility in LBI). *There is a direct cut elimination procedure which proves admissibility of* `Cut` *in* `LBI` *(sketched in [11]; corrected in [1]).*

3 LBIZ: A Structural-Rule-Free BI Sequent Calculus

In this section we present a new BI sequent calculus LBIZ (Figure 2) in which no structural rules appear. We first introduce notations that are necessary to read inference rules in the calculus. First, from now on, whenever we write $\widetilde{\Gamma}$ for any BI structure, we indicate that it may be empty. The emptiness is in the following sense: $\widetilde{\Gamma_1}; \Gamma_2 = \Gamma_2$ if Γ_1 is empty; and $\widetilde{\Gamma_1}, \Gamma_2 = \Gamma_2$ if Γ_1 is empty. Apart from this, we use two other notations.

3.1 Essence of antecedent structures

Co-existence of IL and MILL in BI calls for new contraction-absorption techniques. Possible interferences to one structural rule from the others need considered. To illustrate the technical difficulty, $EqAnt_{2\,\texttt{LBI}}$ for instance interacts directly with $WkL_{\texttt{LBI}}$. When $WkL_{\texttt{LBI}}$ is absorbed into the rest, the effect propagates to one direction of $EqAnt_{2\,\texttt{LBI}}$, resulting in;

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_1, ({}^*\!\top; \widetilde{\Gamma_2})) \vdash H}\ EA_2$$

Hence absorption of $WkL_{\texttt{LBI}}$ must involve analysis of $EqAnt_{2\,\texttt{LBI}}$ as well. To solve this particular problem we define a new notation of 'essence' of BI structures.

Definition 5 (Essence of BI structures). *Let Γ_1 be a BI structure. Then we have a set of its essences as defined in the following inductive rules.*

- *Γ_2 is an essence of Γ_1 if $\Gamma_1 = \Gamma_2$.*[4]
- *$\Gamma(\Gamma', ({}^*\!\top; \widetilde{\Gamma_2}))$*[5] *is an essence of Γ_1 if $\Gamma(\Gamma')$ is an essence of Γ_1.*
- *$\Gamma((\Gamma', ({}^*\!\top; \widetilde{\Gamma_2})); \Gamma'')$ is an essence of Γ_1 if $\Gamma(\Gamma'; \Gamma'')$ is an essence of Γ_1.*

By $\mathbb{E}(\Gamma_1)$ we denote an essence of Γ_1.

The essence takes care of an arbitrary number of EA_2 applications, while nicely retaining a compact representation of a sequent (see the calculus). In each of $\supset L$ and $-\!\!*L$, the essence in the premise(s) and that in the conclusion are the same and identical BI structure. Specifically, in a derivation tree, the use of $\mathbb{E}(\Gamma)$ in multiple sequents in the derivation tree signifies the same BI structure.

Example 1. Given a LBIZ-derivation:

$$\dfrac{\dfrac{}{F_1; (({}^*\!\top; \Gamma_1), F_1 \supset F_2) \vdash F_1}\ id \qquad \dfrac{}{F_2; F_1; (({}^*\!\top; \Gamma_1), F_1 \supset F_2) \vdash F_2}\ id}{F_1; (({}^*\!\top; \Gamma_1), F_1 \supset F_2) \vdash F_2}\ \supset L$$

it can be alternatively written down by;

$$\dfrac{\dfrac{}{\mathbb{E}(F_1; F_1 \supset F_2) \vdash F_1}\ id \qquad \dfrac{}{F_2; \mathbb{E}(F_1; F_1 \supset F_2) \vdash F_2}\ id}{\mathbb{E}(F_1; F_1 \supset F_2) \vdash F_2}\ \supset L$$

[4] For some Γ_2. The equality is of course up to associativity and commutativity.

[5] For some $\widetilde{\Gamma_2}$; similarly in the rest.

$$\frac{}{\mathtt{E}(\widetilde{\Gamma};p) \vdash p}\ id \qquad \frac{}{\Gamma(\bot) \vdash F}\ \bot L \qquad \frac{}{\Gamma \vdash \top}\ \top R \qquad \frac{}{\mathtt{E}(\widetilde{\Gamma};{}^{*}\!\top) \vdash {}^{*}\!\top}\ {}^{*}\!\top R$$

$$\frac{\Gamma(F;G) \vdash H}{\Gamma(F \wedge G) \vdash H}\ \wedge L \qquad \frac{\Gamma \vdash F \quad \Gamma \vdash G}{\Gamma \vdash F \wedge G}\ \wedge R$$

$$\frac{\Gamma(F) \vdash H \quad \Gamma(G) \vdash H}{\Gamma(F \vee G) \vdash H}\ \vee L \qquad \frac{\Gamma \vdash F_i}{\Gamma \vdash F_1 \vee F_2}\ \vee R$$

$$\frac{\mathtt{E}(\widetilde{\Gamma_1}; F \supset G) \vdash F \quad \Gamma(G; \mathtt{E}(\widetilde{\Gamma_1}; F \supset G)) \vdash H}{\Gamma(\mathtt{E}(\widetilde{\Gamma_1}; F \supset G)) \vdash H}\ \supset L \qquad \frac{\Gamma; F \vdash G}{\Gamma \vdash F \supset G}\ \supset R$$

$$\frac{\Gamma(F, G) \vdash H}{\Gamma(F * G) \vdash H}\ *L \qquad \frac{Re_i \vdash F_1 \quad Re_j \vdash F_2}{\Gamma' \vdash F_1 * F_2}\ *R$$

$$\frac{Re_i \vdash F \quad \Gamma((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_1}; F \multimap G))) \vdash H}{\Gamma(\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_1}; F \multimap G)) \vdash H}\ \multimap L \qquad \frac{\Gamma, F \vdash G}{\Gamma \vdash F \multimap G}\ \multimap R$$

Fig. 2: LBIZ: a BI sequent calculus with zero occurrence of explicit structural rules. $i, j \in \{1, 2\}$. $i \neq j$. Structural connectives are fully associative and commutative. In $*R$ and $\multimap L$, if Γ' is not empty, $(Re_1, Re_2) \in \mathtt{Candidate}(\Gamma')$; otherwise, $Re_i = {}^{*}\!\top$ and Re_j is empty. Both E and Candidate are as defined in the main text.

where $\mathtt{E}(F_1; F_1 \supset F_2) = F_1; (({}^{*}\!\top; \Gamma_1), F_1 \supset F_2)$.

$\mathtt{E}'(\Gamma)$ (or $\mathtt{E}_1(\Gamma)$ or any essence that differs from E by the presence of a sub-script, a super-script or both) in the same derivation tree does not have to be coincident with the BI structure that the $\mathtt{E}(\Gamma)$ denotes. However, we do - for prevention of inundation of many super-scripts and sub-scripts - make an exception. In the cases where no ambiguity is likely to arise such as in the following;

$$\frac{\Gamma(\mathbb{E}(\Gamma_1; F; G)) \vdash H}{\Gamma(\mathbb{E}(\Gamma_1; F \wedge G)) \vdash H}\ \wedge L$$

we assume that the essence in the conclusion is the same antecedent structure as the essence in the premise(s) except what the inference rule modifies.

3.2 Correspondence between Re_i/Re_j and Γ'

Definition 6 (Relation $\preceq$). *We define a reflexive and transitive binary relation $\preceq$: $\mathfrak{S} \times \mathfrak{S}$ as follows.*

- $\Gamma_1 \preceq \Gamma_2$ *if* $\Gamma_1 = \Gamma_2$.
- $\Gamma(\Gamma_1) \preceq \Gamma(\Gamma_1; \Gamma')$.
- $[\Gamma_1 \preceq \Gamma_2] \wedge^{\dagger} [\Gamma_2 \preceq \Gamma_3] \rightarrow^{\dagger} [\Gamma_1 \preceq \Gamma_3]$.

Intuitively if $\Gamma_1 \preceq \Gamma_2$, then there exists a LBI-derivation:

$$\frac{\Gamma(\Gamma_1) \vdash H}{\Gamma(\Gamma_2) \vdash H}\, WkL$$

for any $\Gamma(-)$ and any H. Here and elsewhere a double line indicates zero or more derivation steps.

Definition 7 (Candidates). *Let Γ be a* BI *structure, then any of the following pairs is a candidate of Γ.*

- $(\Gamma_x, {}^*\top)$ *if* $\Gamma_x \preceq \Gamma$.
- (Γ_x, Γ_y) *if* $\Gamma_x, \Gamma_y \preceq \Gamma$.

We denote the set of candidates of Γ by $\texttt{Candidate}(\Gamma)$.

Now we see the connection between Re_i/Re_j and Γ' in the two rules $*R/\text{-}\!*L$.

Definition 8 (Re_i/Re_j in $*R/\text{-}\!*L$). *In $*R$ and $\text{-}\!*L$, if Γ' is empty,*[6] *$Re_i = {}^*\top$ and Re_j is empty. If it is not empty, then $(Re_1, Re_2) \in \texttt{Candidate}(\Gamma')$.*

Let us reflect on the purposes of the two notations that we have introduced. An essence absorbs a finite number of EA_2 derivation steps. `Candidate` absorbs a finite number of Wk derivation steps. Then what the inference rules in `LBIZ` are doing should be clear. There are no structural rules. Implicit contraction occurs only in $\supset L$ and $\text{-}\!*L$.[7] In both of the inference rules, a structure than a formula duplicates upwards. This is necessary, for we have the following observation.

Observation 1 (Structural contractions are not admissible)
There exist sequents $\Gamma \vdash F$ which are derivable in `LBI` - `Cut` *but not derivable in* `LBI` - `Cut` *without structural contraction.*

Proof. For $\text{-}\!*L$ use a sequent $\top\text{-}\!*p_1, \top\text{-}\!*(p_1 \supset p_2) \vdash p_2$ and assume that every propositional variable is distinct. Then without contraction, there are several derivations. Two sensible ones are shown below (the rest similar). Here and elsewhere we may label a sequent by D with or without a sub-/super-script just so that we may refer to it by the name.

1. $$\dfrac{\dfrac{}{\top\text{-}\!*(p_1 \supset p_2) \vdash \top}\,\top R \qquad p_1 \vdash p_2}{D : \top\text{-}\!*p_1, \top\text{-}\!*(p_1 \supset p_2) \vdash p_2}\,\text{-}\!*L$$

2. $$\dfrac{\dfrac{}{\top\text{-}\!*p_1 \vdash \top}\,\top R \qquad \dfrac{\dfrac{\top \vdash p_1 \qquad \dfrac{}{p_2 \vdash p_2}\,id}{\top; p_1 \supset p_2 \vdash p_2}\,\supset L}{p_1 \supset p_2 \vdash p_2}\,EqAnt_1L}{D : \top\text{-}\!*p_1, \top\text{-}\!*(p_1 \supset p_2) \vdash p_2}\,\text{-}\!*L$$

[6] This case applies to $\text{-}\!*L$ only.

[7] Implicit weakening and others occur also in other inference rules; but they are not very relevant in backward theorem proving.

In both of the derivation trees above, one branch is open. Moreover, such holds true when only formula-level contraction is permitted in `LBI`. The sequent D cannot be derived under the given restriction. In the presence of structural contraction, however, another construction is possible:

$$\dfrac{\dfrac{\Pi(D_1) \qquad \Pi(D_2)}{(\top\mathrel{-\!*}p_1, \top\mathrel{-\!*}(p_1 \supset p_2)); (\top\mathrel{-\!*}p_1, \top\mathrel{-\!*}(p_1 \supset p_2)) \vdash p_2}\,{\mathrel{-\!*}L}}{D : \top\mathrel{-\!*}p_1, \top\mathrel{-\!*}(p_1 \supset p_2) \vdash p_2}\,CtrL$$

where $\Pi(D_1)$ and $\Pi(D_2)$ are:

$\Pi(D_1)$**:**

$$\dfrac{}{\top\mathrel{-\!*}(p_1 \supset p_2) \vdash \top}\,\top R$$

$\Pi(D_2)$**:**

$$\dfrac{\dfrac{}{\top\mathrel{-\!*}p_1 \vdash \top}\,\top R \qquad \dfrac{\dfrac{}{p_1 \vdash p_1}\,id \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\,id}{p_1; p_2 \vdash p_2}\,WkL}{p_1; p_1 \supset p_2 \vdash p_2}\,{\supset L}}{p_1; (\top\mathrel{-\!*}(p_1 \supset p_2)) \vdash p_2}\,{\mathrel{-\!*}L}$$

where all the derivation tree branches are closed upward.

For $\supset L$, use $({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2$. Without structural contraction we have (only two sensible ones are shown; the rest similar):

1.

$$\dfrac{{}^*\top \vdash p_1 \qquad \dfrac{\dfrac{\dfrac{}{p_2 \vdash p_2}\,id}{{}^*\top; p_2 \vdash p_2}\,WkL}{({}^*\top; p_1), ({}^*\top; p_2) \vdash p_2}\,EA_2}{D : ({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,{\supset L}$$

2.

$$\dfrac{\dfrac{p_1 \vdash p_2}{{}^*\top; p_1 \vdash p_2}\,WkL}{D : ({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,EA_2$$

In the presence of structural contraction, there is a closed derivation.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p_1 \vdash p_1}\,id}{{}^*\top; p_1; {}^*\top \vdash p_1}\,WkL \qquad \dfrac{\dfrac{}{p_2 \vdash p_2}\,id}{{}^*\top; p_1; {}^*\top; p_2 \vdash p_2}\,WkL}{{}^*\top; p_1; {}^*\top; p_1 \supset p_2 \vdash p_2}\,{\supset L}}{(({}^*\top; p_1), ({}^*\top; p_1 \supset p_2)); (({}^*\top; p_1), ({}^*\top; p_1 \supset p_2)) \vdash p_2}\,EA_2}{D : ({}^*\top; p_1), ({}^*\top; p_1 \supset p_2) \vdash p_2}\,CtrL$$

4 Main Properties of LBIZ

In this section we show the main properties of LBIZ, *i.e.* admissibility of weakening, that of EA_2, that of both $EqAnt_{1\,\mathtt{LBI}}$ and $EqAnt_{2\,\mathtt{LBI}}$, that of contraction, and its equivalence to LBI. Cut is also admissible. We will refer to derivation depth very often.

Definition 9 (Derivation depth). *By $\Pi(D)$ we denote a derivation tree of a sequent D. We assume that $\Pi(D)$ is always closed: every derivation branch of the tree has an empty sequent as the leaf node (the premise of an axiom). For derivation depth, let $\Pi(D)$ be a derivation tree. Then the derivation depth of D', a node in $\Pi(D)$, is:*

- *1 if D' is the conclusion node of of an axiom inference rule.*
- *1 + (derivation depth of D_1) if $\Pi(D')$ looks like:*

$$\frac{\Pi(D_1)}{D'}$$

- *1 + (the larger of the derivation depths of D_1 and D_2) if $\Pi(D')$ looks like:*

$$\frac{\Pi(D_1) \qquad \Pi(D_2)}{D'}$$

4.1 Weakening admissibility and EA_2 admissibility

Admissibilities of both weakening and EA_2 are proved depth-preserving. This means in case of weakening that if a sequent $\Gamma(\Gamma_1) \vdash H$ is derivable with derivation depth of k, then $\Gamma(\Gamma_1; \Gamma_2) \vdash H$ is derivable with derivation depth of l such that $l \leq k$.

Proposition 1 (LBIZ weakening admissibility). *If a sequent $D : \Gamma(\Gamma_1) \vdash F$ is LBIZ-derivable, then so is $D' : \Gamma(\Gamma_1; \Gamma_2) \vdash F$, preserving the derivation depth.*

Proof. By induction on derivation depth of D. Details are in Appendix A. □

Proposition 2 (Admissibility of EA_2). *If a sequent $D : \Gamma(\Gamma_1) \vdash F$ is LBIZ-derivable, then so is $D' : \Gamma(\mathtt{E}(\Gamma_1)) \vdash F$, preserving the derivation depth.*

Proof. By induction on derivation depth of D. If it is one, *i.e.* D is the conclusion sequent of an axiom, then so is D'. Inductive cases are straightforward due to a near identical proof approach to the weakening admissibility proof (see Appendix A). □

4.2 Inversion lemma

The inversion lemma below is important in simplification of the subsequent discussion.

Lemma 2 (Inversion lemma for LBIZ). *For the following sequent pairs, if the sequent on the left is LBIZ-derivable at most with the derivation depth of k, then so is* *(are) the sequent(s) on the right.*

$$\begin{array}{rl} \Gamma(F \wedge G) \vdash H, & \Gamma(F; G) \vdash H \\ \Gamma(F_1 \vee F_2) \vdash H, & \textit{both } \Gamma(F_1) \vdash H \textit{ and } \Gamma(F_2) \vdash H \\ \Gamma(F * G) \vdash H, & \Gamma(F, G) \vdash H \\ \Gamma(\Gamma_1; \top) \vdash H, & \Gamma(\Gamma_1) \vdash H \\ \Gamma(\Gamma_1, {}^{*}\!\top) \vdash H, & \Gamma(\Gamma_1) \vdash H \\ \Gamma \vdash F \wedge G, & \textit{both } \Gamma \vdash F \textit{ and } \Gamma \vdash G \\ \Gamma \vdash F \supset G, & \Gamma; F \vdash G \\ \Gamma \vdash F \mathrel{-\!\!*} G, & \Gamma, F \vdash G \end{array}$$

Proof. By induction on derivation depth. Details are in Appendix B.

4.3 Admissibility of $EqAnt_{1,2}$

Proposition 3 (Admissibility of $EqAnt_{1,2}$**).** *$EqAnt_{1\,\mathtt{LBI}}$ and $EqAnt_{2\,\mathtt{LBI}}$ are admissible in* $[\mathtt{LBIZ} + EqAnt_{1,2\,\mathtt{LBI}}]$*, preserving the derivation depth.*

Proof. Follows from inversion lemma,[8] Proposition 1 and Proposition 2. □

4.4 Preparation for contraction admissibility in $*R/\mathrel{-\!\!*}L$ cases

We dedicate one subsection here to prepare for the main proof of contraction admissibility. Based on Proposition 1, we make an observation concerning the set of candidates. The discovery, which is to be stated in Proposition 4, led to the solution to the open problem.

Definition 10 (Representing candidates). *Let $\hat{\preceq} : \mathfrak{S} \times \mathfrak{S}$ be a reflexive and transitive binary relation satisfying:*

- $\Gamma_1 \hat{\preceq} \Gamma_2$ *if* $\Gamma_1 = \Gamma_2$.
- $\Gamma_1 \hat{\preceq} \Gamma_1; \Gamma_3$.
- $[\Gamma_1 \hat{\preceq} \Gamma_2] \wedge^{\dagger} [\Gamma_2 \hat{\preceq} \Gamma_3] \to^{\dagger} [\Gamma_1 \hat{\preceq} \Gamma_3]$.
- $\Gamma_1, \Gamma_2 \hat{\preceq} \Gamma_1, (\Gamma_2; \Gamma_3)$.

Now let Γ be a BI *structure. Then any of the following pairs is a representing candidate of Γ.*

- $(\Gamma_x, {}^{*}\!\top)$ *if* $\Gamma_x \hat{\preceq} \Gamma$.
- (Γ_x, Γ_y) *if* $\Gamma_x, \Gamma_y \hat{\preceq} \Gamma$.

We denote the set of representing candidates of Γ by $\mathtt{RepCandidate}(\Gamma)$.

We trivially have that $\mathtt{RepCandidate}(\Gamma) \subseteq \mathtt{Candidate}(\Gamma)$ for any Γ. More can be said.

Proposition 4 (Sufficiency of `RepCandidate`**).** `LBIZ` *with* `RepCandidate` *instead of* `Candidate` *for (Re_1, Re_2) is as expressive as* `LBIZ` *(with* `Candidate`*).*

[8] Inversion lemma proves one direction.

Proof. The only inference rules in LBIZ that use Candidate are $*R$ and $-\!\!*L$. So it suffices to consider only those.

For $*R$, suppose by way of showing contradiction that LBIZ with RepCandidate is not as expressive as LBIZ, then there exists some LBIZ derivation tree $\Pi(D)$:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : Re_i \vdash F_1\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : Re_j \vdash F_2\end{array}}{D : \Gamma' \vdash F_1 * F_2}\,*R$$

such that (Re_1, Re_2) must be in $\texttt{Candidiate}(\Gamma')\backslash\texttt{RepCandidate}(\Gamma')$. Now, without loss of generality assume $(i, j) = (1, 2)$. Then $D'_1 : Re'_i \vdash F_1$ and $D'_2 : Re'_j \vdash F_2$ for $(Re'_i, Re'_j) \in \texttt{RepCandidate}(\Gamma')$ are also LBIZ derivable (by Proposition 1). But this means that we can choose the (Re'_i, Re'_j) for (Re_1, Re_2), a direct contradiction to the supposition. Similarly for $-\!\!*L$. □

Contraction admissibility in LBIZ follows.

Theorem 1 (Contraction admissibility in LBIZ). *If $D : \Gamma(\Gamma_a; \Gamma_a) \vdash F$ is LBIZ-derivable, then so is $D' : \Gamma(\Gamma_a) \vdash F$, preserving the derivation depth.*

Proof. By induction on the derivation depth of D. For an interesting case, we have $*R$. $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : Re_i \vdash F_1\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : Re_j \vdash F_2\end{array}}{D : \Gamma(\Gamma_a; \Gamma_a) \vdash F_1 * F_2}\,*R$$

By Proposition 4, assume that $(Re_1, Re_2) \in \texttt{RepCandidate}(\Gamma(\Gamma_a; \Gamma_a))$ without loss of generality. Then by the definition of $\hat{\preceq}$ it must be that either (1) $\Gamma_a; \Gamma_a$ preserves completely in Re_1 or Re_2, or (2) it remains neither in Re_1 nor in Re_2. If $\Gamma_a; \Gamma_a$ is preserved in Re_1 (or Re_2), then induction hypothesis on the premise that has Re_1 (or Re_2) and then $*R$ conclude; otherwise, it is trivial to see that only a single Γ_a needs to be present in D. Details are in Appendix C.

4.5 Equivalence of LBIZ to LBI

Theorem 2 (Equivalence between LBIZ and LBI). *$D : \Gamma \vdash F$ is LBIZ-derivable if and only if it is LBI-derivable.*

Proof. Into the *only if* direction, assume that D is LBIZ-derivable, and then show that there is a LBI-derivation for each LBIZ derivation. But this is obvious because each LBIZ inference rule is derivable in LBI.[9]

Into the *if* direction, assume that D is LBI-derivable, and then show that there is a corresponding LBIZ-derivation to each LBI derivation by induction on the derivation depth of D. Details are in Appendix D.

[9] Note that EA_2 is LBI-derivable with $WkL_{\texttt{LBI}}$ and $EqAnt_{2\,\texttt{LBI}}$.

4.6 LBIZ Cut Elimination

Cut is admissible in [LBIZ + Cut]. As a reminder (although already stated under Figure 1) Cut is the following rule:

$$\frac{\Gamma_1 \vdash F \qquad \Gamma_2(F) \vdash G}{\Gamma_2(\Gamma_1) \vdash G}\ \mathtt{Cut}$$

Just as in the case of intuitionistic logic, cut admissibility proof for a contraction-free BI sequent calculus is simpler than that for LBI [1]. Since we have already proved depth-preserving weakening admissibility, the following context sharing cut, $\mathtt{Cut}_{CS}$, is easily verified derivable in LBIZ + Cut:

$$\frac{\widetilde{\Gamma_3}; \Gamma_1 \vdash F \qquad \Gamma_2(F; \Gamma_1) \vdash H}{\Gamma_2(\widetilde{\Gamma_3}; \Gamma_1) \vdash H}\ \mathtt{Cut}_{CS}$$

where Γ_1 appears on both of the premises. F in the above cut instance which is upward introduced on both premises is called the cut formula (for the cut instance). The use of $\mathtt{Cut}_{CS}$ is just because it simplifies the cut elimination proof.

For the proof, we recall the standard notations of cut rank and cut level.

Definition 11 (Cut level/rank). *Given a cut instance in a closed derivation:*

$$\frac{D_1 : \Gamma_1 \vdash F \qquad D_2 : \Gamma_2(F) \vdash H}{D_3 : \Gamma_2(\Gamma_1) \vdash H}\ \mathtt{Cut}$$

The level of the cut instance is: $\mathtt{der\_depth}(D_1) + \mathtt{der\_depth}(D_2)$, *where* $\mathtt{der\_depth}(D)$ *denotes derivation depth of* D*. The rank of the cut instance is the size of the cut formula* F, $\mathtt{f\_size}(F)$*, which is defined as follows:*

- *it is 1 if* F *is a nullary logical connective or a propositional variable.*
- *it is* $\mathtt{f\_size}(F_1) + \mathtt{f\_size}(F_2) + 1$ *if* F *is in the form:* $F_1 \bullet F_2$ *for* $\bullet \in \{\wedge, \vee, \supset, *, -\!\!*\}$.

Theorem 3 (Cut admissibility in LBIZ). Cut *is admissible within* LBIZ + Cut.

Proof. By induction on the cut rank and a sub-induction on the cut level, by making use of $\mathtt{Cut}_{CS}$. Details are in Appendix E. □

5 Conclusion

We solved an open problem of structural rule absorption in BI sequent calculus. This problem stood unsolved for a while. As far back as we can see, the first attempt was made in [9]. References to the problem were subsequently made [2,3,5]. The work that came closest to ours is one by Donnelly *et al.* [3]. They consider weakening absorption in the context of forward theorem proving (where weakening than contraction is a source of non-termination). One inconvenience in their approach, however, is that the effect of weakening is not totally isolated from that of contraction: it is absorbed into contraction as well as into logical rules. But then structural weakening is still possible through the new structural contraction. Also, the coupling of the two structural rules amplifies the difficulty of analysis on the behaviour of contraction. Further, their work is on a subset of BI without units. In comparison, our solution covers the whole BI. And our analysis fully decoupled the effect of structural weakening from the effect of structural contraction. LBIZ comes with no structural rules, in fact. Techniques we used in this work should be useful for deriving a contraction-free sequent calculus of other non-classical logics coming with a non-formula contraction. There are also more recent BI extensions in sequent calculus such as [8], to which this work has relevance.

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Appendix A: Proof of Proposition 1

By induction on derivation depth of D. If it is one, *i.e.* D is the conclusion sequent of an axiom, then so is D'. For inductive cases, assume that the current proposition holds for all the derivations of depth up to k. It must be now demonstrated that it still holds for derivations of depth $k+1$. Consider what the last inference rule is in $\Pi(D)$.

1. $\supset L$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ \mathtt{E}(\widetilde{\Gamma_1}; F{\supset}G) \vdash F\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma(G; \mathtt{E}(\widetilde{\Gamma_1}; F{\supset}G)) \vdash H\end{array}}{\Gamma(\mathtt{E}(\widetilde{\Gamma_1}; F{\supset}G)) \vdash H} \supset L$$

   By induction hypothesis on both of the premises, $\mathtt{E}'(\widetilde{\Gamma_1'}; F{\supset}G) \vdash F$ and $\Gamma'(G; \mathtt{E}'(\widetilde{\Gamma_1'}; F{\supset}G)) \vdash H$ are both LBIZ-derivable. Here we assume that:
   $\mathtt{E}'(\widetilde{\Gamma_1'}; F \supset G) \preceq \mathtt{E}(\widetilde{\Gamma_1'}; F \supset G) \preceq \mathtt{E}(\widetilde{\Gamma_1}; F \supset G)$, and $\Gamma'(-) \preceq \Gamma(-)$.
   Then $\Gamma'(\mathtt{E}'(\widetilde{\Gamma_1'}; F \supset G)) \vdash H$ is also LBIZ-derivable via $\supset L$.
2. $-\!\!*L$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ Re_i \vdash F\end{array} \qquad \begin{array}{c}\vdots\\ \Gamma((\widetilde{Re_j}, G); (\widetilde{\Gamma_1}, \mathbb{E}(\widetilde{\Gamma_2}; F{-\!\!*}G))) \vdash H\end{array}}{\Gamma(\widetilde{\Gamma_1}, \mathbb{E}(\widetilde{\Gamma_2}; F{-\!\!*}G)) \vdash H} -\!\!*L$$

   Assume that $\widetilde{\Gamma_1'} \preceq \widetilde{\Gamma_1}$ and that $\mathtt{E}'(\widetilde{\Gamma_2'}; F{-\!\!*}G) \preceq \mathtt{E}(\widetilde{\Gamma_2'}; F{-\!\!*}G) \preceq \mathtt{E}(\widetilde{\Gamma_2}; F{-\!\!*}G)$. Then by induction hypothesis on the right premise sequent,
   $\Gamma'((\widetilde{Re_j}, G); (\widetilde{\Gamma_1'}, \mathtt{E}'(\widetilde{\Gamma_2'}; F{-\!\!*}G))) \vdash H$ is LBIZ-derivable.
   Then $\Gamma'(\widetilde{\Gamma_1'}, \mathtt{E}'(\widetilde{\Gamma_2'}; F{-\!\!*}G)) \vdash H$ is also LBIZ-derivable via $-\!\!*L$.
3. Other cases are simpler and similar. □

Appendix B: Proof of Lemma 2

By induction on the derivation depth k. We abbreviate $(\Gamma(\Gamma_1))(\Gamma_2)$ by $\Gamma(\Gamma_1)(\Gamma_2)$. And we also do not explicitly show a tilde on top of a possibly empty structure.

1. For a LBIZ sequent $\Gamma(F \wedge G) \vdash H$, the base case is when it is an axiom, and the proof is trivial. For inductive cases, assume that the statement holds true for all the derivation depths up to k, and show that it still holds true at $k+1$. Consider what the last inference rule applied is.
   (a) $\vee L$: The derivation ends in:

$$\dfrac{\Gamma(F \wedge G)(F_1) \vdash H \qquad \Gamma(F \wedge G)(F_2) \vdash H}{\Gamma(F \wedge G)(F_1 \vee F_2) \vdash H} \vee L$$

   By induction hypothesis, both $\Gamma(F; G)(F_1) \vdash H$ and $\Gamma(F; G)(F_2) \vdash H$ are LBIZ-derivable. Then $\Gamma(F; G)(F_1 \vee F_2) \vdash H$ as required via $\vee L$.
   (b) $\wedge L$: Similar, or trivial when the principal should coincide with $F \wedge G$.

(c) $\supset L$: The derivation ends in one of the following:

$$\dfrac{\mathbb{E}(\Gamma_1(F \wedge G); F_1 \supset G_1) \vdash F_1 \qquad \Gamma(G_1; \mathbb{E}(\Gamma_1(F \wedge G); F_1 \supset G_1)) \vdash H}{\Gamma(\mathbb{E}(\Gamma_1(F \wedge G); F_1 \supset G_1)) \vdash H} \supset L$$

$$\dfrac{\mathbb{E}(\Gamma_1'; F_1 \supset G_1) \vdash F_1 \qquad \Gamma'(F \wedge G)(G_1; \mathbb{E}(\Gamma_1'; F_1 \supset G_1)) \vdash H}{\Gamma'(F \wedge G)(\mathbb{E}(\Gamma_1'; F_1 \supset G_1)) \vdash H} \supset L$$

By induction hypothesis, both $\mathbb{E}(\Gamma_1(F; G); F_1 \supset G_1) \vdash F_1$ and $\Gamma(G_1; \mathbb{E}(\Gamma_1(F; G); F_1 \supset G_1)) \vdash H$ in case the former, or $\Gamma'(F; G)(G_1; \mathbb{E}(\Gamma_1'; F_1 \supset G_1)) \vdash H$ in case the latter.
Then $\supset L$ (with the untouched left premise if the latter) produces the required result.

(d) $*L$: The derivation ends in:

$$\dfrac{\Gamma(F \wedge G)(F_1, G_1) \vdash H}{\Gamma(F \wedge G)(F_1 * G_1) \vdash H} *L$$

By induction hypothesis, $\Gamma(F; G)(F_1, G_1) \vdash H$. Then, $\Gamma(F; G)(F_1 * G_1) \vdash H$ as required via $*L$.

(e) $-\!\!*L$: The derivation ends in one of the following, depending on the location at which $F \wedge G$ appears. In the below inference steps, we assume that the particular formula $F \wedge G$ occurs in $Re_{(i,j)}(F \wedge G)$ as the focused substructure, but not in $Re_{(i,j)}$.[10]

$$\dfrac{Re_i \vdash F_1 \qquad \Gamma((Re_j, G_1); (\Gamma', \mathbb{E}(\Gamma_1(F \wedge G); F_1 -\!\!* G_1))) \vdash H}{\Gamma((\Gamma', \mathbb{E}(\Gamma_1(F \wedge G); F_1 -\!\!* G_1))) \vdash H}$$

$$\dfrac{Re_i \vdash F_1 \qquad \Gamma((Re_j, G_1); (\Gamma'(F \wedge G), \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma((\Gamma'(F \wedge G), \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}$$

$$\dfrac{Re_i(F \wedge G) \vdash F_1 \qquad \Gamma((Re_j, G_1); (\Gamma'(F \wedge G), \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma((\Gamma'(F \wedge G), \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}$$

$$\dfrac{Re_i \vdash F_1 \qquad \Gamma((R_j(F \wedge G), G_1); (\Gamma'(F \wedge G), \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma((\Gamma'(F \wedge G), \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}$$

$$\dfrac{Re_i \vdash F_1 \qquad \Gamma(F \wedge G)((Re_j, G_1); (\Gamma', \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}{\Gamma(F \wedge G)((\Gamma', \mathbb{E}(\Gamma_1; F_1 -\!\!* G_1))) \vdash H}$$

$$\dfrac{Re_i \vdash F_1 \qquad \Gamma((Re_j, G_1); \mathbb{E}(\Gamma_2(F \wedge G)(\Gamma', (\Gamma_1; F_1 -\!\!* G_1)))) \vdash H}{\Gamma(\mathbb{E}(\Gamma_2(F \wedge G)(\Gamma', (\Gamma_1; F_1 -\!\!* G_1)))) \vdash H}$$

For each, the required sequent results from induction hypothesis for the particular occurrences of $F \wedge G$ on both of the premises, and then $-\!\!*L$ to recover Γ' (or $\Gamma'(F; G)$) such that $(Re_i, Re_j) \in \texttt{Candidate}(\Gamma')$ (or $\texttt{Candidate}(\Gamma'(F; G))$).

[10] Note, however, that this does not preclude occurrences of $F \wedge G$ in case it occurs multiple times in the conclusion sequent.

(f) $\wedge R$: Similar to $\vee L$ in approach but simpler.
(g) $\vee R$: Similar.
(h) $\supset R$: The derivation ends in:

$$\frac{\Gamma(F \wedge G); F_1 \vdash G_1}{\Gamma(F \wedge G) \vdash F_1 \supset G_1} \supset R$$

By induction hypothesis, $\Gamma(F; G); F_1 \vdash G_1$. Then, $\Gamma(F; G) \vdash F_1 \supset G_1$ as required via $\supset R$.

(i) $*R$: The derivation ends in one of the below:

$$\frac{Re_i \vdash F_1 \qquad Re_j \vdash G_1}{\Gamma'(F \wedge G) \vdash F_1 * G_1}$$

$$\frac{Re_i(F \wedge G) \vdash F_1 \qquad Re_j \vdash G_1}{\Gamma'(F \wedge G) \vdash F_1 * G_1}$$

$$\frac{Re_i \vdash F_1 \qquad Re_j(F \wedge G) \vdash G_1}{\Gamma'(F \wedge G) \vdash F_1 * G_1}$$

Trivial for the first case. For the second, induction hypothesis on the left premise sequent produces $Re_i(F; G) \vdash F_1$.
Then $*R$ such that $(Re_i(F; G), Re_j) \in \texttt{Candidate}(\Gamma'(F; G))$. Similarly for the third case.

(j) $-\!\!*R$: Trivial.

2. A LBIZ sequent $\Gamma(F \vee G) \vdash H$: similar.
3. For a LBIZ sequent $\Gamma(F * G) \vdash H$, the base case is when it is an axiom for which a proof is trivially given. For inductive cases, assume that it holds true for all the derivation depths up to k and show that the same still holds for the derivation depth of $k + 1$. Consider what the last inference rule is.
   (a) $*L$: Trivial if the principal coincides with $F*G$. Otherwise, the derivation looks like:

$$\frac{\Gamma(F * G)(F_1, G_1) \vdash H}{\Gamma(F * G)(F_1 * G_1) \vdash H} *L$$

   By induction hypothesis, $\Gamma(F, G)(F_1, G_1) \vdash H$. Then, $\Gamma(F, G)(F_1 * G_1) \vdash H$ as desired via $*L$.
   (b) The rest: Similar to the previous cases.
4. For a LBIZ sequent $D : \Gamma(\Gamma_1, {}^*\top) \vdash H$, the base case is when it is the conclusion sequent of an axiom.
   (a) id: $D : \mathtt{E}(\Gamma'(\Gamma_1, {}^*\top); p) \vdash p$. Then $D' : \mathtt{E}(\Gamma'(\Gamma_1); p) \vdash p$ is also an axiom.
   (b) $\bot L$, $\top R$: straightforward.
   (c) ${}^*\top R$: similar to id case.

   For inductive cases, assume that the statement holds true for all the derivation depths up to k, and show that it still holds true at $k + 1$. Consider what the last inference rule applied is.
   (a) $\vee L$: The derivation ends in one of the following:

$$\frac{\Gamma(\Gamma_1,{}^*\top)(F_1) \vdash H \qquad \Gamma(\Gamma_1,{}^*\top)(F_2) \vdash H}{\Gamma(\Gamma_1,{}^*\top)(F_1 \vee F_2) \vdash H} \vee L$$

$$\frac{\Gamma(\Gamma_1(F_1),{}^*\top) \vdash H \qquad \Gamma(\Gamma_1(F_2),{}^*\top) \vdash H}{\Gamma(\Gamma_1(F_1 \vee F_2),{}^*\top) \vdash H} \vee L$$

For the former, $\Gamma(\Gamma_1)(F_1) \vdash H$ and $\Gamma(\Gamma_1)(F_2) \vdash H$ (induction hypothesis); then $\Gamma(\Gamma_1)(F_1 \vee F_2) \vdash H$ via $\vee L$ as required. For the latter, $\Gamma(\Gamma_1(F_1)) \vdash H$ and $\Gamma(\Gamma_1(F_2)) \vdash H$ (induction hypothesis); then $\Gamma(\Gamma_1(F_1 \vee F_2)) \vdash H$ via $\vee L$ as required.

(b) $\supset L$: The derivation ends in one of the following:

$$\frac{\mathtt{E}(\Gamma_1; F_1 \supset F_2) \vdash F_1 \qquad \Gamma(F_2; \mathtt{E}(\Gamma_1; F_1 \supset F_2))(\Gamma_2,{}^*\top) \vdash H}{\Gamma(\mathtt{E}(\Gamma_1; F_1 \supset F_2))(\Gamma_2,{}^*\top) \vdash H} \supset L$$

$$\frac{\mathtt{E}(\Gamma_1(\Gamma_2,{}^*\top); F_1 \supset F_2) \vdash F_1 \qquad \Gamma(F_2; \mathtt{E}(\Gamma_1(\Gamma_2,{}^*\top); F_1 \supset F_2)) \vdash H}{\Gamma(\mathtt{E}(\Gamma_1(\Gamma_2,{}^*\top); F_1 \supset F_2)) \vdash H} \supset L$$

$$\frac{\mathtt{E}(\Gamma_1; F_1 \supset F_2) \vdash F_1 \qquad \Gamma(\Gamma_2(F_2; \mathtt{E}(\Gamma_1; F_1 \supset F_2)),{}^*\top) \vdash H}{\Gamma(\Gamma_2(\mathtt{E}(\Gamma_1; F_1 \supset F_2)),{}^*\top) \vdash H} \supset L$$

For the first, $\Gamma(F_2; \mathtt{E}(\Gamma_1; F_1 \supset F_2))(\Gamma_2) \vdash H$ (induction hypothesis); then $\Gamma(\mathtt{E}(\Gamma_1; F_1 \supset F_2))(\Gamma_2) \vdash H$ via $\supset L$ as required.
For the second, $\mathtt{E}(\Gamma_1(\Gamma_2); F_1 \supset F_2) \vdash F_1$ and $\Gamma(\mathtt{E}(\Gamma_1(\Gamma_2); F_1 \supset F_2)) \vdash H$ (induction hypothesis); then $\Gamma(\mathtt{E}(\Gamma_1(\Gamma_2); F_1 \supset F_2)) \vdash H$ via $\supset L$ as required.
For the third, induction hypothesis on the right premise sequent, then $\supset L$ to conclude.

(c) $-\!\!*L$: Suppose the derivation ends in one of the following:

$$\frac{Re_i \vdash F \qquad \Gamma((Re_j, G); (\Gamma_2, \mathbb{E}(\Gamma_3(\Gamma_1,{}^*\top); F -\!\!* G))) \vdash H}{\Gamma((\Gamma_2, \mathbb{E}(\Gamma_3(\Gamma_1,{}^*\top); F -\!\!* G))) \vdash H} -\!\!*L$$

$$\frac{Re_i \vdash F_1 \qquad \Gamma(\Gamma_1,{}^*\top)((Re_j, G); (\Gamma_2, \mathbb{E}(\Gamma_3; F -\!\!* G))) \vdash H}{\Gamma(\Gamma_1,{}^*\top)(\Gamma_2, \mathbb{E}(\Gamma_3; F -\!\!* G)) \vdash H} -\!\!*L$$

$$\frac{Re_i \vdash F_1 \qquad \Gamma(\Gamma_1((Re_j, G); (\Gamma_2, \mathbb{E}(\Gamma_3; F -\!\!* G)),{}^*\top)) \vdash H}{\Gamma(\Gamma_1((\Gamma_2, \mathbb{E}(\Gamma_3; F -\!\!* G)),{}^*\top)) \vdash H} -\!\!*L$$

For each of the above, induction hypothesis, if applicable, and $-\!\!*L$ conclude. Now consider other cases where the ${}^*\top$ occurs in the conclusion sequent as $\Gamma(\mathtt{E}(\Gamma_2(\Gamma_1,{}^*\top), (\Gamma_3; F -\!\!* G))) \vdash H$. Less involved cases are when "$\Gamma_1,{}^*\top$" is entirely retained or entirely discarded upwards:

$$\frac{Re_i(\Gamma_1,{}^*\top) \vdash F \qquad \Gamma((Re_j, G); (\Gamma_2(\Gamma_1,{}^*\top), \mathbb{E}(\Gamma_3; F -\!\!* G))) \vdash H}{\Gamma((\Gamma_2(\Gamma_1,{}^*\top), \mathbb{E}(\Gamma_3; F -\!\!* G))) \vdash H} -\!\!*L$$

$$\frac{Re_i \vdash F \qquad \Gamma((Re_j(\Gamma_1, {}^*\top), G); (\Gamma_2(\Gamma_1, {}^*\top), \mathbb{E}(\Gamma_3; F {-\!\!*} G))) \vdash H}{\Gamma((\Gamma_2(\Gamma_1, {}^*\top), \mathbb{E}(\Gamma_3; F {-\!\!*} G))) \vdash H} {-\!\!*} L$$

$$\frac{Re_i \vdash F \qquad \Gamma((Re_j, G); (\Gamma_2(\Gamma_1, {}^*\top), \mathbb{E}(\Gamma_3; F {-\!\!*} G))) \vdash H}{\Gamma((\Gamma_2(\Gamma_1, {}^*\top), \mathbb{E}(\Gamma_3; F {-\!\!*} G))) \vdash H} {-\!\!*} L$$

The first assumes that the specific "$\Gamma_1, {}^*\top$" does not occur in Re_j; the second that it does not occur in Re_i; the third that it does not occur in Re_i or in Re_j. Each of them is concluded via induction hypothesis and then ${-\!\!*}L$.
Finally, if "$\Gamma_1, {}^*\top$" should be split between the two premises, Re_j is the ${}^*\top$, in which case we have on the right premise sequent:
$\Gamma(({}^*\top, G); (\Gamma_2(\Gamma_1, {}^*\top), \mathtt{E}(\Gamma_3; F {-\!\!*} G))) \vdash H$.
In this case we apply induction hypothesis and obtain
$\Gamma(({}^*\top, G); (\Gamma_2(\Gamma_1), \mathtt{E}(\Gamma_3; F {-\!\!*} G))) \vdash H$.
By the definition of a candidate, however, we have from the sequent that
$\Gamma(\Gamma_2(\Gamma_1), \mathtt{E}(\Gamma_3; F {-\!\!*} G)) \vdash H$
is LBIZ-derivable, as required.

(d) The rest: similar or straightforward.

5. The rest: similar or straightforward.

Appendix C: Proof of Theorem 1

By induction on derivation depth. The base cases are when it is 1, *i.e.* when D is the conclusion sequent of an axiom. Consider which axiom has applied. If it is $\top R$, then it is trivial to show that if $\Gamma(\Gamma_a; \Gamma_a) \vdash \top$, then so is $\Gamma(\Gamma_a) \vdash \top$. Also for $\bot L$, a single occurrence of $\bot$ on the antecedent part of D suffices for the $\bot L$ application, and the current theorem is trivially provable in this case, too. For both id and ${}^*\top R$, $\Pi(D)$ looks like:

$$\frac{}{\mathtt{E}(\widetilde{\Gamma_1}; \alpha) \vdash \alpha}$$

where α is $p \in \mathcal{P}$ for id, ${}^*\top$ for ${}^*\top R$ and $\Gamma(\Gamma_a; \Gamma_a) = \mathtt{E}(\widetilde{\Gamma_1}; \alpha)$. If α is not a sub-structure of either of the occurrences of Γ_a, then D' is trivially derivable. Otherwise, assume that the focused α in $\mathtt{E}(\widetilde{\Gamma_1}; \alpha)$ is a sub-structure of one of the occurrences of Γ_a in $\Gamma(\Gamma_a; \Gamma_a)$. Then there exists some Γ_2 and $\widetilde{\Gamma_3}$ such that $\mathtt{E}(\widetilde{\Gamma_1}; \alpha) = \mathtt{E}(\Gamma_2; \widetilde{\Gamma_3}; \alpha) = \mathtt{E}_1(\Gamma_2); \mathtt{E}_2(\widetilde{\Gamma_3}; \alpha)$ and that Γ_a is an essence of $\widetilde{\Gamma_3}; \alpha$. But then $D' : \Gamma(\Gamma_a)$ is still an axiom.

For inductive cases, suppose that the current theorem holds true for any derivation depth of up to k. We must demonstrated that it still holds for the derivation depth of $k + 1$. Consider what the LBIZ inference rule applied last is, and, in case of a left inference rule, consider where the active structure Γ_b of the inference rule is in $\Gamma(\Gamma_a; \Gamma_a)$.

1. $\wedge L$, and Γ_b is $F_1 \wedge F_2$: if Γ_b does not appear in Γ_a, induction hypothesis on the premise sequent concludes. Otherwise, $\Pi(D)$ looks like:

$$\frac{\vdots \\ D_1 : \Gamma(\Gamma'_a(F_1; F_2); \Gamma'_a(F_1 \wedge F_2)) \vdash H}{D : \Gamma(\Gamma'_a(F_1 \wedge F_2); \Gamma'_a(F_1 \wedge F_2)) \vdash H} \wedge L$$

$D_1' : \Gamma(\Gamma_a'(F_1; F_2); \Gamma_a'(F_1; F_2)) \vdash H$ is LBIZ-derivable (inversion lemma); D_1'' : $\Gamma(\Gamma_a'(F_1; F_2)) \vdash H$ is also LBIZ-derivable (induction hypothesis); then $\wedge L$ on D_1'' concludes.

2. $\supset L$, and Γ_b is $\mathtt{E}(\widetilde{\Gamma'}; F \supset G)$: if Γ_b does not appear in Γ_a, then the induction hypothesis on both of the premises concludes. If it is entirely in Γ_a, then $\Pi(D)$ looks either like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ D_1 : \mathtt{E}(\widetilde{\Gamma'}; F \supset G) \vdash F & D_2\end{array}}{D : \Gamma(\Gamma_a'(\mathtt{E}(\widetilde{\Gamma'}; F \supset G)); \widetilde{\Gamma_a'}(\mathtt{E}(\widetilde{\Gamma'}; F \supset G))) \vdash H} \supset L$$

where $D_2 : \Gamma(\Gamma_a'(G; \mathtt{E}(\widetilde{\Gamma'}; F{\supset}G)); \Gamma_a'(\mathtt{E}(\widetilde{\Gamma'}; F{\supset}G))) \vdash H$, or, in case Γ_a is $\Gamma_a'; F{\supset}G$, like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ D_1 : \Gamma_a'; F{\supset}G; \Gamma_a'; F{\supset}G \vdash F & D_2\end{array}}{D : \Gamma(\Gamma_a'; F{\supset}G; \Gamma_a'; F{\supset}G) \vdash H} \supset L$$

where $D_2 : \Gamma(G; \Gamma_a'; F{\supset}G; \Gamma_a'; F{\supset}G) \vdash H$.
In the former case,
$D_2' : \Gamma(\Gamma_a'(G; \mathtt{E}(\widetilde{\Gamma'}; F{\supset}G)); \Gamma_a'(G; \mathtt{E}(\widetilde{\Gamma'}; F{\supset}G))) \vdash H$ (weakening admissibility);
$D_2'' : \Gamma(\Gamma_a'(G; \mathtt{E}(\widetilde{\Gamma'}; F{\supset}G))) \vdash H$ (induction hypothesis);
then $\supset L$ on D_1 and D_2'' concludes. In the latter, induction hypothesis on D_1 and on D_2; then via $\supset L$ for a conclusion. Finally, if only a substructure of Γ_b is in Γ_a with the rest spilling out of Γ_a, then if the principal formula $F \supset G$ does not occur in Γ_a, then straightforward; otherwise similar to the latter case.

3. $*R$: $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ D_1 : Re_i \vdash F_1 & D_2 : Re_j \vdash F_2\end{array}}{D : \Gamma(\Gamma_a; \Gamma_a) \vdash F_1 * F_2} *R$$

By Proposition 4, assume that $(Re_1, Re_2) \in \mathtt{RepCandidate}(\Gamma(\Gamma_a; \Gamma_a))$ without loss of generality. Then by the definition of $\hat{\preceq}$ it must be that either (1) $\Gamma_a; \Gamma_a$ preserves completely in Re_1 or Re_2, or (2) it remains neither in Re_1 nor in Re_2. If $\Gamma_a; \Gamma_a$ is preserved in Re_1 (or Re_2), then induction hypothesis on the premise that has Re_1 (or Re_2) and then $*R$ conclude; otherwise, it is trivial to see that only a single Γ_a needs to be present in D.

4. $-\!*L$, and Γ_b is $\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!*}G)$: if Γ_b is not in Γ_a, then induction hypothesis on the right premise sequent concludes. If it is in Γ_a, $\Pi(D)$ looks like:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ D_1 : Re_i \vdash F & D_2\end{array}}{D : \Gamma(\Gamma_a'(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!*}G)); \Gamma_a'(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!*}G))) \vdash H} -\!*L_1$$

where D_2 is:

$$\Gamma(\Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G))); \Gamma'_a(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G))) \vdash H$$

$D'_2 : \Gamma(\Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G))); \Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G)))) \vdash H$ via Proposition 1 is also `LBIZ`-derivable. $D''_2 : \Gamma(\Gamma'_a((\widetilde{Re_j}, G); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_1}; F{-\!\!*}G)))) \vdash H$ via induction hypothesis. Then ${-\!\!*}L$ on D_1 and D''_2 concludes. If, on the other hand, Γ_a is in Γ_b, then it is either in Γ_1 or in Γ'. But if it is in Γ_1, then it must be weakened away, and if it is in Γ', similar to the $*R$ case.

5. Other cases are similar to one of the cases already examined.

Appendix D: Proof of Theorem 2

Into the *only if* direction, assume that D is `LBIZ`-derivable, and then show that there is a `LBI`-derivation for each `LBIZ` derivation. But this is obvious because each `LBIZ` inference rule is derivable in `LBI`.[11]

Into the *if* direction, assume that D is `LBI`-derivable, and then show that there is a corresponding `LBIZ`-derivation to each `LBI` derivation by induction on the derivation depth of D.

If it is 1, *i.e.* if D is the conclusion sequent of an axiom, we note that $\bot L_{\text{LBI}}$ is identical to $\bot L_{\text{LBIZ}}$; id_{LBI} and $^*\!\top R_{\text{LBI}}$ via id_{LBIZ} and resp. $^*\!\top R_{\text{LBIZ}}$ with Proposition 1 and Proposition 2; and $\top R_{\text{LBI}}$ is identical to $\top R_{\text{LBIZ}}$. For inductive cases, assume that the *if* direction holds true up to the `LBI`-derivation depth of k, then it must be demonstrated that it still holds true for the `LBI`-derivation depth of $k+1$. Consider what the `LBI` rule applied last is:

1. $\supset L_{\text{LBI}}$: $\Pi_{\text{LBI}}(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \Gamma_1 \vdash F\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : \Gamma(\Gamma_1; G) \vdash H\end{array}}{D : \Gamma(\Gamma_1; F \supset G) \vdash H} \supset L_{\text{LBI}}$$

By induction hypothesis, both D_1 and D_2 are also `LBIZ`-derivable. Proposition 1 on D_1 in `LBIZ`-space results in $D'_1 : \Gamma_1; F \supset G \vdash F$, and on D_2 results in $D'_2 : \Gamma(\Gamma_1; G; F \supset G) \vdash H$. Then an application of $\supset L_{\text{LBIZ}}$ on D'_1 and D_2 concludes in `LBIZ`-space.

2. ${-\!\!*}L_{\text{LBI}}$: $\Pi_{\text{LBI}}(D)$ looks like:

$$\dfrac{\begin{array}{c}\vdots\\ D_1 : \Gamma_1 \vdash F\end{array} \qquad \begin{array}{c}\vdots\\ D_2 : \Gamma(G) \vdash H\end{array}}{D : \Gamma(\Gamma_1, F{-\!\!*}G) \vdash H} {-\!\!*}L_{\text{LBI}}$$

By induction hypothesis, D_1 and D_2 are also `LBIZ`-derivable.

(a) If $\Gamma(G)$ is G, *i.e.* if the antecedent part of D_2 is a formula (G), then Proposition 1 on D_2 results in $D'_2 : G; (\Gamma_1, F{-\!\!*}G) \vdash H$ in `LBIZ`-space. Then ${-\!\!*}L_{\text{LBIZ}}$ on D_1 and D'_2 leads to $D' : \Gamma_1, F{-\!\!*}G \vdash H$ as required.

[11] Note that EA_2 is `LBI`-derivable with WkL_{LBI} and $EqAnt_{2\,\text{LBI}}$.

(b) If $\Gamma(G)$ is $\Gamma'(\Gamma'', G)$, then Proposition 1 on D_2 leads to $D_2' : \Gamma'((\Gamma'', G); (\Gamma'', \Gamma_1, F{-\!*}G)) \vdash H$. Then ${-\!*}L_{\texttt{LBIZ}}$ on D_1 and D_2' leads to $D' : \Gamma'(\Gamma'', \Gamma_1, F{-\!*}G) \vdash H$ as required.

(c) Finally, if $\Gamma(G)$ is $\Gamma'(\Gamma''; G) \vdash H$, then Proposition 1 on D_2 leads to $D_2' : \Gamma'(\Gamma''; G; (\Gamma_1, F{-\!*}G)) \vdash H$. Then ${-\!*}L_{\texttt{LBIZ}}$ on D_1 and D_2' leads to $D' : \Gamma'(\Gamma''; (\Gamma_1, F{-\!*}G)) \vdash H$ as required.

3. $WkL_{\texttt{LBI}}$: Proposition 1.
4. $CtrL_{\texttt{LBI}}$: Theorem 1.
5. $EqAnt_{1\,\texttt{LBI}}$: Proposition 3.
6. $EqAnt_{2\,\texttt{LBI}}$: Proposition 3.
7. The rest: straightforward.

Appendix E: Proof of Theorem 3

By induction on the cut rank and a sub-induction on the cut level, by making use of $\texttt{Cut}_{CS}$. In this proof (X, Y) denotes, for some LBIZ inference rules X and Y, that one of the premises has been just derived with X and the other with Y. As before, $\Gamma(\Gamma_1)(\Gamma_2)$ abbreviates $(\Gamma(\Gamma_1))(\Gamma_2)$. In the pairs of derivations, the first is the derivation tree to be permuted and the second is the permuted derivation tree.

(id, id)**:**

1.

$$\dfrac{\dfrac{}{\texttt{E}(\widetilde{\Gamma_1}; p) \vdash p}\,id \qquad \dfrac{}{\texttt{E}'(\widetilde{\Gamma_2}; p) \vdash p}\,id}{\texttt{E}'(\widetilde{\Gamma_2}; \texttt{E}(\widetilde{\Gamma_1}; p)) \vdash p}\,\texttt{Cut}$$

$\Rightarrow$

$$\dfrac{}{\texttt{E}'(\widetilde{\Gamma_2}; \texttt{E}(\widetilde{\Gamma_1}; p)) \vdash p}\,id$$

Of course, for the above permutation to be correct, we must be able to demonstrate the fact that the antecedent structure is $\texttt{E}''(\widetilde{\Gamma_2}; \widetilde{\Gamma_1}; p)$ such that $[\texttt{E}''(\widetilde{\Gamma_2}; \widetilde{\Gamma_1}; p)] = [\texttt{E}'(\widetilde{\Gamma_2}; \texttt{E}(\widetilde{\Gamma_1}; p))]$. But note that it only takes a finite number of (backward) EA_2 applications (*Cf.* Proposition 2) on $\widetilde{\Gamma_2}; \texttt{E}(\widetilde{\Gamma_1}; p) \vdash p$ to upward derive $\widetilde{\Gamma_2}; \widetilde{\Gamma_1}; p \vdash p$. The implication is that, since $\widetilde{\Gamma_2}; \texttt{E}(\widetilde{\Gamma_1}; p) \vdash p$ results upward from $\texttt{E}'(\widetilde{\Gamma_2}; \texttt{E}(\widetilde{\Gamma_1}; p)) \vdash p$ also in a finite number of backward EA_2 applications, the antecedent structure must be in the form: $\texttt{E}''(\widetilde{\Gamma_2}; \widetilde{\Gamma_1}; p)$.

2.

$$\dfrac{\dfrac{}{\texttt{E}(\widetilde{\Gamma_1}; p) \vdash p}\,id \qquad \dfrac{}{\texttt{E}'(\Gamma_2(p); q) \vdash q}\,id}{\texttt{E}'(\Gamma_2(\texttt{E}(\widetilde{\Gamma_1}; p)); q) \vdash q}\,\texttt{Cut}$$

Other patterns for which one of the premises is an axiom sequent are straightforward.

For the rest, if the cut formula is principal only for one of the premise sequents, then we follow the routine [12] to permute up the other premise sequent for which it is the principal. For example, in case we have the derivation pattern below:

$$\dfrac{\dfrac{D_1 \qquad D_2}{D_5 : \Gamma_1(H_1 \vee H_2) \vdash F_1 \supset F_2} \vee L \qquad \dfrac{D_3 : \mathbb{E}(\widetilde{\Gamma_3}; F_1 \supset F_2) \vdash F_1 \qquad D_4 : \Gamma_2(F_2; \mathbb{E}(\widetilde{\Gamma_3}; F_1 \supset F_2)) \vdash H}{D_6 : \Gamma_2(\mathbb{E}(\widetilde{\Gamma_3}; F_1 \supset F_2)) \vdash H} \supset L}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1(H_1 \vee H_2))) \vdash H} \texttt{Cut}$$

where $D_1 : \Gamma_1(H_1) \vdash F_1 \supset F_2$ and $D_2 : \Gamma_1(H_2) \vdash F_1 \supset F_2$. The cut formula $F_1 \supset F_2$ is not the principal on the left premise. In this case, we simply apply Cut on the pairs: (D_1, D_6) and (D_2, D_6), to conclude:

$$\dfrac{\dfrac{D_1 \qquad D_6}{\Gamma_2(\mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1(H_1))) \vdash H} \texttt{Cut} \qquad \dfrac{D_2 \qquad D_6}{\Gamma_2(\mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1(H_2))) \vdash H} \texttt{Cut}}{\Gamma_2(\mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1(H_1 \vee H_2))) \vdash H} \vee L$$

Of course, for this particular permutation to be correct, we must be able to demonstrate, in the permuted derivation tree, that $\mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1(H_1 \vee H_2)) = \mathtt{E}'(\widetilde{\Gamma_3}) \star \Gamma_1(H_1 \vee H_2)$ with $\star$ either a semi-colon or a comma, that $\mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1(H_1)) = \mathtt{E}'(\widetilde{\Gamma_3}) \star \Gamma_1(H_1)$, and that $\mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1(H_2)) = \mathtt{E}'(\widetilde{\Gamma_3}) \star \Gamma_1(H_2)$. But this is vacuous since the cut formula which is replaced with the structure $\Gamma_1(H_1)$ or $\Gamma_1(H_2)$ is a formula.

Cases that remain are those for which both premises of the cut instance have the cut formula as the principal. We go through each to conclude the proof.

($\wedge L, \wedge R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_1 \qquad D_2 : \Gamma_1 \vdash F_2}{\Gamma_1 \vdash F_1 \wedge F_2} \wedge R \qquad \dfrac{D_3 : \Gamma_2(F_1; F_2) \vdash H}{\Gamma_2(F_1 \wedge F_2) \vdash H} \wedge L}{\Gamma_2(\Gamma_1) \vdash H} \texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\Gamma_2(\Gamma_1; F_2) \vdash H} \texttt{Cut}}{\Gamma_2(\Gamma_1) \vdash H} \texttt{Cut}_{CS}$$

($\vee L, \vee R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1 \vdash F_i \quad (i \in \{1, 2\})}{\Gamma_1 \vdash F_1 \vee F_2} \vee R \qquad \dfrac{D_2 : \Gamma_2(F_1) \vdash H \qquad D_3 : \Gamma_2(F_2) \vdash H}{\Gamma_2(F_1 \vee F_2) \vdash H} \vee L}{\Gamma_2(\Gamma_1) \vdash H} \texttt{Cut}$$

$\Rightarrow$

$$\dfrac{D_1 \qquad D_{(2\ or\ 3)}}{\Gamma_2(\Gamma_1) \vdash H} \texttt{Cut}$$

Whether D_2 or D_3 for the right premise sequent depends on the value of i.

($\supset L, \supset R$):

$$\dfrac{\dfrac{D_1 : \Gamma_3; F_1 \vdash F_2}{D_4 : \Gamma_3 \vdash F_1 \supset F_2}\supset R \qquad \dfrac{D_2 : \mathbb{E}(\widetilde{\Gamma_1}; F_1 \supset F_2) \vdash F_1 \qquad D_3 : \Gamma_2(F_2; \mathbb{E}(\widetilde{\Gamma_1}; F_1 \supset F_2)) \vdash H}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_1}; F_1 \supset F_2)) \vdash H}\supset L}{\Gamma_2(\mathbb{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H}\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{\dfrac{\dfrac{D_4 \qquad D_2}{\mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3) \vdash F_1}\mathtt{Cut} \qquad D_1}{\Gamma_3; \mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3) \vdash F_2}\mathtt{Cut} \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2(F_2; \mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H}\mathtt{Cut}}{\Gamma_2(\Gamma_3; \mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H}\mathtt{Cut}_{CS}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 1}$$

$$\Gamma_2(\widetilde{\Gamma_1}; \Gamma_3; \mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 2}$$

$$\Gamma_2(\mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3); \mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 1}$$

$$\Gamma_2(\mathtt{E}(\widetilde{\Gamma_1}; \Gamma_3)) \vdash H$$

where a dotted line denotes that the derivation step is depth-preserving.

($*L, *R$):

$$\dfrac{\dfrac{D_1 : Re_i \vdash F_1 \qquad D_2 : Re_j \vdash F_2}{\Gamma_1 \vdash F_1 * F_2}*R \qquad \dfrac{D_3 : \Gamma_2(F_1, F_2) \vdash H}{\Gamma_2(F_1 * F_2) \vdash H}*L}{\Gamma_2(\Gamma_1) \vdash H}\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad D_3}{\Gamma_2(Re_i, F_2) \vdash H}\mathtt{Cut}}{\Gamma_2(Re_i, Re_j) \vdash H}\mathtt{Cut}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 1}$$

$$\Gamma_2(\Gamma_1) \vdash H$$

($-\!\!*L, -\!\!*R$):

$$\dfrac{\dfrac{D_1 : \Gamma_1, F_1 \vdash F_2}{D_4 : \Gamma_1 \vdash F_1 -\!\!* F_2}-\!\!*R \qquad \dfrac{D_2 : Re_i \vdash F_1 \qquad D_3 : \Gamma_2((\widetilde{Re_j}, F_2); (\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; F_1 -\!\!* F_2))) \vdash H}{\Gamma_2(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; F_1 -\!\!* F_2)) \vdash H}-\!\!*L_1}{\Gamma_2(\widetilde{\Gamma'}, \mathbb{E}(\widetilde{\Gamma_3}; \Gamma_1)) \vdash H}\mathtt{Cut}$$

$\Rightarrow$

$$\dfrac{D_2 \qquad \dfrac{D_1 \qquad \dfrac{D_4 \qquad D_3}{\Gamma_2((\widetilde{Re_j}, F_2); (\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H}\mathtt{Cut}}{\Gamma_2((\widetilde{Re_j}, \Gamma_1, F_1); (\Gamma', \mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H}\mathtt{Cut}}{\Gamma_2((\widetilde{Re_j}, \Gamma_1, Re_i); (\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H}\mathtt{Cut}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 1}$$

$$\Gamma_2((\widetilde{\Gamma'}, (\widetilde{\Gamma_3}; \Gamma_1)); (\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ Proposition 2}$$

$$\Gamma_2((\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1)); (\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1))) \vdash H$$

$$\cdots\cdots\cdots\cdots\cdots\cdots \text{ Theorem 1}$$

$$\Gamma_2(\widetilde{\Gamma'}, \mathtt{E}(\widetilde{\Gamma_3}; \Gamma_1)) \vdash H$$