Birational 3D free-form deformations of degree 1 x 1 x 1

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Abstract

The construction of birational maps is useful for several applications in geometric modeling, including the deformation of shapes by the free-form deformation technique. In this paper, we present effective methods to manipulate 3D birational tensor-product maps with entries of degree $1 \times 1 \times 1$, which are the first for the construction of non-affine birational maps in 3D. To achieve birationality, specific geometric constraints on the Bézier control points are necessary as they permit the existence of certain syzygies. We provide criteria for the computation of suitable weights for the control points that yield birationality, and give explicit formulas for the inverse rational map. According to the degree of the inverse, we find different classes of birational maps. The first is the class of hexahedral maps, i.e. tensor-product maps whose control net determines the vertices of a quadrilaterally-faced hexahedron. Hexahedral birational maps are the direct generalization to 3D of 2D quadrilateral birational maps. In the remaining classes, the inverse rational map is quadratic for one of the parameters. Interestingly, the geometry of the control points is less constrained and the subsequent tensor-product map becomes more flexible.

Keywords: birational map, free-form deformation, syzygy, hexahedral map, doubly ruled quadric

1. Introduction

Motivation. Free-form deformation (FFD), first introduced by Sederberg and Parry (1986), stands for a simple and intuitive method for the geometric manipulation of 2D and 3D objects with arbitrary shapes, which has revealed a powerful technique with multiple applications in computer-aided geometric design (CAGD) and related fields (see Sederberg et al. (2016) and references therein). More specifically, an FFD is a deformation of the ambient space using a tensor-product rational map that roughly captures the geometry of a complex model, as illustrated in Figure 1. This rational map is defined by a net of control points, or control net, which can be given non-negative weights to gain flexibility. In order to be valid, this rational map must be injective on the input. More interestingly, rational maps with an inverse rational map, known as *birational maps*, are globally injective and permit the computation of preimages avoiding numerical methods, which is convenient for many applications (see Sederberg et al. (2016) and references).

Birational maps have already been studied by algebraic geometers for more than a century, mostly from a theoretical point of view (see for instance Hudson (1927), Alberich-Carramiñana (2002), Cerveau and Déserti (2013), and Déserti (2021)). Historically, birational endomorphisms of the projective space, known as *Cremona maps*, have received special attention. In the last decade, Sederberg and Zheng (2015) introduced the concept of *birational FFD* and exploited birational maps in geometric modeling for the first time. With their methods, a designer can move freely the control points and compute weights that make the FFD birational. Unfortunately, providing criteria to construct birational FFDs is not an easy task, and so far the existing literature only treats 2D birational FFDs.

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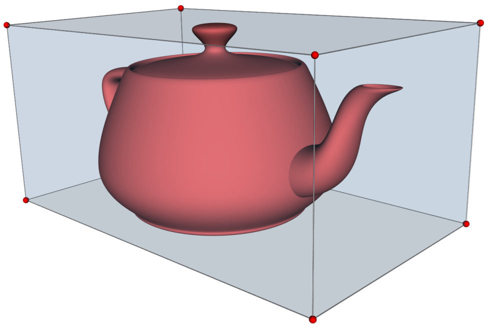

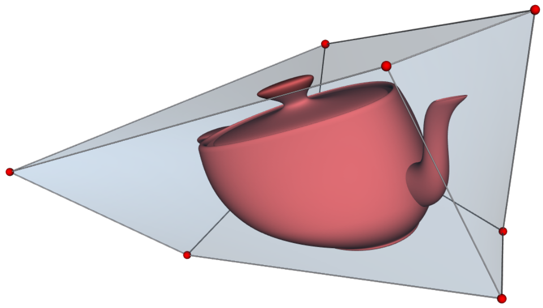

Figure 1: Left: a complex model in the parameter 3D space. Right: a 3D FFD of degree $1 \times 1 \times 1$, determined by $2 \times 2 \times 2 = 8$ control points.

Related works. In Sederberg and Zheng (2015) the authors provide a simple characterization of birational tensor-product maps of degree 1×1, and derive an efficient method to construct birational 2D FFDs without geometric constraints on the control points. Additionally, they give explicit formulas for the inverse rational map. Interestingly, more general inversion formulas for 2D rational maps of degree 1×1 with generic coefficients are studied in Floater (2015). Birational 2D FFDs of degree $1 \times n$ are addressed in Sederberg et al. (2016), and sufficient conditions for birationality are provided. On the other hand, a general characterization of birationality for 2D rational maps of degree $m \times n$ appears in Botbol et al. (2017), but its effectiveness for geometric design is limited. Recently, 2D quadratic birational maps have been considered in Wang et al. (2021) and methods to construct birational triangular patches are proposed, although the geometry of the control points needs to be constrained. All these birationality criteria rely on the analysis of algebraic relations of the entries defining the rational map. Most of them are syzygy-based, i.e. they focus on linear relations in the entries with polynomial coefficients. Unfortunately, these methods are strongly limited by the degree of the entries, giving an insight on the inherent complexity of birational transformations.

As far as we know, there are no results concerning birational 3D FFDs in the existing literature, and only some preliminary observations are reported in (Sederberg et al., 2016, §7).

Contributions. In this paper, we provide for the first time effective methods for the manipulation of birational 3D FFDs of degree $1 \times 1 \times 1$, which is the simplest tensor-product degree in 3D. Geometrically, these transformations are determined by two control points in each direction of the space, i.e. $2 \times 2 \times 2 = 8$ control points (see Figure 1), and their associated weights.

We consider two classes of birational FFDs, according to the degree of the entries defining the inverse rational map. From a geometric point of view, these classes differ by the constraints imposed on the control points. The first is the class of *hexahedral* maps, i.e. rational maps of degree $1 \times 1 \times 1$ where the control points define a quadrilaterally-faced hexahedron, illustrated in Figure 2. Beyond their application to FFDs, hexahedral birational maps can also be useful for the generation of hexahedral meshes. In Theorem 2, we give an effective criterion for the computation of positive weights that make a hexahedral map birational, and we write down explicit formulas for the inverse rational map. Secondly, we study birational maps where one pair of opposite boundary surfaces are doubly ruled quadrics, i.e. the associated control points are not coplanar. These FFDs are more flexible than hexahedral maps, as they can conform general quad patches. Theorem 3 presents an efficient strategy to compute positive weights that ensure that an FFD belongs to this class, and provides formulas for the inverse rational map.

The paper is organized as follows. In §2 we introduce our notation and discuss some preliminary concepts that are necessary in our analysis. In particular, we describe the multi-projective formulation for 3D FFDs of degree $1 \times 1 \times 1$, and explain the connection between birationality and the existence of specific algebraic relations satisfied by the entries of the rational map. More specifically, our contributions build upon a recent syzygy-based birationality criterion derived in Busé et al. (2023). In the remaining sections, we study the two classes of birational maps separately. Namely, §3 is devoted to the study of birational hexahedral maps, whereas §4 deals with birational maps in the second class.

2. Preliminaries

A 3D FFD of degree $1\times1\times1$ is formalized as follows. For each $0 \leq i,j,k \leq 1$, let $\mathbf{p}_{ijk} = (x_{ijk}, y_{ijk}, z_{ijk}) \in \mathbb{R}^3$ and $w_{ijk} \in \mathbb{R}_{\geq 0}$. Additionally, write $B_0^1(s) = B_0(s) = 1-s$ and $B_1^1(s) = B_1(s) = s$ for the polynomials in the Bernstein basis of degree one. These data determine the rational map

$$\begin{aligned} \phi : \mathbb{R}^3 \quad & \dashrightarrow \quad \mathbb{R}^3 \\ (s,t,u) \quad & \mapsto \quad (x,y,z) = \frac{\sum\limits_{0\leq i,j,k\leq 1} w_{ijk}\,\mathbf{p}_{ijk}\,B_i(s)B_j(t)B_k(u)}{\sum\limits_{0\leq i,j,k\leq 1} w_{ijk}\,B_i(s)B_j(t)B_k(u)} \end{aligned} \tag{2.1}$$

that defines a 3D FFD. The $\mathbf{p}_{ijk}$'s are the *control points* of the FFD, and the w_{ijk}'s are their associated *weights*. These control points are the images by ϕ of the vertices of the unit cube $[0,1]\times[0,1]\times[0,1] \subset \mathbb{R}^3$. Moreover, if all the w_{ijk}'s are positive the image of the unit cube lies in the convex hull of the $\mathbf{p}_{ijk}$'s. The FFD is said to be birational if there exists a rational map $\phi^{-1} : \mathbb{R}^3 \dashrightarrow \mathbb{R}^3$ such that $\phi^{-1}\circ\phi$ is the identity.

2.1. Parametric and boundary surfaces

If the image of the FFD lies in a surface, we say that it is *degenerated*. Otherwise, given a general $\lambda \in \mathbb{R}$ the image of the restriction $\phi_\lambda(t,u) = \phi(\lambda,t,u)$ is either a plane or a doubly ruled quadric surface in $\mathbb{R}^3$, that we call *s-surface*. Analogously, we define the *t-* and *u-surfaces*. If ϕ is birational, the degrees of these parametric surfaces yield different possibilities for ϕ^{-1}. More specifically, if the degree of the s-surfaces (resp. t- and u-surfaces) is d_s (resp. d_t, d_u), we define the *type of* ϕ as the triple (d_s, d_t, d_u).

The FFD is often restricted to the unit cube $[0,1]\times[0,1]\times[0,1] \subset \mathbb{R}^3$. Therefore, it makes sense to define the boundary surfaces as the image by ϕ of the planes supporting its facets. Namely, for each $0 \leq i,j,k \leq 1$ we define $\mathbf{\Sigma}_i \subset \mathbb{R}^3$ (resp. $\mathbf{T}_j$ and $\mathbf{Y}_k$) as the (closure of the) image of $\phi(i,t,u)$ (resp. $\phi(s,j,u)$ and $\phi(s,t,k)$).

2.2. Multi-projective formulation

In our study of birational 3D FFDs we use techniques from commutative algebra that are better developed in a multi-projective setting. We briefly transpose the previous framework into this formulation.

The parameter $s \in \mathbb{R}$ (resp. t,u) is replaced by the projective point $(s_0 : s_1) \in \mathbb{P}^1_{\mathbb{R}}$ (resp. $(t_0 : t_1)$, $(u_0 : u_1) \in \mathbb{P}^1_{\mathbb{R}}$). For each $0 \leq i,j,k \leq 1$, we define $\mathbf{P}_{ijk} = (1, x_{ijk}, y_{ijk}, z_{ijk})^T \in \mathbb{R}^4$ and identify it with the projective point $(1 : x_{ijk} : y_{ijk} : z_{ijk})$ in $\mathbb{P}^3_{\mathbb{R}}$, where we consider the homogeneous variables x_0, x_1, x_2, x_3. In particular, $\mathbf{P}_{ijk}$ coincides with $\mathbf{p}_{ijk}$ in the affine chart $\mathbb{R}^3 \subset \mathbb{P}^3_{\mathbb{R}}$ determined by $x_0 \neq 0$. Therefore, we also refer to the $\mathbf{P}_{ijk}$'s as control points. Additionally, we set $R = \mathbb{R}[s_0,s_1]\otimes_{\mathbb{R}}\mathbb{R}[t_0,t_1]\otimes_{\mathbb{R}}\mathbb{R}[u_0,u_1]$ and define

$$\mathbf{P} = \mathbf{P}(s_0,s_1,t_0,t_1,u_0,u_1) = (f_0,f_1,f_2,f_3)^T \coloneqq \sum_{0\leq i,j,k\leq 1} w_{ijk}\,\mathbf{P}_{ijk}\,B_i(s_0,s_1)B_j(t_0,t_1)B_k(u_0,u_1) \in R^4$$

where we maintain the notation for the homogeneous Bernstein polynomials, i.e. $B_0(s_0,s_1) = s_0 - s_1$ and $B_1(s_0,s_1) = s_1$. In this formulation, ϕ can be rewritten as

$$\begin{aligned} \phi : \mathbb{P}^1_{\mathbb{R}}\times\mathbb{P}^1_{\mathbb{R}}\times\mathbb{P}^1_{\mathbb{R}} \quad & \dashrightarrow \quad \mathbb{P}^3_{\mathbb{R}} \\ (s_0:s_1)\times(t_0:t_1)\times(u_0:u_1) \quad & \mapsto \quad (f_0:f_1:f_2:f_3), \end{aligned} \tag{2.2}$$

and is birational if there exists an inverse rational map

$$\begin{array}{rcl} \phi^{-1} : \mathbb{P}^3_{\mathbb{R}} & \dashrightarrow & \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}} \\ (x_0 : x_1 : x_2 : x_3) & \mapsto & (a_0 : a_1) \times (b_0 : b_1) \times (c_0 : c_1), \end{array} \tag{2.3}$$

where the $a_i = a_i(x_0, x_1, x_2, x_3)$ (resp. b_j, c_k) are homogeneous of the same degree. Without loss of generality, we can assume that there is no common factor to the a_i's (resp. b_j's, c_k's). Interestingly, the type of ϕ^{-1} is given by the triple $(\deg(a_i), \deg(b_j), \deg(c_k))$ (see Busé et al. (2023)).

2.3. Syzygies and quadratic relations of $\mathbf{P}$

For each $0 \leq i \leq 3$, let $g_i \in R$ be homogeneous of degree $d_s \times d_t \times d_u$. The tuple $\mathbf{g} = (g_0, g_1, g_2, g_3)$ is a *syzygy* of degree $d_s \times d_t \times d_u$ of $\mathbf{P}$ if

$$\langle \mathbf{g}, \mathbf{P} \rangle = g_0\, f_0 + g_1\, f_1 + g_2\, f_2 + g_3\, f_3 = 0,$$

where $\langle -, - \rangle$ stands for the usual scalar product. On the other hand, we set $\mathbf{X} = (x_0, x_1, x_2, x_3)^T$ and $\mathbf{x} = (1, x, y, z)^T$. The syzygy $\mathbf{g}$ can be identified with the polynomial

$$\langle \mathbf{g}, \mathbf{X} \rangle = g_0\, x_0 + g_1\, x_1 + g_2\, x_2 + g_3\, x_3,$$

which vanishes after the specializations $x_i \mapsto f_i$ for each $0 \leq i \leq 3$, or equivalently after $\mathbf{X} \mapsto \mathbf{P}$.

If ϕ is birational, the composition $\phi^{-1} \circ \phi : (\mathbb{P}^1_{\mathbb{R}})^3 \dashrightarrow (\mathbb{P}^1_{\mathbb{R}})^3$ given by

$$(s_0 : s_1) \times (t_0 : t_1) \times (u_0 : u_1) \mapsto (a_0(f_i) : a_1(f_i)) \times (b_0(f_i) : b_1(f_i)) \times (c_0(f_i) : c_1(f_i))$$

yields the identity in $(\mathbb{P}^1_{\mathbb{R}})^3$, i.e. the 2×2 determinants

$$\begin{vmatrix} B_0(s_0, s_1) & B_1(s_0, s_1) \\ a_0 - a_1 & a_1 \end{vmatrix}, \quad \begin{vmatrix} B_0(t_0, t_1) & B_1(t_0, t_1) \\ b_0 - b_1 & b_1 \end{vmatrix}, \quad \begin{vmatrix} B_0(u_0, u_1) & B_1(u_0, u_1) \\ c_0 - c_1 & c_1 \end{vmatrix} \tag{2.4}$$

vanish after $\mathbf{X} \mapsto \mathbf{P}$. In particular, these determinants represent algebraic relations satisfied by the f_i's. Geometrically, they provide the implicit equations defining the parametric surfaces. More specifically, given a fixed $(s_0 : s_1) = (\lambda_0 : \lambda_1) \in \mathbb{P}^1_{\mathbb{R}}$ the equation of the corresponding s-surface in $\mathbb{P}^3_{\mathbb{R}}$ is

$$\begin{vmatrix} B_0(\lambda_0, \lambda_1) & B_1(\lambda_0, \lambda_1) \\ a_0 - a_1 & a_1 \end{vmatrix} = 0,$$

and similarly for the t- and u-surfaces. In particular, the implicit equations of the boundary surfaces $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_1$ are $a_1 = 0$ and $a_0 - a_1 = 0$, respectively.

If the a_i's (resp. b_j's and c_k's) are linear, the first (resp. second and third) relation in (2.4) is a syzygy of $\mathbf{P}$ of degree $1 \times 0 \times 0$ (resp. $0 \times 1 \times 0$ and $0 \times 0 \times 1$), and the s-surfaces (resp. t- and u-surfaces) form a pencil of planes. On the other hand, if the a_i's (resp. b_j's and c_k's) are quadratic, the s-surfaces (resp. t- and u-surfaces) form a pencil of doubly ruled quadric surfaces. The following is the syzygy-based characterization of birationality that we exploit, for the classes of birational maps that we study.

Theorem 1 (Busé et al. (2023), Theorem 6.1)**.** Assume that ϕ is non-degenerated. Then:

- ϕ is birational of type $(1, 1, 1)$ if and only if $\mathbf{P}$ has syzygies of degree $1 \times 0 \times 0$, $0 \times 1 \times 0$, and $0 \times 0 \times 1$.
- ϕ is birational of type $(1, 1, 2)$ if and only if $\mathbf{P}$ has syzygies of degree $1 \times 0 \times 0$ and $0 \times 1 \times 0$, but not $0 \times 0 \times 1$.

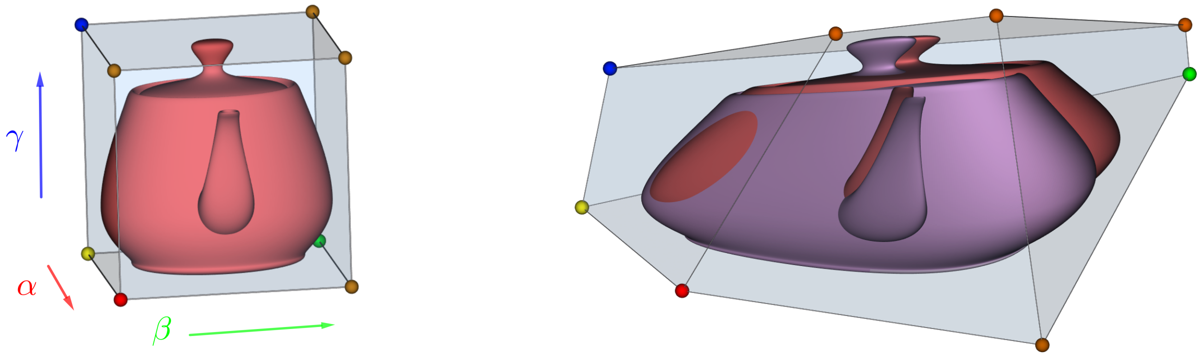

FIGURE 2: Left: we choose freely positive values for w_{000}, w_{100}, w_{010}, w_{001} and compute constants α, β, γ that allow us to derive the remaining w_{ijk}'s while preserving the weight ratio in each direction of the space. Right: A hexahedral FFD with all the w_{ijk}'s equal to 1 (red), and w_{ijk}'s computed using Theorem 2 (purple). In the second case, the FFD is birational although the deformation has not changed significantly.

Control points: $\mathbf{p}_{000} = (\frac{-208}{157}, \frac{-424}{157}, 0)$, $\mathbf{p}_{100} = (\frac{-16}{169}, \frac{-268}{169}, 0)$, $\mathbf{p}_{010} = (\frac{-304}{127}, \frac{200}{127}, 0)$, $\mathbf{p}_{110} = (\frac{272}{259}, \frac{380}{259}, 0)$, $\mathbf{p}_{001} = (\frac{-244}{169}, \frac{-376}{169}, \frac{526}{507})$, $\mathbf{p}_{101} = (\frac{92}{133}, \frac{68}{133}, \frac{982}{399})$, $\mathbf{p}_{011} = (\frac{-316}{131}, \frac{216}{131}, \frac{154}{393})$, $\mathbf{p}_{111} = (\frac{308}{247}, \frac{492}{247}, \frac{2098}{741})$

Weights: $w_{000} = 1$, $w_{100} = 1$, $w_{010} = \frac{2}{3}$, $w_{110} = \frac{81326}{64389}$, $w_{001} = 1$, $w_{101} = \frac{20881}{28561}$, $w_{011} = \frac{41134}{64389}$, $w_{111} = \frac{936662}{837057}$

Boundary surfaces: $\mathbf{\Sigma}_0 : 8 + 4\,x + y = 0$, $\mathbf{\Sigma}_1 : 10 - 20\,x + \frac{15}{2}\,y = 0$, $\mathbf{T}_0 : 24 - \frac{29}{2}\,x + 16\,y - 9\,z = 0$
$\mathbf{T}_1 : 24 - \frac{1}{2}\,x - 16\,y + 3\,z = 0$, $\mathbf{Y}_0 : z = 0$, $\mathbf{Y}_1 : 6 + 2\,x - 3\,z = 0$

3. Hexahedral birational maps

In this section, we study hexahedral birational maps, i.e. birational maps where the control points define a quadrilaterally-faced hexahedron. FIGURE 2 illustrates the variation in a hexahedral FFD after computing weights to achieve birationality, showing that the resulting deformation is very similar to the original.

We require that all the parametric surfaces of a hexahedral birational map are planes. More specifically, the following geometric conditions on the $\mathbf{P}_{ijk}$'s permit the existence of linear syzygies of $\mathbf{P}$.

Property A.1. For each $i = 0, 1$, $\mathbf{\Sigma}_i$ is the plane defined by $\langle \boldsymbol{\sigma}_i, \mathbf{X} \rangle = 0$ for some $\boldsymbol{\sigma}_i = (\sigma_i^0, \sigma_i^1, \sigma_i^2, \sigma_i^3) \in \mathbb{R}^4$.

Property A.2. For each $j = 0, 1$, $\mathbf{T}_j$ is the plane defined by $\langle \boldsymbol{\tau}_j, \mathbf{X} \rangle = 0$ for some $\boldsymbol{\tau}_j = (\tau_j^0, \tau_j^1, \tau_j^2, \tau_j^3) \in \mathbb{R}^4$.

Property A.3. For each $k = 0, 1$, $\mathbf{Y}_k$ is the plane defined by $\langle \boldsymbol{\upsilon}_k, \mathbf{X} \rangle = 0$ for some $\boldsymbol{\upsilon}_k = (\upsilon_k^0, \upsilon_k^1, \upsilon_k^2, \upsilon_k^3) \in \mathbb{R}^4$.

For each $i = 0, 1$, let $\boldsymbol{\pi}_i \in R^4$ and $h_i \in R$. We will use the notation

$$\begin{vmatrix} h_0 & h_1 \\ \boldsymbol{\pi}_1 & \boldsymbol{\pi}_0 \end{vmatrix} := h_0\, \boldsymbol{\pi}_0 - h_1\, \boldsymbol{\pi}_1$$

although the entries in the second row of the determinant above are tuples. Notice that the expression is invariant under linear combinations of the columns.

The following result characterizes the existence syzygies of degree $1 \times 0 \times 0$ of $\mathbf{P}$.

Lemma 3.1. Assume Property A.1 and ϕ non-degenerate. Then, $\mathbf{P}$ has a syzygy of degree $1 \times 0 \times 0$ if and only if

$$\operatorname{rank} \begin{pmatrix} w_{100}\,\langle \boldsymbol{\sigma}_0, \mathbf{P}_{100} \rangle & w_{110}\,\langle \boldsymbol{\sigma}_0, \mathbf{P}_{110} \rangle & w_{101}\,\langle \boldsymbol{\sigma}_0, \mathbf{P}_{101} \rangle & w_{111}\,\langle \boldsymbol{\sigma}_0, \mathbf{P}_{111} \rangle \\ w_{000}\,\langle \boldsymbol{\sigma}_1, \mathbf{P}_{000} \rangle & w_{010}\,\langle \boldsymbol{\sigma}_1, \mathbf{P}_{010} \rangle & w_{001}\,\langle \boldsymbol{\sigma}_1, \mathbf{P}_{001} \rangle & w_{011}\,\langle \boldsymbol{\sigma}_1, \mathbf{P}_{011} \rangle \end{pmatrix} = 1. \tag{3.1}$$

In particular, if (3.1) is satisfied we find $\alpha \in \mathbb{R}$ such that

$$-\,w_{1jk}\,\langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle = \alpha\, w_{0jk}\,\langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle \tag{3.2}$$

for every $0 \leq j,k \leq 1$. Then, any syzygy of degree $1 \times 0 \times 0$ of $\mathbf{P}$ is proportional to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(s_0, s_1) = \begin{vmatrix} B_0(s_0, s_1) & -B_1(s_0, s_1) \\ \alpha\, \boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_0 \end{vmatrix}. \tag{3.3}$$

Proof. First, suppose that $\mathbf{P}$ has a syzygy of the form

$$\boldsymbol{\pi} = \boldsymbol{\pi}(s_0, s_1) = \begin{vmatrix} B_0(s_0, s_1) & -B_1(s_0, s_1) \\ \boldsymbol{\pi}_1 & \boldsymbol{\pi}_0 \end{vmatrix}$$

for some $\boldsymbol{\pi}_0, \boldsymbol{\pi}_1 \in \mathbb{R}^4$. On the other hand, for each $i = 0, 1$ we write

$$\mathbf{P}_i = \mathbf{P}_i(t_0, t_1, u_0, u_1) = \sum_{0 \leq j,k \leq 1} w_{ijk}\, \mathbf{P}_{ijk}\, B_j(t_0, t_1)\, B_k(u_0, u_1).$$

Then, we find

$$\langle \boldsymbol{\pi}, \mathbf{P} \rangle = \begin{vmatrix} B_0(s_0, s_1) & -B_1(s_0, s_1) \\ \langle \boldsymbol{\pi}_1, \mathbf{P} \rangle & \langle \boldsymbol{\pi}_0, \mathbf{P} \rangle \end{vmatrix} =$$

$$\begin{vmatrix} B_0(s_0, s_1) & -B_1(s_0, s_1) \\ B_0(s_0, s_1)\, \langle \boldsymbol{\pi}_1, \mathbf{P}_0 \rangle + B_1(s_0, s_1)\, \langle \boldsymbol{\pi}_1, \mathbf{P}_1 \rangle & B_0(s_0, s_1)\, \langle \boldsymbol{\pi}_0, \mathbf{P}_0 \rangle + B_1(s_0, s_1)\, \langle \boldsymbol{\pi}_0, \mathbf{P}_1 \rangle \end{vmatrix} =$$

$$B_0(s_0, s_1)^2\, \langle \boldsymbol{\pi}_0, \mathbf{P}_0 \rangle + B_0(s_0, s_1)\, B_1(s_0, s_1)\, (\, \langle \boldsymbol{\pi}_0, \mathbf{P}_1 \rangle + \langle \boldsymbol{\pi}_1, \mathbf{P}_0 \rangle\,) + B_1(s_0, s_1)^2\, \langle \boldsymbol{\pi}_1, \mathbf{P}_1 \rangle = 0.$$

It follows that $\langle \boldsymbol{\pi}_i, \mathbf{P}_i \rangle = 0$ for each $i = 0, 1$. By Property A.1, the tuple $\mathbf{P}_i$ defines a parametrization of the boundary plane $\boldsymbol{\Sigma}_i$, so we must have $\boldsymbol{\pi}_i = \lambda_i\, \boldsymbol{\sigma}_i$ for some non-zero $\lambda_i \in \mathbb{R}$. On the other hand,

$$\langle \boldsymbol{\pi}_0, \mathbf{P}_1 \rangle + \langle \boldsymbol{\pi}_1, \mathbf{P}_0 \rangle = \lambda_0\, \langle \boldsymbol{\sigma}_0, \mathbf{P}_1 \rangle + \lambda_1\, \langle \boldsymbol{\sigma}_1, \mathbf{P}_0 \rangle =$$

$$\sum_{0 \leq j,k \leq 1} (\, \lambda_0\, w_{1jk}\, \langle \boldsymbol{\sigma}_0, \mathbf{P}_{1jk} \rangle + \lambda_1\, w_{0jk}\, \langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk} \rangle\,)\, B_j(t_0, t_1)\, B_k(u_0, u_1) = 0,$$

which holds for some λ_0, λ_1 if and only if (3.1) is satisfied. As ϕ is non-degenerate, no row in (3.1) is identically zero. Hence, α is unique and $\lambda_1/\lambda_0 = \alpha$. In particular, $\boldsymbol{\pi}$ is proportional to $\boldsymbol{\sigma}$. □

Remark 3.2. Similarly, we derive analogous results to Lemma 3.1 for syzygies of degree $0 \times 1 \times 0$ and $0 \times 0 \times 1$. Namely, let ϕ be non-degenerate. Assuming Property A.2, $\mathbf{P}$ has a syzygy of degree $0 \times 1 \times 0$ if and only if

$$\operatorname{rank} \begin{pmatrix} w_{010}\, \langle \boldsymbol{\tau}_0, \mathbf{P}_{010} \rangle & w_{110}\, \langle \boldsymbol{\tau}_0, \mathbf{P}_{110} \rangle & w_{011}\, \langle \boldsymbol{\tau}_0, \mathbf{P}_{011} \rangle & w_{111}\, \langle \boldsymbol{\tau}_0, \mathbf{P}_{111} \rangle \\ w_{000}\, \langle \boldsymbol{\tau}_1, \mathbf{P}_{000} \rangle & w_{100}\, \langle \boldsymbol{\tau}_1, \mathbf{P}_{100} \rangle & w_{001}\, \langle \boldsymbol{\tau}_1, \mathbf{P}_{001} \rangle & w_{101}\, \langle \boldsymbol{\tau}_1, \mathbf{P}_{101} \rangle \end{pmatrix} = 1. \tag{3.4}$$

On the other hand, assuming Property A.3 then $\mathbf{P}$ has a syzygy of degree $0 \times 0 \times 1$ if and only if

$$\operatorname{rank} \begin{pmatrix} w_{001}\, \langle \boldsymbol{\upsilon}_0, \mathbf{P}_{001} \rangle & w_{101}\, \langle \boldsymbol{\upsilon}_0, \mathbf{P}_{101} \rangle & w_{011}\, \langle \boldsymbol{\upsilon}_0, \mathbf{P}_{011} \rangle & w_{111}\, \langle \boldsymbol{\upsilon}_0, \mathbf{P}_{111} \rangle \\ w_{000}\, \langle \boldsymbol{\upsilon}_1, \mathbf{P}_{000} \rangle & w_{100}\, \langle \boldsymbol{\upsilon}_1, \mathbf{P}_{100} \rangle & w_{010}\, \langle \boldsymbol{\upsilon}_1, \mathbf{P}_{010} \rangle & w_{110}\, \langle \boldsymbol{\upsilon}_1, \mathbf{P}_{110} \rangle \end{pmatrix} = 1. \tag{3.5}$$

If (3.4) and (3.5) hold, we find $\beta, \gamma \in \mathbb{R}$ such that

$$-\, w_{i1k}\, \langle \boldsymbol{\tau}_0, \mathbf{P}_{i1k} \rangle = \beta\, w_{i0k}\, \langle \boldsymbol{\tau}_1, \mathbf{P}_{i0k} \rangle \;,\; -\, w_{ij1}\, \langle \boldsymbol{\upsilon}_0, \mathbf{P}_{ij1} \rangle = \gamma\, w_{ij0}\, \langle \boldsymbol{\upsilon}_1, \mathbf{P}_{ij0} \rangle, \tag{3.6}$$

and any syzygy of degree either $0 \times 1 \times 0$ or $0 \times 0 \times 1$ of $\mathbf{P}$ is respectively proportional to

$$\boldsymbol{\tau} = \boldsymbol{\tau}(t_0, t_1) = \begin{vmatrix} B_0(t_0, t_1) & -B_1(t_0, t_1) \\ \beta\, \boldsymbol{\tau}_1 & \boldsymbol{\tau}_0 \end{vmatrix} \;,\; \boldsymbol{\upsilon} = \boldsymbol{\upsilon}(u_0, u_1) = \begin{vmatrix} B_0(u_0, u_1) & -B_1(u_0, u_1) \\ \gamma\, \boldsymbol{\upsilon}_1 & \boldsymbol{\upsilon}_0 \end{vmatrix}. \tag{3.7}$$

Given indices $0 \leq i, j, k \leq 1$, we denote their converse by $\hat{i}, \hat{j}, \hat{k}$. More explicitly, we set $\hat{0} = 1$ and $\hat{1} = 0$.

Remark 3.3. With the hypotheses of Lemma 3.1, we find that

$$\langle \boldsymbol{\sigma}_i, \mathbf{P}\rangle = B_{\hat{i}}(s_0,s_1) \sum_{0\le j,k\le 1} w_{\hat{i}jk}\, \langle \boldsymbol{\sigma}_i, \mathbf{P}_{\hat{i}jk}\rangle\, B_j(t_0,t_1)\, B_k(u_0,u_1) = B_{\hat{i}}(s_0,s_1)\, g_i(t_0,t_1,u_0,u_1).$$

We observe that the coefficients of g_i in the Bernstein basis coincide with the entries of the $(i+1)$-th row in (3.1). Therefore, if (3.1) is satisfied we find $\mu_0, \mu_1 \in \mathbb{R}$ such that $g = \mu_i\, g_i$. In particular, the pullback by ϕ of the line $\boldsymbol{\Sigma}_0 \cap \boldsymbol{\Sigma}_1$ is given by the vanishing of $g = 0$ in $(\mathbb{P}^1_{\mathbb{R}})^3$, which defines a surface of degree $0\times 1\times 1$. Geometrically, $\mathbf{P}$ admits a syzygy of degree $1\times 0\times 0$ if and only if there is a surface of degree $0\times 1\times 1$ that is contracted by ϕ to $\boldsymbol{\Sigma}_0 \cap \boldsymbol{\Sigma}_1$. Repeating the argument with the rank condition (3.4) (resp. (3.5)), $\mathbf{P}$ admits a syzygy of degree $0\times 1\times 0$ (resp. $0\times 0\times 1$) if and only if there is a surface of degree $1\times 0\times 1$ (resp. $1\times 1\times 0$) in $(\mathbb{P}^1_{\mathbb{R}})^2$ that is contracted to $\mathbf{T}_0\cap\mathbf{T}_1$ (resp. $\mathbf{Y}_0\cap\mathbf{Y}_1$).

The following configuration will ensure that a hexahedral map is non-degenerate and that the weights are positive.

Configuration 1. Assume Properties A.1, A.2, and A.3. Moreover, for each $0\le i,j,k\le 1$ assume that

$$\langle \boldsymbol{\sigma}_i, \mathbf{P}_{\hat{i}jk}\rangle > 0 \quad , \quad \langle \boldsymbol{\tau}_j, \mathbf{P}_{i\hat{j}k}\rangle > 0 \quad , \quad \langle \boldsymbol{\upsilon}_k, \mathbf{P}_{ij\hat{k}}\rangle > 0. \tag{3.8}$$

Equivalently, the $\mathbf{P}_{ijk}$'s define the vertices of a quadrilaterally-faced hexahedron. Additionally, set

$$\Delta_{ijk} = \begin{vmatrix} \sigma_i^1 & \sigma_i^2 & \sigma_i^3 \\ \tau_j^1 & \tau_j^2 & \tau_j^3 \\ \upsilon_k^1 & \upsilon_k^2 & \upsilon_k^3 \end{vmatrix}. \tag{3.9}$$

By the inequalities (3.8), Δ_{ijk} is non-zero for every $0\le i,j,k\le 1$. Moreover, the vectors $\boldsymbol{\sigma}_i, \boldsymbol{\tau}_j$, and $\boldsymbol{\upsilon}_k$ are linearly independent. In particular, we can write

$$\mathbf{P}_{ijk} = \Delta_{ijk}^{-1}\, \boldsymbol{\sigma}_i \wedge \boldsymbol{\tau}_j \wedge \boldsymbol{\upsilon}_k,$$

where $-\wedge-$ stands for the usual exterior product.

The following is our first main result, which provides a criterion for the computation of positive weights that make a hexahedral map birational of type $(1,1,1)$. More specifically, after an initial choice of weights in each direction of the space, we compute constants that allow us to derive the remaining weights while preserving the ratios required for the existence of the necessary syzygies (see FIGURE 2).

Theorem 2. Assume Configuration 1. Choose freely positive values for

$$w_{000}\ ,\ w_{100}\ ,\ w_{010}\ ,\ w_{001}$$

and define

$$\alpha = \left(\frac{w_{100}}{\Delta_{100}}\right)\left(\frac{w_{000}}{\Delta_{000}}\right)^{-1} \ ,\ \beta = \left(\frac{w_{010}}{\Delta_{010}}\right)\left(\frac{w_{000}}{\Delta_{000}}\right)^{-1} \ ,\ \gamma = \left(\frac{w_{001}}{\Delta_{001}}\right)\left(\frac{w_{000}}{\Delta_{000}}\right)^{-1}.$$

Then, ϕ is birational of type $(1,1,1)$ if and only if

$$w_{ijk} = \alpha^i\, \beta^j\, \gamma^k\, \Delta_{ijk}\left(\frac{w_{000}}{\Delta_{000}}\right) \tag{3.10}$$

for each $0\le i,j,k\le 1$. Moreover, the inverse rational map is given by

$$s = \frac{\langle \boldsymbol{\sigma}_0, \mathbf{x}\rangle}{\langle \boldsymbol{\sigma}_0, \mathbf{x}\rangle - \alpha\,\langle \boldsymbol{\sigma}_1, \mathbf{x}\rangle} \quad , \quad t = \frac{\langle \boldsymbol{\tau}_0, \mathbf{x}\rangle}{\langle \boldsymbol{\tau}_0, \mathbf{x}\rangle - \beta\,\langle \boldsymbol{\tau}_1, \mathbf{x}\rangle} \quad , \quad u = \frac{\langle \boldsymbol{\upsilon}_0, \mathbf{x}\rangle}{\langle \boldsymbol{\upsilon}_0, \mathbf{x}\rangle - \gamma\,\langle \boldsymbol{\upsilon}_1, \mathbf{x}\rangle}. \tag{3.11}$$

Proof. By Configuration 1, the image of ϕ contains the distinct planes $\mathbf{\Sigma}_0, \mathbf{\Sigma}_1$ (as well as $\mathbf{T}_0, \mathbf{T}_1$ and $\mathbf{Y}_0, \mathbf{Y}_1$). As this image is an irreducible variety (in the Zariski topology), ϕ is non-degenerate. Therefore, by Theorem 1 ϕ is birational of type $(1,1,1)$ if and only if $\mathbf{P}$ has syzygies of degrees $1\times 0\times 0$, $0\times 1\times 0$, and $0\times 0\times 1$, or equivalently by Lemma 3.1 and Remark 3.2, if the rank conditions (3.1), (3.4), and (3.5) are simultaneously satisfied. On the other hand, for any $0\le i,j,k\le 1$ we can write

$$\Delta_{0jk}\,\langle \boldsymbol{\sigma}_1, \mathbf{P}_{0jk}\rangle = \boldsymbol{\sigma}_1\wedge\boldsymbol{\sigma}_0\wedge\boldsymbol{\tau}_j\wedge\boldsymbol{\upsilon}_k = -\Delta_{1jk}\,\Delta_{1jk}^{-1}\,\boldsymbol{\sigma}_0\wedge\boldsymbol{\sigma}_1\wedge\boldsymbol{\tau}_j\wedge\boldsymbol{\upsilon}_k = -\Delta_{1jk}\,\langle\boldsymbol{\sigma}_0, \mathbf{P}_{1jk}\rangle. \tag{3.12}$$

Hence, (3.1) can be equivalently written as

$$\operatorname{rank}\begin{pmatrix} w_{100}\,\Delta_{000} & w_{110}\,\Delta_{010} & w_{101}\,\Delta_{001} & w_{111}\,\Delta_{011}\\ w_{000}\,\Delta_{100} & w_{010}\,\Delta_{110} & w_{001}\,\Delta_{101} & w_{011}\,\Delta_{111}\end{pmatrix} = 1. \tag{3.13}$$

With a similar argument, we derive

$$\Delta_{i0k}\,\langle\boldsymbol{\tau}_1, \mathbf{P}_{i0k}\rangle = -\Delta_{i1k}\,\langle\boldsymbol{\tau}_0,\mathbf{P}_{i1k}\rangle \quad , \quad \Delta_{ij0}\,\langle\boldsymbol{\upsilon}_1,\mathbf{P}_{ij0}\rangle = -\Delta_{ij1}\,\langle\boldsymbol{\upsilon}_0,\mathbf{P}_{ij1}\rangle,$$

and (3.4) and (3.5) can be equivalently rewritten as

$$\operatorname{rank}\begin{pmatrix} w_{010}\,\Delta_{000} & w_{110}\,\Delta_{100} & w_{011}\,\Delta_{001} & w_{111}\,\Delta_{101}\\ w_{000}\,\Delta_{010} & w_{100}\,\Delta_{110} & w_{001}\,\Delta_{011} & w_{101}\,\Delta_{111}\end{pmatrix} = 1, \tag{3.14}$$

$$\operatorname{rank}\begin{pmatrix} w_{001}\,\Delta_{000} & w_{101}\,\Delta_{100} & w_{011}\,\Delta_{010} & w_{111}\,\Delta_{110}\\ w_{000}\,\Delta_{001} & w_{100}\,\Delta_{101} & w_{010}\,\Delta_{011} & w_{100}\,\Delta_{111}\end{pmatrix} = 1. \tag{3.15}$$

Therefore, the rank conditions (3.13), (3.14), and (3.15) are simultaneously satisfied if and only if the w_{ijk}'s are as (3.10). Moreover, by Configuration 1 we find the strict inequalities

$$\frac{\Delta_{1jk}}{\Delta_{0jk}} = -\frac{\langle\boldsymbol{\sigma}_1,\mathbf{P}_{0jk}\rangle}{\langle\boldsymbol{\sigma}_0,\mathbf{P}_{1jk}\rangle} < 0 \quad , \quad \frac{\Delta_{i1k}}{\Delta_{i0k}} = -\frac{\langle\boldsymbol{\tau}_1,\mathbf{P}_{i0k}\rangle}{\langle\boldsymbol{\tau}_0,\mathbf{P}_{i1k}\rangle} < 0 \quad , \quad \frac{\Delta_{ij1}}{\Delta_{ij0}} = -\frac{\langle\boldsymbol{\upsilon}_1,\mathbf{P}_{ij0}\rangle}{\langle\boldsymbol{\upsilon}_0,\mathbf{P}_{ij1}\rangle} < 0, \tag{3.16}$$

and from (3.10) and (3.16), it follows that

$$w_{1jk} = w_{0jk}\,\alpha\left(\frac{\Delta_{1jk}}{\Delta_{0jk}}\right) > 0\ ,\ w_{i1k} = w_{i0k}\,\beta\left(\frac{\Delta_{i1k}}{\Delta_{i0k}}\right) > 0\ ,\ w_{ij1} = w_{ij0}\,\gamma\left(\frac{\Delta_{ij1}}{\Delta_{ij0}}\right) > 0.$$

Finally, by Lemma 3.1 and Remark 3.2 the syzygies of degree $1\times 0\times 0$, $0\times 1\times 0$, and $0\times 0\times 1$ are as in (3.3), (3.7), and the inverse rational map follows. □

Remark 3.4. If the $\mathbf{p}_{ijk}$'s define a parallelogram, and assuming $\|\boldsymbol{\sigma}_i\| = \|\boldsymbol{\tau}_j\| = \|\boldsymbol{\upsilon}_k\| = 1$, then we have $\langle\boldsymbol{\sigma}_0,\mathbf{P}_{1jk}\rangle = \langle\boldsymbol{\sigma}_1,\mathbf{P}_{0jk}\rangle$ (resp. $\langle\boldsymbol{\tau}_0,\mathbf{P}_{i1k}\rangle = \langle\boldsymbol{\tau}_1,\mathbf{P}_{i0k}\rangle$ and $\langle\boldsymbol{\upsilon}_0,\mathbf{P}_{ij1}\rangle = \langle\boldsymbol{\upsilon}_1,\mathbf{P}_{ij0}\rangle$). In particular, if $w_{ijk}=1$ for every $0\le i,j,k\le 1$ the conditions (3.1), (3.4), and (3.5) are satisfied, and ϕ is birational of type $(1,1,1)$.

4. Birational maps of type $(1,1,2)$

In this section, we study birational maps of degree $1\times 1\times 1$ and type $(1,1,2)$. Figure 3 illustrates the variation of an FFD after computing weights to achieve birationality, showing that the resulting deformation is very similar to the original. Notice that this class is equivalent to the classes of birational maps of type $(1,2,1)$ and $(2,1,1)$, as they differ simply by a permutation of the parameters.

Now, the u-surfaces form a pencil of doubly ruled quadric surfaces. Namely, we have the following.

Property B.1. For each $k=0,1$, $\mathbf{Y}_k$ is a doubly ruled quadric surface.

In this class of birational maps the geometry of the control points is less constrained than for hexahedral maps, as for any non-coplanar choice of $\mathbf{P}_{00k}$, $\mathbf{P}_{10k}$, $\mathbf{P}_{01k}$, and $\mathbf{P}_{11k}$ the boundary surface $\mathbf{Y}_k$ is a doubly ruled quadric. However, the following geometric condition is required.

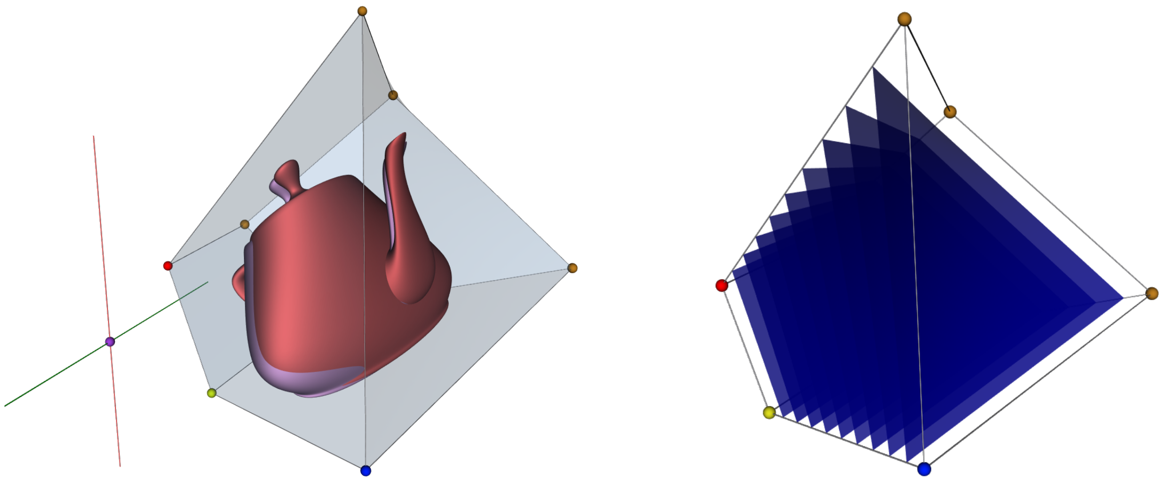

FIGURE 3: Left: A 3D FFD of degree $1 \times 1 \times 1$ with one pair of doubly ruled quadric boundary surfaces, with all the w_{ijk}'s equal to 1 (red) and w_{ijk}'s computed using Theorem 3 (purple). In the second case, the FFD is birational although the deformation remains very similar. The lines $\mathbf{\Sigma}_0 \cap \mathbf{\Sigma}_1$ and $\mathbf{T}_0 \cap \mathbf{T}_1$ intersect at the point $\mathbf{V}$. Equivalently, the lines $\overline{\mathbf{p}_{ij0}\mathbf{p}_{ij1}}$ converge at $\mathbf{V}$, for every $0 \le i, j \le 1$. Right: the u-surfaces define a pencil of doubly ruled quadrics.

Control points: $\mathbf{p}_{000} = (\frac{1}{2}, \frac{-5}{4}, 0)$, $\mathbf{p}_{100} = (-1, \frac{-1}{2}, 3)$, $\mathbf{p}_{010} = (\frac{3}{4}, \frac{23}{4}, 0)$, $\mathbf{p}_{110} = (-1, 4, 3)$, $\mathbf{p}_{001} = (\frac{11}{2}, \frac{-15}{4}, 0)$, $\mathbf{p}_{101} = (\frac{11}{2}, \frac{-15}{4}, \frac{19}{2})$, $\mathbf{p}_{011} = (\frac{15}{2}, \frac{25}{2}, 0)$, $\mathbf{p}_{111} = (\frac{5}{2}, \frac{15}{2}, \frac{13}{2})$

Weights: $w_{000} = 1$, $w_{100} = 1$, $w_{010} = 1$, $w_{110} = \frac{19}{18}$, $w_{001} = 1$, $w_{101} = \frac{2}{3}$, $w_{011} = \frac{361}{414}$, $w_{111} = \frac{361}{351}$

Boundary surfaces: $\mathbf{\Sigma}_0 : z = 0\ ,\ \mathbf{\Sigma}_1 : 4 + x - z = 0\ ,\ \mathbf{T}_0 : 2 + x + 2y = 0\ ,\ \mathbf{T}_1 : 5 + x - y = 0$
$\mathbf{Y}_0 : 244 - 387\,x - 112\,x^2 + 16\,y + 4\,xy - 232\,z - 59\,xz - 4\,yz = 0$
$\mathbf{Y}_1 : 40300 + 3315\,x - 1690\,x^2 + 832\,y + 208\,xy - 760\,z - 380\,xz - 760\,yz = 0$

Property C. Assume Properties A.1 and A.2. The planes $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_1$, $\mathbf{T}_0$, $\mathbf{T}_1$ intersect at a point $\mathbf{V}$ (in $\mathbb{P}^3_{\mathbb{R}}$).

Lemma 4.1. If ϕ is birational of type $(1, 1, 2)$, then Property C is satisfied.

Proof. For the existence of the syzygies of degrees $1 \times 0 \times 0$ and $0 \times 1 \times 0$ defining ϕ^{-1} on the first factors of $(\mathbb{P}^1_{\mathbb{R}})^3$, Properties A.1 and A.2 must be satisfied. On the other hand, with the notation of (2.3) we find polynomials $g = g(t_0, t_1, u_0, u_1)$ and $h = h(s_0, s_1, u_0, u_1)$, of degrees $0 \times 1 \times 1$ and $1 \times 0 \times 1$ respectively, such that for each $0 \le i, j \le 1$ we have

$$a_i \xmapsto{\mathbf{X} \mapsto \mathbf{P}} s_i\, g \quad , \quad b_j \xmapsto{\mathbf{X} \mapsto \mathbf{P}} t_i\, h.$$

In particular, the pullback by ϕ of the line $\mathbf{\Sigma}_0 \cap \mathbf{\Sigma}_1$ (resp. $\mathbf{T}_0 \cap \mathbf{T}_1$), defined by $a_0 = a_1 = 0$ (resp. $b_0 = b_1 = 0$), is the surface given by $g = 0$ (resp. $h = 0$) in $(\mathbb{P}^1_{\mathbb{R}})^3$ (see Remark 3.3). The surfaces $g = 0$ and $h = 0$ have degree respectively $0 \times 1 \times 1$ and $1 \times 0 \times 1$ in $(\mathbb{P}^1_{\mathbb{R}})^3$. It follows that the lines $\mathbf{\Sigma}_0 \cap \mathbf{\Sigma}_1$ and $\mathbf{T}_0 \cap \mathbf{T}_1$ are distinct, as their pullbacks have different degrees. Moreover, these pullbacks intersect either on a surface or a curve of degree $1 \times 1 \times 1$. However, the base locus of ϕ is a curve of degree $1 \times 1 \times 0$ (see Busé et al. (2023)), so the intersection $g = h = 0$ is not contained in it. Hence, $g = h = 0$ must be contracted to a point $\mathbf{V} = \mathbf{\Sigma}_0 \cap \mathbf{\Sigma}_1 \cap \mathbf{T}_0 \cap \mathbf{T}_1 \in \mathbb{P}^3_{\mathbb{R}}$. $\square$

The following result shows that birational maps of type $(1, 1, 2)$ must contract certain surfaces to the lines $\mathbf{\Sigma}_0 \cap \mathbf{\Sigma}_1$ and $\mathbf{T}_0 \cap \mathbf{T}_1$.

Lemma 4.2. Assume Property C, as well as the rank conditions (3.1) and (3.4). Additionally, suppose that no three of the $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_1$, $\mathbf{T}_0$, and $\mathbf{T}_1$ share a common line. Write

$$E_i = \begin{vmatrix} \sigma_i^1 & \sigma_i^2 & \sigma_i^3 \\ \tau_0^1 & \tau_0^2 & \tau_0^3 \\ \tau_1^1 & \tau_1^2 & \tau_1^3 \end{vmatrix} \quad , \quad Z_j = \begin{vmatrix} \sigma_0^1 & \sigma_0^2 & \sigma_0^3 \\ \sigma_1^1 & \sigma_1^2 & \sigma_1^3 \\ \tau_j^1 & \tau_j^2 & \tau_j^3 \end{vmatrix},$$

for each $0 \leq i, j \leq 1$, and set $\epsilon = E_1/E_0$ and $\theta = Z_1/Z_0$. Similarly, write

$$F = F(s_0, s_1) = \begin{vmatrix} B_0(s_0, s_1) & -B_1(s_0, s_1) \\ \alpha\,\epsilon & 1 \end{vmatrix} \quad , \quad G = G(t_0, t_1) = \begin{vmatrix} B_0(t_0, t_1) & -B_1(t_0, t_1) \\ \beta\,\theta & 1 \end{vmatrix}.$$

Then, the surface $F = 0$ (resp. $G = 0$) in $(\mathbb{P}^1_{\mathbb{R}})^3$ is contracted by ϕ to the line $\mathbf{T}_0 \cap \mathbf{T}_1$ (resp. $\mathbf{\Sigma}_0 \cap \mathbf{\Sigma}_1$).

Proof. By the independence of the boundary planes, all the E_i's and Z_j's are non-zero. Moreover, by Property C we can write

$$\mathbf{V} = E_0^{-1}\,\boldsymbol{\sigma}_0 \wedge \boldsymbol{\tau}_0 \wedge \boldsymbol{\tau}_1 = E_1^{-1}\,\boldsymbol{\sigma}_1 \wedge \boldsymbol{\tau}_0 \wedge \boldsymbol{\tau}_1 = Z_0^{-1}\,\boldsymbol{\sigma}_0 \wedge \boldsymbol{\sigma}_1 \wedge \boldsymbol{\tau}_0 = Z_1^{-1}\,\boldsymbol{\sigma}_1 \wedge \boldsymbol{\sigma}_1 \wedge \boldsymbol{\tau}_1.$$

In particular, we find

$$\mathbf{V} - \mathbf{V} = \left(E_0^{-1}\,\boldsymbol{\sigma}_0 - E_1^{-1}\,\boldsymbol{\sigma}_1\right) \wedge \boldsymbol{\tau}_0 \wedge \boldsymbol{\tau}_1 = E_1^{-1}\,(\epsilon\,\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1) \wedge \boldsymbol{\tau}_0 \wedge \boldsymbol{\tau}_1 = 0,$$

so we can find $\lambda_0, \lambda_1 \in \mathbb{R}$ such that $\epsilon\,\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1 = \lambda_0\,\boldsymbol{\tau}_0 + \lambda_1\,\boldsymbol{\tau}_1$. Similarly, we can find $\mu_0, \mu_1 \in \mathbb{R}$ such that $\theta\,\boldsymbol{\tau}_0 - \boldsymbol{\tau}_1 = \mu_0\,\boldsymbol{\sigma}_0 + \mu_1\,\boldsymbol{\sigma}_1$. On the other hand, by (3.1) and (3.4) $\mathbf{P}$ admits the syzygies

$$\begin{vmatrix} B_0(s_0, s_1) & -B_1(s_0, s_1) \\ \alpha\,\boldsymbol{\sigma}_1 & \boldsymbol{\sigma}_0 \end{vmatrix} \quad , \quad \begin{vmatrix} B_0(t_0, t_1) & -B_1(t_0, t_1) \\ \beta\,\boldsymbol{\tau}_1 & \boldsymbol{\tau}_0 \end{vmatrix}.$$

In particular, if $F = 0$ and $\mathbf{P}'$ is the corresponding specialization of $\mathbf{P}$ we obtain

$$\begin{vmatrix} \alpha\,\epsilon & 1 \\ \alpha\,\langle \boldsymbol{\sigma}_1, \mathbf{P}' \rangle & \langle \boldsymbol{\sigma}_0, \mathbf{P}' \rangle \end{vmatrix} = \alpha\,\langle \epsilon\,\boldsymbol{\sigma}_0 - \boldsymbol{\sigma}_1, \mathbf{P}' \rangle = \alpha\,\langle \lambda_0\,\boldsymbol{\tau}_0 + \lambda_1\,\boldsymbol{\tau}_1, \mathbf{P}' \rangle = 0.$$

Then, for a general choice of $(t_0 : t_1) \in \mathbb{P}^1_{\mathbb{R}}$ the restriction of ϕ to $F = 0$ lies in two distinct planes in the pencil defined by $\mathbf{T}_0$ and $\mathbf{T}_1$. Therefore, it must be contained in the line $\mathbf{T}_0 \cap \mathbf{T}_1$. Similarly, the restriction of ϕ to the surface defined by $G = 0$ must lie in the line $\mathbf{\Sigma}_0 \cap \mathbf{\Sigma}_1$. □

As in the class of hexahedral maps, the following configuration will ensure that an FFD of type $(1, 1, 2)$ is non-degenerated and the w_{ijk}'s are positive.

Configuration 2. Assume Properties B.1 and C. Let $\mathbf{Q}_0, \mathbf{Q}_1$ be any two distinct points in the line $\mathbf{T}_0 \cap \mathbf{T}_1$ different from $\mathbf{V}$, and set

$$\boldsymbol{\upsilon}_{ik} = \mathbf{P}_{i0k} \wedge \mathbf{P}_{i1k} \wedge \mathbf{Q}_k.$$

Moreover, assume that

$$\langle \boldsymbol{\sigma}_i, \mathbf{P}_{\hat{i}jk} \rangle > 0 \quad , \quad \langle \boldsymbol{\tau}_j, \mathbf{P}_{i\hat{j}k} \rangle > 0 \quad , \quad \langle \boldsymbol{\upsilon}_{ik}, \mathbf{P}_{ij\hat{k}} \rangle > 0. \tag{4.1}$$

Additionally, set

$$\Delta_{ijk} = \begin{vmatrix} \sigma_i^1 & \sigma_i^2 & \sigma_i^3 \\ \tau_j^1 & \tau_j^2 & \tau_j^3 \\ \upsilon_{ik}^1 & \upsilon_{ik}^2 & \upsilon_{ik}^3 \end{vmatrix} \quad , \quad \delta_{ik} = \langle \boldsymbol{\upsilon}_{ik}, \mathbf{V} \rangle. \tag{4.2}$$

By the inequalities (4.1), Δ_{ijk} is non-zero for every $0 \leq i, j, k \leq 1$. Moreover, the vectors $\boldsymbol{\sigma}_i$, $\boldsymbol{\tau}_j$ and $\boldsymbol{\upsilon}_{ik}$ are independent. In particular, we can write

$$\mathbf{P}_{ijk} = \Delta_{ijk}^{-1}\,\boldsymbol{\sigma}_i \wedge \boldsymbol{\tau}_j \wedge \boldsymbol{\upsilon}_{ik}.$$

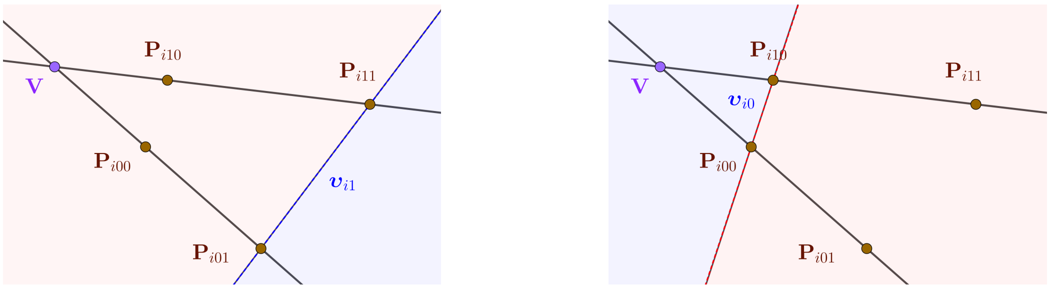

FIGURE 4: Left: positive and negative halfplanes defined by $\boldsymbol{\upsilon}_{i1}$ in $\boldsymbol{\Sigma}_i$. Right: positive and negative halfplanes defined by $\boldsymbol{\upsilon}_{i0}$ in $\boldsymbol{\Sigma}_i$. The point $\mathbf{V}$ lies in different signed regions, implying that δ_{i0}, δ_{i1} have different sign.

The following is our second main result. Namely, it is a criterion for the computation of positive weights that make an FFD birational of type $(1,1,2)$.

Theorem 3. Assume Configuration 2. Choose freely positive values for the weights

$$w_{000}\ ,\ w_{100}\ ,\ w_{010}\ ,\ w_{001}$$

and define

$$\alpha=\left(\frac{\delta_{10}\,w_{100}}{\Delta_{100}}\right)\left(\frac{\delta_{00}\,w_{000}}{\Delta_{000}}\right)^{-1}\ ,\ \beta=\left(\frac{w_{010}}{\Delta_{010}}\right)\left(\frac{w_{000}}{\Delta_{000}}\right)^{-1}\ ,\ \gamma=\left(\frac{\delta_{01}\,w_{001}}{\Delta_{001}}\right)\left(\frac{\delta_{00}\,w_{000}}{\Delta_{000}}\right)^{-1}.$$

Then, ϕ is birational of type $(1,1,2)$ if and only if

$$w_{ijk}=\alpha^i\,\beta^j\,\gamma^k\left(\frac{\Delta_{ijk}}{\delta_{ik}}\right)\left(\frac{w_{000}\,\delta_{00}}{\Delta_{000}}\right) \tag{4.3}$$

for each $0\le i,j,k\le 1$. Moreover, ϕ^{-1} is given by

$$s=\frac{\langle\boldsymbol{\sigma}_0,\mathbf{x}\rangle}{\langle\boldsymbol{\sigma}_0,\mathbf{x}\rangle-\alpha\,\langle\boldsymbol{\sigma}_1,\mathbf{x}\rangle}\quad,\quad t=\frac{\langle\boldsymbol{\tau}_0,\mathbf{x}\rangle}{\langle\boldsymbol{\tau}_0,\mathbf{x}\rangle-\beta\,\langle\boldsymbol{\tau}_1,\mathbf{x}\rangle}\quad,$$

$$u=\frac{\begin{vmatrix}\langle\boldsymbol{\sigma}_1,\mathbf{X}\rangle & \langle\boldsymbol{\sigma}_0,\mathbf{X}\rangle\\ \frac{\epsilon}{\delta_{10}}\,\langle\boldsymbol{\upsilon}_{10},\mathbf{X}\rangle & \frac{1}{\delta_{00}}\,\langle\boldsymbol{\upsilon}_{00},\mathbf{X}\rangle\end{vmatrix}}{\begin{vmatrix}\langle\boldsymbol{\sigma}_1,\mathbf{X}\rangle & \langle\boldsymbol{\sigma}_0,\mathbf{X}\rangle\\ \frac{\epsilon}{\delta_{10}}\,\langle\boldsymbol{\upsilon}_{10},\mathbf{X}\rangle & \frac{1}{\delta_{00}}\,\langle\boldsymbol{\upsilon}_{00},\mathbf{X}\rangle\end{vmatrix}-\gamma\begin{vmatrix}\langle\boldsymbol{\sigma}_1,\mathbf{X}\rangle & \langle\boldsymbol{\sigma}_0,\mathbf{X}\rangle\\ \frac{\epsilon}{\delta_{11}}\,\langle\boldsymbol{\upsilon}_{11},\mathbf{X}\rangle & \frac{1}{\delta_{01}}\,\langle\boldsymbol{\upsilon}_{01},\mathbf{X}\rangle\end{vmatrix}}.$$

Proof. By the same argument in the proof of Theorem 2, ϕ is non-degenerated. Now, from Theorem 1 ϕ is birational of type $(1,1,2)$ if and only if $\mathbf{P}$ has syzygies of degrees $1\times 0\times 0$ and $0\times 1\times 0$, as Property B.1 prevents the existence of a syzygy of degree $0\times 0\times 1$. Equivalently, the rank conditions (3.1) and (3.4) must be satisfied simultaneously. On the other hand, given $0\le i,j,k\le 1$ we can write

$$\langle\boldsymbol{\sigma}_{\hat{i}},\mathbf{P}_{ijk}\rangle=\Delta_{ijk}^{-1}\,\boldsymbol{\sigma}_{\hat{i}}\wedge\boldsymbol{\sigma}_i\wedge\boldsymbol{\tau}_j\wedge\boldsymbol{\upsilon}_{ik}=(-1)^{\hat{i}}\,\Delta_{ijk}^{-1}\,Z_j\,\langle\boldsymbol{\upsilon}_{ik},\mathbf{V}\rangle=(-1)^{\hat{i}}\,\Delta_{ijk}^{-1}\,Z_j\,\delta_{ik}.$$

In particular, we find

$$\left(\frac{\delta_{0k}}{\Delta_{0jk}}\right)^{-1}\left(\frac{\delta_{1k}}{\Delta_{1jk}}\right)=-\frac{\langle\boldsymbol{\sigma}_0,\mathbf{P}_{1jk}\rangle}{\langle\boldsymbol{\sigma}_1,\mathbf{P}_{0jk}\rangle}<0, \tag{4.4}$$

and (3.1) can be equivalently written as

$$\operatorname{rank}\begin{pmatrix}\delta_{10}\,w_{100}\,\Delta_{000} & \delta_{10}\,w_{110}\,\Delta_{010} & \delta_{11}\,w_{101}\,\Delta_{001} & \delta_{11}\,w_{111}\,\Delta_{011}\\ \delta_{00}\,w_{000}\,\Delta_{100} & \delta_{00}\,w_{010}\,\Delta_{110} & \delta_{01}\,w_{001}\,\Delta_{101} & \delta_{01}\,w_{011}\,\Delta_{111}\end{pmatrix}=1. \tag{4.5}$$

With a similar argument to (3.12), we also derive

$$\Delta_{i0k}\,\langle \boldsymbol{\tau}_1, \mathbf{P}_{i0k}\rangle = -\Delta_{i1k}\,\langle \boldsymbol{\tau}_0, \mathbf{P}_{i1k}\rangle \quad , \quad \Delta_{ij0}\,\langle \boldsymbol{v}_{i1}, \mathbf{P}_{ij0}\rangle = -\Delta_{ij1}\,\langle \boldsymbol{v}_{i0}, \mathbf{P}_{ij1}\rangle$$

and (3.4) can be equivalently rewritten as

$$\operatorname{rank}\begin{pmatrix} \Delta_{000}\,w_{010} & \Delta_{100}\,w_{110} & \Delta_{001}\,w_{011} & \Delta_{101}\,w_{111} \\ \Delta_{010}\,w_{000} & \Delta_{110}\,w_{100} & \Delta_{011}\,w_{001} & \Delta_{111}\,w_{101} \end{pmatrix} = 1. \tag{4.6}$$

Therefore, the rank conditions (4.5) and (4.6) are simultaneously satisfied if and only if the w_{ijk}'s are as (4.3). Moreover, by Configuration 2 we find the strict inequalities

$$\frac{\Delta_{i1k}}{\Delta_{i0k}} = -\frac{\langle \boldsymbol{\tau}_1, \mathbf{P}_{i0k}\rangle}{\langle \boldsymbol{\tau}_0, \mathbf{P}_{i1k}\rangle} < 0\ ,\ \frac{\Delta_{ij1}}{\Delta_{ij0}} = -\frac{\langle \boldsymbol{v}_{i1}, \mathbf{P}_{ij0}\rangle}{\langle \boldsymbol{v}_{i0}, \mathbf{P}_{ij1}\rangle} < 0. \tag{4.7}$$

Additionally, by the inequalities in (4.1) δ_{i0}, δ_{i1} have distinct sign (see Figure 4), implying that

$$\left(\frac{\delta_{i1}}{\Delta_{ij1}}\right)^{-1}\left(\frac{\delta_{i0}}{\Delta_{ij0}}\right) > 0\ . \tag{4.8}$$

for all $0 \le i, j \le 1$. Therefore, from (4.3) and the inequalities (4.4), (4.7) and (4.8) it follows that

$$w_{1jk} = w_{0jk}\,\alpha\left(\frac{\delta_{1k}}{\Delta_{1jk}}\right)^{-1}\left(\frac{\delta_{0k}}{\Delta_{0jk}}\right) > 0\,,\ w_{i1k} = w_{i0k}\,\beta\left(\frac{\Delta_{i1k}}{\Delta_{i0k}}\right) > 0\,,\ w_{ij1} = w_{ij0}\,\gamma\left(\frac{\delta_{i1}}{\Delta_{ij1}}\right)^{-1}\left(\frac{\delta_{i0}}{\Delta_{ij0}}\right) > 0.$$

On the other hand, by Lemma 3.1 and Remark 3.2 the syzygies of degree $1\times 0\times 0$ and $0\times 1\times 0$ are $\boldsymbol{\sigma} = \boldsymbol{\sigma}(s_0,s_1)$ and $\boldsymbol{\tau} = \boldsymbol{\tau}(t_0,t_1)$ as defined in (3.3) and (3.7), and ϕ^{-1} follows for the parameters s and t. In order to derive the inverse for u, we prove the identity

$$\mathbf{P} = \left(\frac{w_{000}\,\delta_{00}}{\Delta_{000}}\right)\boldsymbol{\sigma}\wedge\boldsymbol{\tau}\wedge\boldsymbol{v}, \tag{4.9}$$

where

$$\boldsymbol{v} = \boldsymbol{v}(s_0,s_1,u_0,u_1) = \begin{vmatrix} & B_0(u_0,u_1) & -B_1(u_0,u_1) \\ \gamma & \begin{vmatrix} B_0(s_0,s_1) & -B_1(s_0,s_1) \\ \frac{\alpha\epsilon}{\delta_{11}}\,\boldsymbol{v}_{11} & \frac{1}{\delta_{01}}\,\boldsymbol{v}_{01}\end{vmatrix} & \begin{vmatrix} B_0(s_0,s_1) & -B_1(s_0,s_1) \\ \frac{\alpha\epsilon}{\delta_{10}}\,\boldsymbol{v}_{10} & \frac{1}{\delta_{00}}\,\boldsymbol{v}_{00}\end{vmatrix}\end{vmatrix}.$$

In the first place, we have

$$\langle \frac{1}{\delta_{1k}}\,\boldsymbol{v}_{1k} - \frac{1}{\delta_{0k}}\,\boldsymbol{v}_{0k}, \mathbf{V}\rangle = \frac{1}{\delta_{1k}}\,\langle \boldsymbol{v}_{1k}, \mathbf{V}\rangle - \frac{1}{\delta_{0k}}\,\langle \boldsymbol{v}_{0k}, \mathbf{V}\rangle = 1 - 1 = 0,$$

so the plane defined by $\langle \frac{1}{\delta_{1k}}\,\boldsymbol{v}_{1k} - \frac{1}{\delta_{0k}}\,\boldsymbol{v}_{0k}, \mathbf{X}\rangle = 0$ contains the point $\mathbf{V}$. Moreover, maintaining the notation of Lemma 4.2 we can write

$$\begin{vmatrix} B_0(s_0,s_1) & -B_1(s_0,s_1) \\ \frac{\alpha\epsilon}{\delta_{1k}}\,\boldsymbol{v}_{1k} & \frac{1}{\delta_{0k}}\,\boldsymbol{v}_{0k}\end{vmatrix} = \begin{vmatrix} F(s_0,s_1) & -B_1(s_0,s_1) \\ \alpha\epsilon\left(\frac{1}{\delta_{1k}}\,\boldsymbol{v}_{1k} - \frac{1}{\delta_{0k}}\,\boldsymbol{v}_{0k}\right) & \frac{1}{\delta_{0k}}\,\boldsymbol{v}_{0k}\end{vmatrix} = \begin{vmatrix} B_0(s_0,s_1) & -\frac{1}{\alpha\epsilon}\,F(s_0,s_1) \\ \frac{\alpha\epsilon}{\delta_{1k}}\,\boldsymbol{v}_{1k} & -(\frac{1}{\delta_{1k}}\,\boldsymbol{v}_{1k} - \frac{1}{\delta_{0k}}\,\boldsymbol{v}_{0k})\end{vmatrix}.$$

Thus, we can expand

$$\sum_{0\le i,j\le 1} \alpha^i\,\beta^j\,\boldsymbol{\sigma}_i\wedge\boldsymbol{\tau}_j\wedge\begin{vmatrix} B_0(s_0,s_1) & -B_1(s_0,s_1)\\ \frac{\alpha\epsilon}{\delta_{1k}}\,\boldsymbol{\upsilon}_{1k} & \frac{1}{\delta_{0k}}\,\boldsymbol{\upsilon}_{0k}\end{vmatrix}\,B_i(s_0,s_1)\,B_j(t_0,t_1) \;=$$

$$B_0(s_0,s_1)\sum_{0\le j\le 1}\beta^j\,\boldsymbol{\sigma}_0\wedge\boldsymbol{\tau}_j\wedge\begin{vmatrix} F(s_0,s_1) & -B_1(s_0,s_1)\\ \alpha\epsilon\left(\frac{1}{\delta_{1k}}\,\boldsymbol{\upsilon}_{1k}-\frac{1}{\delta_{0k}}\,\boldsymbol{\upsilon}_{0k}\right) & \frac{1}{\delta_{0k}}\,\boldsymbol{\upsilon}_{0k}\end{vmatrix}\,B_j(t_0,t_1)\;+$$

$$\alpha\,B_1(s_0,s_1)\sum_{0\le j\le 1}\beta^j\,\boldsymbol{\sigma}_1\wedge\boldsymbol{\tau}_j\wedge\begin{vmatrix} B_0(s_0,s_1) & -\frac{1}{\alpha\epsilon}\,F(s_0,s_1)\\ \frac{\alpha\epsilon}{\delta_{1k}}\,\boldsymbol{\upsilon}_{1k} & -(\frac{1}{\delta_{1k}}\,\boldsymbol{\upsilon}_{1k}-\frac{1}{\delta_{0k}}\,\boldsymbol{\upsilon}_{0k})\end{vmatrix}\,B_j(t_0,t_1)=$$

$$F(s_0,s_1)\sum_{0\le i,j\le 1}\left(\frac{\alpha^i\,\beta^j}{\delta_{ik}}\right)\,\boldsymbol{\sigma}_i\wedge\boldsymbol{\tau}_j\wedge\boldsymbol{\upsilon}_{ik}\;B_i(s_0,s_1)\,B_j(t_0,t_1)\,+$$

$$\alpha\,B_0(s_0,s_1)\,B_1(s_0,s_1)\sum_{0\le j\le 1}(\epsilon\,\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_1)\wedge\boldsymbol{\tau}_j\wedge\left(\frac{1}{\delta_{1k}}\,\boldsymbol{\upsilon}_{1k}-\frac{1}{\delta_{0k}}\,\boldsymbol{\upsilon}_{0k}\right)\,B_j(t_0,t_1),$$

and the second term in the last identity vanishes because the planes

$$\langle\epsilon\,\boldsymbol{\sigma}_0-\boldsymbol{\sigma}_1,\mathbf{X}\rangle=0\;,\;\langle\boldsymbol{\tau}_j,\mathbf{X}\rangle=0\;,\;\langle\frac{1}{\delta_{1k}}\,\boldsymbol{\upsilon}_{1k}-\frac{1}{\delta_{0k}}\,\boldsymbol{\upsilon}_{0k},\mathbf{X}\rangle=0$$

share the line $\mathbf{T}_0\cap\mathbf{T}_1$. Thus, we conclude

$$\boldsymbol{\sigma}\wedge\boldsymbol{\tau}\wedge\boldsymbol{\gamma}=F(s_0,s_1)\sum_{0\le i,j,k\le 1}\left(\frac{\alpha^i\,\beta^j\,\gamma^k}{\delta_{ik}}\right)\,\boldsymbol{\sigma}_i\wedge\boldsymbol{\tau}_j\wedge\boldsymbol{\upsilon}_{ik}\,B_i(s_0,s_1)\,B_j(t_0,t_1)\,B_k(u_0,u_1)\;=$$

$$\left(\frac{\Delta_{000}}{w_{000}\,\delta_{00}}\right)\sum_{0\le i,j,k\le 1}w_{ijk}\,\mathbf{P}_{ijk}\,B_i(s_0,s_1)\,B_j(t_0,t_1)\,B_k(u_0,u_1)=\left(\frac{\Delta_{000}}{w_{000}\,\delta_{00}}\right)\,\mathbf{P}.$$

In particular, $\boldsymbol{\upsilon}$ is a syzygy of degree $1\times 0\times 1$ of $\mathbf{P}$ as we find

$$\langle\boldsymbol{\upsilon},\mathbf{P}\rangle=\left(\frac{w_{000}\,\delta_{00}}{\Delta_{000}}\right)\,\boldsymbol{\upsilon}\wedge\boldsymbol{\sigma}\wedge\boldsymbol{\tau}\wedge\boldsymbol{\upsilon}=0,$$

which additionally is independent from $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. Therefore, the expression

$$\begin{vmatrix} & B_0(u_0,u_1) & -B_1(u_0,u_1)\\ \gamma & \begin{vmatrix}\langle\boldsymbol{\sigma}_1,\mathbf{X}\rangle & \langle\boldsymbol{\sigma}_0,\mathbf{X}\rangle\\ \frac{\epsilon}{\delta_{11}}\,\langle\boldsymbol{\upsilon}_{11},\mathbf{X}\rangle & \frac{1}{\delta_{01}}\,\langle\boldsymbol{\upsilon}_{01},\mathbf{X}\rangle\end{vmatrix} & \begin{vmatrix}\langle\boldsymbol{\sigma}_1,\mathbf{X}\rangle & \langle\boldsymbol{\sigma}_0,\mathbf{X}\rangle\\ \frac{\epsilon}{\delta_{10}}\,\langle\boldsymbol{\upsilon}_{10},\mathbf{X}\rangle & \frac{1}{\delta_{00}}\,\langle\boldsymbol{\upsilon}_{00},\mathbf{X}\rangle\end{vmatrix}\end{vmatrix}$$

vanishes after the specialization $\mathbf{X}\mapsto\mathbf{P}$, and it defines ϕ^{-1} for the parameter u. □

Remark 4.3. Assume Property B.1. If the boundary surfaces $\boldsymbol{\Sigma}_0,\boldsymbol{\Sigma}_1$ (resp. $\mathbf{T}_0,\mathbf{T}_1$) are parallel planes, and assuming $\|\boldsymbol{\sigma}_i\|=\|\boldsymbol{\tau}_j\|=1$, then we have $\langle\boldsymbol{\sigma}_0,\mathbf{P}_{1jk}\rangle=\langle\boldsymbol{\sigma}_1,\mathbf{P}_{0jk}\rangle$ (resp. $\langle\boldsymbol{\tau}_0,\mathbf{P}_{i1k}\rangle=\langle\boldsymbol{\tau}_1,\mathbf{P}_{i0k}\rangle$). In particular, if $w_{ijk}=1$ for every $0\le i,j,k\le 1$ the rank conditions (3.1) and (3.4) are simultaneously satisfied, and ϕ is birational of type $(1,1,2)$.

5. Discussion and future work

We provide criteria for the computation of suitable weights that, given control points satisfying either Configuration 1 or 2, ensure that a 3D FFD of degree $1\times 1\times 1$ is birational, answering a question raised in (Sederberg et al., 2016, §7). Additionally, we prove that the geometric conditions required in these configurations are not only sufficient but necessary. More specifically, Properties A.1, A.2, and A.3 are necessary for hexahedral maps of type $(1,1,1)$ as the parametric surfaces of the birational maps in this class are planes. Furthermore, in Lemma 4.1 we prove that Property C is always satisfied by birational maps of type $(1,1,2)$. We also require the inequalities in (3.8) and (4.1), but this is only to ensure the computation of positive weights. The same computation of weights still yields birational maps if these conditions are dropped, although some weights may be negative.

Even though the control points must satisfy specific geometric configurations, a designer can still decide new control points intuitively. Namely, for hexahedral maps the designer can move freely any of the boundary planes (instead of the control points) and modify the control points in the corresponding facet accordingly so Configuration 1 is preserved. For birational maps of type $(1,1,2)$, the designer can freely choose a point $\mathbf{V}$ and four distinct lines through it, representing the lines $\overline{\mathbf{p}_{ij0}\mathbf{p}_{ij1}}$. Then, the control points $\mathbf{p}_{ij0}$ and $\mathbf{p}_{ij1}$ can be freely chosen in each of these lines. Remarkably, the computation of weights that we propose is very efficient as the Δ_{ijk}'s and δ_{ik}'s can be computed in real time while the designer moves the control points.

It can also occur that ϕ is birational of type either $(1,2,2)$ (and permutations) or $(2,2,2)$ (see Busé et al. (2023)). In this cases, respectively two and three pairs of boundary surfaces are doubly ruled quadrics, and ϕ^{-1} is given by ratios of quadratic polynomials for the corresponding parameters. Interestingly, (Busé et al., 2023, Theorem 6.1) provides a syzygy-based characterization of birationality for these maps as well, but it relies on the existence of bilinear syzygies which are not as geometrically intuitive as linear syzygies. Moreover, the geometry of doubly ruled quadrics is inherently more complex than the geometry of planes. In particular, for birational maps of type $(2,2,2)$ the existence of syzygies of some degrees is not sufficient to conclude birationality, and further factorization conditions need to be checked. However, for birational maps in these classes the control points are less constrained, making them interesting for CAGD. The derivation of methods for the efficient computation of weights for the birationality of maps in these classes is a work to be done.

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