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Abstract

We consider a general class of diffusive Kermack-McKendrick SIR epidemic models with an age-structured protection phase with limited duration, for example due to vaccination or drugs with temporary immunity. A saturated incidence rate is also considered which is more realistic than the bilinear rate. The characteristics method reduces the model to a coupled system of a reaction-diffusion equation and a continuous difference equation with a time-delay and a nonlocal spatial term caused by individuals moving during their protection phase. We study the existence and non-existence of non-trivial traveling wave solutions. We get almost complete information on the threshold and the minimal wave speed that describes the transition between the existence and non-existence of non-trivial traveling waves that indicate whether the epidemic can spread or not. We discuss how model parameters, such as protection rates, affect the minimal wave speed. The difficulty of our model is to combine a reaction-diffusion system with a continuous difference equation. We deal with our problem mainly by using Schauder's fixed point theorem. More precisely, we reduce the problem of the existence of non-trivial

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traveling wave solutions to the existence of an admissible pair of upper and lower solutions.

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1. Introduction and presentation of the model

One of the most famous epidemic models is Kermack-McKendrick SIR model [1, 2]

$$\begin{cases} S'(t) &= -\beta S(t)I(t), \\ I'(t) &= \beta S(t)I(t) - \mu I(t), \\ R'(t) &= \mu I(t), \end{cases} \tag{1}$$

where $S(t)$, $I(t)$ and $R(t)$ denote the sizes of the susceptible, infected and removed individuals, respectively. The parameter β describes the transmission coefficient of the disease and μ is the recovery rate (permanent immunity acquisition after the infection). The model (1) is without demography, then the total population $N = S(t) + I(t) + R(t)$ is constant. Let $S^0 = S(0)$ be the initial size of the susceptible individuals. The basic reproduction number $\mathcal{R}_0 := \beta S^0/\mu$ is the threshold that completely determines the dynamics of transmission of the epidemic. It was states (see for instance, [1, 2]) that if $\mathcal{R}_0 > 1$, then $I(t)$ first increases to its maximum and then decreases to zero (case of epidemic) and if $\mathcal{R}_0 < 1$, then $I(t)$ always decreases to zero (case of no epidemic).

The model of Kermack-McKendrick (1) has been widely investigated with several modifications, in [3–5] and references therein, by introducing the age dependence, and in [6–15] by considering the space and diffusion effects. In [9], the diffusion with unbounded space was incorporated in Kermack-McKendrick model

$$\begin{cases} \dfrac{\partial}{\partial t}S(t,x) &= \Delta S(t,x) - \beta S(t,x)I(t,x), \\ \dfrac{\partial}{\partial t}I(t,x) &= d\Delta I(t,x) + \beta S(t,x)I(t,x) - \mu I(t,x). \end{cases} \tag{2}$$

As the equations of S and I are independent on R, we omit the equation of R. In [9], the authors proved that if $S(0,x) = S^0$ is constant and if

$\beta S^0/\mu > 1$, then for each $c \geq c^\star := 2\sqrt{d\mu(\beta S^0/\mu - 1)}$ there exists a positive constant $S^\infty < S^0$ such that the system (2) has a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying $S(-\infty) = S^0$, $S(+\infty) = S^\infty$ and $I(\pm) = 0$.

To incorporate the latency period that many infectious diseases have, because for instance the infected individuals do not infect other susceptible individuals until some time later or because vaccination protects susceptible individuals for a while, many papers [11, 12, 16–19], considered reaction-diffusion systems with non-locality and time-delay. In [12], the authors considered a general class of diffusive Kermack-McKendrick SIR models with nonlocal and delayed disease transmission. They obtained full information about the existence and non-existence of traveling wave solutions. In particular, they gave the minimal wave speed and discussed how the model parameters affect this minimal speed. There are many other recent papers that investigated the spatial dynamics of epidemic models with diffusion and latent period, see [10, 20–27] and the references therein.

The vaccination is one of the most efficient way to halt disease transmission through promoting population immunity [28, 29]. The designed of vaccination strategies are based on the type of infectious agent, viruses, bacteria, fungi, protozoa, or worms, and always search for risk groups [30], the thresholds such as the proportion of the population to vaccinate [30, 31], and the optimum age for vaccination [32, 33], with the aim of optimize disease control. The duration of immunity promoted by vaccination and its efficacy determine the number of doses and the interval among them to ensure that the individuals are protected. For example, recently, Human Papillomavirus (HPV) vaccination moves from a 3-dose schedule to a 2-dose schedule [34]. HPV vaccination target-individuals between 9 to 14 years age because exposure to infection is higher at younger ages with a peak after the debut of sexual activity [35]. To prolong the immunity conferred by certain vaccines, it is sometimes necessary to update them. It is easier to focus on the individuals that are already vaccinated to incite them to update their vaccine. Indeed, these individuals are already known and easier to encourage to vaccinate again. In the literature, we can find many study of models involving the protection period effects (vaccination is one of them), [36–40]. Moreover, in most infectious disease models with incorporation of vaccination, the duration of protection has not been much and well considered except some vaccine-age, [41–43], and to our knowledge (except our recent work [44]), no one considered the already vaccinated individuals that need to update their vaccine. In general, they assumed that all vaccinated individuals are in a class without the track of vaccine-protection period. However, the duration of protection provided by vaccination and the update of vaccine could play an important role on the evolution of the epidemic.

The objective of this work is to investigate a new general diffusive Kermack-McKendrick SIR epidemic model that takes into account the temporary protection and the specific individuals that are at the end of their previous period of protection [44]. We will use the saturating incidence rate $\beta SI/(1 + aI)$, [45], rather than the bilinear incidence rate βSI. The saturating incidence rate is more realistic and many researchers have studied diffusive epidemic models with such an incidence rate [14, 46, 47]. In our work, we also investigate the existence of traveling waves. We consider a population of individuals $N(t, x)$ divided into four classes: susceptible $S(t, x)$, infected $I(t, x)$, recovered $R(t, x)$ and protected $P(t, x)$, where $t \geq 0$ is the time and $x \in \mathbb{R}$ is the location. Let $p := p(t, x, a)$ be the age distribution of the population of protected individuals with limited duration τ. In fact, the age $a \in [0, \tau]$ is the time since an individual is temporarily protected. So, the total population of protected individuals is given by

$$P(t, x) = \int_0^\tau p(t, x, a)da.$$

The contact rate per infective individual that result in infection is given by the positive constant β. We suppose that the disease confers a long-lasting immunity with a rate μ (recovering rate). The protection rate, through for instance vaccination or drugs with fixed temporary immunity, is given by the positive constant h. The specific protection rate for individuals at the end of their previous period of protection is given by $\alpha \in (0, 1)$. The diffusion rates for susceptible, infected, removed and protected individuals are respectively given by d_S, d_I, d_R and d_p. The interactions between the compartments of the epidemiological model are described in Figure 1.

The model, for S, I and R, is given, for $t > 0$ and $x \in \mathbb{R}$, by

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \dfrac{\partial^2 S}{\partial x^2} - hS(t, x) - \beta S(t, x) g(I(t, x)) + (1 - \alpha)p(t, x, \tau), \\ \dfrac{\partial I}{\partial t} = d_I \dfrac{\partial^2 I}{\partial x^2} - \mu I(t, x) + \beta S(t, x) g(I(t, x)), \\ \dfrac{\partial R}{\partial t} = d_R \dfrac{\partial^2 R}{\partial x^2} + \mu I(t, x), \end{cases} \tag{3}$$

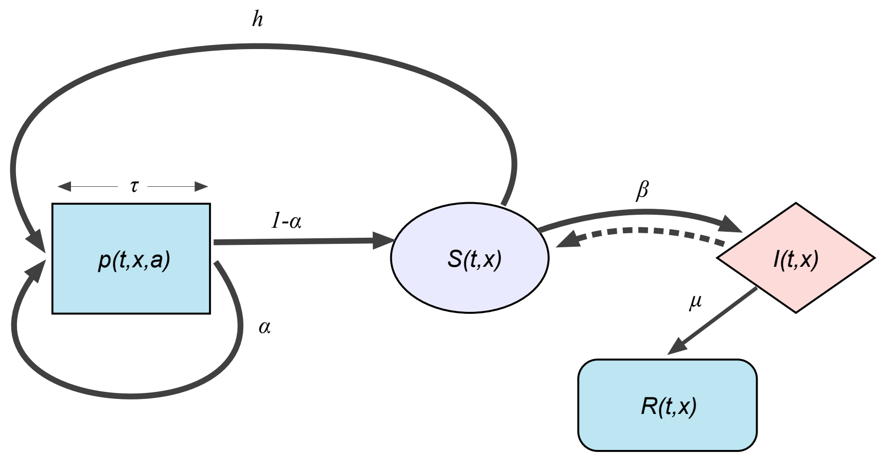

Figure 1: Schematic representation of the interactions between the compartments of the epidemiological model.

where the saturating incidence rate is given by

$$g(I) = \frac{I}{1+aI}, \qquad I \geq 0, \quad a > 0.$$

This rate measures the inhibitory effect which depends on the infected population.

The evolution of the density $p(t,x,a)$ of the protected individuals is given by

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} = d_p \frac{\partial^2 p}{\partial x^2}, \qquad x \in \mathbb{R}, \;\; 0 < a < \tau. \tag{4}$$

The boundary condition for the variable a is given by

$$p(t,x,0) = hS(t,x) + \alpha p(t,x,\tau). \tag{5}$$

The initial conditions, for $x \in \mathbb{R}$ and $a \in [0,\tau]$, are given by

$$S(0,x) = S^0(x),\; I(0,x) = I^0(x),\; R(0,x) = R^0(x) \text{ and } p(0,x,a) = p^0(x,a).$$

We are interested in analyzing the model (3)-(5). In the next section (Section 2), we reduce the model by using the method of characteristics, to a reaction-diffusion system coupled with a delayed difference equation. Section 3 is devoted to the study of the existence and non-existence of traveling wave solutions for the reduced system. Our main result shows that if the initial protected population $p(0, x, a) = u^0$ is constant and if

$$\frac{\beta(1-\alpha)u^0}{h\mu} > 1,$$

then for each

$$c > c^\star := 2\sqrt{d_I\mu\left(\frac{\beta(1-\alpha)u^0}{h\mu} - 1\right)}$$

there exists a positive constant

$$S^\infty < S^0 = \frac{(1-\alpha)u^0}{h},$$

such that the system (3)-(5) has a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying $S(-\infty) = S^0$, $S(+\infty) = S^\infty$ and $I(\pm\infty) = 0$. In Section 4, we finish by some numerical simulations to illustrate our results.

2. Reduction to a differential-difference diffusive model

Let $X = BUC(\mathbb{R}, \mathbb{R})$ be the Banach space of all bounded and uniformly continuous functions from $\mathbb{R}$ to $\mathbb{R}$ with the usual supremum norm $|\cdot|_X$ and $X^+ := \{\phi \in X : \phi(x) \geq 0, \text{ for all } x \in \mathbb{R}\}$. The space X is a Banach lattice under the partial ordering induced by the closed cone X^+. Consider the one-dimensional heat equation

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} &= d_p \dfrac{\partial^2 w(t,x)}{\partial x^2}, \quad t > 0,\ x \in \mathbb{R}, \\ w(0,x) &= w^0(x), \qquad\qquad x \in \mathbb{R}, \end{cases} \tag{6}$$

with $w^0 \in X$. The solution $w(t,x)$ of system (6) can be expressed in terms of the heat kernel

$$\Gamma_p(t,x) = \frac{1}{2\sqrt{d_p \pi t}} \exp\left(-\frac{x^2}{4d_p t}\right), \quad t > 0,\ x \in \mathbb{R}.$$

More precisely,

$$w(t,x) = \int_{-\infty}^{+\infty} \Gamma_p(t, x-y) w^0(y) dy, \quad t > 0,\ x \in \mathbb{R}$$

with

$$\int_{-\infty}^{+\infty} \Gamma_p(t,x)dx = 1 \quad \text{for all} \quad t > 0. \tag{7}$$

It is important to note that the solutions of (6) are well defined for initial conditions w^0 given on the space X. More precisely, if we put

$$(T_p(t)w^0)(x) := \int_{-\infty}^{+\infty} \Gamma_p(t, x-y)w^0(y)dy, \quad t > 0,\ x \in \mathbb{R},$$

we obtain (see for instance [25]) an analytic semigroup $T_p(t) : X \to X$ such that $T_p(t)X^+ \subset X^+$, for all $t \geq 0$.

We solve the age-structured equation (4) using the characteristics method and the heat kernel. We get, for $x \in \mathbb{R}$,

$$p(t,x,\tau) = \begin{cases} \displaystyle\int_{-\infty}^{+\infty} \Gamma_p(t, x-y)p(0,y,\tau-t)dy, & t \leq \tau, \\ \displaystyle\int_{-\infty}^{+\infty} \Gamma_p(\tau, x-y)p(t-\tau,y,0)dy, & t > \tau. \end{cases}$$

Then, for a large time ($t > \tau$), we have

$$p(t,x,\tau) = \int_{-\infty}^{+\infty} \Gamma_p(\tau, x-y)p(t-\tau,y,0)dy, \quad x \in \mathbb{R}.$$

We put $u(t,x) := p(t,x,0)$ and $P(t,x) := \int_0^\tau p(t,x,s)ds$, $t > 0$ and $x \in \mathbb{R}$. The expression of $p(t,x,a)$ becomes, for $t > \tau \geq a \geq 0$ and $x \in \mathbb{R}$,

$$p(t,x,a) = \int_{-\infty}^{+\infty} \Gamma_p(a, x-y)u(t-a,y)dy.$$

Then, we obtain the following system

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \dfrac{\partial^2 S}{\partial x^2} - hS(t,x) - \beta S(t,x) g(I(t,x)) \\ \qquad\qquad\qquad\qquad + (1-\alpha) \displaystyle\int_{-\infty}^{+\infty} \Gamma_p(\tau, x-y) u(t-\tau, y) dy, \\ \dfrac{\partial I}{\partial t} = d_I \dfrac{\partial^2 I}{\partial x^2} - \mu I(t,x) + \beta S(t,x) g(I(t,x)), \\ \dfrac{\partial R}{\partial t} = d_R \dfrac{\partial^2 R}{\partial x^2} + \mu I(t,x), \\ u(t,x) = hS(t,x) + \alpha \displaystyle\int_{-\infty}^{+\infty} \Gamma_p(\tau, x-y) u(t-\tau, y) dy, \\ P(t,x) = \displaystyle\int_0^{\tau} \int_{-\infty}^{+\infty} \Gamma_p(a, x-y) u(t-a, y) dy da. \end{cases} \tag{8}$$

By doing a translation in time ($t \mapsto t - \tau$), we can consider the system (8) for $t > 0$ and $x \in \mathbb{R}$ with the initial conditions

$$S(0,x) = S^0(x), \quad I(0,x) = I^0(x), \quad R(0,x) = R^0(x),$$

and

$$u(t,x) = u^0(t,x), \quad \text{for} \quad -\tau \leq t \leq 0.$$

P and R depend on S, I and u. However, the equations of S, I and u are independent on P and R. Then, we omit the equations of P and R and we study the following system with S, I and u only

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \dfrac{\partial^2 S}{\partial x^2} - hS(t,x) - \beta S(t,x) g(I(t,x)) \\ \qquad\qquad\qquad\qquad + (1-\alpha) \displaystyle\int_{-\infty}^{+\infty} \Gamma_p(\tau, x-y) u(t-\tau, y) dy, \\ \dfrac{\partial I}{\partial t} = d_I \dfrac{\partial^2 I}{\partial x^2} - \mu I(t,x) + \beta S(t,x) g(I(t,x)), \\ u(t,x) = hS(t,x) + \alpha \displaystyle\int_{-\infty}^{+\infty} \Gamma_p(\tau, x-y) u(t-\tau, y) dy. \end{cases} \tag{9}$$

This system is completed by the initial conditions

$$S(0,x) = S^0(x), \;\; I(0,x) = I^0(x) \quad \text{and} \quad u(t,x) = u^0(t,x), \quad \text{for} \;\; -\tau \leq t \leq 0.$$

System (9) shows nonlocal terms in time and space that is caused by individuals moving during the protection phase. If $\alpha = 0$, the system (9) becomes

$$\begin{cases} \dfrac{\partial S}{\partial t} = d_S \dfrac{\partial^2 S}{\partial x^2} - hS(t,x) - \beta S(t,x) g(I(t,x)) \\ \qquad\qquad\qquad\qquad + h \displaystyle\int_{-\infty}^{+\infty} \Gamma_p(\tau, x-y) S(t-\tau, y) dy, \\ \dfrac{\partial I}{\partial t} = d_I \dfrac{\partial^2 I}{\partial x^2} - \mu I(t,x) + \beta S(t,x) g(I(t,x)). \end{cases} \tag{10}$$

Traveling wave solutions for systems with time-delay and non-local spatial term similar to System (10), have been widely studied in [22, 24, 48–51] and the references therein.

3. Traveling wave solutions

Our objective in this section is to show the existence of traveling wave solutions for the system (9). We will use the method based on the notion of upper and lower solutions combined with Schauder's fixed point theorem [12, 22].

Let $S(0,x) = S^0$ and $p(0,x,a) = u^0$ be constant at the beginning of the epidemic with everyone either susceptible or protected. By the translation $t \mapsto t - \tau$, we can see that the initial condition $u^0(t,x) := u^0$, for $t \in [-\tau, 0]$. On the other hand, $(S^0, 0, u^0)$ is a constant steady state of the system (9) if and only if

$$S^0 = \frac{(1-\alpha)u^0}{h}. \tag{11}$$

We are interested in the influence of initial protected population u^0 on the epidemic. Then, we parametrize the system in terms of the constant u^0. We will refer to

$$\left(\frac{(1-\alpha)u^0}{h}, 0, u^0 \right), \qquad u^0 > 0,$$

as the disease free steady states of System (9).

A traveling wave solution of (9) is a special solution of the form

$$(S(t,x), I(t,x), u(t,x)) = (\phi(x+ct), \psi(x+ct), U(x+ct)),$$

where $\phi, \psi, U \in C^2(\mathbb{R}, \mathbb{R}^+)$ and $c > 0$ is a constant corresponding to the wave speed [25, 52]. We will look for positive constants S^∞ and u^∞ such that the system (9) has nontrivial traveling wave solutions satisfying

$$\begin{aligned} &\phi(-\infty) = S^0 := \frac{(1-\alpha)u^0}{h}, \quad \psi(-\infty) = 0, \quad U(-\infty) = u^0, \\ &\phi(+\infty) = S^\infty, \quad \psi(+\infty) = 0, \quad U(+\infty) = u^\infty, \\ &S^\infty < S^0, \quad u^\infty < u^0, \quad \phi(\cdot) \geq 0, \quad \psi(\cdot) \geq 0 \quad \text{and} \quad U(\cdot) \geq 0. \end{aligned} \tag{12}$$

Such solutions describe epidemic waves that may show the spatial propagation of a disease with a speed c. In order to state the mathematical results, we put $z = x + ct$ and we obtain the corresponding wave system

$$\begin{cases} c\phi'(z) = d_S \phi''(z) - h\phi(z) - \beta\phi(z)g(\psi(z)) + (1-\alpha)\left(T_p(\tau)U\right)(z - c\tau), \\ c\psi'(z) = d_I \psi''(z) - \mu\psi(z) + \beta\phi(z)g(\psi(z)), \\ U(z) = h\phi(z) + \alpha\left(T_p(\tau)U\right)(z - c\tau), \end{cases} \tag{13}$$

subject to the following asymptotic conditions

$$\begin{aligned} &\phi(-\infty) = S^0, \quad \psi(-\infty) = 0, \quad U(-\infty) = u^0, \\ &\phi(+\infty) = S^\infty, \quad \psi(+\infty) = 0 \quad \text{and} \quad U(+\infty) = u^\infty. \end{aligned}$$

Let $A_p : X \to X$ be the bounded linear operator defined by

$$\left(A_p U\right)(z) = \alpha\left(T_p(\tau)U\right)(z - c\tau), \quad z \in \mathbb{R}.$$

Lemma 3.1. *The operator $Id - A_p$ is invertible and its inverse is given by*

$$(Id - A_p)^{-1}(U) = \Sigma * U, \quad U \in X,$$

where

$$\Sigma(z) = \sum_{n=0}^{+\infty} \Sigma_n(z), \quad z \in \mathbb{R}, \tag{14}$$

and

$$\Sigma_n(z) = \frac{\alpha^n}{2\sqrt{n d_p \pi \tau}} \exp\left(-\frac{(z - nc\tau)^2}{4 n d_p \tau}\right) = \alpha^n \Gamma_p(n\tau, z - nc\tau), \; z \in \mathbb{R}, \; n \in \mathbb{N}.$$

Proof. Thanks to (7), we have $|T_p(\tau)|_{\mathcal{L}(X)} = 1$. Then, $|A_p|_{\mathcal{L}(X)} = \alpha < 1$. Furthermore,

$$\begin{aligned}\left(A_p^n U\right)(z) &= \alpha^n \left(T_p(n\tau)U\right)(z - nc\tau),\\ &= \frac{\alpha^n}{2\sqrt{nd_p\pi\tau}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(z-y-nc\tau)^2}{4nd_p\tau}\right) U(y)dy, \quad n \in \mathbb{N}^*.\end{aligned}$$

Then, the operator $Id - A_p$ is invertible and its inverse is

$$\begin{aligned}\left((Id - A_p)^{-1}U\right)(z) &= \sum_{n=0}^{+\infty} (A_p^n U)(z),\\ &= \int_{-\infty}^{+\infty} \sum_{n=0}^{+\infty} \frac{\alpha^n}{2\sqrt{nd_p\pi\tau}} \exp\left(-\frac{(z-y-nc\tau)^2}{4nd_p\tau}\right) U(y)dy,\\ &= \int_{-\infty}^{+\infty} \Sigma(z-y)U(y)dy,\\ &= \left(\Sigma * U\right)(z),\end{aligned}$$

where Σ is given by the expression (14). This completes the proof. □

Remark 1. Σ_n, $n \in \mathbb{N}$, has the following fundamental property

$$\int_{-\infty}^{+\infty} \Sigma_n(y)dy = \alpha^n.$$

Thanks to Lemma 3.1, the system (13) becomes

$$\begin{cases} c\phi'(z) = d_S\phi''(z) - h\phi(z) - \beta\phi(z)g(\psi(z)) + (1-\alpha)\left(T_p(\tau)U\right)(z - c\tau),\\ c\psi'(z) = d_I\psi''(z) - \mu\psi(z) + \beta\phi(z)g(\psi(z)),\\ U(z) = h(\Sigma * \phi)(z), \end{cases} \tag{15}$$

where

$$(\Sigma * \phi)(z) := \int_{-\infty}^{+\infty} \Sigma(z-y)\phi(y)dy, \qquad z \in \mathbb{R}.$$

Remark 2. From the third equation of (15) and using (11), we can remark that if $\phi(-\infty) = S^0$, then formally we have

$$U(-\infty) = \lim_{z\to-\infty} h(\Sigma * \phi)(z) = \frac{hS^0}{1-\alpha} = u^0.$$

If $\phi(+\infty) = S^\infty < S^0$, we obtain

$$U(+\infty) = \lim_{z\to+\infty} h(\Sigma * \phi)(z) = \frac{hS^\infty}{1-\alpha} =: u^\infty < u^0.$$

In fact, the system (15) can be reduced to the following system

$$\begin{cases} c\phi'(z) = d_S\phi''(z) - h\phi(z) - \beta\phi(z)g(\psi(z)) + h(1-\alpha)\left[T_p(\tau)\left(\Sigma * \phi\right)\right](z - c\tau), \\ c\psi'(z) = d_I\psi''(z) - \mu\psi(z) + \beta\phi(z)g(\psi(z)). \end{cases} \tag{16}$$

Remember that

$$T_p(\tau)\left(\Sigma * \phi\right) = \Gamma_p(\tau,\cdot) * \left(\Sigma * \phi\right) = \left(\Gamma_p(\tau,\cdot) * \Sigma\right) * \phi.$$

Let τ_a, $a \in \mathbb{R}$, be the translation operator defined on X^+ by $(\tau_a\psi)(z) = \psi(z-a)$. Then,

$$\left(T_p(\tau)\left(\Sigma * \phi\right)\right)(z - c\tau) = \left[\tau_{c\tau}\left(T_p(\tau)\left(\Sigma * \phi\right)\right)\right](z) = \left[\left(\tau_{c\tau}\left(\Gamma_p(\tau,\cdot) * \Sigma\right)\right) * \phi\right](z).$$

We put

$$\chi = (1-\alpha)\,\tau_{c\tau}\left(\Gamma_p(\tau,\cdot) * \Sigma\right). \tag{17}$$

Then, $\chi : \mathbb{R} \to \mathbb{R}^+$ and

$$\left(T_p(\tau)\left(\Sigma * \phi\right)\right)(z - c\tau) = \frac{1}{1-\alpha}\left(\chi * \phi\right)(z).$$

Consequently, the system (16) becomes

$$\begin{cases} c\phi'(z) = d_S\phi''(z) - h\phi(z) - \beta\phi(z)g(\psi(z)) + h\left(\chi * \phi\right)(z), \\ c\psi'(z) = d_I\psi''(z) - \mu\psi(z) + \beta\phi(z)g(\psi(z)). \end{cases} \tag{18}$$

From now on, we focus on this problem with the following asymptotic condition

$$\phi(-\infty) = S^0 := \frac{(1-\alpha)u^0}{h}, \quad \phi(+\infty) = S^\infty < S^0, \quad \psi(\pm\infty) = 0. \tag{19}$$

We have the lemma.

Lemma 3.2. *The function* $\chi : \mathbb{R} \to \mathbb{R}^+$ *defined by* (17) *belongs to* $L^1(\mathbb{R}, \mathbb{R}^+)$ *and satisfies the following properties, for* $z \in \mathbb{R}$, *1.* $\chi(z) = (1-\alpha)\sum_{n=0}^{+\infty} \alpha^n \Gamma_p\left((n+1)\tau, z-(n+1)c\tau\right)$,

2. $\int_{-\infty}^{+\infty} \chi(y)\mathrm{d}y = (1-\alpha)\int_{-\infty}^{+\infty} \Sigma(y)\mathrm{d}y = 1$,

3. there exists $\lambda^-(c) < 0 < \lambda^+(c)$ *given by*

$$\lambda^\pm(c) := \frac{c}{2d_p}\left(1 \pm \sqrt{1+\frac{4d_p}{\tau c^2}\ln\left(\frac{1}{\alpha}\right)}\right), \tag{20}$$

such that, for $\lambda \in (\lambda^-(c), \lambda^+(c))$,

$$1-\alpha e^{d_p\tau\lambda^2 - c\tau\lambda} > 0 \quad \textit{and} \quad \left(\chi * e^{\lambda\cdot}\right)(z) = \frac{(1-\alpha)\, e^{d_p\tau\lambda^2+(z-c\tau)\lambda}}{1-\alpha e^{d_p\tau\lambda^2-c\tau\lambda}}.$$

Proof. 1. and 2. come from the definition of χ and the property (7).
Let $\lambda \in \mathbb{R}$. To prove 3., we first remark that by a simple calculation, we have the following fundamental property

$$\left(\Gamma_p(t,\cdot) * e^{\lambda\cdot}\right)(z) = e^{d_p t\lambda^2+\lambda z}, \quad \text{for } \ t>0, \ \ z \in \mathbb{R}.$$

Then,

$$\begin{aligned}\left(\chi * e^{\lambda\cdot}\right)(z) &= (1-\alpha)\, e^{\lambda z}\sum_{n=0}^{+\infty}\alpha^n e^{d_p(n+1)\tau\lambda^2-(n+1)c\tau\lambda}, \\ &= (1-\alpha)\, e^{d_p\tau\lambda^2+(z-c\tau)\lambda}\sum_{n=0}^{+\infty}\left(\alpha e^{d_p\tau\lambda^2-c\tau\lambda}\right)^n.\end{aligned}$$

To obtain $\chi * e^{\lambda\cdot} < +\infty$, we have to choose $\lambda \in \mathbb{R}$, such that

$$\alpha e^{d_p\tau\lambda^2-c\tau\lambda} < 1.$$

This is equivalent to

$$d_p\tau\lambda^2 - c\tau\lambda - \ln\left(\frac{1}{\alpha}\right) < 0,$$

which is satisfied for all $\lambda \in (\lambda^-(c), \lambda^+(c))$, with $\lambda^-(c)$ and $\lambda^+(c)$ given by (20). This completes the proof. $\square$

Next, we define and construct upper and lower solutions of (18).

Definition 3.3. A continuous function $(\overline{\phi}, \overline{\psi}) \in C(\mathbb{R}, \mathbb{R}^2)$ is called an upper solution of (18) if $\overline{\phi}'$, $\overline{\phi}''$, $\overline{\psi}'$ and $\overline{\psi}''$ exist almost everywhere (a.e.) and satisfy

$$\begin{cases} c\overline{\phi}'(z) \geq d_S\overline{\phi}''(z) - h\overline{\phi}(z) - \beta\overline{\phi}(z)g(\underline{\psi}(z)) + h\left(\chi * \overline{\phi}\right)(z), & \text{a.e. in } \mathbb{R}, \\ c\overline{\psi}'(z) \geq d_I\overline{\psi}''(z) - \mu\overline{\psi}(z) + \beta\overline{\phi}(z)g(\overline{\psi}(z)), & \text{a.e. in } \mathbb{R}. \end{cases}$$

A lower solution $(\underline{\phi}, \underline{\psi}) \in C(\mathbb{R}, \mathbb{R}^2)$ of (18) is defined in a similar way by

$$\begin{cases} c\underline{\phi}'(z) \leq d_S\underline{\phi}''(z) - h\underline{\phi}(z) - \beta\underline{\phi}(z)g(\overline{\psi}(z)) + h\left(\chi * \underline{\phi}\right)(z), & \text{a.e. in } \mathbb{R}, \\ c\underline{\psi}'(z) \leq d_I\underline{\psi}''(z) - \mu\underline{\psi}(z) + \beta\underline{\phi}(z)g(\underline{\psi}(z)), & \text{a.e. in } \mathbb{R}. \end{cases}$$

The existence of traveling wave solutions needs to find suitable upper and lower solutions of (18). For this aim, linearizing the second equation of (18) at the disease free point $(S^0, 0)$. As $g'(0) = 1$, we obtain

$$c\psi'(z) = d_I\psi''(z) - \mu\psi(z) + \beta S^0\psi(z), \quad z \in \mathbb{R}.$$

Define the transcendental characteristic function, for $\lambda \in \mathbb{R}$,

$$\Delta_c(\lambda) = c\lambda - d_I\lambda^2 + \mu - \beta S^0, \tag{21}$$

where S^0 is given by (11). We denote by $\lambda(c)$, $c > 0$, the real positive roots of (21). It is not difficult to prove the following result.

Lemma 3.4. *Assume that*

$$\frac{\beta(1-\alpha)u^0}{h\mu} > 1. \tag{22}$$

Then,

$$c^\star = 2\sqrt{d_I\mu\left(\frac{\beta(1-\alpha)u^0}{h\mu} - 1\right)} \quad \textit{and} \quad \lambda^\star := \lambda(c^\star) = \sqrt{\frac{\beta(1-\alpha)u^0 - h\mu}{hd_I}} \tag{23}$$

are the unique positive reals such that

1. $\Delta_{c^\star}(\lambda^\star) = \dfrac{\partial}{\partial\lambda}\Delta_{c^\star}(\lambda)\big|_{\lambda=\lambda^\star} = 0,$

2. *if $c > c^\star$, there exist two real roots*

$$\lambda_1(c) = \frac{c}{2d_I}\left(1-\sqrt{1-\left(\frac{c^\star}{c}\right)^2}\right) \quad \text{and} \quad \lambda_2(c) = \frac{c}{2d_I}\left(1+\sqrt{1-\left(\frac{c^\star}{c}\right)^2}\right)$$

of the equation $\Delta_c(\lambda) = 0$ such that $0 < \lambda_1(c) < \lambda_2(c)$ and $\Delta_c(\lambda) > 0$ for all $\lambda \in (\lambda_1(c), \lambda_2(c))$,

3. *if $0 < c < c^\star$, $\Delta_c(\lambda) < 0$ for all $\lambda \in \mathbb{R}$.*

Remark 3. From Lemma 3.2, we note that we have to take into account the threshold

$$\lambda^+(c) := \frac{c}{2d_p}\left(1+\sqrt{1+\frac{4d_p}{\tau c^2}\ln\left(\frac{1}{\alpha}\right)}\right).$$

Next, we assume that (22) is satisfied. We fix $c > c^\star$, where $c^\star$ is given by (23), and we put

$$\lambda_1 := \lambda_1(c), \quad \lambda_2 := \lambda_2(c) \quad \text{and} \quad \lambda^+ := \lambda^+(c).$$

The following lemma ensures the existence of continuous upper and lower solutions of (18).

Lemma 3.5. *The function $(\overline{\phi}, \overline{\psi}) : \mathbb{R} \to (\mathbb{R}^+)^2$ defined by*

$$\overline{\phi}(z) = S^0 := \frac{(1-\alpha)u^0}{h} \qquad \text{and} \qquad \overline{\psi}(z) = e^{\lambda_1 z},$$

is an upper solution of (18) and the function $(\underline{\phi}, \underline{\psi}) : \mathbb{R} \to (\mathbb{R}^+)^2$ defined by

$$\underline{\phi}(z) = \max\{0, S^0 - \eta_1 e^{\epsilon_1 z}\} \qquad \text{and} \qquad \underline{\psi}(z) = \max\{0, e^{\lambda_1 z} - \eta_2 e^{(\lambda_1+\epsilon_2)z}\},$$

with $\epsilon_1 \in (0, \min\{\lambda^+, \lambda_1\})$, $\epsilon_2 \in (0, \min\{\epsilon_1, \lambda_2 - \lambda_1\})$, and $\eta_1 > S^0$, $\eta_2 > 0$ well-chosen, is a lower solution of (18). Moreover, we have $\underline{\phi}(z) \le \overline{\phi}(z)$ and $\underline{\psi}(z) \le \overline{\psi}(z)$, for all $z \in \mathbb{R}$.

Proof. First, we will prove that $(\overline{\phi}, \overline{\psi})$ is an upper solution of (18). We have $\overline{\phi}'(z) = \overline{\phi}''(z) = 0$ and $\overline{\psi}'(z) = \lambda_1 e^{\lambda_1 z}$, $\overline{\psi}''(z) = \lambda_1^2 e^{\lambda_1 z}$. By using Lemma 3.2 and the fact that $g(I) \le I$ for all $I \ge 0$, we get, for $z \in \mathbb{R}$,

$$\begin{cases} c\overline{\phi}'(z) - d_S\overline{\phi}''(z) + h\overline{\phi}(z) + \beta\overline{\phi}(z)g(\underline{\psi}(z)) - h\left(\chi * \overline{\phi}\right)(z) = \beta S^0 g(\underline{\psi}(z)) & \ge 0, \\ c\overline{\psi}'(z) - d_I\overline{\psi}''(z) + \mu\overline{\psi}(z) - \beta\overline{\phi}(z)g(\overline{\psi}(z)) \ge e^{\lambda_1 z}\left(c\lambda_1 - d_I\lambda_1^2 + \mu - \beta S^0\right) & = 0. \end{cases}$$

We conclude that $(\overline{\phi}, \overline{\psi})$ is an upper solution of (18).

Now, we focus on proving that $(\underline{\phi}, \underline{\psi})$ is a lower solution of (18). We start by the component $\underline{\phi}$. For $z \in \mathbb{R}$, we have $\underline{\phi}(z) = \max\{0, S^0 - \eta_1 e^{\epsilon_1 z}\}$. As $\epsilon_1 > 0$ there exists $z_1 \in \mathbb{R}$ such that

$$\underline{\phi}(z) = \begin{cases} 0, & z \geq z_1, \\ S^0 - \eta_1 e^{\epsilon_1 z}, & z < z_1. \end{cases}$$

We can choose $z_1 := (1/\epsilon_1) \ln (S^0/\eta_1)$, which is negative because $S^0 < \eta_1$. Suppose that $z \in (z_1, +\infty)$. Then, we have $\underline{\phi}(z) = 0$. Using the fact that $\underline{\phi}$ is nonnegative, then the following inequality holds, for $z \in (z_1, +\infty)$,

$$c\underline{\phi}'(z) \leq d_S \underline{\phi}''(z) - h\underline{\phi}(z) - \beta \underline{\phi}(z) g(\overline{\psi}(z)) + h \left(\chi * \underline{\phi}\right)(z).$$

Suppose that $z \in (-\infty, z_1)$. This means that $\underline{\phi}(z) = S^0 - \eta_1 e^{\epsilon_1 z}$. In this case,

$$\begin{aligned} & c\underline{\phi}'(z) - d_S \underline{\phi}''(z) + h\underline{\phi}(z) + \beta \underline{\phi}(z) g(\overline{\psi}(z)) - h \left(\chi * \underline{\phi}\right)(z) \\ & \leq -c\eta_1 \epsilon_1 e^{\epsilon_1 z} + d_S \eta_1 \epsilon_1^2 e^{\epsilon_1 z} + h(S^0 - \eta_1 e^{\epsilon_1 z}) + \beta (S^0 - \eta_1 e^{\epsilon_1 z}) e^{\lambda_1 z} - h \left(\chi * \underline{\phi}\right)(z). \end{aligned}$$

Obviously, we have

$$\begin{aligned} & c\underline{\phi}'(z) - d_S \underline{\phi}''(z) + h\underline{\phi}(z) + \beta \underline{\phi}(z) g(\overline{\psi}(z)) - h \left(\chi * \underline{\phi}\right)(z) \\ & \leq e^{\epsilon_1 z} \left[-c\eta_1 \epsilon_1 + d_S \eta_1 \epsilon_1^2 - \eta_1 h + \beta S^0 e^{(\lambda_1 - \epsilon_1) z} + \eta_1 h \frac{(1-\alpha)\, e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}}{1 - \alpha e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}} \right]. \end{aligned}$$

The objective is to find $\epsilon_1 > 0$ and $\eta_1 > S^0$ such that, for all $z \in (-\infty, z_1)$

$$-c\eta_1 \epsilon_1 + d_S \eta_1 \epsilon_1^2 - \eta_1 h + \beta S^0 e^{(\lambda_1 - \epsilon_1) z} + \eta_1 h \frac{(1-\alpha)\, e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}}{1 - \alpha e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}} < 0.$$

In fact, we have for $z \in (-\infty, z_1)$

$$\begin{aligned} & -c\eta_1 \epsilon_1 + d_S \eta_1 \epsilon_1^2 - \eta_1 h + \beta S^0 e^{(\lambda_1 - \epsilon_1) z} + \eta_1 h \frac{(1-\alpha)\, e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}}{1 - \alpha e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}} \\ & < -c\eta_1 \epsilon_1 + d_S \eta_1 \epsilon_1^2 + \beta S^0 \left(\frac{S^0}{\eta_1} \right)^{(\lambda_1 - \epsilon_1)/\epsilon_1} + \eta_1 h \frac{-1 + e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}}{1 - \alpha e^{d_p \tau \epsilon_1^2 - c\tau\epsilon_1}}. \end{aligned}$$

We can choose $\eta_1\epsilon_1 = 1$, with ϵ_1 small (or η_1 large) to obtain

$$-c\eta_1\epsilon_1 + d_S\eta_1\epsilon_1^2 + \beta S^0\left(\frac{S^0}{\eta_1}\right)^{(\lambda_1-\epsilon_1)/\epsilon_1} + \eta_1 h\frac{-1+e^{d_p\tau\epsilon_1^2-c\tau\epsilon_1}}{1-\alpha e^{d_p\tau\epsilon_1^2-c\tau\epsilon_1}} < 0.$$

Or, in terms of $\epsilon_1 = 1/\eta_1$,

$$-c + \epsilon_1 d_S + \beta S^0\left(\epsilon_1 S^0\right)^{(\lambda_1-\epsilon_1)/\epsilon_1} + h\frac{-1+e^{d_p\tau\epsilon_1^2-c\tau\epsilon_1}}{\epsilon_1\left(1-\alpha e^{d_p\tau\epsilon_1^2-c\tau\epsilon_1}\right)} < 0.$$

Then, it suffices to choose

$$\epsilon_1 < \min\left\{\frac{c^\star}{d_p}, \frac{1}{S^0}\right\},$$

to get, for $z \in (-\infty, z_1)$,

$$c\underline{\phi}'(z) \leq d_S\underline{\phi}''(z) - h\underline{\phi}(z) - \beta\underline{\phi}(z)g(\overline{\psi}(z)) + h\left(\chi * \underline{\phi}\right)(z).$$

Now, we will prove that, for $z \in \mathbb{R}$,

$$c\underline{\psi}'(z) \leq d_I\underline{\psi}''(z) - \mu\underline{\psi}(z) + \beta\underline{\phi}(z)g(\underline{\psi}(z)).$$

Obviously, if $\underline{\psi}$ is null, then the inequality holds. Let's focus on the case where $\underline{\psi}(z) = e^{\lambda_1 z} - \eta_2 e^{(\lambda_1+\epsilon_2)z}$, $z \in \mathbb{R}$. We distinguish two cases:

- Suppose that $z \in [z_1, +\infty)$. Then, it is needed to prove that

$$c\underline{\psi}'(z) \leq d_I\underline{\psi}''(z) - \mu\underline{\psi}(z). \tag{24}$$

  Using the expression of $\underline{\psi}$, we have

$$\begin{aligned} c\underline{\psi}'(z) - d_I\underline{\psi}''(z) + \mu\underline{\psi}(z) &= c\lambda_1 e^{\lambda_1 z} - c(\lambda_1+\epsilon_2)\eta_2 e^{(\lambda_1+\epsilon_2)z} \\ &\quad -d_I\lambda_1^2 e^{\lambda_1 z} + d_I(\lambda_1+\epsilon_2)^2\eta_2 e^{(\lambda_1+\epsilon_2)z} + \mu e^{\lambda_1 z} - \mu\eta_2 e^{(\lambda_1+\epsilon_2)z}, \\ &\leq e^{\lambda_1 z}\left(\beta S^0 - \eta_2 e^{\epsilon_2 z_1}\left[\Delta_c(\lambda_1+\epsilon_2) + \beta S^0\right]\right). \end{aligned}$$

  For a sufficiently large $\eta_2 > 1$, we get the inequality (24).

- Suppose that $z \in (-\infty, z_1)$. We need to prove that

$$c\underline{\psi}'(z) \leq d_I \underline{\psi}''(z) - \mu\underline{\psi}(z) + \beta(S^0 - \eta_1 e^{\epsilon_1 z}) g(\underline{\psi}(z)).$$

The function g satisfies the following estimation

$$g(I) \geq I(1 - aI) = I - aI^2, \qquad \text{for all} \quad I \geq 0.$$

Again, if we use the expression of $\underline{\psi}$, we have

$$\begin{aligned}
& c\underline{\psi}'(z) - d_I\underline{\psi}''(z) + \mu\underline{\psi}(z) - \beta\underline{\phi}(z)g(\underline{\psi}(z)) \\
\leq\ & c\underline{\psi}'(z) - d_I\underline{\psi}''(z) + \mu\underline{\psi}(z) - \beta\underline{\phi}(z)\underline{\psi}(z) + a\beta\underline{\phi}(z)(\underline{\psi}(z))^2 \\
\leq\ & c\lambda_1 e^{\lambda_1 z} - c(\lambda_1 + \epsilon_2)\eta_2 e^{(\lambda_1+\epsilon_2)z} - d_I\lambda_1^2 e^{\lambda_1 z} + d_I(\lambda_1 + \epsilon_2)^2\eta_2 e^{(\lambda_1+\epsilon_2)z} \\
& +\mu e^{\lambda_1 z} - \mu\eta_2 e^{(\lambda_1+\epsilon_2)z} - \beta(S^0 - \eta_1 e^{\epsilon_1 z})(e^{\lambda_1 z} - \eta_2 e^{(\lambda_1+\epsilon_2)z}) + a\beta S^0 e^{2\lambda_1 z}, \\
=\ & e^{\lambda_1 z}\left(\beta S^0 - \eta_2 e^{\epsilon_2 z}\left[\Delta_c(\lambda_1 + \epsilon_2) + \beta S^0\right]\right) - \beta S^0 e^{\lambda_1 z} + \beta S^0\eta_2 e^{(\lambda_1+\epsilon_2)z} \\
& +\beta\eta_1 e^{(\lambda_1+\epsilon_1)z} - \beta\eta_1\eta_2 e^{(\lambda_1+\epsilon_2+\epsilon_1)z} + a\beta S^0 e^{2\lambda_1 z}, \\
=\ & e^{\lambda_1 z}\left(-\eta_2 e^{\epsilon_2 z}\Delta_c(\lambda_1 + \epsilon_2) + \beta\eta_1 e^{\epsilon_1 z} - \beta\eta_1\eta_2 e^{(\epsilon_2+\epsilon_1)z} + a\beta S^0 e^{\lambda_1 z}\right), \\
=\ & e^{(\lambda_1+\epsilon_2)z}\left(-\eta_2\Delta_c(\lambda_1 + \epsilon_2) + \beta\eta_1 e^{(\epsilon_1-\epsilon_2)z} - \beta\eta_1\eta_2 e^{\epsilon_1 z} + a\beta S^0 e^{(\lambda_1-\epsilon_2)z}\right), \\
\leq\ & e^{(\lambda_1+\epsilon_2)z}\left(-\eta_2\Delta_c(\lambda_1 + \epsilon_2) + \beta\eta_1 - \beta\eta_1\eta_2 e^{\epsilon_1 z} + a\beta S^0\right).
\end{aligned}$$

We choose η_2 large enough such that

$$-\eta_2\Delta_c(\lambda_1 + \epsilon_2) + \beta\eta_1 - \beta\eta_1\eta_2 e^{\epsilon_1 z} + a\beta S^0 < 0.$$

Then, we obtain, for $z \in (-\infty, z_1)$,

$$c\underline{\psi}'(z) \leq d_I\underline{\psi}''(z) - \mu\underline{\psi}(z) + \beta\underline{\phi}(z)g(\underline{\psi}(z)).$$

The proof is completed.

□

Let $(\overline{\phi}, \overline{\psi})$ and $(\underline{\phi}, \underline{\psi})$ be the upper and lower solutions, respectively, given by Lemma 3.5. We define the profile set of traveling wave fronts as

$$\mathit{\Theta} = \left\{ \begin{array}{l} (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2): \quad \text{for all } z \in \mathbb{R}, \\ \underline{\phi}(z) \leq \phi(z) \leq \overline{\phi}(z) \ \text{ and } \ \underline{\psi}(z) \leq \psi(z) \leq \overline{\psi}(z) \end{array} \right\}.$$

In particular, we have $(\underline{\phi}, \underline{\psi})$, $(\overline{\phi}, \overline{\psi}) \in \Theta$ ($\Theta \neq \emptyset$). Define the operators $H_1, H_2 : \Theta \to C(\mathbb{R}, \mathbb{R})$ by

$$\begin{pmatrix} H_1(\phi,\psi)(z) \\ H_2(\phi,\psi)(z) \end{pmatrix} = \begin{pmatrix} B_1\phi(z) - \phi(z)g(\psi(z)) + \dfrac{h}{\beta}\left(\chi * \phi\right)(z) \\ \phi(z)g(\psi(z)) + \dfrac{B_2-\mu}{\beta}\psi(z) \end{pmatrix},$$

where $B_1 > 0$ and $B_2 > 0$ are large constants such that

$$B_1 > \frac{1}{a} = \sup_{I \geq 0} g(I) \quad \text{and} \quad B_2 > \mu. \tag{25}$$

We also consider the operator $F : \Theta \to C(\mathbb{R}, \mathbb{R}^2)$ defined by

$$F(\phi,\psi)(z) = \begin{pmatrix} F_1(\phi,\psi)(z) \\ F_2(\phi,\psi)(z) \end{pmatrix},$$

with

$$F_1(\phi,\psi)(z) = \frac{\beta}{\nu_1}\int_{-\infty}^{z} e^{\kappa_{11}(z-s)}H_1(\phi,\psi)(s)ds + \frac{\beta}{\nu_1}\int_{z}^{+\infty} e^{\kappa_{12}(z-s)}H_1(\phi,\psi)(s)ds,$$

and

$$F_2(\phi,\psi)(z) = \frac{\beta}{\nu_2}\int_{-\infty}^{z} e^{\kappa_{21}(z-s)}H_2(\phi,\psi)(s)ds + \frac{\beta}{\nu_2}\int_{z}^{+\infty} e^{\kappa_{22}(z-s)}H_2(\phi,\psi)(s)ds,$$

where

$$\kappa_{11} := \frac{c}{2d_S}\left(1 - \sqrt{1 + \frac{4d_S(\beta B_1 + h)}{c^2}}\right),\ \kappa_{12} := \frac{c}{2d_S}\left(1 + \sqrt{1 + \frac{4d_S(\beta B_1 + h)}{c^2}}\right),$$

$$\kappa_{21} := \frac{c}{2d_I}\left(1 - \sqrt{1 + \frac{4d_I B_2}{c^2}}\right), \quad \kappa_{22} := \frac{c}{2d_I}\left(1 + \sqrt{1 + \frac{4d_I B_2}{c^2}}\right),$$

$$\nu_1 := d_S(\kappa_{12} - \kappa_{11}) \quad \text{and} \quad \nu_2 := d_I(\kappa_{22} - \kappa_{21}).$$

We can easily see that the operator F is well defined. The existence of solutions for the system (18) can be transformed to the existence of fixed points of the operator F. The parameters κ_{11} and κ_{21} are negative. We choose $B_1 > 0$ and $B_2 > 0$ large enough such that

$$2\lambda_1 < \min\{-\kappa_{11}, -\kappa_{21}\}.$$

Let $\nu \in (2\lambda_1, \min\{-\kappa_{11}, -\kappa_{21}\})$. We define the Banach space $(B_\nu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\nu)$ with

$$\begin{aligned} B_\nu(\mathbb{R}, \mathbb{R}^2) = &\big\{\Psi = (\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) : \\ &\sup_{z\in\mathbb{R}} |\phi(z)|e^{-\nu|z|} < +\infty, \ \sup_{z\in\mathbb{R}} |\psi(z)|e^{-\nu|z|} < +\infty\big\} \end{aligned}$$

and $|\cdot|_\nu$ is the exponential decay norm

$$|\Psi|_\nu = \max\left\{\sup_{z\in\mathbb{R}} |\phi(z)|e^{-\nu|z|}, \ \sup_{z\in\mathbb{R}} |\psi(z)|e^{-\nu|z|}\right\}, \quad \text{for} \quad \Psi \in B_\nu(\mathbb{R}, \mathbb{R}^2).$$

The next lemma states some results that ensure the existence of fixed points for the operator F in Θ.

Lemma 3.6. *1. The set Θ is nonempty, bounded, closed and convex in $(B_\nu(\mathbb{R}, \mathbb{R}^2), |\cdot|_\nu)$.*

2. The operator F maps Θ into Θ.

Proof. The proof of 1. is obvious and it is omitted.
For the proof of 2., we consider $(\phi, \psi) \in \Theta$. It is sufficient to show that, for all $z \in \mathbb{R}$,

$$\begin{aligned} \underline{\phi}(z) &\le F_1(\phi, \psi)(z) \le \overline{\phi}(z), \\ \underline{\psi}(z) &\le F_2(\phi, \psi)(z) \le \overline{\psi}(z). \end{aligned}$$

The operator F_2 (given by the second equation of (18)) looks like the one in [12] and we can do the same study. Then, in a similar way, we can show that $\underline{\psi}(z) \le F_2(\phi, \psi)(z) \le \overline{\psi}(z)$ for all $z \in \mathbb{R}$.
Let's focus on the operator F_1 given, for $z \in \mathbb{R}$, by

$$F_1(\phi, \psi)(z) = \frac{\beta}{\nu_1}\int_{-\infty}^{z} e^{\kappa_{11}(z-s)} H_1(\phi, \psi)(s)ds + \frac{\beta}{\nu_1}\int_{z}^{+\infty} e^{\kappa_{12}(z-s)} H_1(\phi, \psi)(s)ds.$$

Since $(\phi, \psi) \in \Theta$, then $\underline{\phi}(z) \le S^0$, for all $z \in \mathbb{R}$. Thus

$$H_1(\phi, \psi)(z) = B_1\phi(z) - \phi(z)g(\psi(z)) + \frac{h}{\beta}\left(\chi * \phi\right)(z) \le \left(B_1 + \frac{h}{\beta}\right) S^0.$$

So, we obtain, for $z \in \mathbb{R}$,

$$F_1(\phi, \psi)(z) \le \frac{\left(\beta B_1 + h\right) S^0}{\nu_1}\left[\int_{-\infty}^{z} e^{\kappa_{11}(z-s)}ds + \int_{z}^{+\infty} e^{\kappa_{12}(z-s)}ds\right] = S^0 = \overline{\phi}(z).$$

In another hand, by using (25) we have, for $z \neq z_1$,

$$\begin{array}{rcl} c\underline{\phi}'(z) & \leq & d_S\underline{\phi}''(z) - h\underline{\phi}(z) - \beta\underline{\phi}(z)g(\overline{\psi}(z)) + h\left(\chi * \underline{\phi}\right)(z), \\ & \leq & d_S\underline{\phi}''(z) - h\underline{\phi}(z) - \beta B_1\underline{\phi}(z) + \beta\underline{\phi}(z)(B_1 - g(\psi(z))) + h\left(\chi * \phi\right)(z), \\ & \leq & d_S\underline{\phi}''(z) - h\underline{\phi}(z) - \beta B_1\underline{\phi}(z) + \beta\phi(z)(B_1 - g(\psi(z))) + h\left(\chi * \phi\right)(z). \end{array}$$

It follows that, for $z \neq z_1$,

$$\begin{array}{rcl} \beta H_1(\phi,\psi)(z) & = & \beta\phi(z)(B_1 - g(\psi(z))) + h\left(\chi * \phi\right)(z), \\ & \geq & c\underline{\phi}'(z) - d_S\underline{\phi}''(z) + \left(\beta B_1 + h\right)\underline{\phi}(z). \end{array}$$

Hence, we have

$$\begin{aligned} F_1(\phi,\psi)(z) \geq & \frac{1}{\nu_1}\int_{-\infty}^{z} e^{\kappa_{11}(z-s)}\left[c\underline{\phi}'(s) - d_S\underline{\phi}''(s) + \left(\beta B_1 + h\right)\underline{\phi}(s)\right] ds \\ & + \frac{1}{\nu_1}\int_{z}^{+\infty} e^{\kappa_{12}(z-s)}\left[c\underline{\phi}'(s) - d_S\underline{\phi}''(s) + \left(\beta B_1 + h\right)\underline{\phi}(s)\right] ds. \end{aligned}$$

If $z \geq z_1$, we have

$$\begin{aligned} & F_1(\phi,\psi)(z) \geq \frac{1}{\nu_1}\int_{-\infty}^{z_1} e^{\kappa_{11}(z-s)}\left[c\underline{\phi}'(s) - d_S\underline{\phi}''(s) + \left(\beta B_1 + h\right)\underline{\phi}(s)\right] ds \\ & \quad + \frac{1}{\nu_1}\int_{z_1}^{z} e^{\kappa_{11}(z-s)}\left[c\underline{\phi}'(s) - d_S\underline{\phi}''(s)\right] ds + \frac{1}{\nu_1}\int_{z}^{+\infty} e^{\kappa_{12}(z-s)}\left[c\underline{\phi}'(s) - d_S\underline{\phi}''(s)\right] ds, \\ & = \frac{1}{\nu_1}e^{\kappa_{11}(z-z_1)}\left[c\underline{\phi}(z_1) - d_S\underline{\phi}'(z_1-0) - c\underline{\phi}(z_1) + d_S\underline{\phi}'(z_1+0)\right] + \frac{1}{\nu_1}\left[-d_S\underline{\phi}'(z) + d_S\underline{\phi}'(z)\right] \\ & \quad + \frac{1}{\nu_1}\int_{-\infty}^{z_1} e^{\kappa_{11}(z-s)}\left[\kappa_{11}c\underline{\phi}(s) - \kappa_{11}d_S\underline{\phi}'(s) + \left(\beta B_1 + h\right)\underline{\phi}(s)\right] ds \\ & \quad - \frac{1}{\nu_1}\int_{z_1}^{z} e^{\kappa_{11}(z-s)}\kappa_{11}d_S\underline{\phi}'(s)ds - \frac{1}{\nu_1}\int_{z}^{+\infty} e^{\kappa_{12}(z-s)}\kappa_{12}d_S\underline{\phi}'(s)ds, \\ & = \frac{d_S}{\nu_1}e^{\kappa_{11}(z-z_1)}\left[\underline{\phi}'(z_1+0) - \underline{\phi}'(z_1-0)\right] \\ & \quad + \frac{1}{\nu_1}\int_{-\infty}^{z_1} e^{\kappa_{11}(z-s)}\left[-d_S\kappa_{11}^2 + c\kappa_{11} + \beta B_1 + h\right]\underline{\phi}(s)ds \ \geq \ 0 \ = \ \underline{\phi}(z). \end{aligned}$$

Note that by definition, $\underline{\phi}'(z_1+0) = 0 > \underline{\phi}'(z_1-0) = -\eta_1\epsilon_1 e^{\epsilon_1 z_1}$.

If $z < z_1$, we have in a similar way

$$\begin{aligned}
F_1(\phi,\psi)(z) &\geq \frac{d_S}{\nu_1} e^{\kappa_{12}(z-z_1)} \left[\underline{\phi}'(z_1+0) - \underline{\phi}'(z_1-0)\right] \\
&\quad - \frac{d_S}{\nu_1}\kappa_{11}\underline{\phi}(z) + \frac{1}{\nu_1}\int_{-\infty}^{z} e^{\kappa_{11}(z-s)} \left[-d_S\kappa_{11}^2 + c\kappa_{11} + \beta B_1 + h\right] \underline{\phi}(s)ds \\
&\quad + \frac{d_S}{\nu_1}\kappa_{12}\underline{\phi}(z) + \frac{1}{\nu_1}\int_{z}^{z_1} e^{\kappa_{12}(z-s)} \left[-d_S\kappa_{12}^2 + c\kappa_{12} + \beta B_1 + h\right] \underline{\phi}(s)ds, \\
&\geq \frac{d_S(\kappa_{12}-\kappa_{11})}{\nu_1}\underline{\phi}(z) = \underline{\phi}(z).
\end{aligned}$$

This completes the proof. □

We establish in the following lemma the continuity and the compactness of the operator F. Except few details for the convolution term and for the function g, the proof is similar to the proof of the corresponding result in [12], see also [14, 47] (the convolution term is linear and then can be easy managed).

Lemma 3.7. *1. The operator $F : \Theta \to C(\mathbb{R}, \mathbb{R}^2)$ is continuous with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^2)$.*

2. The operator $F : \Theta \to \Theta$ is compact with respect to the norm $|\cdot|_\nu$ in $B_\nu(\mathbb{R}, \mathbb{R}^2)$.

In conclusion, we get the following theorem that states the existence of traveling wave front for the system (18) .

Theorem 3.8. *Assume that*

$$\frac{\beta(1-\alpha)u^0}{h\mu} > 1.$$

Then, for every

$$c > c^\star := 2\sqrt{d_I\mu\left(\frac{\beta(1-\alpha)u^0}{h\mu} - 1\right)},$$

the system (9) *has a traveling wave front*

$$(S(t,x), I(t,x), u(t,x)) = (\phi(x+ct), \psi(x+ct), U(x+ct)),$$

satisfying the condition (12). *On the other hand, if $c \in (0, c^\star)$ there is no traveling wave front of the system* (9)-(12).

Proof. It suffice to prove the existence of a nonnegative and nontrivial solution for the wave system (18) satisfying the condition (19). Let $c > c^\star$. From lemmas 3.6 and 3.7, and thanks to Schauder's fixed point theorem, we have the existence of a fixed point (ϕ, ψ) of the operator F belonging to Θ. That is,

$$\phi(z) = \frac{\beta}{\nu_1} \int_{-\infty}^{z} e^{\kappa_{11}(z-s)} H_1(\phi, \psi)(s) ds + \frac{\beta}{\nu_1} \int_{z}^{+\infty} e^{\kappa_{12}(z-s)} H_1(\phi, \psi)(s) ds, \tag{26}$$

and

$$\psi(z) = \frac{\beta}{\nu_2} \int_{-\infty}^{z} e^{\kappa_{21}(z-s)} H_2(\phi, \psi)(s) ds + \frac{\beta}{\nu_2} \int_{z}^{+\infty} e^{\kappa_{22}(z-s)} H_2(\phi, \psi)(s) ds.$$

Then, the existence of nonnegative and nontrivial solutions for (18) is obtained. On the other hand, we have $\underline{\phi}(z) \leq \phi(z) \leq \overline{\phi}(z)$ and $\underline{\psi}(z) \leq \psi(z) \leq \overline{\psi}(z)$, for all $z \in \mathbb{R}$. These last inequalities imply

$$\lim_{z \to -\infty} \phi(z) = S^0 := \frac{(1-\alpha)u^0}{h} \quad \text{and} \quad \lim_{z \to -\infty} \psi(z) = 0.$$

Moreover,

$$\lim_{z \to -\infty} e^{-\lambda_1 z} \psi(z) = 1 \qquad \text{and} \qquad 0 \leq \phi(z) \leq S^0, \quad \text{for all } z \in \mathbb{R}.$$

Next, we show that $\lim_{z \to +\infty} \psi(z) = 0$ and $\lim_{z \to +\infty} \phi(z) = S^\infty < S^0$. We first show that $\lim_{z \to -\infty} \phi'(z) = 0$. In fact, by the expression (26) we have

$$\begin{aligned} \phi'(z) &= \kappa_{11} \frac{\beta}{\nu_2} \int_{-\infty}^{z} e^{\kappa_{11}(z-s)} H_1(\phi, \psi)(s) ds + \kappa_{12} \frac{\beta}{\nu_2} \int_{z}^{+\infty} e^{\kappa_{12}(z-s)} H_1(\phi, \psi)(s) ds, \\ &= \kappa_{11} \frac{\beta}{\nu_2} \int_{0}^{+\infty} e^{\kappa_{11}\tau} H_1(\phi, \psi)(z-\tau) d\tau + \kappa_{12} \frac{\beta}{\nu_2} \int_{-\infty}^{0} e^{\kappa_{12}\tau} H_1(\phi, \psi)(z-\tau) d\tau. \end{aligned} \tag{27}$$

Note that, for any $\tau > 0$,

$$\begin{aligned} \lim_{z \to -\infty} H_1(\phi, \psi)(z-\tau) &= \lim_{z \to -\infty} \left[B_1 \phi(z-\tau) - \phi(z-\tau) g(\psi(z-\tau)) + \frac{h}{\beta} (\chi * \phi)(z-\tau) \right], \\ &= \left(B_1 + \frac{h}{\beta} \right) S^0. \end{aligned}$$

Hence, from the equation (27), we get

$$\lim_{z\to-\infty}\phi'(z) = \left(B_1 + \frac{h}{\beta}\right) S^0 \frac{\beta}{\nu_2}\left\{[e^{\kappa_{11}\tau}]_0^{+\infty} + [e^{\kappa_{12}\tau}]_{-\infty}^0\right\} = 0.$$

Integrating the first equation of (18) from $-\infty$ to z, we obtain

$$c\left[\phi(z) - S^0\right] = d_S\phi'(z) - \int_{-\infty}^{z} \beta\phi(s)g(\psi(s))ds + \mathcal{F}(z), \tag{28}$$

where

$$\mathcal{F}(z) := -h\int_{-\infty}^{z}\phi(s)ds + h\int_{-\infty}^{z}\left(\chi * \phi\right)(s)ds.$$

Note that

$$\begin{aligned}
|\mathcal{F}(z)| &= \left|-h\int_{-\infty}^{z}\phi(s)ds + h\int_{-\infty}^{z}\int_{-\infty}^{+\infty}\chi(y)\phi(s-y)dyds\right|,\\
&= \left|-h\int_{-\infty}^{z}\phi(s)ds + h\int_{-\infty}^{+\infty}\int_{-\infty}^{z}\phi(s-y)ds\chi(y)dy\right|,\\
&= \left|-h\int_{-\infty}^{z}\phi(s)ds + h\int_{-\infty}^{+\infty}\int_{-\infty}^{z-y}\phi(\tau)d\tau\chi(y)dy\right|,\\
&= \left|h\int_{-\infty}^{+\infty}\int_{z-y}^{z}\phi(\tau)d\tau\chi(y)dy\right|,\\
&\le hS^0\int_{-\infty}^{+\infty}|y|\chi(y)dy < +\infty.
\end{aligned}$$

Thus,

$$\lim_{z\to+\infty}\mathcal{F}(z) = -h\int_{-\infty}^{+\infty}\phi(s)ds + h\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\chi(s-y)ds\phi(y)dy = 0.$$

Suppose that

$$\int_{-\infty}^{+\infty}\beta\phi(s)g(\psi(s))ds = +\infty.$$

Then, we have from the equation (28)

$$\phi'(z) = \frac{c}{d_S}\left[\phi(z) - S^0\right] + \frac{1}{d_S}\left\{\int_{-\infty}^{z}\beta\phi(s)g(\psi(s))ds - \mathcal{F}(z)\right\}.$$

Hence, there exists $\delta > 0$ such that $\phi'(z) > \delta$, for all sufficiently large z. This is a contradiction with the fact that $\phi(z) \leq S^0$, for all $z \in \mathbb{R}$. We conclude that

$$\int_{-\infty}^{+\infty} \beta\phi(s)g(\psi(s))ds = \tilde{M} < +\infty.$$

From (18), we have, for $z \in \mathbb{R}$,

$$\psi(z) = \frac{\beta}{\nu_2'} \int_0^{+\infty} e^{\kappa_{21}' s}\phi(z-s)g(\psi(z-s))ds + \frac{\beta}{\nu_2'} \int_{-\infty}^{0} e^{\kappa_{22}' s}\phi(z-s)g(\psi(z-s))ds, \tag{29}$$

with

$$\kappa_{21}' := \frac{c}{2d_I}\left(1 - \sqrt{1 + \frac{4d_I\mu}{c^2}}\right), \quad \kappa_{22}' := \frac{c}{2d_I}\left(1 + \sqrt{1 + \frac{4d_I\mu}{c^2}}\right)$$

and

$$\nu_2' := d_I(\kappa_{22}' - \kappa_{21}').$$

It follows from Fubini's theorem that

$$\int_{-\infty}^{+\infty} \psi(z)dz = \frac{\beta}{\nu_2'}\left[\int_0^{+\infty} e^{\kappa_{21}' s}ds + \int_{-\infty}^{0} e^{\kappa_{22}' s}ds\right]\tilde{M} < +\infty.$$

Furthermore, one can easily show, from (29), that $\psi'(z)$ is uniformly bounded. This, together with the fact that $\psi(z) > 0$ is integrable on $\mathbb{R}$, implies that $\lim_{z\to+\infty} \psi(z) = 0$.

Now, we show that $\lim_{z\to+\infty} \phi(z) = S^\infty < S^0$. Since $\lim_{z\to+\infty} \mathcal{F}(z) = 0$, we have from (28),

$$\lim_{z\to+\infty}\left[c\phi(z) - d_S\phi'(z)\right] = cS^0 - \int_{-\infty}^{+\infty} \beta\phi(s)g(\psi(s))ds =: \mathcal{K}. \tag{30}$$

Then, for any $\epsilon > 0$, there exists $z^* \in \mathbb{R}$ such that, for all $z \geq z^*$,

$$c\phi(z) - d_S\phi'(z) \geq \mathcal{K} - \epsilon.$$

Consequently, for any $\zeta \geq z^*$,

$$\begin{aligned}\phi(z) &\leq \phi(\zeta)e^{\frac{c}{d_S}(z-\zeta)} - \frac{1}{d_S}\int_\zeta^z e^{\frac{c}{d_S}(z-s)}\left[\mathcal{K} - \epsilon\right]ds,\\ &= \left[\phi(\zeta) - \frac{\mathcal{K}-\epsilon}{c}\right]e^{\frac{c}{d_S}(z-\zeta)} + \frac{\mathcal{K}-\epsilon}{c}.\end{aligned}$$

We conclude that

$$\phi(\zeta) \geq \frac{\mathcal{K}-\epsilon}{c}.$$

Otherwise, $\phi(z) \to -\infty$ as $z \to +\infty$, which gives a contradiction. Since $\zeta \geq z^*$ is chosen arbitrarily, we have $\phi(z) \geq (\mathcal{K}-\epsilon)/c$, for all $z \geq z^*$. Furthermore, (30) implies that there exists $\tilde{z} \geq z^*$ such that, for all $z \geq \tilde{z}$,

$$c\phi(z) - d_S\phi'(z) \leq \mathcal{K}+\epsilon.$$

In a similar way as above, we have, for any $\zeta \geq \tilde{z}$,

$$\phi(z) \geq \left[\phi(\zeta) - \frac{\mathcal{K}+\epsilon}{c}\right] e^{\frac{c}{d_S}(z-\zeta)} + \frac{\mathcal{K}+\epsilon}{c}.$$

Hence, $\phi(\zeta) \leq (\mathcal{K}+\epsilon)/c$. Otherwise, $\phi(z) \to +\infty$ as $z \to +\infty$, which is a contradiction. Since $\zeta \geq \tilde{z}$ is arbitrary, we have $\phi(z) \leq (\mathcal{K}+\epsilon)/c$ for all $z \geq \tilde{z}$. Since $\tilde{z} \geq z^*$, then we have for all $z \geq \tilde{z}$

$$\frac{\mathcal{K}}{c} - \frac{\epsilon}{c} \leq \phi(z) \leq \frac{\mathcal{K}}{c} + \frac{\epsilon}{c}.$$

Note that $\epsilon > 0$ is also arbitrary. Then, these last inequalities imply that $\lim\limits_{z\to+\infty} \phi(z) = \mathcal{K}/c =: S^\infty$. This gives the existence of S^∞. Since $\phi(z) \geq \underline{\phi}(z) \geq 0$, for all $z \in \mathbb{R}$, we have $S^\infty \geq 0$. Furthermore, by definition we have

$$S^\infty = \frac{\mathcal{K}}{c} = S^0 - \frac{1}{c}\int_{-\infty}^{+\infty} \beta\phi(s)g(\psi(s))ds < S^0.$$

For $c \in (0, c^\star)$, we can use the same general techniques applying the Laplace transform as in [12, 53, 54] to show that there is no traveling wave front of the system (9)-(12). The proof of this part is classical and can be adapted from [12, 53, 54]. This is left to the reader. □

Remark 4. - Contrary to many previous works, for instance in the paper [12] the authors have proved the monotonicity of traveling wave fronts, in our model the traveling wave fronts are not always monotonic. This is due of the presence of the the difference equation. To support this statement, we have numerically drawn the solutions of our model, Section 4, and we have clearly obtained non-monotonic waves.
- The condition

$$\frac{\beta(1-\alpha)u^0}{h\mu} > 1 \qquad \text{and} \qquad c > c^\star := 2\sqrt{d_I\mu\left(\frac{\beta(1-\alpha)u^0}{h\mu} - 1\right)},$$

that ensures the existence of traveling wave fronts is controlled by the density of the initial distribution $p(0, x, a) := u^0$ of the protected individuals, the parameters β, α, h, μ and the diffusion coefficient d_I.

As we can see in the next result, the condition $\frac{\beta(1-\alpha)u^0}{h\mu} > 1$ is necessary to have the existence of traveling wave fronts.

Theorem 3.9. *Assume that*

$$\frac{\beta(1-\alpha)u^0}{h\mu} < 1.$$

Then, for every $c > 0$, the system (9)-(12) *has no traveling wave front.*

Proof. Assume by contradiction that such wave exists. Then, it satisfies (18)-(19). We have from the expression of ψ, for $z \in \mathbb{R}$,

$$\psi(z) = \frac{\beta}{\nu_2'} \int_0^{+\infty} e^{\kappa_{21}' s} \phi(z-s) g(\psi(z-s)) ds + \frac{\beta}{\nu_2'} \int_{-\infty}^{0} e^{\kappa_{22}' s} \phi(z-s) g(\psi(z-s)) ds, \tag{31}$$

with

$$\kappa_{21}' := \frac{c}{2d_I}\left(1 - \sqrt{1 + \frac{4d_I\mu}{c^2}}\right), \quad \kappa_{22}' := \frac{c}{2d_I}\left(1 + \sqrt{1 + \frac{4d_I\mu}{c^2}}\right)$$

and

$$\nu_2' := d_I(\kappa_{22}' - \kappa_{21}').$$

Now, we are in a position to show that

$$\int_{-\infty}^{+\infty} \psi(z) dz < +\infty.$$

In fact, as in the proof of Theorem 3.8, we can see that $\lim\limits_{z \to -\infty} \psi'(z) = 0$, $\lim_{z \to +\infty} \psi'(z) = 0$ and $\int_{-\infty}^{+\infty} \beta\phi(s) g(\psi(s)) ds < +\infty$. Then, by integrating the second equation of (18) from $-\infty$ to $+\infty$, we obtain

$$\int_{-\infty}^{+\infty} \psi(z) dz = \frac{1}{\mu} \int_{-\infty}^{+\infty} \beta\phi(z) g(\psi(z)) dz < +\infty.$$

By integrating (31), we get

$$
\begin{aligned}
\int_{-\infty}^{+\infty} \psi(z)dz &= \frac{\beta}{\nu_2'} \int_0^{+\infty} e^{\kappa_{21}' s} \int_{-\infty}^{+\infty} \phi(z-s) g(\psi(z-s)) dz ds \\
&\quad + \frac{\beta}{\nu_2'} \int_{-\infty}^{0} e^{\kappa_{22}' s} \int_{-\infty}^{+\infty} \phi(z-s) g(\psi(z-s)) dz ds, \\
&= \frac{\beta}{\nu_2'} \left[\int_0^{+\infty} e^{\kappa_{21}' s} ds + \int_{-\infty}^{0} e^{\kappa_{22}' s} ds \right] \int_{-\infty}^{+\infty} \phi(z) g(\psi(z)) dz, \\
&\leq \frac{\beta S^0}{\nu_2'} \left[\frac{1}{\kappa_{22}'} - \frac{1}{\kappa_{21}'} \right] \int_{-\infty}^{+\infty} g(\psi(z)) dz, \\
&\leq \frac{\beta S^0}{\nu_2'} \left[\frac{1}{\kappa_{22}'} - \frac{1}{\kappa_{21}'} \right] \int_{-\infty}^{+\infty} \psi(z) dz, \\
&= \frac{\beta S^0}{\mu} \int_{-\infty}^{+\infty} \psi(z) dz < \int_{-\infty}^{+\infty} \psi(z) dz.
\end{aligned}
$$

This leads to a contradiction. Consequently, no traveling wave solution of (9)-(12) exists. □

To finish this section, we study the dependence of the minimum wave speed $c^\star$ on the parameters of the model. It is not difficult to see from the explicit expression

$$c^\star := 2\sqrt{d_I \mu \left(\frac{\beta(1-\alpha)u^0}{h\mu} - 1 \right)} \quad \text{with} \quad \frac{\beta(1-\alpha)u^0}{h\mu} > 1,$$

that $c^\star$ depends on the diffusion rate of the infectious individuals d_I, the removed rate of the infectious individuals μ, the transmission rate β, the protection rate of the susceptible individuals h, the specific protection rate for individuals at the end of their previous period of protection α, and the initial protected population u^0. We have the following proposition.

Proposition 1. *Under the condition*

$$\frac{\beta(1-\alpha)u^0}{h\mu} > 1,$$

the minimum wave speed $c^\star$ is an increasing function of d_I, β, u_0, and a decreasing function of μ, h, α.

4. Numerical simulations and discussion

In this section, we numerically solve the system (9) to show the existence and attractivity of the traveling wave fronts. We discretize the problem (9) by implicit finite difference method (Euler schema for the time derivative and a central schema for the spatial derivative). The nonlocal term is approximated by one of the composite integration formulas.

Let us take the following numerical values:

$$d_S = 0.3, \;\; d_I = 0.2, \;\; h = 0.2, \;\; \tau = 1, \;\; \alpha = 0.1, \;\; \mu = 0.1, \;\; \beta = 0.5, \;\; a = 0.1,$$

and

$$S^0 = 1 \qquad \left(\Rightarrow \;\; u^0 = \frac{hS^0}{1-\alpha} = \frac{2}{9} \right).$$

Moreover, we have

$$\frac{\beta(1-\alpha)u^0}{h\mu} = 5 > 1, \qquad \text{and} \qquad c^\star \approx 0.57.$$

These numerical values correspond to the situation of the existence of traveling waves. Figures 2, 3 and 4 show the attractivity of the waves at least when the initial distributions around these waves are small enough. This leads to the local stability of the traveling waves which plays an important role in the description of the long-time behavior. It should be a challenging task for our system to prove this attractivity. We leave this question to a future work.

In this paper, we examined the existence and non-existence of traveling wave solutions for a new model: differential-difference diffusive Kermack-McKendrick system. We showed that, if $u(0,x) = u^0 > 0$, $S(0,x) = (1-\alpha)u^0/h$, with $\beta(1-\alpha)u^0/h\mu > 1$, then for each

$$c > c^\star := 2\sqrt{d_I\mu\left(\frac{\beta(1-\alpha)u^0}{h\mu} - 1\right)},$$

there exists a positive constant $u^\infty \in (0, u^0)$ such that the system (9) has a traveling wave solution $(\phi(x+ct), \psi(x+ct), U(x+ct))$ satisfying $\phi(-\infty) = (1-\alpha)u^0/h$, $U(-\infty) = u^0$, $\phi(+\infty) = (1-\alpha)u^\infty/h$, $U(+\infty) = u^\infty$, $\psi(\pm\infty) = 0$. The proof of the main results is mainly based on Schauder's fixed point theorem .

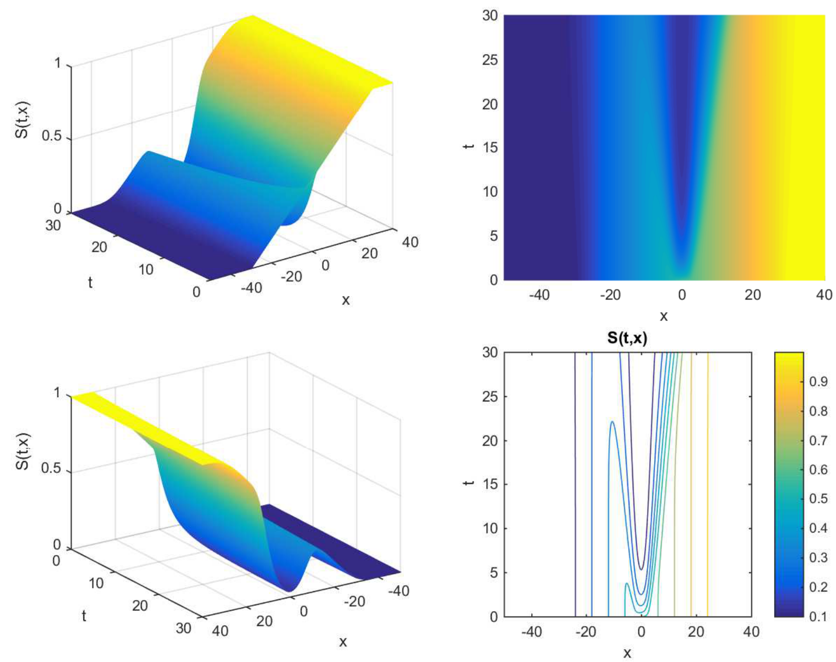

Figure 2: Numerical simulations of the solution $S(t, x)$. Parameters are $d_S = 0.3$, $d_I = 0.2$, $h = 0.2$, $\tau = 1$, $\alpha = 0.1$, $\mu = 0.1$ and $\beta = 0.5$. As we can see in the figure, the wave is not monotone.

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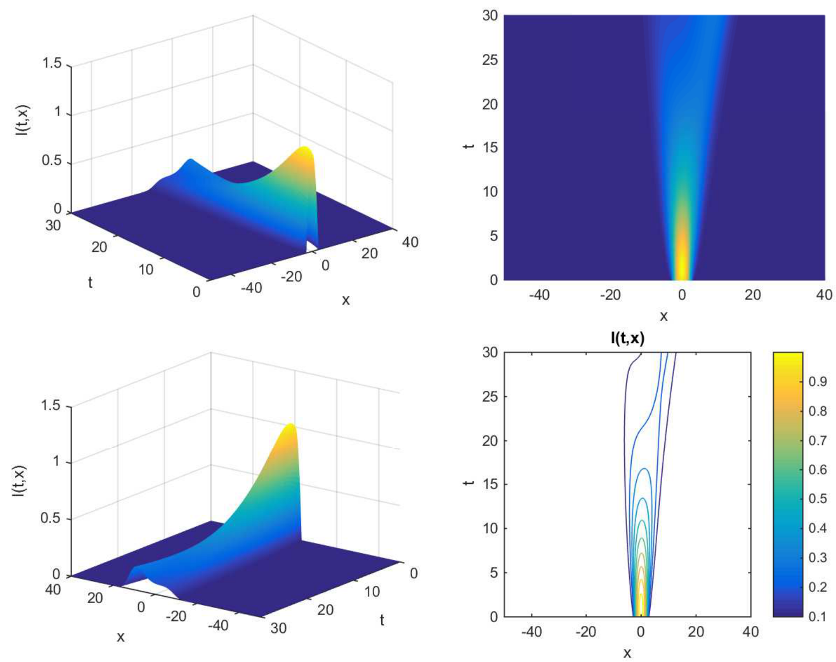

Figure 3: Numerical simulations of the solution $I(t,x)$. Parameters are $d_S = 0.3$, $d_I = 0.2$, $h = 0.2$, $\tau = 1$, $\alpha = 0.1$, $\mu = 0.1$ and $\beta = 0.5$.

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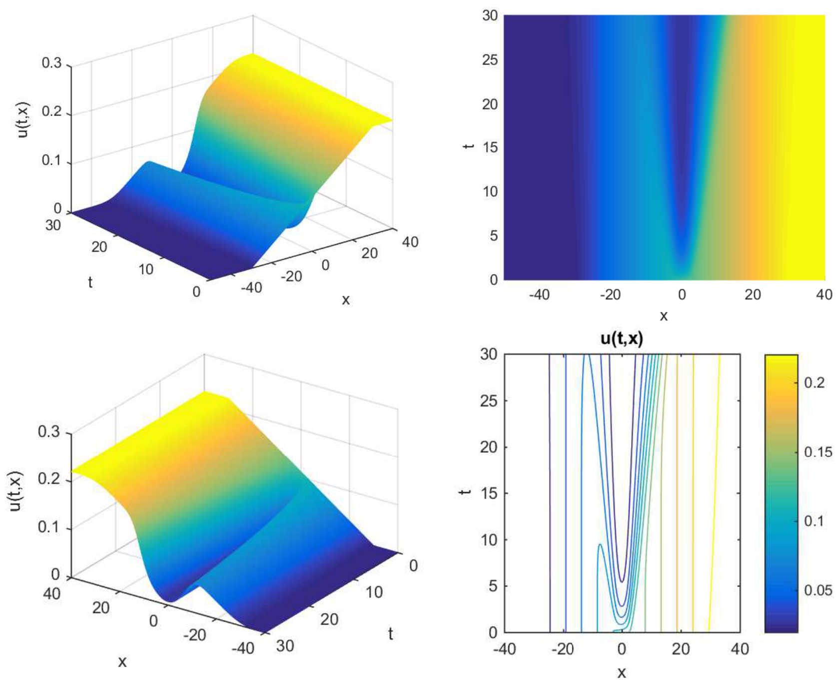

Figure 4: Numerical simulations of the solution $u(t, x)$. Parameters are $d_S = 0.3$, $d_I = 0.2$, $h = 0.2$, $\tau = 1$, $\alpha = 0.1$, $\mu = 0.1$ and $\beta = 0.5$.

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