Local practical stabilization for a class of discrete-time switched affine systems

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Abstract: This paper deals with the study of discrete-time switched affine systems. While generally the literature is focused on the global (practical) stabilization problem, there are classes of switched affine systems that can be stabilized only locally. In this paper we deal with such a class of switched affine systems. More precisely, we consider switched affine systems with decoupled switching in the state matrix and the affine input term. For this particular class of systems, a switching control law is designed and conditions ensuring the system local practical stability are provided. A qualitative approach is given based on the existence of a stabilizing linear state feedback. These results are illustrated using a numerical example.

Keywords: Switched affine systems, switching control, local practical stabilization

1. INTRODUCTION

Switched systems (Liberzon [2003]) represent a class of hybrid dynamical systems (Goebel et al. [2012]). They consist on a family of subsystems and a switching law that orchestrates the switching among them. A particular important class of these systems is the class of switched affine systems (Seatzu et al. [2006]), (Bolzern and Spinelli [2004]), (Hetel and Bernuau [2014]) that have several practical applications, particularly in the domain of power converters (Deaecto et al. [2010]), (Corona et al. [2007]), (Albea et al. [2015]), (Serieye et al. [2019]). In this paper, we are interested in the design of locally stabilizing switching laws for switched affine systems. This means finding conditions to select the active subsystem according to the system state so that stability is ensured. The presence of the affine terms makes that each subsystem has its own equilibrium point. However, the stabilization of the overall system is generally looked towards a desired equilibrium point which is usually different than the equilibria of the subsystems. The problem, is even more challenging in the discrete-time case (Deaecto and Egidio [2016]), (Albea Sanchez et al. [2020]), which is the main object of study of this paper. In general, for discrete-time switched affine systems, the stabilization cannot be addressed to an equilibrium point but only to a neighborhood containing it, or to a limit cycle. Such discrete-time switched systems have been studied for example in (Deaecto and Geromel [2017]), (Egidio and Deaecto [2019]), (Serieye et al. [2021]) and (Serieye et al. [2020]). In (Deaecto and Geromel [2017]), a stabilizing min-type switching state feedback function and a characterization of the attractive invariant set is provided based on general quadratic Lyapunov functions. In (Egidio and Deaecto [2019]), stability conditions are given based on Lyapunov-Metzler inequalities in order to ensure practical stability of an equilibrium point for discrete-time switched affine systems. In (Serieye et al. [2020]) conditions for designing a stabilizing switching function are also given based on Lyapunov-Metzler conditions. However differently from (Egidio and Deaecto [2019]), the positive invariant set is characterized by the union of several potentially disjoint ellipsoids. LMI conditions for the existence of the stabilizing switching law are provided based on the existence of multiple shifted Lyapunov functions. In (Egidio et al. [2020]), conditions for stabilization towards a limit cycle are given. The extension to the case of uncertain switched affine systems has been recently addressed in (Serieye et al. [2021]).

Note that all these results provide global stabilization conditions. However, there are important classes of switched affine systems that can be only locally stabilized. Consider for instance a scalar switched affine system $x_{k+1} = ax_k + b$ where the state matrix a may take two values in the set $\mathcal{A} = \{1.02, 1.03\}$ and where the affine term b switches among $\mathcal{B}$={0.05, -0.05}. Clearly, since all the values of the state matrix are greater than 1, the system cannot be globally stabilized. No matter what the switching law is, if $|x_k| > max\left\{\frac{5}{2}, \frac{5}{3}\right\}$ then $|x_{k+1}| > |x_k|$. However, one may easily check in simulations that when $|x_0| < min\left\{\frac{5}{2}, \frac{5}{3}\right\}$ the solutions of the system

$$x_{k+1} = \begin{cases} 1.02x_k - 0.05, & \text{when} \quad x_k \geqslant 0, \\ 1.03x_k + 0.05, & \text{when} \quad x_k < 0, \end{cases} \tag{1}$$

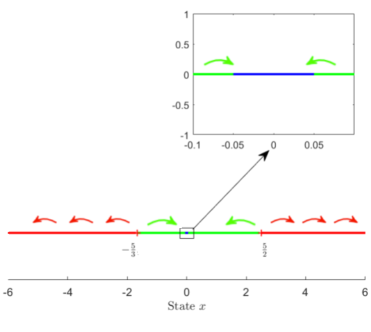

Fig. 1. Behaviour of the system depending on different initial conditions

are bounded (see figure 1 for an illustration). By simple computations, one may show that if $|x_k| < \frac{5}{3}$ and $|x_k| > \frac{0.05}{1+1.02}$ then $|x_{k+1}| < |x_k|$, that is the system can be locally practically stabilized. The local practical stabilization is clearly ensured by the affine terms b. The existing literature cannot address the example under study. For this reason, in this work, we are interested in the local practical stabilization problem. More specifically, we consider a special class of discrete-time switched affine systems with decoupled switchings in the state matrix and the affine term b. Methods for designing a switching control law that locally practically stabilizes the system to some neighborhood of the origin are given. A qualitative approach is provided, based on the existence of a stabilizing linear state feedback. Numerical examples are given to illustrate the approach. The research in this work is related to studies on sampled data controllers for switched affine systems (Hetel and Fridman [2013]), (Sanchez et al. [2019]) and on relay systems in (Govindaswamy et al. [2014]).

This article is organized as follows: in Section 2, the system under study is presented with related assumptions. In Section 3 some necessary preliminary results needed for the synthesis of the control are given. The main theorem of the article is given in Section 4. Numerical examples are then presented in Section 5.2. Proofs are in the appendix.

Notations. The following notations will be used all along the paper. For a given symmetric matrix P, $P \succ 0$ (resp. $P \prec 0$) indicates it is positive definite (resp. negative definite), and the notation $\mathcal{E}(P,\gamma)$ corresponds to the ellipsoid $\mathcal{E}(P,\gamma) = \{x \in \mathbb{R}^n : x^T P x \leqslant \gamma\}$ for some $\gamma > 0$. $\mathcal{B}(\epsilon, 0)$, for some $\epsilon > 0$ describes the space ball centered at the origin with radius $\sqrt{\epsilon}$. $conv(\mathcal{V})$ denotes the convex hull of a set $\mathcal{V}$ and $int(\mathcal{V})$ its interior. $\|X\|$ is the Euclidean norm either for a vector $X \in \mathbb{R}^n$ or a matrix $X \in \mathbb{R}^{n\times n}$. For a finite set of vectors $\mathcal{V} = \{v_1, v_2, \ldots, v_N\}$, where $v_i \in \mathbb{R}^m$, $\forall i \in \{1, \cdots, N\}$ and a function $f : \mathbb{R}^m \to \mathbb{R}$ we denote

$$\underset{v\in\mathcal{V}}{argmin} f(v) = \{v \in \mathcal{V} : f(v) \leqslant f(w), \forall w \in \mathcal{V}\}$$

and

$$\overline{\underset{v\in\mathcal{V}}{argmin}} f(v) = v_i \text{ with}$$
$$i = \min\{j \in \{1, \cdots, N\} : v_j \in \underset{w\in\mathcal{V}}{argmin} f(w)\}.$$

2. PROBLEM STATEMENT

Consider the following discrete time system:

$$x_{k+1} = A_{\lambda_k} x_k + B u_k, \qquad x_0 \in \mathbb{R}^n, \qquad k \in \mathbb{N} \tag{2}$$

where x_k is the system state. $A_i \in \mathbb{R}^{n\times n}$, $i \in \Lambda = \{1, \cdots, L\}$ are L known matrices, $\lambda_k \in \Lambda$ is the index of the active subsystem at time k and $B \in \mathbb{R}^{n\times m}$. $u_k \in \mathbb{R}^m$ is the control input such that it takes values in a finite set of vectors $\mathcal{V} = \{v_1, v_2, \cdots, v_N\}$, $v_i \in \mathbb{R}^m$, $\forall i \in \{1, \cdots, N\}$.

With respect to the classical literature on switched affine systems, system (2) can be represented in the form:

$$x_{k+1} = A_{\lambda_k} x_k + b_{\sigma_k}, \qquad x_0 \in \mathbb{R}^n, \qquad k \in \mathbb{N} \tag{3}$$

where $\sigma_k \in \{1, \cdots, N\}$ and $b_i = B v_i$, $i \in \{1, \cdots, N\}$. σ_k is the classical switching control, i.e. $\sigma_k = i$ when $u_k = v_i$ for some $i \in \{1, \cdots, N\}$. However, this switched affine system has the particularity that the state matrix switches according to another parameter λ_k which is arbitrarily switching. In this sense, we say that system (3) is a switched affine system with decoupled switching in the state matrix and the affine term. The main reason for studying this system is the need to emphasize the role played by the affine terms $b_i = B v_i$ in the local stabilization. Here we assume that λ_k is arbitrarily varying to show the importance of the affine terms. However, we may always formalize another problem statement where λ_k is also a control parameter (maybe also related to σ_k).

All along the article the following Assumptions are considered:

Assumption 2.1. The set $\mathcal{V}$ is non empty and $0 \in int\,(conv(\mathcal{V}))$.

Assumption 2.2. At every time k, the active index λ_k is known.

Assumption 2.3. There exist $K \in \mathbb{R}^{m\times n}$, $0 < \beta < 1$ and a symmetric positive definite matrix $P = P^T \succ 0$ such that the function $V(z) = z^T P z$, $V : \mathbb{R}^n \to \mathbb{R}_+$, satisfies the condition

$$V\Big((A_i + BK)\, x\Big) < (1-\beta) V(x), \tag{4}$$

$\forall x \in \mathbb{R}^n \backslash \{0\}$, $\forall i \in \Lambda$.

Assumption 2.3 implies that there exists a linear static state feedback gain K which ensures the quadratic stability of the system $x_{k+1} = (A_{\lambda_k} + BK) x_k$ uniformly with respect to the switching law $\lambda_k \in \Lambda$.

The objective is to design the control law $u_k = \Pi(x_k, \lambda_k)$, $\Pi : \mathbb{R}^n \times \Lambda \to \mathcal{V}$ such that the closed loop system is locally practically stable for any switching sequence $\{\lambda_k\}_{k\in\mathbb{N}}$.

3. PRELIMINARY RESULTS

Before presenting our main results, some useful preliminary results are given:

Proposition 1. Given $K \in \mathbb{R}^{m\times n}$, $P = P^T \succ 0$ and a finite set $\mathcal{V} = \{v_1, \cdots, v_N\} \subset \mathbb{R}^m$, such that Assumption 2.1 holds, then

$$\exists \gamma > 0 : \; x^T P x < \gamma \implies Kx \in conv(\mathcal{V}), \; \forall x \in \mathbb{R}^n. \tag{5}$$

Proof. See Appendix A.

Proposition 1 indicates that for any given linear state feedback Kx, there exists a sufficiently small neighborhood of the state space such that Kx belongs to the convex polytope described by the switching vectors v_i, $i \in \{1, \cdots, N\}$.

The following theorem from (Ziegler [2012]) - Fundamental Theorem of Polytopes - establishes the link between the vertex and affine representations of a convex polytope.

Theorem 2. (Ziegler [2012]) Consider the convex polytope $conv\,(\mathcal{V})$ whose vertices are the vectors v_i, $i \in \{1, \cdots, N\}$. There exist vectors $h_i \in \mathbb{R}^m$, $i \in \{1, \cdots, n_h\}$, such that

$$conv\,(\mathcal{V}) = \left\{ y \in \mathbb{R}^m : \; h_i^T y \leqslant 1, \; \forall i \in \{1, \cdots, n_h\} \right\}. \tag{6}$$

Using Theorem 2, we may compute the largest level set $\mathcal{E}(P,\gamma)$ of the quadratic Lyapunov Function $V(x) = x^T Px$ where Kx belongs to the convex polytope described by the controls v_i.

Lemma 3. Consider the convex polytope $conv(\mathcal{V})$

$$conv(\mathcal{V}) = \left\{y \in \mathbb{R}^m : h_i^T y \leqslant 1,\ \forall i \in \{1, \cdots, n_h\}\right\}, \quad (7)$$

and a matrix $K \in \mathbb{R}^{m\times n}$. Then for any positive γ^* such that

$$\gamma^* \leqslant \frac{1}{h_i^T KP^{-1}K^T h_i},\ \forall i \in \{1, \cdots, n_h\},$$

$x \in \mathcal{E}(P,\gamma^*)$ implies $Kx \in conv(\mathcal{V})$.

Proof. See Appendix B.

The convexity relation described in Lemma 3 will be further used in the following section in order to design a stabilizing switching control.

4. MAIN RESULT

In this section we present our main results. The following theorem provides methods for designing a locally (practically) stabilizing switching law based on the existence of a stabilizing linear state feedback (Assumption 2.3). In addition, an estimation of the domain of attraction and of the attractor is provided.

Theorem 4. Consider system (2) such that Assumptions 2.2 and 2.1 hold. Let $h_i \in \mathbb{R}^m$, $i \in \{1, \cdots, n_h\}$ be vectors such that

$$conv\,(\mathcal{V}) = \left\{y \in \mathbb{R}^m :\ h_i^T y \leqslant 1,\ \forall i \in \{1, \cdots, n_h\}\right\}. \quad (8)$$

Assume that there exist $P = P^T \succ 0$, $\beta > 0$ and K such that Assumption 2.3 holds. Let

$$\gamma = \min_{i\in\{1,\cdots,n_h\}} \frac{1}{h_i^T KP^{-1}K^T h_i}, \quad (9)$$

$$\delta = \max_{v\in\mathcal{V}} \frac{v^T B^T PBv}{\beta - \epsilon}, \quad \text{with } 0 < \epsilon < \beta < 1 \quad (10)$$

$$\rho = \max_{i\in\Lambda, v\in\mathcal{V}} \left(\|A_i\| \sqrt{\frac{\delta}{\lambda_{min}(P)}} + \|Bv\| \right)^2 \lambda_{max}(P). \quad (11)$$

If $\rho < \gamma$, then system (2) with the control law

$$u_k = \overline{\operatorname{argmin}_{v\in\mathcal{V}}}\ (x_k^T A_{\lambda_k}^T PBv) \quad (12)$$

is locally practically stabilizable from $\mathcal{E}(P,\gamma)$ to $\mathcal{E}(P,\rho)$, that is

$$x_0 \in \mathcal{E}(P,\gamma) \implies \lim_{k\to\infty} x_k \in \mathcal{E}(P,\rho). \quad (13)$$

Proof. The proof of this theorem follows from results of Propositions 5 and 6 given in the Appendix C. Proposition 5 shows that whenever $x_k \in \mathcal{E}(P,\gamma)\backslash\mathcal{E}(P,\delta)$ we have $\Delta V(x_k) < 0$, which means that trajectories starting in $\mathcal{E}(P,\gamma)\backslash\mathcal{E}(P,\delta)$ are all decreasing toward the ellipsoid $\mathcal{E}(P,\delta)$. However, a trajectory that is inside this ellipsoid might leave it. Proposition 6 give an estimation of $\mathcal{E}(P,\rho)$ that contains the trajectories leaving $\mathcal{E}(P,\delta)$.

Remark. Theorem 4 shows that when Assumption 2.3 is satisfied, the switching law (12) ensures the local practical stabilization of system (2). Any solution with initial condition in the ellipsoid $\mathcal{E}(P,\gamma)$ (the estimate of the domain of attraction) with γ given in (9) converges towards the ellipsoid $\mathcal{E}(P,\rho)$ (the attractor) with ρ given in (11) (provided that $\rho < \gamma$).

Remark 2. The method presented here can be adapted for the case where λ_k is also a control parameter. Considering the same conditions as in Theorem 4, a pair of state dependent switching laws $\theta : \mathbb{R}^n \to \Lambda$ and $\Pi : \mathbb{R}^n \to \mathcal{V}$ can be derived such that system (2) with $\lambda_k = \theta(x_k)$ and $u_k = \Pi(x_k)$ is locally practically stable. More precisely the following switching functions are obtained:

$$(\theta(x), \Pi(x)) = \overline{argmin_{\lambda\in\Lambda, v\in\mathcal{V}}} \left(x^T A_\lambda^T PA_\lambda x + 2x^T A_\lambda^T PBv\right) \quad (14)$$

This pair of switching laws can be derived using the same developments used in Theorem 4 (in the steps given in (C.9)-(C.11) we choose the controls v and λ which minimize the expression $M(x,\lambda,v)$).

5. NUMERICAL EXAMPLES

5.1 Example 1

Consider the motivating example given in the Introduction. For this example, the vectors h_1, h_2 in (8) are $\{\frac{1}{0.05}, -\frac{1}{0.05}\}$. The conditions of Theorem 4 are satisfied for

$$\beta = 0.01, \qquad P = 0.4904, \qquad K = -0.0350. \quad (15)$$

With these parameters we obtain $\gamma = 1$ and $\rho = 0.1566$. This means that trajectories with initial condition $|x_0| < 1.4280$ converge to the set $[-0.5650, 0.5650]$.

5.2 Example 2

Consider a system of the form (2) such that A_{λ_k} switches arbitrarily between the two matrices

$$A_1 = \begin{bmatrix} 1 & 0.01 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.02 & 0 \\ -0.01 & 0.99 \end{bmatrix} \quad (16)$$

and B is defined by $B = \begin{bmatrix} 0.005 & 0 \\ 0 & 0.005 \end{bmatrix}$. Both A_1 and A_2 have eigenvalues outside the unit circle. The control u_k takes values in the set

$$\mathcal{V} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}. \quad (17)$$

For this example the vectors $h_i, i = 1, \ldots, 4$, are given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}. \quad (18)$$

It can be checked numerically that Assumption 2.3 is satisfied with $\beta = 10^{-2}$,

$$P = \begin{bmatrix} 25.2785 & 0.3057 \\ 0.3057 & 6.0956 \end{bmatrix}, \quad K = \begin{bmatrix} -5.0102 & 0.1452 \\ 0.7314 & -2.4331 \end{bmatrix},$$

$\gamma = 1$, $\rho = 0.3921$. The shape of the estimation of the domain of attraction $\mathcal{E}(P,\gamma)$ and of the attractor $\mathcal{E}(P,\rho)$ can be seen in Figure 2 together with the trajectories of x_1 and x_2 for the initial condition $x_0 = [0.1866\ 0.1312]^T$.

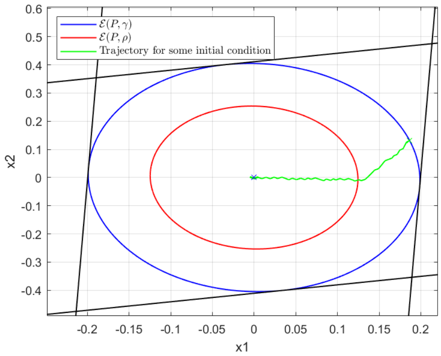

Fig. 2. Domain of attraction $\mathcal{E}(P,\gamma)$ and attractor $\mathcal{E}(P,\rho)$ for the system in Example 5.2. The black lines delimit the region in the state space for which $Kx \in conv(\mathcal{V})$.

As we can see in the simulations, the estimation of the attractor $\mathcal{E}(P,\rho)$ is quite conservative. This leaves space for improvement in the estimation of ρ.

6. CONCLUSION

This work investigated the local stabilization properties of discrete-time switched affine systems. A particular class of these systems, characterized by a decoupling in the state matrix and the affine term has been treated. A stabilizing switching law design has been proposed for the affine term whatever the switching of the state matrix is. Sufficient conditions for the existence of a locally practically stabilizing switching law have been given. In addition, a method for estimating the domain of attraction and the attractor has been given. These results have been tested on numerical examples. In the future, we intend to provide methods for simultaneously design switching laws for the switching matrix and the affine terms in order to improve the local stabilization properties.

Appendix A. PROOF OF PROPOSITION 1

Proof. $\mathcal{V}$ is a non empty subset of $\mathbb{R}^m$ containing 0 in its interior, which means that there exists an $\mathbb{R}^m$ ball $\mathcal{B}(\epsilon_1, 0)$ of sufficiently small radius $\sqrt{\epsilon_1}$, centered in 0, contained in the convex hull of $\mathcal{V}$. The function $Kx, K \in \mathbb{R}^{m\times n}$ is linear and continuous from $\mathbb{R}^n$ to $\mathbb{R}^m$. Then there exists an $\mathbb{R}^n$ ball $\mathcal{B}(\epsilon_2, 0)$ centered in 0 with radius $\sqrt{\epsilon_2}$, such that

$$x \in \mathcal{B}(\epsilon_2, 0) \implies Kx \in \mathcal{B}(\epsilon_1, 0). \tag{A.1}$$

Since $\mathcal{B}(\epsilon_2, 0) \subset \mathbb{R}^n$ exists and is non empty, then an ellipsoid $\mathcal{E}(P,\gamma)$ can be found inside it, with a γ that satisfies $\gamma \leqslant \epsilon_2\lambda_{min}(P)$. This comes from the fact that for all $x \in \mathcal{E}(P,\gamma)$ we have $x^Tx \leqslant \frac{\gamma}{\lambda_{min}(P)}$ and for all $x \in \mathcal{B}(\epsilon_2, 0)$ we have $x^Tx \leqslant \epsilon_2$. Then, in order to have $\mathcal{E}(P,\gamma) \subseteq \mathcal{B}(\epsilon_2, 0)$, $\frac{\gamma}{\lambda_{min}(P)} \leqslant \epsilon_2$ needs to hold.

Appendix B. PROOF OF LEMMA 3

Proof. The proof of this lemma is adopted from (Hindi and Boyd [1998]). We give it in what follows for the sake of self-containment. From equation (6), the condition $Kx \in conv(\mathcal{V})$ is equivalent to the existence of $n_h > 0$ vectors h_i from $\mathbb{R}^m$ such that $h_i^T Kx \leqslant 1$, $\forall i \in \{1, \cdots, n_h\}$.

We want to show that whenever $x \in \mathcal{E}(P,\gamma)$, $h_i^T Kx \leqslant 1$, $\forall i \in \{1, \cdots, n_h\}$. Note that this is implied by the assumption

$$x^T K^T h_i h_i^T Kx \leqslant 1 \quad \text{whenever} \quad x^T \frac{P}{\gamma} x \leqslant 1, \tag{B.1}$$

$\forall i \in \{1, \cdots, n_h\}$. Let us remark that if $K^T h_i h_i^T K \leqslant \frac{P}{\gamma}$, $\forall i \in \{1, \cdots, n_h\}$ then condition (B.1) is satisfied. Rewriting

$$K^T h_i h_i^T K \leqslant \frac{P}{\gamma} \quad \text{as} \quad P - K^T h_i \gamma h_i^T K \geqslant 0 \tag{B.2}$$

and using the Schur Complement Lemma, we can show that this is equivalent to:

$$\begin{bmatrix} P & K^T h_i \\ h_i^T K & \dfrac{1}{\gamma} \end{bmatrix} \geqslant 0. \tag{B.3}$$

Then using the reverse formula of the Schur Complement, gives:

$$\frac{1}{\gamma} - h_i^T K P^{-1} K^T h_i \geqslant 0, \tag{B.4}$$

which is equivalent to:

$$\gamma \leqslant \frac{1}{h_i^T K P^{-1} K^T h_i}, \qquad \forall i \in \{1, \cdots, n_h\}. \tag{B.5}$$

Therefore, we have shown that if $0 < \gamma \leqslant \frac{1}{h_i^T K P^{-1} K^T h_i}$, $\forall i \in \{1, \cdots, n_h\}$, then (B.1) holds, and therefore the Proposition holds.

Appendix C

Proposition 5. Consider system (2) such that Assumptions 2.2 and 2.1 hold. Let h_i, $i \in \{1, \cdots, n_h\}$ be $\mathbb{R}^m$ vectors such that:

$$conv\,(\mathcal{V}) = \left\{y \in \mathbb{R}^m : \ h_i^T y \leqslant 1 \qquad \forall i \in \{1, \cdots, n_h\}\right\}. \tag{C.1}$$

Assume that there exist $P = P^T > 0$, $\beta > 0$ and K such that Assumption 2.3 holds. Let:

$$\gamma \leqslant \frac{1}{h_i^T K P^{-1} K^T h_i}, \qquad \forall i \in \{1, \cdots, n_h\} \tag{C.2}$$

$$\delta > \max_{v \in \mathcal{V}} \left\{ \frac{v^T B^T P B v}{\beta - \epsilon} \right\}, \quad 0 < \epsilon < \beta < 1 \tag{C.3}$$

If $\delta < \gamma$, then for system (2) with the switching control

$$u_k = \Pi(x_k, \lambda_k) = \underset{v \in \mathcal{V}}{\overline{\operatorname{argmin}}}\,(x_k^T A_{\lambda_k}^T P B v)$$

we have

$$V(A_{\lambda_k} x_k + B u_k) < V(x_k), \qquad \forall \lambda_k \in \Lambda, \tag{C.4}$$

whenever

$$x_k \in \mathcal{E}(P,\gamma) \backslash \mathcal{E}(P,\delta), \qquad \forall k \in \mathbb{N}.$$

Proof. Consider $K \in \mathbb{R}^{m\times n}$, $0 < \beta < 1$ and $P = P^T > 0$ such that Assumption 2.3 holds. Then we have:

$$\begin{aligned}(A_{\lambda_k} x_k + BKx_k)^T P (A_{\lambda_k} x_k + BKx_k) - x_k^T P x_k < \\ -\beta x_k^T P x_k, \qquad \forall \lambda_k \in \Lambda, \qquad x_k \neq 0.\end{aligned} \tag{C.5}$$

Proposition 1 states that when $conv(\mathcal{V})$ is a nonempty subset of $\mathbb{R}^m$ and that $0 \in int(conv(\mathcal{V}))$, then $\exists \gamma > 0$ such that for any $x \in \mathcal{E}(P,\gamma)$, $Kx \in conv(\mathcal{V})$. The latter means that

$\forall x_k \in \mathcal{E}(P,\gamma),\ \exists \alpha_j(x_k) \geqslant 0,\ \forall j \in \{1,\cdots,N\}$, satisfying

$$\sum_{j=1}^{N} \alpha_j(x_k) = 1, \quad \text{such that} \quad Kx_k = \sum_{j=1}^{N} \alpha_j(x_k) v_j. \quad \text{(C.6)}$$

Inequality (C.5) implies that

$$x_k^T A_{\lambda_k}^T P A_{\lambda_k} x_k + 2x_k^T A_{\lambda_k}^T PBKx_k + x_k^T K^T B^T PBKx_k \\ -(1-\beta)x_k^T P x_k < 0, \qquad \forall \lambda_k \in \Lambda, \qquad \forall x_k \neq 0. \tag{C.7}$$

Then taking into account (C.6) leads to

$$\sum_{j=1}^{N} \alpha_j(x_k) \left(x_k^T A_{\lambda_k}^T P A_{\lambda_k} x_k + x_k^T K^T B^T PBKx_k \right. \\ \left. -(1-\beta)x_k^T P x_k + 2x_k^T A_{\lambda_k}^T PBv_j \right) < 0, \tag{C.8}$$

For all $\lambda_k \in \Lambda$, $x_k \in \mathcal{E}(P,\gamma)$. This is the same as

$$\sum_{j=1}^{N} \alpha_j(x_k) M(x_k, \lambda_k, v_j) < 0,\ \forall \lambda_k \in \Lambda,\ \forall x_k \in \mathcal{E}(P,\gamma), \tag{C.9}$$

with

$$M(x_k,\lambda_k,v_j) = x_k^T A_{\lambda_k}^T P A_{\lambda_k} x_k + x_k^T K^T B^T PBKx_k \\ -(1-\beta)x_k^T P x_k + 2x_k^T A_{\lambda_k}^T PBv_j.$$

Since all $\alpha_j(x) \geqslant 0$, then there exists at least one index $i^* \in \{1,\cdots,N\}$ such that $M(x_k,\lambda_k,v_{i^*}) < 0$. This means that the minimum of $M(x_k,\lambda_k,v)$ is always negative for $v \in \mathcal{V}$. Let us remark that

$$\overline{\underset{v\in\mathcal{V}}{\text{argmin}}}\, M(x_k,\lambda_k,v) = \overline{\underset{v\in\mathcal{V}}{\text{argmin}}}\, (x_k^T A_{\lambda_k}^T PBv), \qquad \forall \lambda_k \in \Lambda \tag{C.10}$$

with matrix A_{λ_k} corresponding to the active state matrix at instant k. Therefore the following is satisfied:

$$M(x_k,\lambda_k,v^*) < 0, \qquad \forall x_k \in \mathcal{E}(P,\gamma), \qquad \forall \lambda_k \in \Lambda \tag{C.11}$$

with $v^* = \overline{\underset{v\in\mathcal{V}}{\text{argmin}}}\, (x_k^T A_{\lambda_k}^T PBv) \equiv u_k$.

Let us recall that u_k is a function of x_k and λ_k, i.e. $u_k = \Pi(x_k,\lambda_k) = \overline{\underset{v\in\mathcal{V}}{\text{argmin}}}\, (x_k^T A_{\lambda_k}^T PBv)$. Using the fact that

$$\begin{aligned} \Delta V_{x_k} &\equiv V(x_{k+1}) - V(x_k) \\ &= V(A_{\lambda_k}x_k + B\Pi(x_k,\lambda_k)) - V(x_k) \\ &= (A_{\lambda_k}x_k + B\Pi(x_k,\lambda_k))^T P (A_{\lambda_k}x_k + B\Pi(x_k,\lambda_k)) \\ &\quad - x_k^T P x_k \\ &= x_k^T A_{\lambda_k}^T P A_{\lambda_k} x_k + 2x_k^T A_{\lambda_k}^T PB\Pi(x_k,\lambda_k) \\ &\quad + \Pi(x_k,\lambda_k)^T B^T PB\Pi(x_k,\lambda_k) - x_k^T P x_k \end{aligned}$$

and adding and subtracting $\Pi^T(x_k,\lambda_k)B^T PB\Pi(x_k,\lambda_k)$ to $M(x_k,\lambda_k,\Pi(x_k,\lambda_k))$ lead to the following $\forall x_k \in \mathcal{E}(P,\gamma)$:

$$M(x_k,\lambda_k,\Pi(x_k,\lambda_k)) = \Delta V(x_k) + x_k^T K^T B^T PBKx_k \\ + \beta x_k^T P x_k - \Pi(x_k,\lambda_k)^T B^T PB\Pi(x_k,\lambda_k) < 0, \tag{C.12}$$

In what follows we use (C.12) in order to characterize the set of states x_k such that $\Delta V(x_k) < 0$. More precisely, we try to find an ellipsoid $\mathcal{E}(P,\delta)$ inside $\mathcal{E}(P,\gamma)$ such that $\Delta V(x_k) < 0$ when $x_k \in \mathcal{E}(P,\gamma)\backslash\mathcal{E}(P,\delta)$. A sufficient condition to prove that $\Delta V(x_k) < 0$, $\forall x \in \mathcal{E}(P,\gamma)\backslash\mathcal{E}(P,\delta)$ is to show that

$$x_k^T \left(K^T B^T PBK + \beta P \right) x_k - v_i^T B^T PBv_i > 0,$$

for all $i \in \{1,\cdots,N\}$ whenever $x \in \mathcal{E}(P,\gamma)\backslash\mathcal{E}(P,\delta)$.

Using the S-procedure, we can show that the condition

$$x_k^T \left(K^T B^T PBK + \beta P \right) x_k - v_i^T B^T PBv_i > 0, \\ \forall i \in \{1,\cdots,N\} \text{ whenever } \quad x_k^T P x_k > \delta, \tag{C.13}$$

is satisfied if and only if there exists $c > 0$ such that

$$x_k^T \left(K^T B^T PBK + \beta P \right) x_k - v_i^T B^T PBv_i \\ -c(x_k^T P x_k - \delta) > 0. \tag{C.14}$$

The latter is the same as

$$x_k^T \left(K^T B^T PBK + (\beta - c)P \right) x_k + c\delta - v_i^T B^T PBv_i > 0 \tag{C.15}$$

Let us choose $c < \beta$ in order to have

$$x_k^T \left(K^T B^T PBK + \beta P - cP \right) x_k > 0.$$

Let us take $c = \beta - \epsilon$, for some $\epsilon > 0$ sufficiently small such that $0 < \epsilon < \beta < 1$. Then a feasible δ is found such that $c\delta - v_i^T B^T PBv_i > 0$, $\forall i \in \{1,\cdots,N\}$ which means that

$$\delta > \frac{v_i^T B^T PBv_i}{c}, \qquad \forall i \in \{1,\cdots,N\}. \tag{C.16}$$

In order to take into account all possible switches, δ should be chosen such that $\delta > \underset{v\in\mathcal{V}}{max} \frac{v^T B^T PBv}{\beta - \epsilon}$.

Proposition 6. Consider system (2) with $u_k \in \mathcal{V}$, $\forall k \in \mathbb{N}$. Given $P = P^T > 0$ and $\delta > 0$. If $x_k \in \mathcal{E}(P,\delta)$, then $x_{k+1} \in \mathcal{E}(P,\rho)$ with

$$\rho = \underset{i\in\Lambda\ v_j\in\mathcal{V}}{max} \left(\|A_i\| \sqrt{\frac{\delta}{\lambda_{min}(P)}} + \|Bv_j\| \right)^2 \lambda_{max}(P). \tag{C.17}$$

Proof. Let $x_k \in \mathcal{E}(P,\delta)$, which means that $x_k^T P x_k < \delta$. Since P is symmetric and positive definite:

$$\lambda_{min}(P)\|x_k\|^2 \leqslant x_k^T P x_k \leqslant \lambda_{max}(P)\|x_k\|^2 \tag{C.18}$$

where $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are the minimum and maximum eigenvalues of P. The last equation leads to

$$||x_k||^2 < \frac{\delta}{\lambda_{min}(P)}. \tag{C.19}$$

We are looking for a lower bound of $\rho > 0$ such that:

$$x_{k+1}^T P x_{k+1} < \rho, \qquad \forall x_k \in \mathcal{E}(P,\delta), \qquad \forall \lambda_k \in \Lambda. \tag{C.20}$$

We know that :

$$x_{k+1} = A_{\lambda_k} x_k + Bv, \qquad \lambda_k \in \Lambda, \qquad v \in \mathcal{V}. \tag{C.21}$$

It follows

$$\|x_{k+1}\| \leqslant \underset{i\in\Lambda, v\in\mathcal{V}}{max} \|A_i\|\|x_k\| + \|Bv\|, \tag{C.22}$$

$$\|x_{k+1}\|^2 \leqslant \underset{i\in\Lambda, v\in\mathcal{V}}{max} \left(\|A_i\|\|x_k\| + \|Bv\| \right)^2. \tag{C.23}$$

From (C.19) we can have :

$$\|x_k\| < \sqrt{\frac{\delta}{\lambda_{min}(P)}}, \tag{C.24}$$

which gives :

$$\|x_{k+1}\|^2 < \max_{i\in\Lambda, v\in\mathcal{V}} \left(\|A_i\| \sqrt{\frac{\delta}{\lambda_{min}(P)}} + \|Bv\| \right)^2, \quad \text{(C.25)}$$

using the fact that $\lambda_{min}(P)\|x_{k+1}\|^2 \leqslant x_{k+1}^T P x_{k+1} \leqslant \lambda_{max}(P)\|x_{k+1}\|^2$, equation (C.25) gives:

$$x_{k+1}^T P x_{k+1} < \max_{i\in\Lambda, v\in\mathcal{V}} \left(\|A_i\| \sqrt{\frac{\delta}{\lambda_{min}(P)}} + \|Bv\| \right)^2 \lambda_{max}(P). \quad \text{(C.26)}$$

Then, it is clear that $x_{k+1} \in \mathcal{E}(P, \rho)$ with

$$\rho = \max_{i\in\Lambda, v\in\mathcal{V}} \left(\|A_i\| \sqrt{\frac{\delta}{\lambda_{min}(P)}} + \|Bv\| \right)^2 \lambda_{max}(P).$$

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