Sampled-data Control for a Class of Linear Hyperbolic System via the Lyapunov-Razumikhin Technique

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***Abstract*— This work investigates the stability for a class of linear hyperbolic systems with distributed sampled-data controllers. First, we convert the original system into an equivalent system in which the sampling induced error is modeled as a reset integrator. Then by means of an appropriate Lyapunov function coupled with the Razumikhin technique, sufficient conditions are given for the Rε - stability of the system. Finally, our results are validated by a numerical example.**

I. INTRODUCTION

The application of digital computer in control system has become a general trend, which makes sampled-data control an active field of research in the past decades [1], [2], [21]. Stability and control design for finite-dimentional systems have been considered in many research works: see e.g. the survey [16], [22], [28]. Compared with the research method of finite dimensional system, the analysis and control of infinite dimensional systems is more challenging. Few results exist for sampled-data infinite dimensional system [25], [26].

In general, sampled-data systems can be analyzed using discrete-time, time-delay and Input-Output methods (see [16] and references therein). For the class of partial differential equations (PDEs), using discrete-time finite-dimensional approximate models, [34] proposed a methodology for the design of sampled-data controller with practical stability guarantees. In references [14], [18], [33], the time-delay approach has been used for the analysis of parabolic PDEs. Hold boundary feedback control in one-dimensional linear hyperbolic systems were considered in [20]. In [10], [13], event-triggered sampled-data control with controller on the boundaries was developed. The boundary feedback control of a 2×2 hyperbolic system was implemented by backstepping method in [6], [12].

It can be seen from the literature review that the analysis of sampled-data controller for hyperbolic PDEs is a wide-open area of research, and there are still many topics worth studying. The present paper aims at studying the distributed sampled-control for a class of hyperbolic PDEs. The idea is to generalize the Input-Output approach [15], [19], [30] for finite dimensional systems, to the case of hyperbolic PDEs. An interconnected equivalent system consisting of a continuous-time PDE and a reset-integral operator is derived from the original system. In our previous work [11], the stability of linear hyperbolic systems with sampled-data controller has been ensured for a sufficiently small sampling period. In the present paper, new stability conditions are proposed by using Lyapunov-Razumikhin stability criteria (e.g. [23], [24]), the estimation of the maximum sampling interval for the system stability is improved.

The paper is structured as follows: Section II presents the systems and the problem under study. In Section III, we propose the equivalent remodelling of system, followed by the concrete stability analysis process. A numerical example is given to illustrate the feasibility of our method in Section IV. The paper is ended with conclusions and perspectives.

Notation: $\mathbb{N}$ is the set of nonnegative integers from 0 to infinity, $\mathbb{R}_+$ is the set of nonnegative reals, $\mathbb{R}^n$ is used to denote the set of n-dimensional Euclidean space with the norm $|\cdot|$. $L^2(0,L)$ stands for the Hilbert space of square integrable scalar functions on $(0,L)$ with the norm $\|\cdot\|_{L^2(0,L)}$, defined by $\|\tau\|_{L^2(0,L)} = \sqrt{\int_0^L |\tau(x)|^2 dx}$. The associated norm to Sobolev space $H^1(0,L)$ is defined as $\|\tau\|_{H^1(0,L)} = \sqrt{\int_0^L \left(|\tau(x)|^2 + |\tau_x(x)|^2\right) dx}$. Given a functional $V : H^1([0,L];\mathbb{R}^n) \to \mathbb{R}_+$ such that $\mathfrak{L}_{V\leq C} = \left\{y \in H^1([0,L];\mathbb{R}^n) : V(y) \leq C\right\}$. The notation $W \leq 0$ denotes that W is symmetric and negative semidefinite. The symmetric elements are denoted by $*$ in the symmetric matrix. The identity matrix is denoted by I and $\lambda_{\min}(\Theta)$ and $\lambda_{\max}(\Theta)$ are the minimum and maximum eigenvalues of the matrix Θ. $\mathcal{C}^0$ is the space of continuous functions, whereas $\mathcal{C}^1$ is the space of continuously differentiable functions. $\lceil\cdot\rceil$ is the ceiling function.

II. SYSTEM DESCRIPTION AND PROBLEM FORMULATION

A. System Description

We consider the following sampled-data controlled hyperbolic system (1)

$$\begin{cases} \partial_t z(t,x) + \Lambda\partial_x z(t,x) + \Gamma z(t,x) + u(t,x) = 0, & \text{(1a)}\\ u(t,x) = F z(t_k,x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & \text{(1b)}\\ z(t,0) = 0, \forall t \geq 0, & \text{(1c)}\\ z(0,x) = z_0(x), \forall x \in [0,L], & \text{(1d)} \end{cases}$$

where $z : [0,+\infty) \times [0,L] \to \mathbb{R}^n$, $\Lambda = diag\{\lambda_1, \lambda_2, ..., \lambda_n\}$ with $\lambda_1, \lambda_2, ..., \lambda_n > 0$, Γ and F are real $n\times n$ constants matrices. The sampling instants are defined as a sequence $\{t_k\}_{k\in\mathbb{N}}$ where

$$t_0 = 0, t_{k+1} - t_k \in [\underline{h}, \bar{h}], \tag{2}$$

and $\bar{h}$, $\underline{h}$ are the given bounds of the sampling intervals satisfying $\bar{h} \geq \underline{h} > 0$.

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To address the issue under consideration, we need the compatibility condition given below:

Condition 1. The initial condition $z_0(x)$, satisfies

$$z_0(0) = 0, \forall x \in [0, L]. \tag{3}$$

Remark 1. We explain the concept of the solution and rewrite the system (1)-(2) as a first order system

$$\begin{cases} \frac{dz(t)}{dt} = \Upsilon z(t) + f(z(t_k)), t \in [t_k, t_{k+1}), k \in \mathbb{N}, \\ z(0) = z_0, \end{cases}$$

where $f(z(t_k)) = -\digamma z(t_k)$, and Υ is an operator defined by $\Upsilon z = -\Lambda \partial_x z(t, x) - \Gamma z(t, x)$, with domain

$$\mathcal{D}(\Upsilon) = \left\{ z \in H^1(0, L; \mathbb{R}^n) \,\middle|\, \quad z(0) = 0. \right\} \tag{4}$$

A stable C_0 semigroup is produced by the operator Υ (see the proof of theorem A.1. in [3]). Moreover, the fact is that $f_k : H^1(0, L) \to H^1(0, L)$ is continuously differentiable for $t \in [t_k, t_{k+1})$. If $z_0 \in \mathcal{D}(\Upsilon)$, then in the light of Theorem 6.1.5 of [31], there is a classical solution for each $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Consequently, a solution can be constructed by selecting the last value of the previous sampling interval as the initial condition for the next sampling interval so that it is continuous at each sampling instant.

B. Problem Formulation

In the present work, we adopt the Rε-stability for the system (1)-(2), which is defined as follows.

Definition 1. Rε-stability [32] Consider positive scalars R and ε, such that $0 < \varepsilon < R$, and a Lyapunov function $V : H^1([0, L]; \mathbb{R}^n) \to \mathbb{R}_+$. If for all solutions of system (1) with $z_0(x) \in \mathfrak{L}_{V<R}$, the trajectory of the state $z(t, x)$ converges to $\mathfrak{L}_{V \le \varepsilon}$ as t goes to infinity, then, system (1) is called Rε-stable from $\mathfrak{L}_{V<R}$ to $\mathfrak{L}_{V \le \varepsilon}$.

Our main goal is to ensure that the closed-loop system (1)-(2) is Rε-stable due to Input-Output method.

III. MAIN RESULT

This section consists of two parts. Firstly, the sampled-data system is equivalently expressed as a continuous hyperbolic PDE with sampling induced error as disturbances in the input. Secondly, we provide constructive Rε-stability criteria based on the provided model.

A. System Remodelling

System (1) can be rewritten equivalently as

$$\begin{cases} \partial_t z(t, x) + \Lambda \partial_x z(t, x) + (\Gamma + \digamma) z(t, x) \\ \qquad + \digamma \varpi(t, x) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & \text{(5a)} \\ z(t, 0) = 0, \forall t \ge 0, & \text{(5b)} \\ z(0, x) = z_0(x), \forall x \in [0, L], & \text{(5c)} \end{cases}$$

with the sampling error

$$\varpi(t, x) = z(t_k, x) - z(t, x). \tag{6}$$

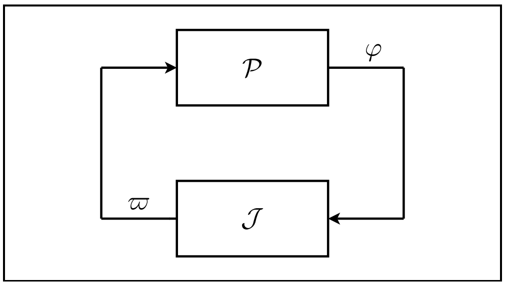

Fig. 1. Alternative representation of the closed-loop system.

Define the function $\varphi(t, x) = \frac{\partial z(t,x)}{\partial t}, \forall t \ge 0, x \in [0, L]$. Note that for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0, L]$, we have

$$\varpi(t, x) = -\int_{t_k}^{t} \frac{\partial z(\theta, x)}{\partial \theta} d\theta = -\int_{t_k}^{t} \varphi(\theta, x) d\theta. \tag{7}$$

Therefore, the closed-loop system can be seen as the interconnection of two systems $\mathcal{P}$ and $\mathcal{J}$ shown in Fig. 1, where the operator $\mathcal{P} : L^2(0, L) \to L^2(0, L)$ is defined by

$$\mathcal{P} : \begin{cases} \partial_t z(t, x) = -\Lambda \partial_x z(t, x) - (\Gamma + \digamma) z(t, x) \\ \qquad - \digamma \varpi(t, x), \\ z(t, 0) = 0, \forall t \ge 0, \\ z(0, x) = z_0(x), \forall x \in [0, L], \\ \varphi(t, x) = -\Lambda \partial_x z(t, x) - (\digamma + \Gamma) z(t, x) \\ \qquad - \digamma \varpi(t, x) = \partial_t z(t, x), \end{cases} \tag{8}$$

and the operator $\mathcal{J} : L^2(0, L) \to L^2(0, L)$ is defined by

$$\mathcal{J} : \begin{cases} \varpi(t, x) = (\mathcal{J} z)(t, x) = -\int_{t_k}^{t} \varphi(\theta, x) d\theta, \\ \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, x \in [0, L]. \end{cases} \tag{9}$$

Remark 2. The operator $\mathcal{P}$ is a nominal continuous-time control-loop since we use continuous-time sampling error instead of sampled-data controller in the system (8), $\mathcal{J}$ is an integral operator representing the sampling error. For the simplicity of the closed-loop system structure, we choose this form of $\mathcal{J}$ so that we need one output φ instead of two $z(t, x)$ and $z(t_k, x)$, which is why we wrote (6) as (7).

B. Stability Analysis

In the following, we present our primary results.

Proposition 1. Consider systems (8)-(9) with (2) and a function $V : H^1([0, L]; \mathbb{R}^n) \to \mathbb{R}_+$ which is differentiable w.r.t. its argument and such that there exists $0 < \iota_1 < \iota_2$ satisfying $\iota_1 \|\varrho\|^2_{H^1([0,L];\mathbb{R}^n)} \le V(\varrho) \le \iota_2 \|\varrho\|^2_{H^1([0,L];\mathbb{R}^n)}$.

Suppose that along the trajectories of the system (8)-(9), the corresponding solution $z(t, \cdot)$ satisfies $\dot{V}(z) + 2\delta V(z) \le 0$, for some $\delta > 0$, whenever

1) $R > V(z(t, \cdot)) \ge \max\{\varepsilon, V(z(t_k, \cdot))/\alpha\}$, with some $\alpha > 1$,
2) $z(t_k, \cdot) \in \mathfrak{L}_{V<R}$.

Then the system is Rε-stable from $\mathfrak{L}_{V<R}$ to $\mathfrak{L}_{V \le \varepsilon}$.

The proof of Proposition 1 can be found in the appendix.

Remark 3. Proposition 1 is based on the generalization of the Razumikhin technique to get the Rε-stability for hyperbolic systems. In the following theorem, we will show how it can be used in constructed manner.

$$W(x)=\begin{bmatrix} -e^{-2\mu x}\left[(F+\Gamma)^T\Theta_1+\Theta_1(F+\Gamma)\right] & -e^{-2\mu x}\Theta_1 F & 0 & 0\\ * & -\gamma I & 0 & 0\\ * & * & -e^{-2\mu x}\left[\Gamma^T\Theta_2+\Theta_2\Gamma+\beta\Theta_2\right] & -e^{-2\mu x}\Theta_2 F\\ * & * & * & -\gamma I \end{bmatrix} \quad (11)$$

$$N(x)=e^{-2\mu x}\begin{bmatrix} (\alpha-1)\Theta_1 & -\Theta_1 & 0 & 0\\ * & -\Theta_1 & 0 & 0\\ * & * & \alpha\Theta_2 & 0\\ * & * & * & -\Theta_2 \end{bmatrix} \quad (12)$$

Theorem 1. Consider systems (8)-(9) with (2) and an initial condition satisfying (3):

1) Let $\underline{\lambda}=\min_{i\in\{1,\dots,n\}}\lambda_i$. Assume that there exist $\mu,\gamma,\kappa>0$, $\alpha>1$ and symmetric positive matrices $\Theta_1\in\mathbb{R}^{n\times n}$, $\Theta_2\in\mathbb{R}^{n\times n}$ satisfying the commutativity conditions: $\Lambda\Theta_1=\Theta_1\Lambda, \Lambda\Theta_2=\Theta_2\Lambda$ and

$$W(0)+\kappa N(0)\preceq 0,\quad W(L)+\kappa N(L)\preceq 0, \quad (10)$$

with $W(x)$ and $N(x)$ defined for all $x\in[0,L]$ as (11)-(12).

2) For given decay rate $\delta>0$, $\exists\varepsilon\in\mathbb{R}_+,\ R\in\mathbb{R}_+$ s.t. $0<\varepsilon<R$ and it holds

$$\gamma 3\bar{h}\left(|\Lambda|^2\Omega_1+\left(|\Gamma|^2+|F|^2\right)\Omega_2\right)+\gamma\Omega_1\leq\ (2\sigma-\beta)\varepsilon-2\delta R, \quad (13)$$

for some $0<\beta<2\sigma$ with$\Omega_1=\frac{R}{\lambda_{\min}(\Theta_2)e^{-2\mu L}},\Omega_2=\frac{R}{\lambda_{\min}(\Theta_1)e^{-2\mu L}}$, $\sigma=\mu\underline{\lambda}$.

Then the considered system (1) is Rε-stable from $\mathfrak{L}_{V<R}$ to $\mathfrak{L}_{V\leq\varepsilon}$ for any sampling sequence satisfying (2), with the Lyapunov function defined by

$$V(z)=V_1(z)+V_2(z), \quad (14)$$

where $V_1(z)=\int_0^L z^Te^{-2\mu x}\Theta_1 z dx$, $V_2(z)=\int_0^L z_x^Te^{-2\mu x}\Theta_2 z_x dx$.

Proof. Consider the Lyapunov function (14). It can be bounded as $\Phi\left\|z\left(t,\cdot\right)\right\|^2_{H^1([0,L];\mathbb{R}^n)}\leq V\left(z\left(t,\cdot\right)\right)\leq\Psi\left\|z\left(t,\cdot\right)\right\|^2_{H^1([0,L];\mathbb{R}^n)}$, where $\Phi=\min\{\lambda_{\min}\left(\Theta_1\right),\lambda_{\min}\left(\Theta_2\right)\}e^{-2\mu L}$, $\Psi=\max\{\lambda_{\max}\left(\Theta_1\right),\lambda_{\max}\left(\Theta_2\right)\}$.

Step 1: In this step, we clarify that the function V defined in (14) is continuous by using the construction method [11].

Remark 4. V_1 is used to bound z, and V_2 is used to deal with the term z_x that appears in the derivative of V_1.

Step 2: In this step we study the time derivative of $V(z)$ defined in (14). Thanks to commutativity condition: $\Lambda\Theta_1=\Theta_1\Lambda$, we first compute the time derivative of $V_1(z)$ along the solutions to (8)-(9),$\forall t\in\left[t_k,t_{k+1}\right),k\in\mathbb{N}$,

$$\begin{aligned}\dot{V}_1(z)&=\int_0^L\Big(\partial_t z^Te^{-2\mu x}\Theta_1 z+z^Te^{-2\mu x}\Theta_1\partial_t z\Big)dx\\ &=\int_0^L\Big(\left(-\Lambda\partial_x z-(F+\Gamma)z-F\varpi\right)^Te^{-2\mu x}\Theta_1 z\\ &\quad+z^Te^{-2\mu x}\Theta_1\left(-\Lambda\partial_x z-(F+\Gamma)z-F\varpi\right)\Big)dx\\ &=-\left[z^T\Lambda e^{-2\mu x}\Theta_1 z\right]_0^L+\int_0^L\Big(-z^T\Big((F+\Gamma)^Te^{-2\mu x}\Theta_1\\ &\quad+e^{-2\mu x}\Theta_1(F+\Gamma)\Big)z-\varpi^TF^Te^{-2\mu x}\Theta_1 z\\ &\quad-z^Te^{-2\mu x}\Theta_1F\varpi\Big)dx-2\mu\int_0^L z^T\Lambda e^{-2\mu x}\Theta_1 z dx. \end{aligned} \quad (15)$$

In order to get the time derivative of z_x in V_2, we refer to the original system (1). Since $z:[0,+\infty)\times[0,L]\to\mathbb{R}^n$ has consecutive partial derivatives in $[0,+\infty)\times[0,L]$, according to Schwartz's theorem [17] we can obtain $\forall t\in\left(t_k,t_{k+1}\right)$

$$\begin{aligned}&\partial_{xt}z\left(t,x\right)=\partial_{tx}z\left(t,x\right)\\ &=-\Lambda\partial_{xx}z\left(t,x\right)-\Gamma\partial_x z\left(t,x\right)-F\partial_x z\left(t_k,x\right). \end{aligned} \quad (16)$$

For the next calculation of the time derivative of V_2, we use Lemma 1 in the appendix. According to (16) and Lemma 1, we have

$$\begin{cases}\partial_x z(t,0)=0,\forall t\in[t_k,t_{k+1}),k\in\mathbb{N}, & \text{(17a)}\\ z_0(0)=0,\quad \partial_x z_0(0)=0. & \text{(17b)}\end{cases}$$

Similarly to the computation of $\dot{V}_1$, by using the commutativity condition: $\Lambda\Theta_2=\Theta_2\Lambda$, the time derivative of $V_2(z)$ along the solutions to (16)-(17), $\forall t\in\left[t_k,t_{k+1}\right),k\in\mathbb{N}$ is shown as follows

$$\begin{aligned}&\dot{V}_2(z)=-\left[\partial_x z^T\Lambda e^{-2\mu x}\Theta_2\partial_x z\right]_0^L\\ &+\int_0^L\left(-\partial_x z^T\left(\Gamma^Te^{-2\mu x}\Theta_2+e^{-2\mu x}\Theta_2\Gamma\right)\partial_x z\right.\\ &\left.-\partial_x z^T\left(t_k,\cdot\right)F^Te^{-2\mu x}\Theta_2\partial_x z-\partial_x z^Te^{-2\mu x}\Theta_2F\partial_x z\left(t_k,\cdot\right)\right)dx\\ &-2\mu\int_0^L\partial_x z^T\Lambda e^{-2\mu x}\Theta_2\partial_x z dx. \end{aligned} \quad (18)$$

Adding $\gamma\|\varpi\left(s,\cdot\right)\|^2_{L^2([0,L];\mathbb{R}^n)}-\gamma\|\varpi\left(s,\cdot\right)\|^2_{L^2([0,L];\mathbb{R}^n)}$ to (15) and $\gamma\|\partial_x z\left(t_k,\cdot\right)\|^2_{L^2([0,L];\mathbb{R}^n)}-\gamma\|\partial_x z\left(t_k,\cdot\right)\|^2_{L^2([0,L];\mathbb{R}^n)}$, $\beta\int_0^L z_x^Te^{-2\mu x}\Theta_2 z_x dx-\beta\int_0^L z_x^Te^{-2\mu x}\Theta_2 z_x dx$ to (18) for some $\gamma>0,\beta>0$, and using boundary condition (5b) and (17a) we have

$$\begin{aligned}\dot{V}(z)=&\dot{V}_1(z)+\dot{V}_2(z)\\ \leq&-2\sigma V_1(z)-(2\sigma-\beta)V_2(z)\\ &+\int_0^L\eta^TW(x)\eta dx+\gamma\left\|\varpi\left(s,\cdot\right)\right\|^2_{L^2([0,L];\mathbb{R}^n)}\\ &+\gamma\left\|\partial_x z\left(t_k,\cdot\right)\right\|^2_{L^2([0,L];\mathbb{R}^n)}. \end{aligned} \quad (19)$$

with $\sigma = \mu\underline{\lambda}$, $\eta = [z^T \ \varpi^T \ (\partial_x z)^T \ (\partial_x z(t_k,\cdot))^T]^T$, and $W(x)$ defined in (11).

Step 3: In this step, we show that $\dot{V}(z) + 2\delta V(z) \leq 0$, whenever

$$\begin{cases} R > V(z(t,\cdot)) \geq \max\{\varepsilon, V(z(t_k,\cdot))/\alpha\}, & \text{(20a)} \\ z(\theta,\cdot) \in \mathfrak{L}_{V<R}, \forall \theta \in [t_k, t), k \in \mathbb{N}. & \text{(20b)} \end{cases}$$

Let us assume that conditions (20) hold. Since condition (10) is linear in $e^{-2\mu x}$ and $0 \leq x \leq L$, by convexity, we have $W(x) + \kappa N(x) \leq 0$, for $x \in [0, L]$. Therefore, we get

$$\int_0^L \eta^T (W(x) + \kappa N(x)) \eta dx \leq 0, \tag{21}$$

with $W(x)$ and $N(x)$ given in (11) and (12).

Now, consider $t \in [t_k, t_{k+1})$ and a trajectory z satisfying (20). Since condition (20a) is satisfied, we have $V(z(t,\cdot)) \geq V(z(t_k,\cdot))/\alpha$ with some $\alpha > 1$, which can be rewritten as

$$\int_0^L \eta^T N(x) \eta dx \geq 0. \tag{22}$$

In view of (21), (22) and $\kappa > 0$, by S-procedure, it implies

$$\int_0^L \eta^T W(x) \eta dx \leq 0. \tag{23}$$

According to condition (20b), the following inequalities are further derived for all $\theta \in [t_k, t]$:

$$\begin{aligned} \|z(\theta,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)} &< \frac{R}{\lambda_{\min}(\Theta_1) e^{-2\mu L}} = A_1, \\ \|\partial_x z(\theta,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)} &< \frac{R}{\lambda_{\min}(\Theta_2) e^{-2\mu L}} = A_2. \end{aligned} \tag{24}$$

Using (7) and (24) for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, the upper bound of $\|\varpi(s,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)}$ can be calculated

$$\begin{aligned} &\|\varpi(s,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)} \\ &= \int_0^L |\varpi(s,x)|^2 dx = \int_0^L \left| \int_{t_k}^t \frac{\partial z(\theta,x)}{\partial \theta} d\theta \right|^2 dx \\ &\leq 3 \int_0^L \int_{t_k}^t \Big(|\Lambda|^2 |\partial_x z(\theta,x)|^2 + |\Gamma|^2 |z(\theta,x)|^2 \\ &\quad + |F|^2 |z(t_k,x)|^2 \Big) d\theta dx \\ &\leq 3\bar{h} \left(|\Lambda|^2 A_2 + \left(|\Gamma|^2 + |F|^2 \right) A_1 \right) = \omega. \end{aligned} \tag{25}$$

In addition, since condition (20a) is satisfied, we have

$$\begin{aligned} -2\sigma V_1(z) - (2\sigma - \beta) V_2(z) &\leq -(2\sigma - \beta)(V_1(z) + V_2(z)) \\ &< -(2\sigma - \beta)\varepsilon. \end{aligned} \tag{26}$$

Therefore, substituting (23), (25) and (26) into (19), we have for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$,

$$\dot{V}(z) < -(2\sigma - \beta)\varepsilon + \gamma\omega + \gamma \frac{R}{\lambda_{\min}(\Theta_2) e^{-2\mu L}}. \tag{27}$$

Since (13) holds, we deduce from (27),

$$\dot{V}(z) < -2\delta R \leq -2\delta V(z). \tag{28}$$

Therefore, we have shown that $\dot{V}(z) + 2\delta V(z) \leq 0$, whenever conditions (20) are satisfied.

Step 4: In this step, we show that if $z(t_k,\cdot) \in \mathfrak{L}_{V<R}$, then $z(t,\cdot) \in \mathfrak{L}_{V<R}, \forall t \in [t_k, t_{k+1})$. Consider z such that $z(t_k,\cdot) \in \mathfrak{L}_{V<R}$, assume that $\exists\ t^\circ \in (t_k, t_{k+1})$ s.t. $V(z(t^\circ,\cdot)) \geq R$. Let us then call T° the minimum of such t°, then $\forall t \in [t_k, T^\circ)$, $V(z(t,\cdot)) < R$. Therefore conditions (20) are going to be satisfied for any $t \in [t_k, T^\circ)$. From step 3, we know that V is going to decrease during that time interval, either continuously, or until V reaches below $\max\{\varepsilon, V(z(t_k,\cdot))/\alpha\}$ and when it reaches that region, it never gets back out. Therefore, we have $V(z(T^\circ,\cdot)) < V(z(t_k,\cdot)) < R$, which contradicts the assumption that there exists $t^\circ \in (t_k, t_{k+1})$ such that $V(z(t^\circ,\cdot)) \geq R$.

Summary: From step 3 and step 4, it is clear that $\dot{V}(z) + 2\delta V(z) \leq 0$ wherever

$$\begin{cases} R > V(z(t,\cdot)) \geq \max\{\varepsilon, V(z(t_k,\cdot))/\alpha\}, & \text{(29a)} \\ z(t_k,\cdot) \in \mathfrak{L}_{V<R}, & \text{(29b)} \end{cases}$$

and therefore, the conditions of Theorem 1 are satisfied, which concludes the proof of Rε-stability. ■

Remark 5. It is worth pointing out that we use several parameters and now we summarize each parameter in detail. For Rε-stability, R is the domain of attraction for a given Lyapunov function, ε specifies the positive invariant level set of V. They satisfy $0 < \varepsilon < R$. In this paper, we can fix R then compute ε or vice versa. α is a parameter introduced in the Lyapunov-Razumikhin method to define level set in which the time derivative of $V(z(t,\cdot))$ should be negative between two sampling interval, we choose it greater than 1. The closer α is to 1, the greater the values of $V(z(t_k,\cdot))/\alpha$ are, and the less conservative the conditions of V-convergence are. μ is related to the decay rate of V_1, V_2, and δ is related to the decay rate of V. γ and κ are found by line search to realize the conditions given in Theorem 1. First, the algorithm of Theorem 1 is implemented in Matlab using Yalmip [29] to solve the condition 1). Then we use the same parameters to test the condition 2). Due to (13), we adjust γ, β to be the smallest possible and μ to be the largest possible. In the numerical section, we sort out their relationship: $\mu > 0$, $\delta > 0$, $\underline{\lambda} = \min_{i \in \{1,\ldots,n\}} \lambda_i > 0$, $\sigma = \mu\underline{\lambda}$, $0 < \beta < 2\sigma$.

IV. NUMERICAL SIMULATION

In this section, we present a numerical example to illustrate the method we proposed in Section III.

Consider system (1) where $\Lambda = \begin{bmatrix} 11 & 0 \\ 0 & 11 \end{bmatrix}$, $\bar{h} = 0.1$, $\Gamma = \begin{bmatrix} 20 & 15 \\ 20 & 25 \end{bmatrix}$, $F = \begin{bmatrix} 2 & 0 \\ 5 & 4 \end{bmatrix}$, $L = 1$, $z_0(x) = \begin{bmatrix} 0.2(1 - \cos 2\pi x) \sin 4\pi x \\ 0.15(1 - \cos 4\pi x) \sin 2\pi x \end{bmatrix}$.

According to Remark 5, the parameters in condition (10) are selected as: $\mu = 0.09$, $\kappa = 1.8$, $\gamma = 0.001$, $\alpha = 1.001$, then we choose $\beta = 0.01$, $\delta = 0.001$ satisfying condition (13). We fix $R = 20$, and choose appropriate Θ_1, Θ_2 to observe the evolution of states.

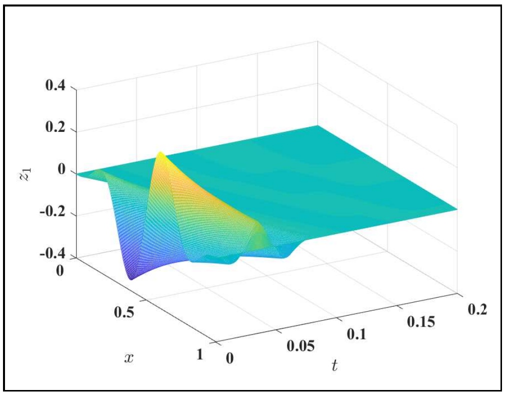

Fig. 2. Response of state z_1.

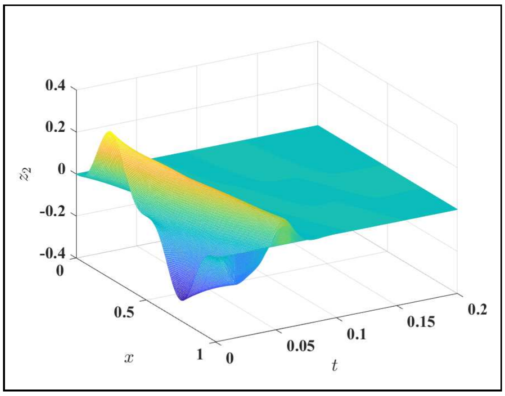

Fig. 3. Response of state z_2.

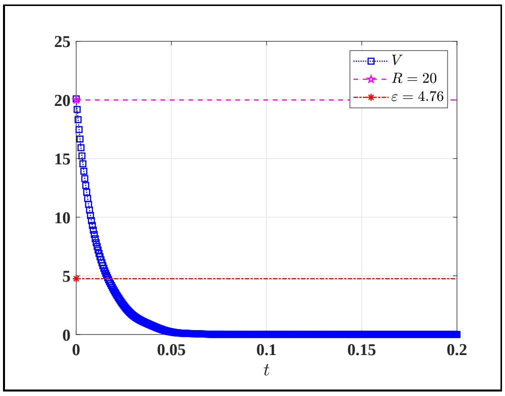

Fig. 4. Time-evolution of function V.

The simulation results are introduced in Figs. 2-4. Figs. 2-3 present that both state trajectories converge to near the origin with the controller and the initial conditions satisfying the compatibility condition (3). As can be seen from Fig. 4, the time-evolution of Lyapunov function $V(z(t,\cdot))$ decreases when $R > V(z(t,\cdot)) \geq \max\{\varepsilon, V(z(t_k,\cdot))/\alpha\}, \alpha > 1$. Please note that using the method proposed in [11] is not feasible in the case presented here.

V. CONCLUSIONS

The main work of this paper is to use a sampling controller for distributed control of linear hyperbolic balance laws. The closed-loop system is reformulated based on Input-Output approach. New stability condition has been obtained by means of the Lyapunov-Razumikhin method. In the future, we will consider the global stability with controller discretized both in time and in space.

APPENDIX

Lemma 1. Consider the system (1)-(2) with initial condition z_0 satisfying Condition 1. Then $\forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$, $\partial_x z(t,0) = 0$.

Proof: We recall system (1), the time derivative of the boundary condition leads to $\partial_t z\,(t,0) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Then combining (1a) with $\partial_t z\,(t,0) = 0$, we obtain $0 = \partial_t z\,(t,0) = -\Lambda \partial_x z\,(t,0) - \Gamma z\,(t,0) - \digamma z\,(t_k, 0)$. Since $z(t,0) = 0, \forall t \geq 0$, we have $\partial_x z(t,0) = 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}$. ■

The proof of Proposition 1: During a sampling interval $[t_k, t_{k+1})$ with initial state $z(t_k,\cdot)$:

(1) If $V(z(t_k,\cdot)) \leq \varepsilon$, $V(z(t,\cdot))$ will remain in ε during $[t_k, t_{k+1})$.

(2) If $R > V(z(t_k,\cdot)) \geq \varepsilon$:

(a) We have $V(z(t,\cdot)) \leq V(z(t_k,\cdot))$ during $[t_k, t_{k+1})$. (Otherwise, we will have $\dot{V}(z) > 0 > -2\delta V(z)$ at some point when $V(z(t,\cdot)) \geq V(z(t_k,\cdot)) \geq V(z(t_k,\cdot))/\alpha$, which would contradict the proposition in Theorem 1.)

(b)We can further show that during $[t_k, t_{k+1})$

$$V(z(t,\cdot)) \leq \max\{\varepsilon, \tfrac{V(z(t_k,\cdot))}{\alpha}, e^{-2\delta(t-t_k)} V(z(t_k,\cdot))\}. \quad (30)$$

Then we will discuss two possibilities in case (b):

(b1) If there exists $t' \in [t_k, t_{k+1})$ such that $V(z(t',\cdot)) = \max\{\varepsilon, V(z(t_k,\cdot))/\alpha\}$. If $t \in [t_k, t')$, $\dot{V}(z) + 2\delta V(z) \leq 0$ holds, and we have $V(z(t,\cdot)) \leq e^{-2\delta(t-t_k)} V(z(t_k,\cdot)), \forall t \in [t_k, t')$. If $t \in [t', t_{k+1})$, $V(z(t,\cdot))$ cannot go back above $\max\{\varepsilon, V(z(t_k,\cdot))/\alpha\}$ otherwise, according to the same principle, it would contradict the proposition in Theorem 1. So, over the whole sampling interval $t \in [t_k, t_{k+1})$, we can get inequality (30).

(b2) When $V(z(t,\cdot)) > \max\{\varepsilon, V(z(t_k,\cdot))/\alpha\}, \forall t \in [t_k, t_{k+1})$, since $\dot{V}(z) + 2\delta V(z) \leq 0$, we have $V(z(t,\cdot)) \leq e^{-2\delta(t-t_k)} V(z(t_k,\cdot))$. Then it is not hard to get (30).

Consider $z(t_k,\cdot) \in \mathfrak{L}_{V<R}$, $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. We have

$$\begin{aligned} V(z(t,\cdot)) &\leq \max\{\varepsilon, V(z(t_k,\cdot))/\alpha, e^{-2\delta(t-t_k)} V(z(t_k,\cdot))\} \\ &= \max\{\varepsilon, \xi V(z(t_k,\cdot))\}, \end{aligned} \quad (31)$$

with $\xi = \max\{1/\alpha, e^{-2\delta(t-t_k)}\} \leq 1$, then we can derive $V(z(t_k,\cdot)) \leq \max\{\varepsilon, \zeta V(z(t_{k-1},\cdot))\}$, with $\zeta = \max\{1/\alpha, e^{-2\delta \underline{h}}\} < 1, \forall k \in \mathbb{N} \setminus \{0\}$, where $\underline{h}$ is the lower bound of the sampling interval.

By recursion, the following inequality holds if $z(t_0,\cdot) \in \mathfrak{L}_{V<R}$, $\forall k \in \mathbb{N}$, we have

$$\begin{aligned} & V(z(t_k,\cdot)) \leq \max\{\varepsilon, \zeta \max\{\varepsilon, \zeta V(z(t_{k-2},\cdot))\}\} \\ & \leq \max\{\varepsilon, \zeta^2 V(z(t_{k-2},\cdot))\} \leq \cdots \leq \max\{\varepsilon, \zeta^k V(z(t_0,\cdot))\}. \end{aligned} \quad (32)$$

Then combining (31) and (32), we get that

$$V(z(t,\cdot)) \leq \max\{\varepsilon, \zeta^k V(z(t_0,\cdot))\} = \varepsilon \quad (33)$$

when k is large enough. Therefore, there $\exists\ \bar{k} = \lceil \log_{\zeta} {}^{\varepsilon}/_{V(z(t_0,\cdot))} \rceil$, such that $z(t,\cdot) \in \mathfrak{L}_{V \leq \varepsilon}, \forall\ t \geq t_{\bar{k}}$, which leads the proof of Rε-stability. ■

REFERENCES

[1] K. J. Åström and B. Wittenmark, Computer-controlled systems: theory and design. Courier Corporation, 2013.

[2] T. Chen and B. A. Francis, Optimal sampled-data control systems. Springer Science & Business Media, 2012.

[3] G. Bastin and J. M. Coron, Stability and boundary stabilization of 1-D hyperbolic systems. Springer, vol. 88, 2016.

[4] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear matrix inequalities in system and control theory. Siam, vol. 15, 1994.

[5] C. Fiter, L. Hetel, W. Perruquetti, and J. P. Richard, A state dependent sampling for linear state feedback. Automatica, vol. 48, no. 8, pp. 1860–1867, 2012.

[6] M. A. Davó, D. Bresch-Pietri, C. Prieur, and F. Di Meglio, Stability analysis of a 2×2 linear hyperbolic system with a sampled-data controller via backstepping method and looped-functionals. IEEE Transactions on Automatic Control, vol. 64, no. 4, pp. 1718–1725, 2018.

[7] L. Hetel, A. Kruszewski, W. Perruquetti, and J. P. Richard, Discrete and intersample analysis of systems with aperiodic sampling. IEEE Transactions on Automatic Control, vol. 56, no. 7, pp. 1696–1701, 2011.

[8] J. Louisell, Delay differential systems with time-varying delay: New directions for stability theory. Kybernetika, vol. 37, no. 3, pp. 239–251, 2001.

[9] L. Mirkin, Some remarks on the use of time-varying delay to model sample-and-hold circuits. IEEE Transactions on Automatic Control, vol. 52, no. 6, pp. 1109–1112, 2007.

[10] N. Espitia, A. Girard, N. Marchand, and C. Prieur, Event-based control of linear hyperbolic systems of conservation laws. Automatica, vol. 70, pp. 275–287, 2016.

[11] X. Y. Wang, Y. Tang, C. Fiter, and Laurentiu Hetel, Stability Analysis for A Class of Linear Hyperbolic System of Balance Laws with Sampled-data Control. IFAC World Congress. Berlin, Germany, 2020, hal-02491857v2.

[12] N. Espitia, A. Girard, N. Marchand, and C. Prieur, Event-based boundary control of a linear 2×2 hyperbolic system via backstepping approach. IEEE Transactions on Automatic Control, vol. 63, no. 8, pp. 2686–2693, 2017.

[13] N. Espitia, A. Tanwani, and S. Tarbouriech, Stabilization of boundary controlled hyperbolic pdes via Lyapunov-based event triggered sampling and quantization. 2017 IEEE 56th Annual Conference on Decision and Control (CDC), pp. 1266–1271, 2017.

[14] E. Fridman and A. Blighovsky, Robust sampled-data control of a class of semilinear parabolic systems. Automatica, vol. 48, no. 5, pp. 826–836, 2012.

[15] H. Fujioka, Stability analysis of systems with aperiodic sample-and-hold devices. Automatica, vol. 45, no. 3, pp. 771–775, 2009.

[16] L. Hetel, C. Fiter, H. Omran, A. Seuret, E. Fridman, J.P. Richard, and S.I. Niculescu, Recent developments on the stability of systems with aperiodic sampling: An overview. Automatica, vol. 76, pp. 309–335, 2017.

[17] R. C. James, Advanced calculus belmont. Wadsworth Pub. Co., 1966.

[18] W. Kang and E. Fridman, Distributed sampled-data control of Kuramoto–Sivashinsky equation. Automatica, vol. 95, pp. 514–524, 2018.

[19] C. Y. Kao and A. Rantzer, Stability analysis of systems with uncertain time-varying delays. Automatica, vol. 43, no. 6, pp. 959–970, 2007.

[20] I. Karafyllis and M. Krstic, Sampled-data boundary feedback control of 1-D linear transport pdes with non-local terms. Systems & Control Letters, vol. 107, 68–75, 2017.

[21] B. C. Kuo, Discrete-data control systems. Prentice-Hall Englewood Cliffs, NJ, 1970.

[22] D. S. Laila, D. Nešić, and A. Astolfi, Sampled-data control of nonlinear systems. Springer, pp. 91–137, 2006.

[23] E. Fridman, Lyapunov-based stability analysis. Springer, pp. 51–133, 2014.

[24] K. Gu and S. I. Niculescu, Survey on recent results in the stability and control of time-delay systems. J. Dyn. Sys., Meas., Control, vol. 125, no. 2, pp. 158–165, 2003.

[25] H. Logemann, R. Rebarber, and S. Townley, Stability of infinite-dimensional sampled-data systems. Transactions of the American mathematical society, vol. 355, no. 8, pp. 3301–3328, 2003.

[26] H. Logemann, R. Rebarber, and S. Townley, Generalized sampled-data stabilization of well-posed linear infinite-dimensional systems. SIAM journal on control and optimization, vol. 44, no. 4, pp. 1345–1369, 2005.

[27] L. Mirkin, Some remarks on the use of time-varying delay to model sample-and-hold circuits. IEEE Transactions on Automatic Control, vol. 52, no. 6, pp. 1109–1112, 2007.

[28] S. Monaco and D. Normand-Cyrot, Issues on nonlinear digital control. European Journal of Control, vol. 7, no. 2-3, pp. 160–177, 2001.

[29] J. Lofberg, YALMIP: A toolbox for modeling and optimization in MATLAB. 2004 IEEE international conference on robotics and automation (IEEE Cat. No. 04CH37508), pp. 284–289, 2004.

[30] H. Omran, L. Hetel, J. P. Richard, and F. Lamnabhi-Lagarrigue, Stability analysis of bilinear systems under aperiodic sampled-data control. Automatica, vol. 50, no. 4, pp. 1288–1295, 2014.

[31] A. Pazy, Semigroups of linear operators and applications to partial differential equations. Springer Science & Business Media, vol.44, 1983.

[32] A. Polyakov, Practical stabilization via relay delayed control. 2008 47th IEEE Conference on Decision and Control, pp. 5306–5311, 2008.

[33] A. Selivanov and E. Fridman, Sampled-data relay control of diffusion pdes. Automatica, vol. 82, pp. 59–68, 2017.

[34] Y. Tan, E. Trélat, Y. Chitour, and D. Nešić, Dynamic practical stabilization of sampled-data linear distributed parameter systems. Proceedings of the 48h IEEE Conference on Decision and Control (CDC) held jointly with 2009 28th Chinese Control Conference, pp. 5508–5513, 2009.