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MINIMIZING THE IN-PROCESS INVENTORY IN STOCHASTIC ENVIRONMENT

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MULTI-LEVEL, CONVERGENT SYSTEMS :
MINIMIZING THE INVENTORY COST
IN STOCHASTIC ENVIRONMENT

by

Camille LIBOSVAR *
Jean Marie PROTH **
Philippe VARIN*

ABSTRACT

This paper presents a set of flow-control policies for industrial systems described as networks of processors and buffers.The main results are first described in the case of a "single processor, single buffer" system facing a normal distributed demand : for such a system, the replenishment policy that minimizes the inventory while keeping the probability of going out of stock less than a given ε is presented ; it appears that this policy leads to replenishments that can be forecasted as normal variables, just as the final demand is... Hence the possibility to extend the model to multi - level systems.

It is also pointed out that in a multilevel system, the control parameters ε that measure, for each buffer, the maximal probability of going out of stock can be computed in order to ensure that the final demand be satisfied with a given probability and that the global inventory cost be minimized. At last, the assumptions subtending this model are discussed and adjustment is proposed for cases of limited capacities.

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Introduction :

In a hierarchical production-management system, flow control is generally achieved in deterministic environment because, at this level, only families of products and groups of machines are considered and it is assumed that their stochastic behaviours balance one another and that mean-value forecasts are enough... The consequence when these forecasts are not very reliable is that the policy computed does not fit actual needs.

We assume in this paper that a second information on forecasts can always be determined to characterize their reliability and we present a set of algorithms defining a flow control policy in stochastic environment.

The first part is devoted to a descripition of the replenishment policy for a "single processor , single buffer" system (that is, the basic module of our models) with infinite capacities and normal distributed demands : it is proved that the replenishments have the same probabilistic characteristics as the demand, which allows us to generalize this policy to multi-stage systems . It is also demonstrated that in case of two-stage systems, the safety stock can be shared between the two levels in order to minimize the global cost, just by modifying the parameters of replenishment policy at each level.

The second one presents the most general formulae describing the policy previously characterized in case of multilevel and multiproduct systems. The problems of computation of the optimal parameters of this policy and of assignment in case of insufficient in-process at a given level are posed.

The third part, at last, discusses the main hypothesis of this heuristic method (i.e. the demands probability law being assumed normal) and presents a pragmatic approach to take limited capacities into account : it is pointed out that, though they have prooved to give good results in practice, these algorithms lead to replenishments that cannot be considered as normal distributed.

1/ Basic results :

a/ Replenishment policy for a " kernel system " :

If we consider the system consisting of a mono-product processor and a finished-product buffer and assume the demand (d_i) it has to face (see fig. 1) is normal distributed, then the replenishment policy leading to the lowest inventory level subject to :

$$\forall \, t \geq 1, \, Pr \, (y_{t+N} < 0 \,) < \varepsilon$$

(which means that the probability of the demand's exceeding the inventory level has to remain less than a given parameter ε).

is described by the formulae :

$$v_1 = \Sigma_1 \cdot F^{-1} (\, 1 - \varepsilon \,) \; + \; \varphi_{1+N} \; + \; [\; \sum_{i=1}^{N} \varphi_i - \sum_{i=1-N}^{0} v_i^{*} - y_0 \,]$$

$$v_t = F^{-1} (\, 1 - \varepsilon \,) \cdot (\, \Sigma_t - \Sigma_{t-1} \,) \; + \; \varphi_{t+N} \; + a_{t-1}^{*}$$

and leads to the inventory levels :

$$y_{t+N} = \Sigma_t \cdot F^{-1} (\, 1 - \varepsilon \,) - \sum_{i=t}^{t+N} a_i$$

where :

- y_t is the inventory level at the end of period t.
- v_t is the replenishment decided at the beginning of period t.
- N is the lead-time of the processor, supposed to be constant : if a replenishment $v_t = x$ occurs at the beginning of period t, then the quantity x of finished product will become available at the end of period t+N in the buffer.
- φ_t is the estimate of the demand expected for period t :
  → the actual demand will then be $d_t = \varphi_t + a_t$ where a_t is the value taken by a random normal variable of nil mean value and standard deviation σ_t supposed to be known ; if i≠j then a_i and a_j are independent.
- $\Sigma_t = \sqrt{\sigma_t^2 + ... + \sigma_{t+N}^2}$ is the standard deviation of the demand cumulated over periods t to t+N, that is, the demand that has not yet occured and will affect the inventory level y_{t+N} .
- F is the distribution of the standard normal variable ; F^{-1} is the reciprocal function.
- The star convention indicates that the values taken by the referred variables are known when the formula has to be applied (ie at the beginning of period t).

replenishment   production   final demand

v_t → PROCESSOR → $P_t = v_{t-N}$ → BUFFER $d_t = \varphi_t + a_t$

lead time : N periods
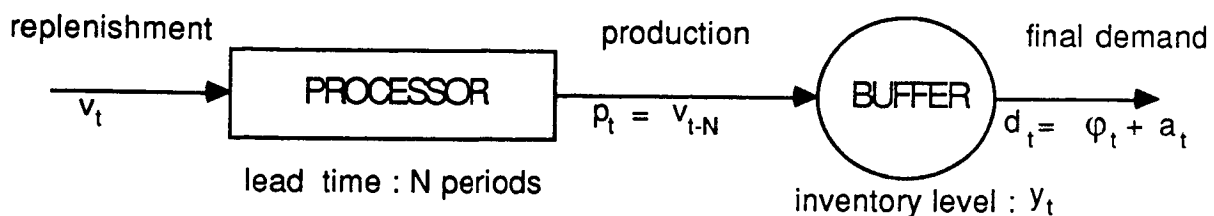
inventory level : y_t

FIGURE 1.

It appears then that the replenishment will be computed as the sum of three different terms :

→ The first one makes it possible to keep a security margin, taking into account the foreseeable difference between demands and forecasts : this margin is set in the first replenishment to build a safety stock and the following replenishments just modulate it according to the evolution of the expected quality of the forecast...

e.g. if the standard deviation of the difference between demand and forecast is constant over time, then this term is nil in all replenishments but the first one.

→ The second one is the forecasted demand for the period during which manufacturing of the reordered raw material will be completed ;

→ The last one takes into account the state reached by the system : in the first replenishment, this term is equal to the inventory level that would be reached just in N periods (so the last inventory level that will not be affected by the replenishment v_1) if demands were equal to their forecasts for the N periods... and in the following replenishments, this term is equal to the last difference established between demand and forecast, in order to update the last virtual inventory level that the replenishment will not affect.

It thereby appears that, besides the first one which is determined by the choice of a time origin and the measure of the initial inventory level, **the replenishments will be the sum of a term depending only on the demands-forecast and the value taken by a random normal variable** (i.e. a quantity that is statistically known)

This result demonstrates that it is possible to apply the same replenishment policy to any level of a multi-stage system consisting in several so-called "kernel systems" in series : the demands the upper level will have to face are the replenishments decided at the lower level but the result stated above ensures that they will have the same characteristics as the final demand... and that upper-level forecasts can be inferred from lower-level's.

b/ computation of the control parameters in order to minimize the global inventory cost :

In the replenishment policy described above, the only control parameter is ε, i.e. the upper bound for the probability of going out of stock : its value determines the safety stock Σ_t. $F^{-1}(1-\varepsilon)$ which is also the mean-value of inventory level y_{t+N} since $\Sigma\, a_i$ is a normal variable with nil mean value.

In the case of the two-stage system, there is one control parameter ε^u for the upper level and one for the lower level : ε^l , and the probability of not going out of stock at the lower level is :

$$Pr\,(\,y^l_t > 0\,) = Pr\,(\,y^l_t > 0\,/\,y^u_{t-N^u} > 0)\, x\, Pr\,(\,y^u_{t-N^u} > 0\,)$$
$$> 1 - \varepsilon^l \qquad\qquad > 1 - \varepsilon^u$$

We can then state another control parameter ε for the finished product such that :
$$\forall\, t\, ,\, Pr\,(y_t < 0\,) < \varepsilon$$
and thereafter compute ε^u and ε^l, since we have established that :
$$1 - \varepsilon = (\,1 - \varepsilon^u\,)\, x\, (\,1 - \varepsilon^l\,) \quad i.e. \quad \varepsilon = \varepsilon^u . \varepsilon^l - (\,\varepsilon^u + \varepsilon^u\,).$$

The point is then, with given inventory cost functions k^u and k^l at the upper and lower level, to compute the parameters ε^u and ε^l in order to minimize the global cost of the safety stock :

$$K = k^u\,(\,F^{-1}\,(1-\varepsilon^u\,)\, x\, \sum_t \Sigma_t^u\,) + k^l\,(\,F^{-1}\,(1-\varepsilon^l\,)\, x\, \sum_t \Sigma_t^l\,)$$

subject to : $\quad \varepsilon^u . \varepsilon^l - (\,\varepsilon^u + \varepsilon^u\,) = \varepsilon\, ,\quad$ where $\quad \forall\, j \in \{\, u\, ,\, l\,\}\, ,\quad \Sigma_t^j = \sqrt{\sum_{i=t}^{t+N^j} (\sigma_i^j)^2}$

(N^j being the lead time at level j and σ_i^j the standard deviation of the difference between demand and forecast for period i at level j).

2/ Generalization to systems modelized as networks of tandem-working processors and buffers:

The system is supposed to be described as a sequence of p "levels" - (the pth level has to face the final demand whereas the first one transforms raw material and has an infinite replenishment capacity) - each level "l" grouping a number "n(l)" of mono-product processors and the buffers associated, so that the ith processor picks parts in different buffers of level "l -1" to manufacture products in a given amount of time corresponding to "N (l,i)" periods and replenishes its buffer :

level   l-1

N (l-1,1)   N (l-1,2)   N (l-1,3) lead times   N (l-1,n(l-1))

BUFFER (l-1,1)   BUFFER (l-1,2)   BUFFER (l-1,3)   BUFFER (l-1,n(l-1))

$$d\,(l\text{-}1,i,t) \;=\; \sum_{j=1}^{n(l)} M\,(l,i,j) \times v\,(l,j,t)$$

level   l

N (l,1)   N (l,2)   lead times   N (l,n(l))

BUFFER (l, 1)   BUFFER (l, 2)   BUFFER (l,n (l))

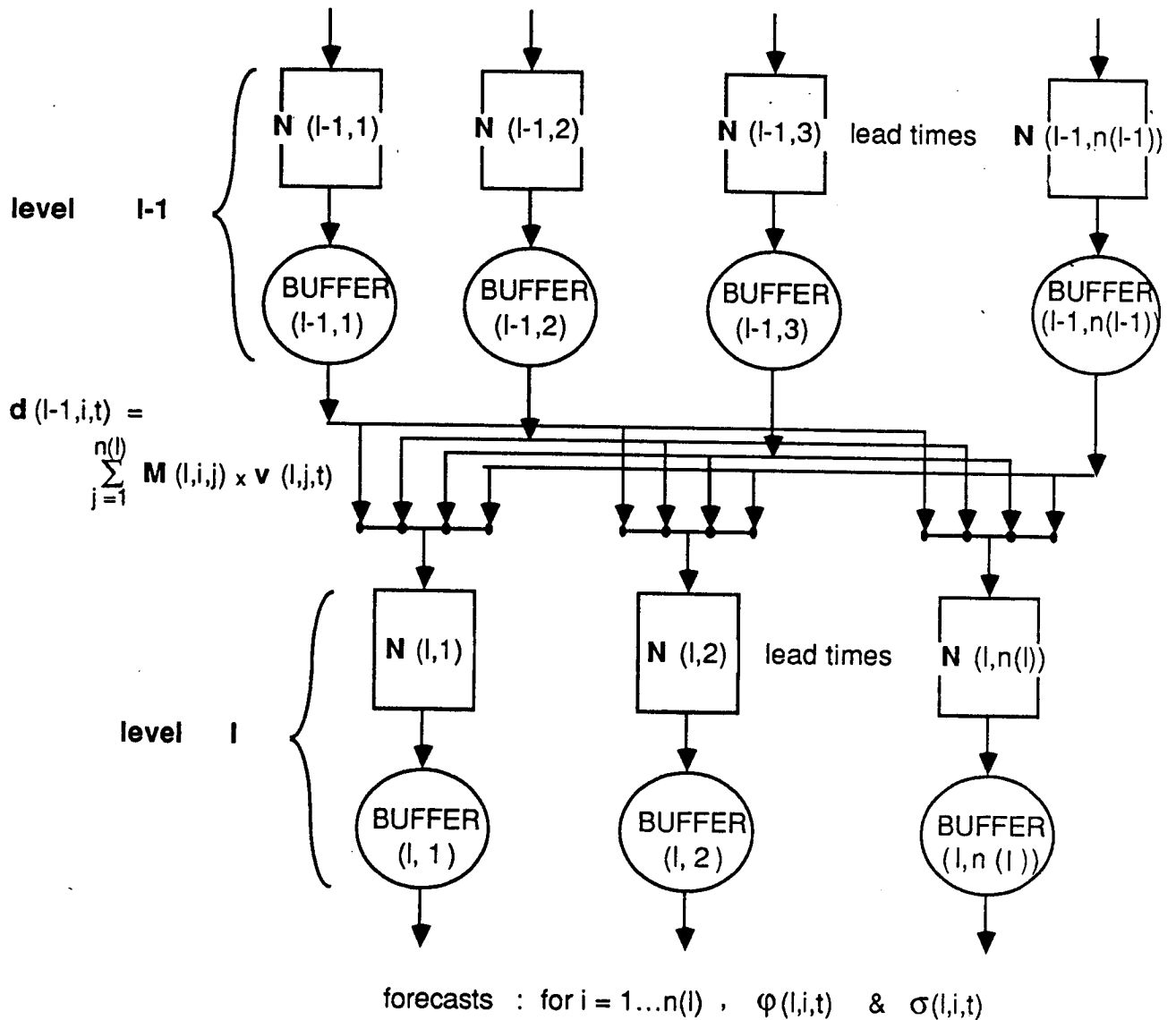forecasts : for i = 1...n(l) , $\varphi\,(l,i,t)$ & $\sigma\,(l,i,t)$

FIGURE 2.

a/ Replenishment policy :

We assume that informations are given concerning the final products (i.e. the products available in buffers of pth level) :

$\forall j \in \{ 1,2...n(p) \}$ - i.e. for each final product -,

→ demands forecasts $\varphi (p,j,t)_{t=1..h}$ with the standard deviations associated : $\sigma (p,j,t)_{t=1..h}$ that are expected for the differences between demands and forecasts - h is the planning horizon and t is the period index -.

→ upper bound of the probability of going out of stock, while facing the final demand : $\varepsilon (p,j)$.

We also assume that the composition of the products at each level is given by means of the matrix **M** : the production of one unit of product by processor i at level l requires $M(l,j,i)$ units of product taken in the buffer of the jth buffer of level l-1.

The replenishment policy for the whole system then consists in an iteration of the computations described above, beginning with the lowest level : once the control parameters $\varepsilon (l,i)$ are determined for each buffer - we shall see hereunder how -, the only difference with the algorithm suggested above is the need to compute the forecasts at each level ;

for any level $l \in \{ p, p-1,...1\}$ and for any processor $j \in \{1,2...n(l)\}$ at this level, we will compute the replenishments according to the formulae :

$$\rightarrow v(l,j,1) = F^{-1} [1-\varepsilon (l,j)] \times \Sigma (l,j,1) + \varphi (l,j,1+N(l,j)) + \sum_{t=1}^{N(l,j)} \varphi(l,j,t) - \sum_{t=1-N(l,j)}^{0} v^*(l,j,t) - y^* (l,j,0)$$

$$\rightarrow \text{and}, \quad \forall t \geq 2 \quad v(l,j,t) = F^{-1} [1-\varepsilon (l,j)] \times [\Sigma (l,j,t) - \Sigma (l,j,t-1)] + \varphi (l,j,t+N(l,j)) + a^* (l,j,t-1)$$

Hence the demands forecast for next upstream level :

$$\varphi (l-1,i,1) = \sum_{j=1}^{n(l)} M(l,i,j) \times v(l,j,1) \quad \text{and} \quad \sigma (l-1,i,1) = 0 \quad \text{whereas} \quad \forall t \geq 2 ,$$

$$\varphi (l-1,i,t) = \sum_{j=1}^{n(l)} M(l,i,j) \times \{ F^{-1} [1-\varepsilon (l,j)] \times [\Sigma (l,j,t) - \Sigma (l,j,t-1)] + \varphi (l,j,t+N(l,j)) \}$$

$$\text{and} \quad \sigma (l-1,i,t) = \sqrt{\sum_{j=1}^{n(l)} [M(l,i,j) \; \sigma (l,i,t-1)]^2}$$

with the same conventions as in part one for Σ :

$$\forall l \in \{1,2...p\}, \forall i \in \{1,2...n(l)\}, \forall t \in \{1,2...h\}, \Sigma (l,i,t) = \sqrt{\sum_{k=t}^{k+N(l,i)} (\sigma (l,i,k))^2}$$

This algorithm makes it possible to compute the forecasts at any level and then the replenishments that lead to the lowest in-process inventory subject to the conditions stated by the control parameters $\varepsilon (l,j)$ chosen ; as the replenishments are computed by means of the same formulae as in part 1, the inventory levels will have the same expressions too and the safety stocks will depend on the values given to the control parameters. Hence the possibility of an optimization.

b/ computation of the control parameters :

As the safety stocks can be computed according to the same formulae as in part one, computation of parameters ε (l,j) will be achieved on the basis of the same principles as in the case of the simple two-stage system in order to minimize the inventory cost ; an additional asumption concerning the replenishment of in-process parts when the demand from down-stream processors (i.e. the replenishments required in order to reach the objectives) exceeds the current quantity in stock (e.g. in buffer (l-1,i)) is required :

we assume that in this case, the parts available in buffer (l-1,i) should be shared between <u>all</u> the processors that demand them in order that the ratio demand / supply be the same for all of them (we shall see that this assignment policy is not optimal). The replenishments for these processors would then be cut down to the values given by following formulae :

$$\forall \, j \in \{1,2...n(l)\} , \quad v''(l,j,t) = v(l,j,t) - \delta(l-1,i,t) \times [\ M(l,i,j) / \sum_k M(l,i,k)\ ]$$

where $v(l,j,t)$ is the replenishment required for the targets to be reached and defined by the policy previously described, and δ (l-1, i, t) represents the lack of components in buffer (l-1, i) at the beginning of period t :

$$\delta(l-1,i,t) = d(l-1,i,t) - [\ y(l-1,i,t-1) + v(l-1,i,t-N(l-1,i))\ ]$$

$$= \sum_{j=1}^{n(l)} M(l,i,j) \times v(l,j,t) - [\ y(l-1,i,t-1) + v(l-1,i,t-N(l-1,i))\ ]$$

Then the probability of not going out of stock for any final part can be computed as the product of the relative probabilities of not going out of stock for any component, relative meaning that these probabilities just take into account the randomness of the demand and assume that the replenishment policy is applied ; we will then state that the upper bounds 1- ε (l,j) of these probabilities are subject to :

$$\forall \, j \in \{1,2...n(l)\} , \quad \pi(p,j) = 1 - \varepsilon(p,j)$$

where ε (p,j) is given as the upper bound for probability of final product j of going out of stock and matrix π is defined by the formulae :

$\rightarrow \forall \, j \in \{1,2...n(l)\} , \quad \pi(1,j) = 1 - \varepsilon(1,j)$

$\rightarrow \forall \, l \in \{2,3...p\} , \forall \, j \in \{1,2...n(l)\} , \quad \pi(l,j) = [1 - \varepsilon(l,j)] \times \prod \pi(l-1,i)$

$$i \in \{1,2...n(l-1)\}$$
$$\text{and } M(l,i,j) \neq 0$$

Exactly as we have seen it for the two-stage system, the level of the safety stock for any processor (l,j) depends on the so-called control parameter ε (l,j) and, if it is possible to determine inventory cost functions $k_{l,j}$ (y) for all the stocks, then computation of all these control parameters becomes a rather complex problem of non-linear programming .hat can be posed as :

$$\underset{\varepsilon \in m_{[0,1]}(p,L)}{\text{Min}} \quad \sum_{l=1}^{p} \sum_{i=1}^{n(l)} k_{l,j} \left(F^{-1}(1 - \varepsilon(l,j)) \times [\ \sum_{t=1}^{h} \sum (l,j,t)\ ] \right)$$

subject to : $\forall \, j \in \{1,2...n(l)\} , \quad \pi(p,j) = 1 - \varepsilon(p,j)$

where $m_{[0,1]}$ (p,L) is the set of pxL matrixes with real elements ranging in [0,1]

and $L = \max \{ n(l) \}$.
$$l \in \{1,2...p\}$$

c/ actual allocation policy :

As we pointed it out in previous paragraph, the rule stated for allocation in case of lack of components is nothing but a convenient assumption in order to compute the control parameters ; it is however obvious that the actual allocation policy in that situation will not be based on this rule which can lead to a misusing of insufficient resources :

for example, let us assume that a given component C_0 is required for production of parts P_1 and P_2 which result respectively of assembly of components C_0 and C_1 , and C_0 and C_2 : $P_1 = C_0 + C_1$ and $P_2 = C_0 + C_2$ ...
If the demands of parts P_1 and P_2 amount to five units whereas the quantities in stock are eight units of component C_0 , five of component C_1 and three of component C_2 , then we notice that the rule previously stated leads to the assignment of four units of component C_0 for assembly of each part P_1 and P_2 . However, only three units of P_2 can be assembled, due to the lack of component C_2 ...The optimal policy in this case would have been to assign five units of C_0 for assembly of part P_1 and only three for assembly of part P_2 .

In any case, when the replenishment policy cannot be applied because of a lack of components, the assignments will be fixed so as to minimize the difference between objective and actual replenishments : if , at the beginning of period t, the stocks of components at level l -1 happen to be too low, then the replenishments v(l, j, t) will have to be cut down to v**(l, j, t) while these quantities are computed by means of any linear programming algorithm in order to achieve :

Min $\displaystyle\sum_{j=1}^{n(l)} [v (l, j, t) - v^{**} (l, j, t)]$

subject to : $\forall j \in \{ 1,2...n(l) \}$, $v^{**} (l, j, t) \in [0, v (l, j, t)]$

and also : $\forall i \in \{ 1,2...n(l-1) \}$, $\delta^{**} (l-1, i, t) \leq 0$

where $\displaystyle\delta^{**} (l-1, i, t) = \sum_{j=1}^{n(l)} M (l, i, j) \times v^{**} (l, j, t) - [y (l-1, i, t-1) + v (l-1, i , t -N(l-1,i))]$

And this assignment policy would ensure the best use of components in case of shortages ...

Remarks :

1° This computation will be achieved rather seldom if the probability of going out of stock is bounded to a low value and only in order that no resource be misused .

2° Computation of the control parameters is achieved according to an equation that conects them to the probability of going out of stock for final products ; as this equation is based on a sub-optimal assignment rule, this probability is slightly overestimated in the computation of the parameters. These ones will then be a little stricter than necessary.

3° All the theory developped above is based on the result which shows that if the demand is normal-distributed, then the optimal replenishment will be too. Two major objections can be raised against the implicit assumptions subtending this approach : first, neither demands nor replenishments can physically take negative values and then, this approach does not take the finite capacities of processors and buffers into account.

3/ A pragmatic approach in case of finite capacities :

We will examine first in what extent the assumption we have done concerning the distribution of the demands can be maintained and then how to cope with anticipate demands increases in case of finite capacities.

a/ truncated normal-distributed variables :

Let us consider an "internal" demand, that is, for a given buffer, the effect of the replenishments decided at next downstream level : their theoretical value is normal distributed but their actual value has to be positive .
We can then state that this latter will be defined by equations :

$$\forall l \in \{1,2...p\}, \forall i \in \{1,2...n(l)\}, \forall t \in \{1,2...h\}, \quad v^a(l,i,t) = \text{Max}\{0, v(l,i,t)\}$$

and we will abbreviate it : $v^a(l,i,t) = v(l,i,t)^+$

Hence the actual ditribution of the replenishments values :
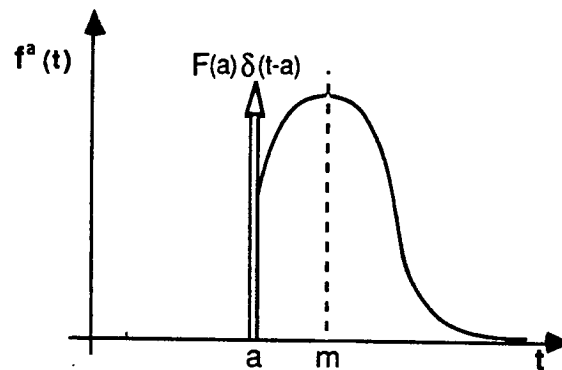
where δ is the Dirac function and parameter 'a' is nil. This 'pseudo-normal' distribution is then defined (in the most general case) by following equations :

$$\begin{cases} \forall t > a, & f^a(t) = f(t) \\ \forall t < a, & f^a(t) = 0 \\ \text{and} & f^a(a) = F(a) \times \delta(0) \end{cases}$$

where f is the distribution of the normal variable of mean - value m and standard deviation σ and F(a) is its integral over] - ∞ a].Then the probability for a pseudo-normal distributed variable to take the value a is no longer nil as it would be for any continuous distribution...

This shows that in considering the demands as normal-distributed whereas they never take negative values, one systematically underestimates them. Unfortunately, this 'pseudo-normal' distribution has not the good properties of the normal distribution : the sum of two pseudo normal variables is not a pseudo-normal variable... and the policy developped above cannot be extended to the case of 'pseudo-normal' distributed demands, which would be a better model.

We will then go on assuming that demands are normal-distributed, which is a legitimate hypothesis as regards the problem of negative values of demands as long as the ratio mean-value / standard deviation is greater than two : the probability for a demands taking a negative value will then be less than 2.5% ... As the errors on demands estimates are not cumulative (computation of replenishments is based on the last inventory level actually reached), they will not be of great impact .

Rather than know if the demands are normal distributed or not, the actual problem in case of finite capacities is in fact how to compute the replenishments in order to take into account future increases in demands which could require objective replenishments exceeding the actual capacity : as there is no miraculous solution to such problems, the only one lies in storing products in prevision of high demands ; our point must then be to take "long term previsions" (i.e. not only these N ones concerning the demands that will take place during the time the replenished quantity is manufactured) into account in our replenishment policy .

b/ forecasting in replenishment policy :

If we consider once more our kernel system under the assumption that demands are normal distributed, be they positive or not, then we have proved that, at a given point in time, objective values of future replenishments are functions of next replenishment to be decided :

in fact,
$$\forall\, t > 0,\ \forall\, u \geq t,\ v_u = \Sigma_u \times F^{-1}(1 - \varepsilon) + \sum_{i=1}^{u+N} \varphi_i + \sum_{i=1}^{u-1} a_i - \sum_{i=1-N}^{u-1} v_i - y_0$$

that is :
$$v_u = \Sigma_u \times F^{-1}(1 - \varepsilon) + \sum_{i=1}^{u+N} \varphi_i + \sum_{i=1}^{t-1} a_i^* - \sum_{i=1-N}^{t-1} v_i^* - \sum_{i=t}^{u-1} a_i - \sum_{i=t}^{u-1} v_i - y_0$$

These formulae demonstrate clearly that, at the beginning of period t, when replenishment v_t has to be decided, the values that future replenishments v_u will take can be represented as sums of three different sorts of terms :

- one term is itself the sum of (u-t) values that will be taken by the random normal variables a_i ; then its probability law is known.
- another one is the sum of (u-t) future replenishments and among them, the replenishment to be decided.
- the remainder is exactly known at that point of time : demands forecasts, previous replenishments etc.

We will then appply the same type of logic as in part one and compute the minimal value v_t^u this replenishment v_t should take in order to ensure that the probability for theoretical value of future replenishment v_u to exceed the capacity M be less than a given parameter ε_M .

This minimal value will be obtained as the value to give to replenishment v_t in order that Pr ($v_u > M$) < ε_M , provided that all intermediate replenishments are set to their maximal possible value, that is, are equal to the capacity M (this assumption ensures that v_t^u is actually the minimal value for the inequation to be satisfied).

We can then state :

$$\Pr(\, v_u > M\, /\ v_{t+1} = v_{t+2} = \ldots = v_{u-1} = M\,)$$

$$= \Pr\left[\, \left(\Sigma_u \times F^{-1}(1 - \varepsilon) + \sum_{i=1}^{u+N} \varphi_i + \sum_{i=1}^{t-1} a_i^* - \sum_{i=1-N}^{t-1} v_i^* - \sum_{i=t}^{u-1} a_i - \sum_{i=t}^{u-1} v_i - y_0 \right) > M \,\right]$$

$$= \Pr\left[\, \sum_{i=t}^{u-1} a_i < \left(\Sigma_u \times F^{-1}(1 - \varepsilon) + \sum_{i=1}^{u+N} \varphi_i + \sum_{i=1}^{t-1} a_i^* - \sum_{i=1-N}^{t-1} v_i^* - y_0 - (u-t) \times M - v_t \right) \right]$$

As the probability law of variables a_i is known and the right term of the inequality is the sum of constants and unknown v_t, the minimal value v_t^u to give to v_t in order that this probability remain less than ε_M can be computed as :

$$v_t^u = \Sigma_t^{u-1} \times F^{-1}(1 - \varepsilon_M) + \Sigma_u \times F^{-1}(1 - \varepsilon) + \sum_{i=1}^{u+N} \varphi_i + \sum_{i=1}^{t-1} a_i^* - \sum_{i=1-N}^{t-1} v_i^* - y_0 - (u-t) \times M$$

$$\text{where } \Sigma_t^{u-1} = \sqrt{\sigma_t^2 + \sigma_{t+1}^2 + \ldots + \sigma_{u-1}^2}$$

(we state that $\Sigma_t^u = 0$ if $u < t$ and we observe that v_t^t is then equal to the objective value for v_t, according to the policy defined in case of infinite capacities, that is, in cases when there is no need to take future demands into account...)

and the value to give to replenishment v_t will be : $\quad v_t^a = [\text{ Min } \{ M, \max_{u=t,t+1,\ldots h} \{v_t^u\} \}]^+$

where h is the horizon of the problem (i.e. the number of forecast periods) .

The same method can be applied in case of limited storage capacity S : replenishment v_t is fixed so that probability for inventory level y_{t+N} to be negative remain less than ε ; in the same way, v_t^S can be computed in order that probability for y_{t+N} to exceed S remain less than ε_S ...

$$y_{t+N} = y_0 + \sum_{i=1-N}^{t-1} v_i^* - \sum_{i=1}^{t+N} \varphi_i - \{ \sum_{i=1}^{t-1} a_i^* - \sum_{i=t}^{t+N} a_i \} + v_t$$

Hence : $\text{Pr}(y_{t+N} > S) = \text{Pr}([y_0 + \sum_{i=1-N}^{t-1} v_i^* - \sum_{i=1}^{t+N} \varphi_i - \{\sum_{i=1}^{t-1} a_i^* + \sum_{i=t}^{t+N} a_i \} + v_t] > S)$

$$= \text{Pr}(\sum_{i=t}^{t+N} a_i < [v_t + y_0 + \sum_{i=1-N}^{t-1} v_i^* - \sum_{i=1}^{t+N} \varphi_i - \sum_{i=1}^{t-1} a_i^* - S])$$

and $\{\text{Pr}(y_{t+N} > S) < \varepsilon_S\} \Leftrightarrow \{v_t > [\Sigma_t \times F^{-1}(1 - \varepsilon_S) + \sum_{i=1}^{t+N} \varphi_i + \sum_{i=1}^{t-1} a_i^* + S - \sum_{i=1-N}^{t-1} v_i^* - y_0]\}$

$$\Leftrightarrow \{v_t > v_t^S\}$$

The replenishment at the beginning of period t will finally be computed as :

$$v_t^a = [\text{ Min } \{ M, \max_{u=t,t+1,\ldots,h} \{v_t^u\}, v_t^S \}]^+$$

That means in particular there is no legitimate reason for any replenishment to be considered as a normal-distributed variable ; this is however the hypothesis we will keep when dealing with multi-level systems : even if it is not, the demands inferred at each internal level by the replenishments decided at next downstream level will be supposed to be normal for computation of upstream replenishments ... because it is the only way to compute them and because the method has prooved to give good results when applied to a computer assembly plant .

Conclusion :

In the end, the approach presented in this paper can be considered as a cross between mean-value analysis techniques for systems modelled as queuing networks and dynamic programming methods to solve such problems as inventory planning, in deterministic environment :

In fact, the scope is the same as for the first class of algorithms, the assumptions on demands are restrictive too and valuation of the proposed policy in case of limited capacity systems must be achieved by means of simulation techniques (valuation parameters could be for example inventory level mean value, service to client rate actually reached...).

On another hand, the hierarchical level concerned is the one of flow-control, that is, the same level as for the second class of techniques evoked, and the entities modelled will be groups of machines and product families rather than parts and machine tools, the main difference consisting in the probabilistic approach of the criterion to optimize.

REFERENCES

Kimemia, J. G. (1982) "Hierarchical Control of Production in Flexible Manufacturing Systems" , Massachusetts Institute of Technology Laboratory for Information and Decision Systems . Report LIDS - TH - 1215

Akella, R., Y. Choong, and S.B. Gershwin (1984) "Performance of Hierarchical Production Scheduling Policy ", Massachusetts Institute of Technology Laboratory for Information and Decision Systems . Report LIDS - FR - 1357

Bensoussan, A. and J.M.Proth (1982) "Inventory planning in a deterministic environment : Continuous time models with concave costs ." European Journal of Operational Research.

Hildebrandt, R. and R. Suri "Modelling Flexible Manufacturing Systems Using Mean Value Analysis".