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A Hybrid System Approach to Exponential Stability with Sampled-data Control for a Class of Linear Hyperbolic Systems ⋆

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Abstract: In this paper, we address the problem of exponential stability for a class of linear hyperbolic systems with distributed sampled-data control. First, we recast the original system into a hybrid model via an augmented system approach. Using this model, the link between the sampling interval, the system state and its sampled vector is characterized by an Integral Quadratic Constraint (IQC). The obtained IQC is used for deriving numerically tractable stability criteria.

Keywords: Linear hyperbolic systems, sampled-data control, hybrid approach, exponential stability, Lyapunov function.

1. INTRODUCTION

Many physical and chemical phenomena can be described by partial differential equations (PDEs). These dynamics take place in an infinite dimensional space, which makes it harder to analyze than finite dimensional systems. This prompted many scholars to study the control of PDEs (See Krstic and Smyshlyaev (2008); Necas et al. (1996); Majda (2003)). In practice, controllers are implemented numerically with algorithms on computers. At present, it is a challenging research topic to analyze infinite dimensional systems with sampled-data controllers (Logemann (2013); Ke et al. (2009)). Here we study the sampled-data controller for hyperbolic systems.

Typically, sampled-data control can be handled by discrete time (Kao and Fujioka (2013)), time-delay (Fridman et al. (2004)), Input/Output (Fujioka (2009)), and hybrid system (Postoyan and Nesic (2011)) methods (see e.g., Hetel et al. (2017) for a survey). For infinite dimensional systems, fewer results are available. In Logemann and Mawby (2002), the sampled-data low-gain control is studied for systems with input hysteresis. (see also Logemann et al. (2003, 2005) for other sampled-data control laws). For parabolic PDEs, the time-delay method has been used in Selivanov and Fridman (2016, 2017); Fridman and Blighovsky (2012); Kang and Fridman (2018) for systems with distributed sampled-data control. Event-triggered control of hyperbolic PDEs was developed in Espitia et al. (2016, 2017b). In Espitia et al. (2017a); Davó et al. (2018), the authors introduced the backstepping approach to stabilize a class of event-triggered hyperbolic systems. Compared with the existing paper addressing the boundary control case (Diagne et al. (2012); Safi et al. (2017); Tang and Mazanti (2017)), few results addressing the case of distributed sampled-data control for hyperbolic systems. Recently, in our previous work (Wang et al. (2020b) and Wang et al. (2020a)), based on the Lyapunov-Razumikhin method, new stability conditions have been provided. The method therein allows to check local practical stability for hyperbolic PDEs with distributed sampled-data control.

In summary, it can be seen for the literature survey that there is a wide open research space for the analysis of sampled hyperbolic PDEs. In this paper, we proposed a new hybrid system approach for the analysis of hyperbolic PDEs with distributed sampled-data control. The idea of this paper is to use an augmented state model with an impulsive form (see Haddad et al. (2006); Naghshtabrizi et al. (2008)) for the finite dimensional case in order to derive Integral Quadratic Constraints (IQCs) (Megretski and Rantzer (1997)) which characterise the effect of the sampling. Using the obtained IQCs, we derive numerical criteria for analyzing stability. Compared with our previous local practical stability conditions (Wang et al. (2020b)), this work provides global exponential stability conditions in the form of linear matrix inequalities (LMIs).

The structure of this paper is as follows. Section 2 re-models the system to be studied into an augmented hybrid system and states the purpose of our research. In Section 3, a useful preliminary result is proposed in order to obtain an IQC characteristic of the sampling effect. Next the main stability result is proposed. A numerical example is shown in Section 4. Finally, we summarize the paper with a conclusion.

Notations: $\mathbb{N}$ is a nonnegative integer from 0 to infinity, the set of real numbers is denoted by $\mathbb{R}$, $\mathbb{R}_+$ is the set of non-negative reals, $\mathbb{R}^n$ is used to denote the set of n-dimensional Euclidean space with the norm $|\cdot|$. $L^2(0,L)$ stands for the Hilbert space of square integrable scalar functions on $(0,L)$ with the corresponding norm $\|\cdot\|$, defined by $\|\rho\|_{L^2(0,L)} = \sqrt{\left(\int_0^L |\rho(x)|^2 dx\right)}$. The set of functions $\psi : [0,L] \to \mathbb{R}^n$ such that $\int_0^L |\psi(x)|^2 dx < \infty$ is denoted by $L^2([0,L];\mathbb{R}^n)$. The notation $M \preceq 0$ denotes that M is a symmetric negative semidefinite matrix, and the symmetric elements are denoted by $\star$. The identity matrix is denoted by I.

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2. PROBLEM STATEMENT

We consider the following hyperbolic systems:

$$\begin{cases} \partial_t y(t,x) + \Lambda \partial_x y(t,x) + \Upsilon y(t,x) + u(t,x) = 0, & \text{(1a)} \\ u(t,x) = K y(t_k,x), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, & \text{(1b)} \\ y(t,0) = y(t,L), \forall t \geq 0, & \text{(1c)} \\ y(0,x) = y_0(x), \forall x \in [0,L], & \text{(1d)} \end{cases}$$

where $y : [0,+\infty) \times [0,L] \to \mathbb{R}^n$, $\Lambda = \operatorname{diag}\{\Lambda_+,\ \Lambda_-\}$, $\Lambda_+ = \operatorname{diag}\{\lambda_1, \cdots, \lambda_m\}$, $\Lambda_- = \operatorname{diag}\{\lambda_{m+1}, \cdots, \lambda_n\}$ with $\lambda_1 > \cdots \lambda_m > 0 > \lambda_{m+1} > \cdots > \lambda_n$, K and Υ are real $n \times n$ constants matrices.

The sampling sequence is defined as $\{t_k\}_{k \in \mathbb{N}}$ where

$$t_0 = 0, t_{k+1} - t_k \in (0, \bar{h}]. \tag{2}$$

and $\bar{h} > 0$. Let $\hat{y}$ indicate a piecewise constant signal representing the latest state measurement of the plant available at the controller, $\hat{y}(t,x) = y(t_k,x)$, for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Using the augmented system state $\eta = [y^T(t,x),\ \ \hat{y}^T(t,x)]^T \in \mathbb{R}^{2n}$, we recast system (1) into an augmented hybrid model with the following structure:

$$\begin{cases} \partial_t \eta(t,x) + A \partial_x \eta(t,x) + B \eta(t,x) = 0, \\ \forall t \in (t_k, t_{k+1}), k \in \mathbb{N}, & \text{(3a)} \\ \eta(t_k,x) = C \eta(t_k^-,x), t = t_k, k \in \mathbb{N}, & \text{(3b)} \\ \eta(t,0) = \eta(t,L), \forall t \geq 0, & \text{(3c)} \\ \eta(0,x) = \eta_0(x) = [y_0(x) \quad y_0(x)]^T, \forall x \in [0,L]. & \text{(3d)} \end{cases}$$

with $A = \begin{bmatrix} \Lambda & \mathbf{0}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}, B = \begin{bmatrix} \Upsilon & K \\ \mathbf{0}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}, C = \begin{bmatrix} \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \\ \mathbf{I}_{n\times n} & \mathbf{0}_{n\times n} \end{bmatrix}$, $\eta(t_k^-, \cdot) = \lim_{t \uparrow t_k} \eta(t, \cdot)$.

Note that the initial condition (3d) is chosen the same as the condition (1d), so that all the solution of (1) are characterized by the first component of the augment state. As a result, the closed-loop system can be regarded as an augmented impulsive system of the $y(t,x)$-variable and the $y(t_k,x)$-variable. In this article, we intend to find the exponential stability criteria of the original system with the help of the augmented one.

3. STABILITY ANALYSIS

Before the statement of the main result, a technical lemma is first given based on model (3) to characterize the link between the system state y of system (1) and its sampled vector $\hat{y}$ by an IQC. The idea of Lemma 1 is to use a norm Φ depending on the augmented system state and study its growth along the solution of hybrid system (3) during one sampling interval $[t_k, t_{k+1})$. Then we overbound the growth of exponential function with maximum growth rate α. Next, stability conditions for system (1) are derived using the obtained IQC.

3.1 IQC Condition

Lemma 1: Consider system (1) and augmented system (3). Let $\alpha \in \mathbb{R}$ and $\Theta \in \mathbb{R}^{2n\times 2n}$ a diagonal matrix satisfying

$$2\alpha\Theta - B^T\Theta - \Theta B \preceq 0. \tag{4}$$

Then, the inequality

$$\int_0^L \eta^T(t,x) N(t-t_k) \eta(t,x)\, dx \geq 0, t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{5}$$

holds along the solutions $\eta \in L^2\left([0,L]; \mathbb{R}^{2n}\right)$ which is the solution of (3), with $N(h)$ defined for all $h \in [0,\bar{h}]$ as

$$N(h) = e^{-2\alpha h} \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \end{bmatrix}^T \Theta \begin{bmatrix} \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n} \end{bmatrix} - \Theta. \tag{6}$$

Proof: Let us consider the following functional Φ defined for all $\eta \in L^2([0,L]; \mathbb{R}^{2n})$:

$$\Phi(\eta) = \int_0^L \eta^T \Theta \eta\, dx, \tag{7}$$

with $\eta \in L^2\left([0,L]; \mathbb{R}^{2n}\right)$ solution of (3).

Computing the time derivative of Φ along the solution to (3)

$$\begin{aligned} \dot{\Phi}(\eta) &= \int_0^L \left(\eta_t{}^T \Theta \eta + \eta^T \Theta \eta_t\right) dx \\ &= \int_0^L \Big((-A\partial_x \eta - B\eta)^T \Theta \eta + \eta^T \Theta (-A\partial_x \eta - B\eta) \Big) dx \\ &= \int_0^L -\partial_x \left[\eta^T A \Theta \eta\right] dx + \int_0^L \left(-\eta^T \left(B^T\Theta + \Theta B\right)\ \eta\right) dx, \end{aligned} \tag{8}$$

using the boundary condition (3c), we get

$$\dot{\Phi}(\eta) = \int_0^L \left(\eta^T \left(-B^T\Theta - \Theta B\right)\ \eta\right) dx. \tag{9}$$

Since the condition (4) holds, then we have

$$\dot{\Phi}(\eta) + 2\alpha\Phi(\eta) = \int_0^L \left(\eta^T \left(2\alpha\Theta - B^T\Theta - \Theta B\right)\ \eta\right) dx \leq 0. \tag{10}$$

Then, according to the comparison lemma we have

$$\Phi(\eta(t,\cdot)) \leq e^{-2\alpha(t-t_k)} \Phi(\eta(t_k,\cdot)), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}. \tag{11}$$

The inequality (11) can be rewritten as:

$$\begin{aligned} & e^{-2\alpha(t-t_k)} \Phi(\eta(t_k,\cdot)) - \Phi(\eta(t,\cdot)) \\ &= \int_0^L \eta^T(t,x) N(t-t_k) \eta(t,x)\, dx \geq 0, \end{aligned} \tag{12}$$

with $N(t-t_k)$ defined in (6). ■

Remark 1. The parameter α in the above lemma represents an upper bound on the growth rate of the function Φ (norm of the state η) between two sampling points. This upper bound is shown in (11). The following example provides a more intuitive explanation. The function Φ captures the growth of the norm of η.

Example 1. Consider system (3) with

$$L = 1, A = \begin{bmatrix} 1.2 & 0 & 0 & 0 \\ \star & -0.8 & 0 & 0 \\ \star & \star & 0 & 0 \\ \star & \star & \star & 0 \end{bmatrix}, B = \begin{bmatrix} 1.3 & 1.5 & 1.1 & 0.5 \\ -0.5 & 2.5 & 1 & 0.9 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$y_0(x) = \begin{bmatrix} 2.5 - 1.5\cos 4\pi x \\ -0.8\sin 6\pi x - 1 \end{bmatrix},$$

and a maximum sampling interval $\bar{h} = 0.1$. Condition (4) in Lemma 1 is an LMI that can be used in order to characterize the link between the system state and sampled version based on the IQC (5). Using Lemma 1 with $\alpha = 3$, we can derive an IQC of the form (5) and (6) with (13) given below:

$$\Theta = \begin{bmatrix} -0.25 & 0 & 0 & 0 \\ \star & -0.39 & 0 & 0 \\ \star & \star & -0.19 & 0 \\ \star & \star & \star & -0.19 \end{bmatrix}. \tag{13}$$

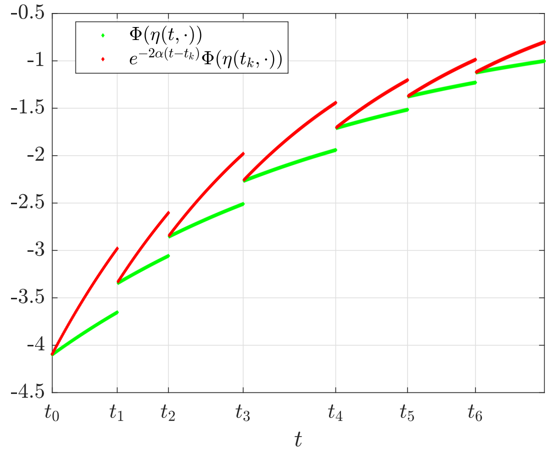

Fig. 1. Time-evolution of the norm Φ along the trajectory of system (3) for the system describing Example 1 (green) and its upper bound in each sampling interval (red).

The bounded growth performance of function $\Phi(\eta(t,\cdot))$ and the upper bound $e^{-2\alpha(t-t_k)}\Phi(\eta(t_k,\cdot))$ as stated in (11) for all sampling intervals $[t_k,t_{k+1}), k\in\mathbb{N}$ are shown in Fig. 1.

3.2 Main Stability Result

Next, the following theorem gives two LMIs allowing to check the exponential stability of system (1).

Theorem 1: Consider the system (1)-(2). Assume that there exist $\varepsilon>0, \lambda>0$, $\alpha\in\mathbb{R}$, a diagonal positive definite matrix $Q\in\mathbb{R}^{n\times n}$ and a diagonal matrix $\Theta\in\mathbb{R}^{2n\times 2n}$ satisfying (4) in Lemma 1

$$2\alpha\Theta - B^T\Theta - \Theta B \preceq 0,$$

and

$$M+\varepsilon N(0)\preceq 0,\ \ M+\varepsilon N(\bar h)\preceq 0, \tag{14}$$

with M defined as

$$M=\begin{bmatrix} 2\lambda Q-\Upsilon^T Q - Q\Upsilon & -QK \\ \star & \mathbf{0}_{n\times n}\end{bmatrix}, \tag{15}$$

and $N(h)$ defined for all $h\in[0,\bar h]$ as in (6):

$$N(h)=e^{-2\alpha h}\begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{I}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n}\end{bmatrix}^T\Theta\begin{bmatrix}\mathbf{0}_{n\times n} & \mathbf{I}_{n\times n}\\ \mathbf{0}_{n\times n} & \mathbf{I}_{n\times n}\end{bmatrix}-\Theta.$$

Then system (1) is exponentially stable in L^2-norm for any sampling sequence satisfying (2), with a decay-rate larger than λ.

Proof: We consider the original system (1) and choose a Lyapunov function as

$$V(y)=\int_0^L y^T Q y dx. \tag{16}$$

For simplicity, we use the notation y instead of $y(t,x)$. For $t\in[t_k,t_{k+1}), k\in\mathbb{N}$, the time derivative of $V(y)$ along the trajectories of (1) is

$$\begin{aligned}\dot V(y) &= \int_0^L \left(y_t{}^T Q y + y^T Q y_t\right) dx\\ &= \int_0^L \Big(\left(-\Lambda\partial_x y - \Upsilon y - K y(t_k,\cdot)\right)^T Q y\\ &\quad + y^T Q\left(-\Lambda\partial_x y - \Upsilon y - K y(t_k,\cdot)\right)\Big) dx\\ &= -\left[y^T\Lambda Q y\right]_0^L\\ &\quad+\int_0^L \Big(-y^T\left(\Upsilon^T Q + Q\Upsilon\right) y - y^T(t_k,\cdot) K^T Q y\\ &\quad - y^T Q K y(t_k,\cdot)\Big) dx.\end{aligned} \tag{17}$$

Using the boundary condition (1c) and adding the term $2\lambda V(y)$ to both sides of (17), then, we have

$$\begin{aligned}&\dot V(y)+2\lambda V(y)\\ &= \int_0^L \Big(y^T\left(2\lambda Q - \Upsilon^T Q - Q\Upsilon\right) y \qquad (18)\\ &\quad - y^T(t_k,\cdot) K^T Q y - y^T Q K y(t_k,\cdot)\Big) dx\\ &= \int_0^L \eta^T M \eta dx \qquad (19)\end{aligned}$$

with M defined in (15) and $\eta=\left[y^T(t,x)\ \ \hat y^T(t,x)\right]^T$, $\forall t\in[t_k,t_{k+1}), k\in\mathbb{N}$.

Since condition (14) is linear in $e^{-2\alpha h}$ and $t-t_k<\bar h$, by using a convexity argument, we have $M+\varepsilon N(t-t_k)\preceq 0$, for $t\in[t_k,t_{k+1}), k\in\mathbb{N}$ and $t_{k+1}-t_k\in(0,\bar h]$. Therefore, we get

$$\int_0^L \eta^T\left(M+\varepsilon N(t-t_k)\right)\eta dx\le 0, \tag{20}$$

recalling the condition (5)

$$\int_0^L \eta^T N(t-t_k)\eta dx \ge 0.$$

Since condition (14) holds and the Integral S-procedure in the appendix implies that if (20) and (5) are satisfied then we obtain

$$\int_0^L \eta^T M \eta dx \le 0. \tag{21}$$

In view of (18) and (21), we have

$$\dot V(y)+2\lambda V(y)\le 0 \tag{22}$$

for $\forall t\in[t_k,t_{k+1}), k\in\mathbb{N}$, which means that the following inequality is obtained:

$$V(y(t,\cdot))\le e^{-2\lambda(t-t_k)}V(y(t_k,\cdot)), \forall t\in[t_k,t_{k+1}), k\in\mathbb{N},$$

by recursion, we get

$$V(y(t,\cdot))\le e^{-2\lambda t}V(y_0),$$

i.e. $\forall t\in[t_k,t_{k+1}), k\in\mathbb{N}$, we have

$$\|\, y(t,\cdot)\,\|^2_{L^2([0,L];\mathbb{R}^n)}\le\frac{\lambda_{\max}(Q)}{\lambda_{\min}(Q)}e^{-2\lambda t}\,\|y_0\|^2_{L^2([0,L];\mathbb{R}^n)}.$$

Hence, according to Proposition 1, we can conclude that the system is exponentially stable. ■

Remark 2. Theorem 1 provides a numerical method for checking the stability of system (1). In order to ease the applicability of the results, we now summarize each parameter in detail. α is a scalar which is an upper bound of the growth rate of functional Φ used in Lemma 1. λ is a positive scalar which is an lower bound of the decay rate of V. ε is found by line search to realize (14). With a fixed α, Θ, Q can be found by solving the LMIs in (4) and (14) in Matlab using Yalmip Lofberg (2004). An example illustrating the tuning of these different parameters is given in section 4.

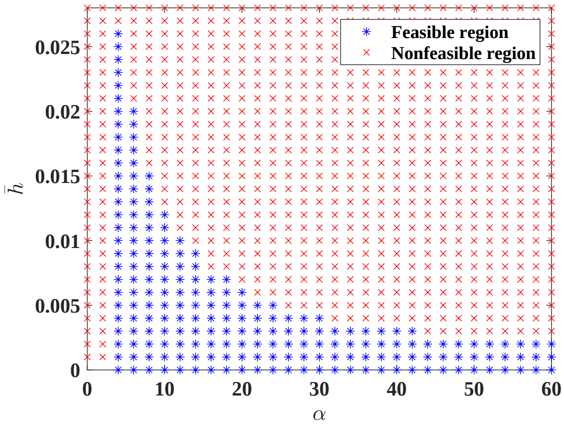

Fig. 2. Feasible region (blue area) and nonfeasible region (red area) guaranteed by Theorem 1 with $\lambda = 10^{-5}$, $\varepsilon = 1$.

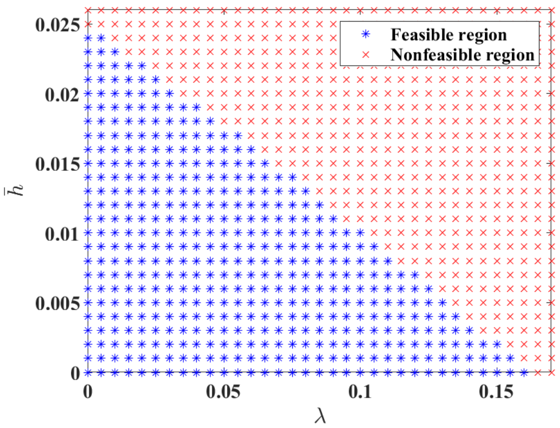

Fig. 3. Feasible region (blue area) and nonfeasible region (red area) guaranteed by Theorem 1 with $\alpha = 3$, $\varepsilon = 1$.

4. NUMERICAL EXAMPLE

Consider system (1) and (2) with

$$L = 1, \ \ \Lambda = \begin{bmatrix} 1.2 & 0 \\ 0 & -0.8 \end{bmatrix}, \ \ \Upsilon = \begin{bmatrix} 1.3 & 1.5 \\ -0.5 & 2.5 \end{bmatrix},$$

$$y_0(x) = \begin{bmatrix} 2.5 - 1.5\cos 4\pi x \\ -0.8\sin 6\pi x - 1 \end{bmatrix}.$$

Then, we consider the closed-loop system under the sampled-data controller with

$$K = \begin{bmatrix} 1.1 & 0.5 \\ 1 & 0.9 \end{bmatrix}.$$

Following Remark 2, Theorem 1 was tested for several values of α and $\bar{h}$. The results are illustrated in Fig. 2, where the growth rate α of Φ can be infinite, however, there is a minimum value and as the growth rate α decreases, $\bar{h}$ increases. This is consistent with Lemma 1. Fig. 3 illustrates that the sampling interval $\bar{h}$ decreases as the decay rate λ increases for some constant parameters:

$$\alpha = 3, \quad \varepsilon = 1.$$

Such figure can be used in order to find a tradeoff between system performance (in terms of decay rate) and robustness

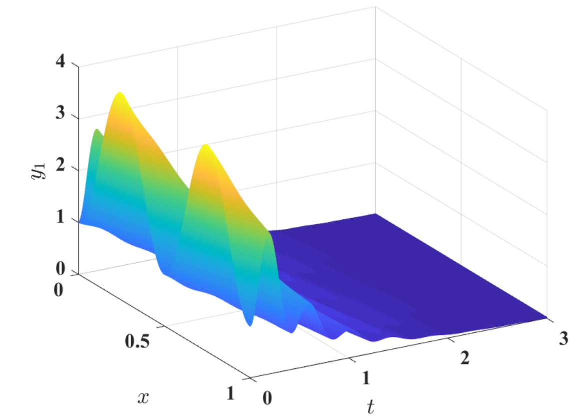

Fig. 4. Response of state y_1 for the closed-loop system.

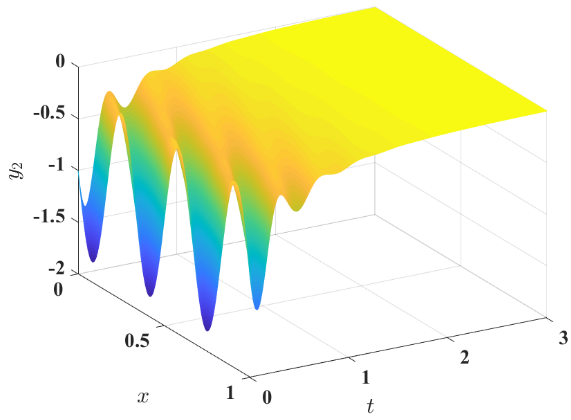

Fig. 5. Response of state y_2 for the closed-loop system.

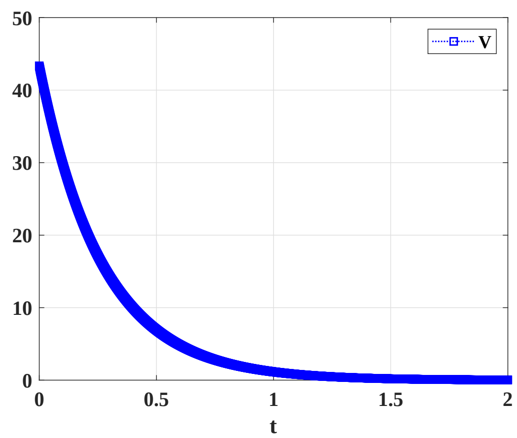

Fig. 6. Time-evolution of function V.

to the sampled-data implementation (in terms of maximum allowable sampling interval). In order to illustrate the response of the state, we choose one point

$$\bar{h} = 0.01, \quad \lambda = 0.1,$$

and Lyapunov function in Theorem 1 with

$$Q = \begin{bmatrix} 5.06 & 0 \\ \star & 4.54 \end{bmatrix},$$

which satisfy the conditions (4) and (14). The results for the closed-loop system are presented in Figs. 4-6. Figs. 4-5 show that the states converge to the origin with the controller. The time-evolution of Lyapunov function V is shown in Fig. 6.

5. CONCLUSION

This paper focused on the distributed sampled-data control for a class of hyperbolic systems using hybrid system approach. The closed-loop system is first represented as an augmented impulsive hybrid system. In addition, by means of the IQCs, we prove the exponential stability of the system. In the future, we will pay more attention to sampled-data control using space and time discretization for different types of hyperbolic systems.

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Appendix A. A LEMMA AND PROOF

Lemma 2 (Integral S-procedure): The property of Integral S-procedure is given as follows:

Let F and G be symmetric matrices. Assume that the strict integral inequality

$$\int_0^L y^T(x)Gy(x)dx \geq 0,$$

holds. Then the implication

$$\int_0^L y^T(x)Fy(x)dx \leq 0$$

holds if there exists some nonnegative number ϖ such that

$$F + \varpi G \preceq 0$$

Proof: It is immediately proved by rewriting the inequality above as $F \preceq -\varpi G$ and multiplying on the left by $y^T(x)$ and on the right by $y(x)$, then taking the integral operation on both sides of the inequality. We have

$$\int_0^L y^T(x)Fy(x)dx \leq \int_0^L y^T(x)(-\varpi G)y(x)dx.$$

Since $\int_0^L y^T(x)Gy(x)dx \geq 0$ and $\varpi \geq 0$ hold, one can obtain

$$\int_0^L y^T(x)(-\varpi G)y(x)dx \leq 0,$$

which leads to

$$\int_0^L y^T(x)Fy(x)dx \leq 0.$$

Hence, this lemma is proved.

Appendix B. A PROPOSITION AND PROOF

Proposition 1 : Consider the systems (1)-(2) and a candidate Lyapunov function $V : L^2([0,L];\mathbb{R}^n) \to \mathbb{R}_+$ which is differentiable for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ w.r.t. its argument $b \in L^2([0,L];\mathbb{R}^n)$ and there exist $0 < a_1 < a_2$, such that:

$$a_1 \|b\|^2_{L^2([0,L];\,\mathbb{R}^n)} \leq V(b) \leq a_2 \|b\|^2_{L^2([0,L];\,\mathbb{R}^n)}. \tag{B.1}$$

Assume that along the trajectories of the system (1)-(2), the corresponding solution $y(t,\cdot)$ satisfies

$$\dot{V}(y(t,\cdot)) + 2\lambda V(y(t,\cdot)) \leq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{B.2}$$

for some $\lambda > 0$. Then the system is *exponentially stable* in L^2-norm with a decay-rate larger than λ, that is for any initial condition $y_0 \in L^2(0,L)$ for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$

$$\|y(t,\cdot)\|^2_{L^2([0,L];\,\mathbb{R}^n)} \leq \frac{a_2}{a_1} e^{-2\lambda t} \|y_0\|^2_{L^2([0,L];\,\mathbb{R}^n)}. \tag{B.3}$$

Proof: First, we declare that V defined in (B.1) is continuous (One can consult Bastin and Coron (2016)). Since $y(t,x)$ is continuous with respect to t for all $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, and continuous at sampling instants by choosing the last value of the previous sampling interval as the initial condition of the following sampling interval, then V is continuous for all $t \geq 0$.

Then, we consider the differentiable Lyapunov function $V : L^2([0,L];\mathbb{R}^n) \to \mathbb{R}_+$ for which

$$\dot{V}(y(t,\cdot)) + 2\lambda V(y(t,\cdot)) \leq 0, \forall t \in [t_k, t_{k+1}), k \in \mathbb{N} \tag{B.4}$$

for some $\lambda > 0$.

Applying the comparison lemma, we have

$$V(y(t,\cdot)) \leq e^{-2\lambda(t-t_k)} V(y(t_k,\cdot)), \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}, \tag{B.5}$$

then we can derive

$$V(y(t_k,\cdot)) \leq e^{-2\lambda(t_k-t_{k-1})} V(y(t_{k-1},\cdot)), \forall k \in \mathbb{N}\setminus\{0\}, \tag{B.6}$$

by recursion, the following inequality holds

$$\begin{aligned} V(y(t_k,\cdot)) \leq & e^{-2\lambda(t_k-t_{k-1})} e^{-2\lambda(t_{k-1}-t_{k-2})} V(y(t_{k-2},\cdot)) \\ \leq & \cdots \\ \leq & e^{-2\lambda(t_k-t_{k-1})} e^{-2\lambda(t_{k-1}-t_{k-2})} \\ & \cdots e^{-2\lambda(t_1-t_0)} V(y(t_0,\cdot)), \end{aligned} \tag{B.7}$$

Then instituting (B.7) into (B.5), we obtain

$$\begin{aligned} V(y(t,\cdot)) \leq & e^{-2\lambda(t-t_k)} V(y(t_k,\cdot)) \\ \leq & e^{-2\lambda(t-t_k)} e^{-2\lambda(t_k-t_{k-1})} e^{-2\lambda(t_{k-1}-t_{k-2})} \cdots \\ & e^{-2\lambda(t_1-t_0)} V(y(t_0,\cdot)) \\ = & e^{-2\lambda(t-t_0)} V(y(t_0,\cdot)) \\ = & e^{-2\lambda t} V(y_0). \end{aligned} \tag{B.8}$$

Combining (B.1) and (B.8), we get

$$\|y(t,\cdot)\|^2_{L^2([0,L];\mathbb{R}^n)} \leq \frac{a_2}{a_1} e^{-2\lambda t} \|y_0\|^2_{L^2([0,L];\mathbb{R}^n)}, \quad \forall t \in [t_k, t_{k+1}), k \in \mathbb{N}.$$

This concludes the proof of Proposition 1. ■