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Abstract: We study the behaviour of the steady-state voltage potential in a material composed of a two-dimensional object surrounded by a rough thin layer and embedded in an ambient medium. The roughness of the layer is supposed to be ε^α–periodic, ε being the magnitude of the mean thickness of the layer, and α a positive parameter describing the degree of roughness. For ε tending to zero, we determine the appropriate boundary layer correctors which lead to approximate transmission conditions equivalent to the effect of the rough thin layer. We also provide an explicit characterization of the polarization tensor as defined by Capdeboscq and Vogelius in [10]. This paper revisits the previous works of the author [17, 16, 11, 13], and it also provides new results for the very rough case $\alpha > 1$.

Key-words: Boundary Layer Correctors, Conductivity problem, Equivalent transmission condition

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Correcteurs de couche limite et tenseur de polarization généralisé pour des couches minces rugueuses.

Résumé : Dans cet article, nous considérons le problème de conduction dans un domaine bidimensionnel composé d'une fine membrane rugueuse entourant un domaine conducteur, le tout plongé dans un milieu ambiant de conductivité différente. La rugosité de la membrane est supposée ε^{α}–périodique, ε étant l'épaisseur moyenne de la membrane, et α un paramètre positif décrivant le degré de rugosité. Nous déterminons des correcteurs de couche limite conduisant à la construction de conditions de transmission approchées lorsque le paramètre ε tend vers zero. Nous donnons aussi une caractérisation explicite du tenseur de polarisation défini par Capdeboscq and Vogelius dans [10]. Cet article revisite des résultats précédents de l'auteur obtenus dans [17, 16, 11, 13], et présente de nouveaux résultats pour le cas très rugueux $\alpha > 1$.

Mots-clés : Correcteur de couches limites, Equation de conduction, Conditions de transmissions équivalentes

Contents

1 Introduction and notation

This paper is a review of boundary layer correctors and generalized polarization tensor for the steady-state potential in a dielectric material with a rough thin layer. The roughness of the layer is supposed to be ε^{α}–periodic, and the mean thickness of the layer is of order ε, ε being a small positive parameter, and α a positive parameter describing the degree of roughness.

Several papers are devoted to rough boundaries and derivations of equivalent boundary conditions [1, 2, 3, 14]. In a recent article Basson and Gérard-Varet [7] derive approximate boundary condition for a boundary with random roughness. The analysis of these previous papers is essentially based on the construction of the so-called "wall law", which is a boundary condition imposed on an artificial boundary inside the domain. The wall law only reflects the large-scale effect on the oscillations. Note also that in chapter 8 of the book [15], Marchenko and Khrushlov presented equivalent boundary conditions in the very general framework of elliptic operators with even degree using homogenization techniques.

The core of this paper consists in deriving *transmission* conditions equivalent to the rough thin layer. Compared to equivalent boundary conditions, the derivation of equivalent transmission conditions leads to several difficulties in the definition of the boundary layer correctors. In particular a naive approach coming from wall law derivation techniques can lead to ill-posed problems satisfied by the boundary layer correctors. In addition, we aim at deriving accurate global error estimates, valid in the vicinity of the rough layer, by treating in the same way the weakly oscillating thin layer and the very rough one. We also provide explicit characterization of the so-called polarization tensor defined in [5, 6]. Actually we aim at showing that the derivation of the boundary layer correctors is very efficient since it provides simultaneously an explicit characterization of the polarization tensor and an accurate description of the electric potential in the vicinity of the inhomogeneity, while the variational techniques as used by Capdeboscq and Vogelius [10] provide estimates only far from the roughness. For $\alpha \leq 1$ several results have been obtained for the conductivity problem [17, 16, 11]. We aim at revisiting these results using a general framework that allows to treat similarly the three cases $\alpha < 1$, $\alpha = 1$ and $\alpha > 1$. We emphasize that for $\alpha > 1$ only weak results have been obtained in [13], using two-scale convergence techniques. In this paper we push-forward the analysis by defining the boundary layer corrector for $\alpha > 1$ and by proving error estimates for $\alpha \in (0, 2)$: far from the layer, we recover the results proved in [13], while in the neighborhood of the roughness the boundary layer corrector provides an accurate description of the potential.

Notation 1.1 *Present now the notations used throughout the paper.*

- *All the closed curves are trigonometrically (counterclockwise) oriented.*
- *We generically denote by n the normal to a closed smooth curve of $\mathbb{R}^2$ outwardly directed from the domain enclosed by the curve to the outside.*
- *Let $\mathcal{C}$ be a curve of $\mathbb{R}^2$, and let u be a sufficiently smooth function defined in a tubular neighbourhood of $\mathcal{C}$. We define $u|_{\mathcal{C}^{\pm}}$ by*

$$\forall x \in \mathcal{C}, \quad u|_{\mathcal{C}^{\pm}}(x) = \lim_{t \to 0^+} u(x \pm t n(x)),$$

and $\partial_n u|_{\mathcal{C}^\pm}$ *denotes*

$$\forall x \in \mathcal{C}, \quad \partial_n u|_{\mathcal{C}^\pm}(x) = \lim_{t \to 0^+} \nabla u(x \pm t n(x)) \cdot n(x).$$

where $\cdot$ *denotes the Euclidean scalar product of* $\mathbb{R}^2$.

- *The jump* $[u]_{\mathcal{C}}$ *of a function* u *defined in a neighbourhood of the curve* $\mathcal{C}$ *is defined by*

$$[u]_{\mathcal{C}} = u|_{\mathcal{C}^+} - u|_{\mathcal{C}^-}.$$

Let Ω be a smooth bounded domain of $\mathbb{R}^2$ with connected boundary $\partial\Omega$. Let $\varepsilon > 0$ and $\alpha > 0$ be small positive parameters, we split Ω into three subdomains: $\mathcal{D}^1$, $\mathcal{D}^m_\varepsilon$ and $\mathcal{D}^0_\varepsilon$. The domain $\mathcal{D}^1$ is a smooth domain strictly embedded in Ω (see figure 1), and we denote by Γ its boundary, which is a connected smooth curve.

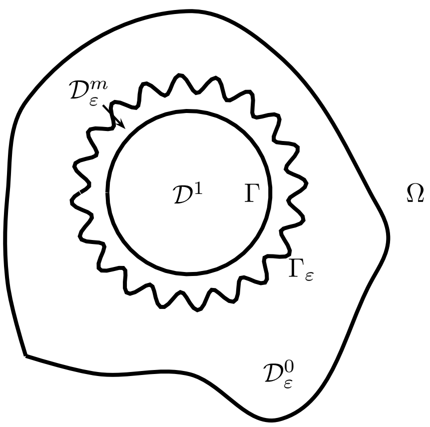

Figure 1: Geometry of the problem

Parameterize Γ by its curvilinear abscissa denoted by Ψ. Let κ be the curvature of Γ and n its normal vector. Let f be a smooth 1–periodic and positive function. For sake of simplicity, we suppose that $1/2 \leq f \leq 3/2$. The domain $\mathcal{D}^m_\varepsilon$ is a thin oscillating layer surrounding $\mathcal{D}^1$. We denote by $\Gamma^\alpha_\varepsilon$ the oscillating boundary of $\mathcal{D}^m_\varepsilon$:

$$\Gamma^\alpha_\varepsilon = \partial\mathcal{D}^m_\varepsilon \setminus \Gamma = \left\{\Psi(\theta) + \varepsilon f(\theta/\varepsilon^\alpha) n(\theta), \quad \theta \in \mathbb{T}\right\}.$$

The domain $\mathcal{D}^0_\varepsilon$ is defined by

$$\mathcal{D}^0_\varepsilon = \Omega \setminus \overline{(\mathcal{D}^1 \cup \mathcal{D}^m_\varepsilon)}.$$

We also write

$$\mathcal{D}^0 = \Omega \setminus \overline{\mathcal{D}^1}.$$

We define the piecewise-constant functions σ^ε, $\sigma : \Omega \to \mathbb{R}$ by

$$\sigma^\varepsilon(z) = \begin{cases} \sigma_1, \text{ if } z \in \mathcal{D}^1, \\ \sigma_m, \text{ if } z \in \mathcal{D}^m_\varepsilon, \\ \sigma_0, \text{ if } z \in \mathcal{D}^0_\varepsilon, \end{cases} \qquad \sigma(z) = \begin{cases} \sigma_1, \text{ if } z \in \mathcal{D}^1, \\ \sigma_0, \text{ if } z \in \mathcal{D}^0, \end{cases}$$

where σ_1, σ_m and σ_0 are given positive constants. Let g be sufficiently smooth and denote by u^ε the unique function satisfying

$$\nabla \cdot (\sigma^\varepsilon \nabla u^\varepsilon) = 0, \text{ in } \Omega, \tag{1a}$$

$$u^\varepsilon|_{\partial\Omega} = g, \tag{1b}$$

and by u^0 the background solution:

$$\nabla \cdot \left(\sigma \nabla u^0\right) = 0, \text{ in } \Omega, \tag{2a}$$

$$u^0|_{\partial\Omega} = g, \tag{2b}$$

For the sake of simplicity we suppose that g is a smooth function, but this assumption can be weakened to $g \in H^s(\partial\Omega)$ for s large enough. According to Capdeboscq and Vogelius [10], there exists a matrix $\mathcal{M}_\alpha$, the so-called generalized polarization tensor, which is symmetric definite positive and such that almost everywhere far from the inhomogeneity, which is here the rough thin layer, we have:

$$u^\varepsilon(y) - u^0(y) = \varepsilon(\sigma_m - \sigma_0) \int_\Gamma \mathcal{M}_\alpha \begin{pmatrix} \partial_n u^0 \\ \nabla_\Gamma u^0 \end{pmatrix} \cdot \begin{pmatrix} \partial_n G \\ \nabla_\Gamma G \end{pmatrix} (., y) \, \mathrm{d}s + o(\varepsilon),$$

where G is the Dirichlet function defined by

$$\begin{cases} \nabla_x \cdot \Big(\sigma \nabla_x G(x, y) \Big) = -\delta_y, \text{ in } \Omega, \\ G(x, y) = 0, \quad \forall x \in \partial\Omega. \end{cases}$$

In this paper we provide an explicit characterization of the polarization for rough thin layer for any $\alpha \geq 0$. The parameter α can be seen as a roughness parameter: for $\alpha = 0$, the layer is not rough, for $\alpha = 1$ the roughness is of same order of the thickness of the layer, and $\alpha > 1$ describes very rough thin layers.

For $\alpha > 0$, denoting by $\beta = \min(\alpha, 1)$, we define the boundary layer corrector $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ by (11), which explicitly characterizes $\mathcal{M}_\alpha$:

$$\mathcal{M}_\alpha = \varepsilon^{\beta-1} \begin{pmatrix} \sigma_0 \, {}^\alpha\mathrm{a}_X & \sigma_0 \, {}^\alpha\mathrm{a}_Y \\ D_X^2 & D_Y^2 \end{pmatrix},$$

D^2 being defined by (26). We emphasize that even for $\alpha < 1$ the matrix $\mathcal{M}_\alpha$ is of order 1 since according to estimate (12) the quantities ${}^\alpha\mathrm{a}$ and D^2 are of order $\varepsilon^{1-\beta}$. Note the boundary layer corrector $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ will be useful to obtain estimates in a neighborhood of the layer (see theorem 3.1), while the variational techniques of Vogelius *et al.* only provided estimates far from the inhomogeneity.

The outline of the paper is the following. We first rewrite the problem in appropriate coordinates to make appear clearly the boundary layer corrector. We then derive formally the asymptotic expansion of u^ε at the first order in section 2. This section is the core of the paper since it precisely describes the boundary layer correctors for any $\alpha > 0$. Section 3 is devoted to the proof of the error estimates for $\alpha \in (0, 2)$, and we also link first order boundary layer and generalized polarization tensor in this part. We end the paper by providing the leading term of the boundary layer corrector for $\alpha \neq 1$ in section 4. In particular for very rough thin layers, using the two-scale limit of the boundary layer corrector, we retrieve the characterization of the polarization tensor given by Ciuperca *et al.* in [13] and an accurate description of the potential near the roughness is also available, as proved by theorem 3.1.

1.1 The equivalent problem in a tubular neighbourhood of Γ

It is convenient to write problem (1) in a smooth tubular neighbourhood Ω_{d_0} of Γ (see figure 2), given for some distance d_0 such that

$$d_0 \leq \frac{1}{\sup_{\theta\in\mathbb{T}} |\kappa(\theta)|}, \text{by} \quad \Omega_{d_0} = \Big\{ z \in \mathbb{R}^2, \quad dist(z,\Gamma) < d_0 \Big\},$$

where κ is the curvature of Γ, which is a smooth function of the curvilinear abscissa θ.

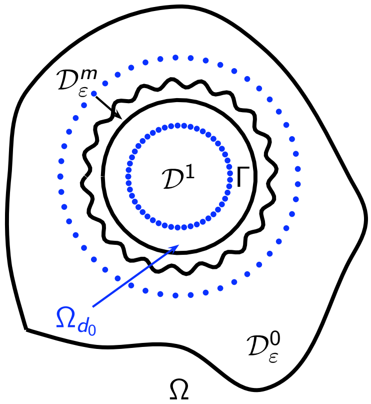

Figure 2: Tubular neighborhood $\Omega_{d_0}$of the layer

Denote by Γ_{-d_0} and Γ_{d_0} the closed curves respectively defined by

$$\Gamma_{-d_0} = \partial\Omega_{d_0} \cap \mathcal{D}^1, \quad \Gamma_{d_0} = \partial\Omega_{d_0} \cap \mathcal{D}^0_\varepsilon.$$

We consider the following Steklov-Poincaré operators $\mathcal{L}_0$, $\mathcal{L}_1$ and $\mathscr{T}$:

$$\begin{aligned} \mathcal{L}_1 &: H^{1/2}(\Gamma_{-d_0}) \longrightarrow H^{-1/2}(\Gamma_{-d_0}), \\ \mathcal{L}_0 &: H^{1/2}(\Gamma_{d_0}) \longrightarrow H^{-1/2}(\Gamma_{d_0}), \\ \mathscr{T} &: H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\Gamma_{d_0}). \end{aligned}$$

Using the convention of the direction of the normals (see notation 1.1), we define the operator $\mathcal{L}_1$ by

$$\forall\phi \in H^{1/2}(\Gamma_{-d_0}),\, \mathcal{L}_1(\phi) = \left.\frac{\partial u}{\partial n}\right|_{\Gamma_{-d_0}},$$

where u is the harmonic function in $\Omega \setminus \overline{(\mathcal{D}^0 \cup \Omega_{d_0})}$ equal to ϕ on Γ_{-d_0}. The operator $\mathcal{L}_0$ is defined by

$$\forall\phi \in H^{1/2}(\Gamma_{d_0}),\, \mathcal{L}_0(\phi) = -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{d_0}},$$

where u is the harmonic function in $\Omega \setminus \overline{(\mathcal{D}^1 \cup \Omega_{d_0})}$ equal to ϕ on Γ_{d_0} and vanishing on $\partial\Omega$. Similarly $\mathscr{T}$ equals:

$$\forall \chi \in H^{1/2}(\partial\Omega),\ \mathscr{T}(\chi) = -\left.\frac{\partial u}{\partial n}\right|_{\Gamma_{d_0}},$$

where u is the harmonic function in $\Omega \setminus \overline{(\mathcal{D}^1 \cup \Omega_{d_0})}$ equal to χ on $\partial\Omega$ and vanishing on Γ_{d_0}. Moreover the operators $\mathscr{T}$, $\mathcal{L}_0$ and $\mathcal{L}_1$ satisfy the following inequalities, for a d_0-independent constant C:

$$\forall g \in H^{1/2}(\partial\Omega), \quad |\mathscr{T} g|_{H^{-1/2}(\Gamma_{d_0})} \leq C\, |g|_{H^{1/2}(\partial\Omega)}, \tag{3a}$$

$$\forall u \in H^{1/2}(\Gamma_{d_0}), \quad |\mathcal{L}_0 u|_{H^{-1/2}(\Gamma_{d_0})} \leq C\, |u|_{H^{1/2}(\Gamma_{d_0})}, \tag{3b}$$

$$\forall u \in H^{1/2}(\Gamma_{-d_0}), \quad |\mathcal{L}_1 u|_{H^{-1/2}(\Gamma_{-d_0})} \leq C\, |u|_{H^{1/2}(\Gamma_{-d_0})}, \tag{3c}$$

and the following coercivity inequalities hold:

$$\forall u \in H^{1/2}(\Gamma_{d_0}), \quad (\mathcal{L}_0 u, u)_{L^2(\Gamma_{d_0})} \geq C\, |u|^2_{H^{1/2}(\Gamma_{d_0})}, \tag{3d}$$

$$\forall u \in H^{1/2}(\Gamma_{-d_0}), \quad (\mathcal{L}_1 u, u)_{L^2(\Gamma_{-d_0})} \geq C\, |u|^2_{H^{1/2}(\Gamma_{-d_0})}. \tag{3e}$$

Problem (1) is then equivalent to

$$\nabla.\,(\sigma \nabla u^\varepsilon) = 0, \text{ in } \Omega_{d_0}, \tag{4a}$$

$$\partial_n u^\varepsilon|_{\Gamma_{d_0}} + \mathcal{L}_0 u^\varepsilon|_{\Gamma_{d_0}} = -\mathscr{T} g, \text{ on } \Gamma_{d_0}, \tag{4b}$$

$$\partial_n u^\varepsilon|_{\Gamma_{-d_0}} - \mathcal{L}_1 u^\varepsilon|_{\Gamma_{-d_0}} = 0, \text{ on } \Gamma_{-d_0}. \tag{4c}$$

1.2 The problem in local coordinates

Denote by Φ the smooth diffeomorphism

$$\forall\, (\eta, \theta) \in (-d_0, d_0) \times \mathbb{T}, \quad \Phi(\eta, \theta) = \Psi(\theta) + \eta n(\theta).$$

Since $d_0 < 1/\|\kappa\|_\infty$, the open neighbourhood of Γ denoted by Ω_{d_0} can be parameterized as follows:

$$\Omega_{d_0} = \Big\{ \Phi(\eta, \theta),\ (\eta, \theta) \in (-d_0, d_0) \times \mathbb{T} \Big\}.$$

Let $\mathcal{O} = (-d_0, d_0) \times \mathbb{T}$ and denote respectively by $\mathcal{O}^1$, $\mathcal{O}^m_\varepsilon$ and $\mathcal{O}^0_\varepsilon$ the domains:

$$\mathcal{O}^1 = (-d_0, 0) \times \mathbb{T},$$

$$\mathcal{O}^m_\varepsilon = \Big\{ \big(\eta f(\theta/\varepsilon^\alpha), \theta\big) : \quad (\eta, \theta) \in (0, \varepsilon) \times \mathbb{T} \Big\},$$

$$\mathcal{O}^0_\varepsilon = \mathcal{O} \setminus \overline{\mathcal{O}^1 \cup \mathcal{O}^m_\varepsilon}.$$

We also denote by $\mathcal{O}^0$ the domain $\mathcal{O} \setminus \mathcal{O}^1$. Define the oscillating curve γ_ε by

$$\gamma_\varepsilon = \{ (\varepsilon f(\theta/\varepsilon^\alpha), \theta),\ \theta \in \mathbb{T} \}.$$

We write $\gamma^s = \{s\} \times \mathbb{T}$ for any $s \in \mathbb{R}$. The Laplacian written in (η, θ)–coordinates equals

$$\Delta_{\eta,\theta} = \frac{1}{1+\eta\kappa(\theta)}\partial_\eta\Big((1+\eta\kappa(\theta))\partial_\eta\Big) + \frac{1}{1+\eta\kappa(\theta)}\partial_\theta\left(\frac{1}{1+\eta\kappa(\theta)}\partial_\theta\right).$$

We also need the normal derivatives on Γ and Γ_ε in (η, θ)–coordinates. In the following, the notation $\nabla_{\eta,\theta}$ denotes the derivative operator:

$$\nabla_{\eta,\theta} = \begin{pmatrix} & \partial_\eta \\ (1+\eta\kappa)^{-1} & \partial_\theta \end{pmatrix}.$$

Let u be defined on Ω, and define v on $(-d_0, d_0) \times \mathbb{T}$ by

$$\forall (\eta,\theta) \in (-d_0, d_0) \times \mathbb{T}, \quad v(\eta,\theta) = u \circ \Phi(\eta,\theta).$$

We denote by $\partial_n^\Phi v$ the following normal derivative on γ_ε in the local coordinates

$$\partial_n^\Phi v\big|_{\gamma_\varepsilon} = \nabla_{\eta,\theta} v|_{\gamma_\varepsilon} \cdot n_{\gamma_\varepsilon}$$

where n_{γ_ε} is defined by

$$n_{\gamma_\varepsilon} = \frac{1}{\sqrt{1+\left(\dfrac{\varepsilon^{1-\alpha} f'(\theta/\varepsilon^\alpha)}{1+\varepsilon\kappa f(\theta/\varepsilon^\alpha)}\right)^2}} \begin{pmatrix} 1 \\ -\dfrac{\varepsilon^{1-\alpha} f'(\theta/\varepsilon^\alpha)}{1+\varepsilon\kappa f(\theta/\varepsilon^\alpha)} \end{pmatrix}.$$

Moreover, we define the bounded linear operators respectively corresponding to $\mathcal{L}_0$ and $\mathcal{L}_1$, and denoted by $\Lambda_0 : H^{1/2}(\gamma^{d_0}) \to H^{-1/2}(\gamma^{d_0})$ and $\Lambda_1 : H^{1/2}(\gamma^{-d_0}) \to H^{-1/2}(\gamma^{-d_0})$ as follows:

$$\langle \Lambda_0 \varphi, \psi \rangle = \langle \mathcal{L}_0(\varphi \circ \Phi^{-1}),\ \psi \circ \Phi^{-1} \rangle, \quad \forall\, \varphi, \psi \in H^{1/2}(\gamma^{d_0}),$$
$$\langle \Lambda_1 \varphi, \psi \rangle = \langle \mathcal{L}_1(\varphi \circ \Phi^{-1}),\ \psi \circ \Phi^{-1} \rangle, \quad \forall\, \varphi, \psi \in H^{1/2}(\gamma^{-d_0}).$$

According to (3), there exists an ε-independent constant $C > 0$ such that

$$\forall u \in H^{1/2}(\mathbb{T}), \quad (\Lambda_0 u, u)_{L^2(\mathbb{T})} \geq C\, |u|^2_{H^{1/2}(\mathbb{T})}, \tag{5a}$$
$$\forall u \in H^{1/2}(\mathbb{T}), \quad (\Lambda_1 u, u)_{L^2(\mathbb{T})} \geq C\, |u|^2_{H^{1/2}(\mathbb{T})}. \tag{5b}$$

With these notations, we can write our initial problem (1) in local coordinates. Denoting by v^ε the solution to problem (1) in (η, θ)–coordinates,

$$v^\varepsilon = u^\varepsilon \circ \Phi,$$

then v^ε is continuous and satisfies

$$\Delta_{\eta,\theta} v^\varepsilon = 0, \text{ in } \mathcal{O}^1 \cup \mathcal{O}^m_\varepsilon \cup \mathcal{O}^0_\varepsilon, \tag{6a}$$
$$(1+d_0\kappa)\partial_\eta v^\varepsilon|_{\eta=d_0} + \Lambda_0 v^\varepsilon|_{\eta=d_0} = -(\mathscr{T} g) \circ \Phi, \tag{6b}$$
$$(1-d_0\kappa)\partial_\eta v^\varepsilon|_{\eta=-d_0} - \Lambda_1 v^\varepsilon|_{\eta=-d_0} = 0, \tag{6c}$$

with the following transmission conditions:

$$\sigma_0\, \partial_n^\Phi v^\varepsilon\big|_{\gamma_\varepsilon^+} = \sigma_m\, \partial_n^\Phi v^\varepsilon\big|_{\gamma_\varepsilon^-}, \tag{6d}$$
$$\sigma_m\, \partial_\eta v^\varepsilon\big|_{\eta=0^+} = \sigma_1\, \partial_\eta v^\varepsilon\big|_{\eta=0^-}. \tag{6e}$$

From now on, all the results will be obtained on v^ε, but the results for u^ε can be straightforwardly derived using the map Φ.

1.3 Localization of the roughness

The derivation of the expansion of v^ε is mainly based on the localization of the oscillation, which leads to solve a boundary value problem in a domain with one oscillation. In order to make appear this "profile" problem, we introduce the following notations. Denote by $\beta = \min(1,\alpha)$. We aim at tackling in the same way all the cases, from the weakly oscillating and the very rough case. This is the reason why we use the parameter β and we perform the rescaling

$$(\eta,\theta) \to (X,Y) = (\eta/\varepsilon^\beta, \theta/\varepsilon^\beta).$$

For sake of simplicity we suppose that the length of Γ equals 1, and we define $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Let $\mathcal{C}_0$ and $\mathcal{C}_{\alpha;1}$ be the curves

$$\mathcal{C}_0 = \{(0,Y), Y \in \mathbb{T}\}, \quad \mathcal{C}_{\alpha;1} = \left\{(f(Y/\varepsilon^{\alpha-\beta}), Y), Y \in \mathbb{T}\right\}. \tag{7}$$

Denote by $n_{\mathcal{C}_{\alpha;1}}$ the normal to $\mathcal{C}_{\alpha;1}$, which is equal on $\mathbb{T}$ to

$$\forall Y \in \mathbb{T}, \quad n_{\mathcal{C}_{\alpha;1}}(Y) = \frac{1}{\sqrt{1 + \left(\varepsilon^{1-\alpha} f'(Y/\varepsilon^{\alpha-\beta})\right)^2}} \begin{pmatrix} 1 \\ -\varepsilon^{1-\alpha} f'(Y/\varepsilon^{\alpha-\beta}) \end{pmatrix},$$

and let $n_{\mathcal{C}_0}$ be the normal to $\mathcal{C}_0$ defined by

$$n_{\mathcal{C}_0}(Y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

It is convenient to denote by $n^\perp_{\mathcal{C}_0}$ and $n^\perp_{\mathcal{C}_{\alpha;1}}$ the two following unitary vectors:

$$n^\perp_{\mathcal{C}_0}(Y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad n^\perp_{\mathcal{C}_{\alpha;1}}(Y) = \frac{1}{\sqrt{1 + \left(\varepsilon^{1-\alpha} f'(Y/\varepsilon^{\alpha-\beta})\right)^2}} \begin{pmatrix} \varepsilon^{1-\alpha} f'(Y/\varepsilon^{\alpha-\beta}) \\ 1 \end{pmatrix}.$$

Before performing the asymptotic expansion we would like to make the following remark.

Remark 1.2 *According to the definition of n_{γ_ε}, the slow and the fast variables (respectively the variables θ and $\theta/\varepsilon^\alpha$) are mixed in $\partial_n^\Phi v|_{\gamma_\varepsilon}$. In order to derive easily the formal asymptotics of the potential, it is convenient to uncouple these variables. Denote by $\mathfrak{f}_\varepsilon$ and M_ε the following function*

$$\forall Y \in \mathbb{T}, \quad \mathfrak{f}_\varepsilon(Y) = f(Y/\varepsilon^{\alpha-\beta}), \text{and} \quad M_\varepsilon(Y) = \frac{\varepsilon^{1-\alpha} f'(Y/\varepsilon^{\alpha-\beta})}{\sqrt{1 + \left(\varepsilon^{1-\alpha} f'(Y^{\alpha-\beta})\right)^2}};$$

we emphasize that M_ε is uniformly bounded with respect to ε. Using these notations, the normal vector n_{γ_ε} writes:

$$n_{\gamma_\varepsilon} = n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta) + \varepsilon\kappa\mathfrak{f}_\varepsilon(\theta/\varepsilon^\beta)\left[M_\varepsilon^2(\theta/\varepsilon^\beta) n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta) + M_\varepsilon(\theta/\varepsilon^\beta) n^\perp_{\mathcal{C}_0}\right] + O(\varepsilon^2),$$

and for any sufficiently smooth function φ

$$\nabla_{\eta,\theta}\varphi|_{\gamma_\varepsilon} = \nabla_{\eta,\theta}\varphi|_{\eta=0} + \varepsilon\mathfrak{f}_\varepsilon(\theta/\varepsilon^\beta) \left.\begin{pmatrix} \partial^2_\eta\varphi \\ -\kappa\partial_\theta\varphi + \partial_\eta\partial_\theta\varphi \end{pmatrix}\right|_{\eta=0^+} + O(\varepsilon^2).$$

hence the two following formulae will be useful in the derivation of the asymptotic expansion:

$$
\begin{aligned}
\partial_n^\Phi \varphi|_{\gamma_\varepsilon} &= \nabla_{\eta,\theta}\varphi|_{\gamma_\varepsilon}\cdot n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta) \\
&+ \varepsilon\kappa\mathfrak{f}_\varepsilon(\theta/\varepsilon^\beta)\nabla_{\eta,\theta}\varphi|_{\gamma_\varepsilon}\cdot\left[M_\varepsilon^2(\theta/\varepsilon^\beta)n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta)+M_\varepsilon(\theta/\varepsilon^\beta)n_{\mathcal{C}_0}^\perp\right] \\
&+ O(\varepsilon^2).
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\partial_n^\Phi \varphi|_{\gamma_\varepsilon} &= \nabla_{\eta,\theta}\varphi|_{\eta=0}\cdot n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta) \\
&+ \varepsilon\mathfrak{f}_\varepsilon(\theta/\varepsilon^\beta)\Bigg\{\left.\begin{pmatrix}\partial_\eta^2\varphi\\ \partial_\eta\partial_\theta\varphi\end{pmatrix}\right|_{\eta=0^+}\cdot n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta) \\
&+ \kappa\nabla_{\eta,\theta}\varphi|_{\eta=0^+}\cdot\left[M_\varepsilon^2(\theta/\varepsilon^\beta)n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta)+M_\varepsilon(\theta/\varepsilon^\beta)n_{\mathcal{C}_0}^\perp\right]\Bigg\} \\
&+ O(\varepsilon^2).
\end{aligned} \tag{9}
$$

We emphasize that the terms in $O(\varepsilon^2)$ involve third order derivatives of the function φ, and they are bounded in L^∞ by ε^2, up to a multiplicative constant since φ is assumed to be smooth enough.

2 Formal asymptotics

2.1 Zeroth-order approximation

Let v^0 be the continuous "background" solution defined by

$$
\begin{aligned}
&\Delta_{\eta,\theta}v^0 = 0, \text{ in } \mathcal{O}^1\cup\mathcal{O}^0,\\
&(1+d_0\kappa)\partial_\eta v^0|_{\eta=d_0} + \Lambda_0 v^0|_{\eta=d_0} = -(\mathscr{T}g)\circ\Phi,\\
&(1-d_0\kappa)\partial_\eta v^0|_{\eta=-d_0} - \Lambda_1 v^0|_{\eta=-d_0} = 0,\\
&\sigma_0\partial_\eta v^0|_{\eta=0^+} = \sigma_1\partial_\eta v^0|_{\eta=0^-}.
\end{aligned}
$$

Since $g\in\mathcal{C}^\infty(\partial\Omega)$, the potential v^0 belongs to $H^1(\Omega)$ and it is smooth in each subdomain $\mathcal{O}^1$ and $\mathcal{O}^0$.

Denote by w^0 the error $v^\varepsilon - v^0$. This continuous function is the unique solution to

$$
\begin{aligned}
&\Delta_{\eta,\theta}w^0 = 0, \text{ in } \mathcal{O}^1\cup\mathcal{O}^0,\\
&(1+d_0\kappa)\partial_\eta w^0|_{\eta=d_0} + \Lambda_0 w^0|_{\eta=d_0} = 0,\\
&(1-d_0\kappa)\partial_\eta w^0|_{\eta=-d_0} - \Lambda_1 w^0|_{\eta=-d_0} = 0,\\
&\sigma_0\partial_n^\Phi w^0|_{\gamma_\varepsilon^+} = \sigma_m\partial_n^\Phi w^0|_{\gamma_\varepsilon^-} + (\sigma_m-\sigma_0)\partial_n^\Phi v^0|_{\gamma_\varepsilon},\\
&\sigma_m\partial_\eta w^0|_{\eta=0^+} = \sigma_1\partial_\eta w^0|_{\eta=0^-} + (\sigma_0-\sigma_m)\partial_\eta v^0|_{\eta=0^+},\\
&w^0|_{\partial\Omega} = 0, \text{ on } \partial\Omega.
\end{aligned}
$$

Since we are interested in the derivation of terms up to order 1, we throw away all the terms, which are *a priori* of order greater than ε. This approximation will be rigorously justified in the next section for the case $\alpha\in(0,2)$. From the above equality (9), we infer

$$
\left[\sigma\partial_n^\Phi w^0\right]_{\gamma_\varepsilon} = (\sigma_0-\sigma_m)\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta) + O(\varepsilon) \tag{10}
$$

Observe also that

$$\left[\sigma \partial_n v^0\right]_{\eta=0} = (\sigma_0 - \sigma_m)\nabla_{\eta,\theta} v^0|_{\eta=0^+} \cdot n_{\mathcal{C}_0}.$$

Therefore it is natural to introduce a boundary layer corrector in order to correct the Neumann transmission conditions.

2.1.a Boundary layer corrector

We split the infinite strip $\mathbb{R} \times \mathbb{T}$ into three domains:

$$\mathbb{R} \times \mathbb{T} = \{X < 0, Y \in \mathbb{T}\} \cup \left\{0 < X < f(Y/\varepsilon^{\alpha-\beta}), Y \in \mathbb{T}\right\} \\ \cup \left\{X > f(Y/\varepsilon^{\alpha-\beta}), Y \in \mathbb{T}\right\} \cup \mathcal{C}_0 \cup \mathcal{C}_{\alpha;1},$$

where $\mathcal{C}_0$ and $\mathcal{C}_{\alpha;1}$ are defined by (7) and let $\sigma^\flat$ be the conductivity corresponding to σ in the strip $\mathbb{R} \times \mathbb{Z}$:

$$\sigma^\flat = \begin{cases} \sigma_0, & \text{in } Y_0 = \left\{(X,Y), \quad X > f(Y/\varepsilon^{\alpha-\beta}), Y \in \mathbb{T}\right\}, \\ \sigma_m, & \text{in } Y_m^\beta = \left\{(X,Y), \quad 0 < X < f(Y/\varepsilon^{\alpha-\beta}), Y \in \mathbb{T}\right\}, \\ \sigma_1, & \text{in } Y_1 = \{(X,Y), \quad X < 0, Y \in \mathbb{T}\}. \end{cases}$$

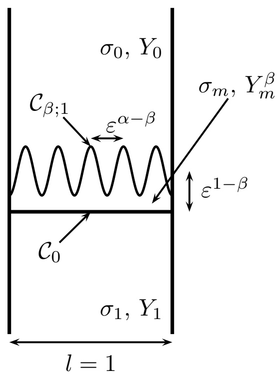

Figure 3: Rescaled strip

Observe that the case $\alpha < 1$ ($\beta = \alpha$) is very different from the case $\alpha > 1$ ($\beta = 1$): in the first case, the domain Y_m^β is a thin domain with non constant (but non oscillating) thickness, while in the second case Y_m^β is a domain of measure $O(1)$ with oscillating thickness. These two cases will lead to different leading term, as it will be proved in section 4.

Define the couple $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$, where ${}^\alpha\mathrm{A}$ is a vector field of $\mathbb{R} \times \mathbb{T}$ and ${}^\alpha\mathrm{a}$ is a constant vector, by

$$\Delta_{X,Y}{}^\alpha\mathrm{A} = 0, \text{ in } \mathbb{R} \times \mathbb{T} \setminus (\mathcal{C}_0 \cup \mathcal{C}_{\alpha;1}), \tag{11a}$$

$$\left[\sigma^\flat \nabla_{X,Y}{}^\alpha\mathrm{A}|_{\mathcal{C}^+_{\alpha;1}} \cdot n_{\mathcal{C}_{\alpha;1}}\right]_{\mathcal{C}_{\alpha;1}} = n_{\mathcal{C}_{\alpha;1}}, \tag{11b}$$

$$\left[\sigma^\flat \nabla_{X,Y}{}^\alpha\mathrm{A}|_{\mathcal{C}^+_0} \cdot n_{\mathcal{C}_0}\right]_{\mathcal{C}_0} = -n_{\mathcal{C}_0}, \tag{11c}$$

$${}^\alpha\mathrm{A} \to_{X\to-\infty} 0, \; {}^\alpha\mathrm{A} \to_{X\to+\infty} {}^\alpha\mathrm{a}. \tag{11d}$$

We emphasize that ${}^\alpha\mathrm{a}$ is inherent to the problem in the sense that it cannot be imposed as a kind of a Dirichlet condition: the unknowns of the above problem are the vector-field ${}^\alpha\mathrm{A}$ and the constant vector ${}^\alpha\mathrm{a}$. Denote by $\mathscr{E}$ the space of functions defined by

$$\mathscr{E} = \left\{ \begin{array}{l} \phi \in \left(H^1_{loc}(\mathbb{R}\times\mathbb{T})\right)^2 : \phi \text{ is } y\text{–periodic,} \\ \nabla\phi \in L^2(\mathbb{R}\times\mathbb{T}); \quad \phi \to_{X\to-\infty} 0 \end{array} \right\}.$$

Endowed with the norm

$$\|\phi\|^2_{\mathscr{E}} = \int_{\mathbb{R}\times\mathbb{T}} \|\phi(X,Y)\|^2 \,\mathrm{d}X\,\mathrm{d}Y + \int_{\mathbb{R}\times\mathbb{T}} \|\nabla\phi(X,Y)\|^2 \,\mathrm{d}X\,\mathrm{d}Y,$$

$\mathscr{E}$ is a Hilbert space. Since the function $\phi \in \mathscr{E}$ vanishes as X tends to $-\infty$, the semi-norm

$$\sqrt{\int_{\mathbb{R}\times\mathbb{T}} \|\nabla\phi(X,Y)\|^2 \,\mathrm{d}X\,\mathrm{d}Y},$$

is equivalent to the norm $\|\cdot\|_{\mathscr{E}}$ on $\mathscr{E}$. The couple $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ satisfies the following properties.

Property 2.1 *There exists an unique solution* $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ *to Problem* (11) *in* $\mathscr{E}$ *and there exists an* ε*-independent constant* C *such that*

$$\sqrt{\int_{\mathbb{R}\times\mathbb{T}} \|\nabla\, {}^\alpha\mathrm{A}(X,Y)\|^2 \,\mathrm{d}X\,\mathrm{d}Y} \leq C\varepsilon^{1-\beta}, \quad |{}^\alpha\mathrm{a}| \leq C\varepsilon^{1-\beta}. \tag{12}$$

Moreover, the constant vector ${}^\alpha\mathrm{a}$ *is linked to the vector-field* ${}^\alpha\mathrm{A}$ *through the following formula:*

$$\sigma_0 {}^\alpha\mathrm{a} = (\sigma_0 - \sigma_m) \int_0^1 {}^\alpha\mathrm{A}\left(\varepsilon^{1-\beta} f(Y/\varepsilon^{\alpha-\beta}), Y\right) \,\mathrm{d}Y + (\sigma_m - \sigma_1) \int_0^1 {}^\alpha\mathrm{A}(0,Y)\,\mathrm{d}Y$$
$$+ \varepsilon^{1-\beta} \int_0^1 f(Y/\varepsilon^{\alpha-\beta})\,\mathrm{d}Y\, n_{\mathcal{C}_0}. \tag{13}$$

Remark 2.2 *The* Y*-periodicity and the equations satisfied by* ${}^\alpha\mathrm{A}$ *imply that* ${}^\alpha\mathrm{A}$ *decays exponentially fast for* $X \to -\infty$ *and similarly* ${}^\alpha\mathrm{A} - {}^\alpha\mathrm{a}$ *decays exponentially fast as* $X \to +\infty$*, as described in [11].*

Proof 1 *The variational formulation of Problem* (11) *writes:*

Find ${}^\alpha\mathrm{A} \in \mathscr{E}$:

$$\forall v \in \mathscr{E}, \quad \int_{\mathbb{R}\times\mathbb{T}} \sigma^\flat \nabla\, {}^\alpha\mathrm{A} \nabla v \,\mathrm{d}X\,\mathrm{d}Y = \int_0^1 \int_0^{\varepsilon^{1-\beta} f(Y/\varepsilon^{\alpha-\beta})} \nabla\cdot v(X,Y)\,\mathrm{d}X\,\mathrm{d}Y,$$

which straightforwardly leads to existence and uniqueness of ${}^\alpha\mathrm{A} \in \mathscr{E}$ *and to inequalities* (12).

Prove now equality (13)*. Let* $M > 2$*, by integrating by parts* (11) ${}^\alpha\mathrm{A}$ *satisfies the following variational formulation:*

$$\forall v \in \left(H^1([-M,M]\times\mathbb{T})\right)^2, \quad \mathscr{A}({}^\alpha\mathrm{A}, v) = \mathscr{B}(v),$$

where $\mathscr{A}$ and $\mathscr{B}$ are given by

$$\begin{aligned}\mathscr{A}({}^{\alpha}\mathrm{A}, v) = &-\int_{[-M,M]\times\mathbb{T}} \sigma^{\flat}\nabla\, {}^{\alpha}\mathrm{A}.\nabla v \,\mathrm{d}X\,\mathrm{d}Y + \sigma_0 \int_0^1 \partial_X {}^{\alpha}\mathrm{A}|_{X=M}\, v|_{X=M}\,\mathrm{d}Y\\ &- \int_0^1 \sigma_1 \partial_X {}^{\alpha}\mathrm{A}|_{X=-M}\, v|_{X=-M}\,\mathrm{d}Y,\\ \mathscr{B}(v) = &\int_0^1 \int_0^{\varepsilon^{1-\beta} f(Y/\varepsilon^{\alpha-\beta})} \nabla\cdot v(X,Y)\,\mathrm{d}X\,\mathrm{d}Y\\ = &\int_{\mathcal{C}_{1;\beta}} n_{\mathcal{C}_{\alpha;1}}(s) v(s)\,\mathrm{d}s - \int_{\mathcal{C}_0} n_{\mathcal{C}_0}(s) v(s)\,\mathrm{d}s.\end{aligned}$$

Moreover, for any function $v \in H^2([-M,M]\times\mathbb{T})$, by integrating by parts once again we infer

$$\begin{aligned}\mathscr{A}({}^{\alpha}\mathrm{A}, v) = &-\sigma_0 \int_0^1 {}^{\alpha}\mathrm{A}(M,Y)\partial_X v|_M\,\mathrm{d}Y + \sigma_1 \int_0^1 {}^{\alpha}\mathrm{A}(-M,Y)\partial_X v|_{-M}\,\mathrm{d}Y\\ &+ \sigma_0 \int_0^1 \partial_X {}^{\alpha}\mathrm{A}(M,Y) v|_M\,\mathrm{d}Y - \sigma_1 \int_0^1 \partial_X {}^{\alpha}\mathrm{A}(-M,Y) v|_{-M}\,\mathrm{d}Y\\ &+ \int_{\mathcal{C}_{\alpha;1}} {}^{\alpha}\mathrm{A}\left[\sigma^{\flat}\partial_n v\right]_{\mathcal{C}_{\alpha;1}}\mathrm{d}s + \int_{\mathcal{C}_0} {}^{\alpha}\mathrm{A}\left[\sigma^{\flat}\partial_n v\right]_{\mathcal{C}_0}\mathrm{d}s\\ &+ \int_{[-M,M]\times\mathbb{T}} \sigma^{\flat}\, {}^{\alpha}\mathrm{A}\cdot\Delta v\,\mathrm{d}X\,\mathrm{d}Y.\end{aligned} \tag{14}$$

Let ψ be the function defined by

$$\forall (X,Y) \in (0,M)\times\mathbb{T}, \quad \psi(X,Y) = X,$$

and choose $v = (\psi,\psi)^T$ in (14). Denoting by R_M the terms

$$\begin{aligned}R_M = \sigma_1 \int_0^1 {}^{\alpha}\mathrm{A}(-M,Y)\,\mathrm{d}Y + \sigma_0 M \int_0^1 \partial_X {}^{\alpha}\mathrm{A}(M,Y)\,\mathrm{d}Y\\ + \sigma_1 M \int_0^1 \partial_X {}^{\alpha}\mathrm{A}(-M,Y)\,\mathrm{d}Y,\end{aligned}$$

we infer that for all $M > 2$,

$$\begin{aligned}\sigma_0 \int_0^1 {}^{\alpha}\mathrm{A}(M,Y)\,\mathrm{d}Y = (\sigma_0 - \sigma_m) &\int_0^1 {}^{\alpha}\mathrm{A}\left(\varepsilon^{1-\beta} f(Y/\varepsilon^{\alpha-\beta}), Y\right)\mathrm{d}Y\\ &+ (\sigma_m - \sigma_1)\int_0^1 {}^{\alpha}\mathrm{A}(0,Y)\,\mathrm{d}Y\\ &+ \varepsilon^{1-\beta}\int_0^1 f(Y/\varepsilon^{\alpha-\beta})\,\mathrm{d}Y\, n_{\mathcal{C}_0} + R_M.\end{aligned}$$

Let M tend to infinity to end the proof.

We now define the boundary layer corrector of order 0, v^0_{BL} on $\mathcal{O}$ by

$$v^0_{BL}(\eta,\theta) = (\sigma_m - \sigma_0)\varepsilon^{\beta}\nabla_{\eta,\theta} v^0|_{\Gamma^+}\cdot \begin{cases}\left({}^{\alpha}\mathrm{A}(\eta/\varepsilon^{\beta}, \theta/\varepsilon^{\beta}) - {}^{\alpha}\mathrm{a}\right), \text{ if } \eta > 0,\\ {}^{\alpha}\mathrm{A}(\eta/\varepsilon^{\beta}, \theta/\varepsilon^{\beta}), \text{ if } \eta < 0.\end{cases}$$

From equality (8) with φ replaced by v^0_{BL}, and according to (12) we deduce the jump of $\partial^\Phi_n v^0_{BL}$ across γ_ε:

$$\begin{aligned}\frac{1}{\sigma_m-\sigma_0}\left[\sigma\partial^\Phi_n v^0_{BL}\right]_{\gamma_\varepsilon} &= \nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta)\\ &+\varepsilon^\beta\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot\left\{(\sigma_m-\sigma_0)M_\varepsilon(Y)\left({}^\alpha\mathrm{A}|_{\mathcal{C}_{\alpha;1}}-{}^\alpha\mathrm{a}\right)\right\}_{Y=\theta/\varepsilon^\beta}\\ &+\varepsilon\kappa\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot\left\{\mathfrak{f}_\varepsilon(Y)\Big(M^2_\varepsilon(Y)n_{\mathcal{C}_{\alpha;1}}(Y)\right.\\ &\left.+M_\varepsilon(Y)\left[\sigma^\flat\partial_Y{}^\alpha\mathrm{A}\right]_{\mathcal{C}_{\alpha;1}}\Big)\right\}_{Y=\theta/\varepsilon^\beta}+O(\varepsilon^2).\end{aligned}\tag{15}$$

In addition, observe that

$$\begin{aligned}\partial^2_\eta v^0|_{\eta=0^+}&=-\kappa\partial_\eta v^0|_{\eta=0^+}-\partial^2_\theta v^0|_{\eta=0^+},\\ \left.\begin{pmatrix}-\kappa\partial_\eta v^0-\partial^2_\theta v^0\\ \partial_\eta\partial_\theta v^0\end{pmatrix}\right|_{\eta=0^+}\cdot n_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta)&=-\kappa\nabla_{\eta,\theta}v^0\cdot n_{\mathcal{C}_0}\\ &\quad-\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot n^\perp_{\mathcal{C}_{\alpha;1}}(\theta/\varepsilon^\beta)\end{aligned}$$

Therefore, using (9)–(15) we infer

$$\begin{aligned}[\sigma\partial^\Phi_n(v^0+v^0_{BL})]_{\gamma_\varepsilon}&=(\sigma_m-\sigma_0)\varepsilon^\beta\partial_\theta\nabla_{\eta,\theta}v^0\cdot\Big[\varepsilon^{1-\beta}\mathfrak{f}_\varepsilon(Y)n^\perp_{\mathcal{C}_{\alpha;1}}(Y)\\ &+(\sigma_m-\sigma_0)M_\varepsilon(Y)\left({}^\alpha\mathrm{A}|_{\mathcal{C}_{\alpha;1}}-{}^\alpha\mathrm{a}\right)\Big]_{Y=\theta/\varepsilon^\beta}\\ &+(\sigma_m-\sigma_0)\varepsilon^\beta\kappa\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot\Big[\varepsilon^{1-\beta}\mathfrak{f}_\varepsilon(Y)\Big(M_\varepsilon(Y)\left[\sigma^\flat\partial_Y{}^\alpha\mathrm{A}\right]_{\mathcal{C}_{\alpha;1}}\\ &+n_{\mathcal{C}_{\alpha;1}}(Y)\Big)\Big]_{Y=\theta/\varepsilon^\beta}\\ &+O(\varepsilon^2).\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{1}{\sigma_m-\sigma_0}\Delta_{\eta,\theta}v^0_{BL}=&\frac{1}{\varepsilon^\beta}\left(\frac{1}{(1+\eta\kappa)^2}-1\right)\partial^2_Y{}^\alpha\mathrm{A}|_{\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+}\\ &+\frac{2}{(1+\eta\kappa)^2}\partial_Y{}^\alpha\mathrm{A}|_{\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\cdot\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}\\ &+\frac{1}{1+\eta\kappa}\left(\kappa\partial_X{}^\alpha\mathrm{A}|_{\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}-\frac{\eta\kappa'}{(1+\eta\kappa)^2}\partial_Y{}^\alpha\mathrm{A}|_{\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\right)\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+}\\ &+\frac{\varepsilon^\beta}{(1+\eta\kappa)^2}{}^\alpha\mathrm{A}|_{\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\cdot\left(\partial^2_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}-\frac{\eta\kappa'}{(1+\eta\kappa)^2}\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}\right).\end{aligned}$$

Decoupling the slow and fast variables, and denoting by $\tilde{O}(\varepsilon)$ any function which is of order ε as $|\eta|/\varepsilon^\beta$ is bounded and which decays exponentially as $|\eta|/\varepsilon^\beta$ tends to infinity, we rewrite the previous equality as follows

$$\frac{1}{\sigma_m-\sigma_0}\Delta_{\eta,\theta}v^0_{BL} = \left(\partial_X {}^\alpha\mathrm{A} - 2X\partial_Y^2\,{}^\alpha\mathrm{A}\right)|_{(X,Y)=\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\cdot\left(\kappa\nabla_{\eta,\theta}v^0|_{\eta=0^+}\right) + 2\partial_Y {}^\alpha\mathrm{A}|_{(X,Y)=\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\cdot\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+} + \tilde{O}(\varepsilon), \tag{16}$$

Therefore $W^0 = v^\varepsilon - (v^0 + v^0_{BL})$ satisfies

$$-\frac{1}{\sigma_m-\sigma_0}\Delta_{\eta,\theta}W^0 = \left(\partial_X {}^\alpha\mathrm{A} - 2X\partial_Y^2\,{}^\alpha\mathrm{A}\right)|_{(X,Y)=\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\cdot\left(\kappa\nabla_{\eta,\theta}v^0|_{\eta=0^+}\right) + 2\partial_Y {}^\alpha\mathrm{A}|_{(X,Y)=\eta/\varepsilon^\beta,\theta/\varepsilon^\beta}\cdot\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+} + \tilde{O}(\varepsilon), \tag{17}$$

$$\frac{1}{\sigma_m-\sigma_0}[\sigma\partial_n^\Phi W^0]_{\gamma_\varepsilon} = -\varepsilon^\beta\partial_\theta\nabla_{\eta,\theta}v^0\cdot\left[(\sigma_m-\sigma_0)M_\varepsilon(Y)\left({}^\alpha\mathrm{A}|_{\mathcal{C}_{\alpha;1}} - {}^\alpha\mathrm{a}\right) + \varepsilon^{1-\beta}\mathfrak{f}_\varepsilon(Y)n^\perp_{\mathcal{C}_{\alpha;1}}(Y)\right]_{Y=\theta/\varepsilon^\beta} - \varepsilon^\beta\kappa\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot\left[\varepsilon^{1-\beta}\mathfrak{f}_\varepsilon(Y)\Big(M_\varepsilon(Y)\left[\sigma^\flat\partial_Y{}^\alpha\mathrm{A}\right]_{\mathcal{C}_{\alpha;1}} + n_{\mathcal{C}_{\alpha;1}}(Y)\Big)\right]_{Y=\theta/\varepsilon^\beta}, + O(\varepsilon^2), \tag{18}$$

Denoting by $(G^j)_{j=1,2}$ and $(B^j)_{j=1,2}$ the following terms:

$$G^1 = -\left(\partial_X{}^\alpha\mathrm{A} - 2X\partial_Y^2\,{}^\alpha\mathrm{A}\right), \quad G^2 = -2\partial_Y{}^\alpha\mathrm{A}, \tag{19a}$$

$$B^1 = -\varepsilon^{1-\beta}\mathfrak{f}_\varepsilon\Big(M_\varepsilon\left[\sigma^\flat\partial_Y{}^\alpha\mathrm{A}\right]_{\mathcal{C}_{\alpha;1}} + n_{\mathcal{C}_{\alpha;1}}\Big), \tag{19b}$$

$$B^2 = -\left[(\sigma_m-\sigma_0)M_\varepsilon\left({}^\alpha\mathrm{A}|_{\mathcal{C}_{\alpha;1}} - {}^\alpha\mathrm{a}\right) + \varepsilon^{1-\beta}\mathfrak{f}_\varepsilon n^\perp_{\mathcal{C}_{\alpha;1}}\right]. \tag{19c}$$

we infer

$$\Delta_{\eta,\theta}W^0 = (\sigma_m-\sigma_0)\kappa\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot G^1(\eta/\varepsilon^\beta,\theta/\varepsilon^\beta) + (\sigma_m-\sigma_0)\partial_\theta\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot G^2(\eta/\varepsilon^\beta,\theta/\varepsilon^\beta) + \tilde{O}(\varepsilon), \tag{20}$$

$$[\sigma\partial_n^\Phi W^0]_{\gamma_\varepsilon} = (\sigma_m-\sigma_0)\varepsilon^\beta\kappa\nabla_{\eta,\theta}v^0|_{\eta=0^+}\cdot B^1(\theta/\varepsilon^\beta) + (\sigma_m-\sigma_0)\varepsilon^\beta\partial_\theta\nabla_{\eta,\theta}v^0\cdot B^2(\theta/\varepsilon^\beta) + O(\varepsilon^2), \tag{21}$$

$$[\sigma\partial_n W^0]_{\gamma_0} = 0, \tag{22}$$

$$[W^0]_{\gamma_\varepsilon} = 0, \tag{23}$$

$$[W^0]_{\gamma_0} = (\sigma_m-\sigma_0)\varepsilon^\beta\cdot{}^\alpha\mathrm{a}\cdot\nabla_{\eta,\theta}v^0|_{\eta=0^+}. \tag{24}$$

with the boundary conditions:

$$(1+d_0\kappa)\partial_\eta W^0|_{\eta=d_0} + \Lambda_0 W^0|_{\eta=d_0} = -v^0_{BL}|_{d_0},$$
$$(1-d_0\kappa)\partial_\eta W^0|_{\eta=-d_0} - \Lambda_1 W^0|_{\eta=-d_0} = -v^0_{BL}|_{-d_0}$$

Define $(D^j)_{j=1,2}$ by

$$D^j = \int_{\mathbb{R}\times\mathbb{T}}\sigma^\flat G^j\,\mathrm{d}X\,\mathrm{d}Y + \int_{\mathcal{C}_{\alpha;1}} B^j(s)\,\mathrm{d}s,. \tag{25}$$

Simple calculations implies

$$D^2 = (\sigma_m - \sigma_0)\int_{\mathcal{C}_{\alpha;1}} M_\varepsilon(s)^\alpha\mathrm{A}\,\mathrm{d}s - \varepsilon^{1-\beta}\int_0^1 f(y/\varepsilon^{\alpha-\beta})\,\mathrm{d}y\, n_{\mathcal{C}_0}^\perp. \qquad (26)$$

Property 2.3 *The following properties hold.*

1. *The coefficient D^1 vanishes.*
2. *The first component D^2_X of the vector D^2 satisfies*

$$D^2_X = \sigma_0{}^\alpha\mathrm{a}_Y,$$

where the index X (resp. Y) denotes the first (resp. the second) component of the corresponding vector.

Proof 2 *According to* (11)*, D^1 writes*

$$D^1 = -\int_{\mathbb{R}\times\mathbb{T}} \sigma^\flat\partial_X\left(X\partial_X{}^\alpha\mathrm{A}\right)\mathrm{d}X\,\mathrm{d}Y - \int_{\mathbb{R}\times\mathbb{T}} \sigma^\flat X\partial_X^2{}^\alpha\mathrm{A}\,\mathrm{d}X\,\mathrm{d}Y$$
$$- \varepsilon^{1-\beta}\int_{\mathcal{C}_{\alpha;1}}\left(\mathfrak{f}_\varepsilon\left(M_\varepsilon\left[\sigma^\flat\partial_Y{}^\alpha\mathrm{A}\right]_{\mathcal{C}_{\alpha;1}} + n_{\mathcal{C}_{\alpha;1}}\right)\right)\mathrm{d}s,$$

Then integrating by parts the first two integrands and using (13) *implies that D^1 equals zero.*

For the second property, let w be

$$\forall X\in\mathbb{R},\quad w(X) = \begin{cases} X, & \text{if } X>0,\\ \dfrac{\sigma_0}{\sigma_1}X, & \text{if } X<0,\end{cases} \qquad (27)$$

and defined V by

$$\forall (X,Y)\in\mathbb{R}\times\mathbb{T},\quad V(X,Y) = \begin{pmatrix} -{}^\alpha\mathrm{A}_Y(X,Y)\\ w(X) + {}^\alpha\mathrm{A}_X(X,Y)\end{pmatrix}.$$

Observe that according to (11)*, V satisfies the following properties:*

$$\Delta V = 0,\ \text{in } \mathbb{R}\times\mathbb{T}\setminus(\mathcal{C}_0\cup\mathcal{C}_{\alpha;1})$$
$$\lim_{M\to+\infty}\partial_X V|_M = \begin{pmatrix}0\\1\end{pmatrix}$$
$$\left[\sigma^\flat\partial_n V\right]_{\mathcal{C}_{\alpha;1}} = (\sigma_m-\sigma_0)M_\varepsilon n_{\mathcal{C}_0},\quad \left[\sigma^\flat\partial_n V\right]_{\mathcal{C}_0} = 0.$$

Take $v = V$ in (14)*, use the expression of D^2 given by* (26) *and let M tend to infinity to infer*

$$-\sigma_0{}^\alpha\mathrm{a}_Y + D^2_X = \varepsilon^{1-\beta}\left(\int_0^1\left(V_X(\varepsilon^{1-\beta}f(Y/\varepsilon^{\alpha-\beta}),Y) - V_X(0,Y)\right)\mathrm{d}Y - \int_0^1 V_Y(\varepsilon^{1-\beta}f(Y/\varepsilon^{\alpha-\beta}),Y)\,\mathrm{d}Y\right).$$

But similar calculations lead to

$$-\int_{\mathcal{C}_{\alpha;1}}{}^\alpha\mathrm{A}_Y\left[\sigma^\flat\partial_n{}^\alpha\mathrm{A}_X\right]_{\mathcal{C}_{1;\beta}} - \int_{\mathcal{C}_0}{}^\alpha\mathrm{A}_Y\left[\sigma^\flat\partial_n{}^\alpha\mathrm{A}_X\right]_{\mathcal{C}_0} = \varepsilon^{1-\beta}\int_0^1{}^\alpha\mathrm{A}_X(\varepsilon^{1-\beta}f(Y/\varepsilon^{\alpha-\beta}),Y)\,\mathrm{d}Y,$$
$$= \varepsilon^{1-\beta}\int_0^1 V_Y(\varepsilon^{1-\beta}f(Y/\varepsilon^{\alpha-\beta}),Y)\,\mathrm{d}Y,$$

and on the other hand using the jumps (11b)–(11c) *we infer*

$$-\int_{\mathcal{C}_{\alpha;1}} {}^{\alpha}\mathrm{A}_Y \left[\sigma^\flat \partial_n {}^{\alpha}\mathrm{A}_X\right]_{\mathcal{C}_{\alpha;1}} - \int_{\mathcal{C}_0} {}^{\alpha}\mathrm{A}_Y \left[\sigma^\flat \partial_n {}^{\alpha}\mathrm{A}_X\right]_{\mathcal{C}_0} = -\varepsilon^{1-\beta}\int_0^1 \Big({}^{\alpha}\mathrm{A}_Y(\varepsilon^{1-\beta} f(Y/\varepsilon^{\alpha-\beta}), Y) - {}^{\alpha}\mathrm{A}_Y(0,Y)\Big)\,\mathrm{d}Y,$$
$$= \varepsilon^{1-\beta}\int_0^1 \Big(V_X(\varepsilon^{1-\beta} f(Y/\varepsilon^{\alpha-\beta}), Y) - V_X(0,Y)\Big)\,\mathrm{d}Y,$$

hence

$$D_X^2 = \sigma_0 {}^{\alpha}\mathrm{a}_Y. \tag{28}$$

2.2 First order approximation

It is then possible to obtain the first order coefficient of the expansion. Define v^1 by

$$\Delta_{\eta,\theta} v^1 = 0, \text{ in } \mathcal{O}^1 \cup \mathcal{O}^0,$$
$$(1 + d_0\kappa)\partial_\eta v^1|_{\eta=d_0} + \Lambda_0 v^1|_{\eta=d_0} = 0,$$
$$(1 - d_0\kappa)\partial_\eta v^1|_{\eta=-d_0} - \Lambda_1 v^1|_{\eta=-d_0} = 0,$$

with the following transmission conditions:

$$\sigma_0\partial_\eta v^1|_{\eta=0^+} - \sigma_1\partial_\eta v^1|_{\eta=0^-} = (\sigma_m - \sigma_0) D^2 \cdot \partial_\theta \nabla_{\eta,\theta} v|^0_{\eta=0^+},$$
$$v^1|_{\eta=0^+} - v^1|_{\eta=0^-} = (\sigma_m - \sigma_0){}^{\alpha}\mathrm{a} \cdot \left(\nabla_{\eta,\theta} v^0\right)|_{\eta=0^+}.$$

Remark 2.4 *Observe that the term* ${}^{\alpha}\mathrm{a}\cdot\left(\nabla_{\eta,\theta} v^0\right)|_{\eta=0^+}$ *is natural, since it comes from the jump of* v^0_{BL} *across the curve* γ_0*. However the introduction of the slow variable term* D^2*, which is obtained by integrating a fast-variable term can be seen as artificial. Actually the jump* (21) *only involves fast variable terms and therefore we could have thought that they would be corrected in the next boundary layer correctors.*

We emphasize that it is necessary to introduce D^2 *so that the next boundary layer correctors can be well-defined. Without this term, the next boundary layer correctors would have satisfied ill-posed partial differential equations.*

Denote by w^1 the following quantity:

$$w^1 = W^0 - \varepsilon^\beta v^1.$$

Since we have

$$\partial_n^\Phi v^1|_{\gamma_\varepsilon} = n_{\mathcal{C}_1} \cdot \nabla_{\eta,\theta} v^1|_{\eta=0^+} + O(\varepsilon),$$

and since $\partial_\eta v^1|_{\eta=0^+} = n_{\mathcal{C}_0}.\nabla_{\eta,\theta} v^1|_{\eta=0^+}$ then w^1 satisfies:

$$\frac{1}{\sigma_m - \sigma_0}\Delta_{\eta,\theta} w^1 = \kappa(\theta) G_1(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta)\cdot\nabla_{\eta,\theta} v^0|_{\eta=0^+} + G_2(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta)\cdot\partial_\theta\nabla_{\eta,\theta} v^0|_{\eta=0^+} + \tilde{O}(\varepsilon),$$
$$(1 + d_0\kappa)\partial_\eta w^1|_{\eta=d_0} + \Lambda_0 w^1|_{\eta=d_0} = g^\varepsilon_{0+},$$
$$(1 - d_0\kappa)\partial_\eta w^1|_{\eta=-d_0} - \Lambda_1 w^1|_{\eta=-d_0} = g^\varepsilon_{0-},$$

with the following transmission conditions:

$$\frac{1}{\sigma_m - \sigma_0}\left[\sigma \partial_n^{\Phi} w^1\right]_{\gamma_\varepsilon} = \varepsilon^\beta \left(\kappa(\theta) B_1(\theta/\varepsilon^\beta).\nabla_{\eta,\theta} v^0|_{\eta=0^+} + B_2(\theta/\varepsilon^\beta).\partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+}\right)$$
$$- \varepsilon^\beta n_{\mathcal{C}_{\alpha;1}}.\nabla_{\eta,\theta} v^1|_{\eta=0^+} + O(\varepsilon^2),$$
$$\frac{1}{\sigma_m - \sigma_0}\left[\sigma \partial_\eta w^1\right]_{\eta=0} = -\varepsilon^\beta D^2.\partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + \varepsilon^\beta n_{\mathcal{C}_0}.\nabla_{\eta,\theta} v^1|_{\eta=0^+},$$

and

$$[w^1]|_{\gamma_\varepsilon} = 0, \quad [w^1]|_{\eta=0} = 0.$$

We now introduce two boundary layer correctors at the order 1 defined in $\mathbb{R}\times\mathbb{T}$. For $j = 1, 2$, the couple $\left({}^{\alpha}\mathrm{A}^{1,j}, {}^{\alpha}\mathrm{a}^{1,j}\right)$ with ${}^{\alpha}\mathrm{A}^{1,j}$ continuous, satisfies

$$\Delta {}^{\alpha}\mathrm{A}^{1,j} = G^j, \text{ in } \mathbb{R}\times\mathbb{T} \setminus (\mathcal{C}_0 \cup \mathcal{C}_1), \tag{29a}$$
$$\left[\sigma^\flat \partial_n {}^{\alpha}\mathrm{A}^{1,j}\right]_{\mathcal{C}_{\alpha;1}} = B^j, \tag{29b}$$
$$\left[\sigma^\flat \partial_n {}^{\alpha}\mathrm{A}^{1,j}\right]_{\mathcal{C}_0} = -D^j, \tag{29c}$$
$${}^{\alpha}\mathrm{A}^{1,j} \to_{X\to-\infty} 0,\ {}^{\alpha}\mathrm{A}^{1,j} \to_{X\to+\infty} {}^{\alpha}\mathrm{a}^{1,j}. \tag{29d}$$

According to the definition of $(D^j)_{j=1,2}$, the compatibility condition that ensures both existence and uniqueness of the solution to Problem (29),

$$\int_{\mathbb{R}\times\mathbb{T}} \sigma^\flat G^j \,\mathrm{d}X\,\mathrm{d}Y + \int_{\mathcal{C}_{\alpha;1}} B^j \,\mathrm{d}s - \int_{\mathcal{C}_0} D^j \,\mathrm{d}s = 0,$$

is satisfied. Define the boundary-layer corrector of order 1 on $\mathcal{O}$ by

$$\forall \eta > 0, \quad v^1_{BL}(\eta,\theta) = (\sigma_m - \sigma_0)\varepsilon^{2\beta}\Bigg(\kappa(\theta)\left[{}^{\alpha}\mathrm{A}^{1,1}(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta) - {}^{\alpha}\mathrm{a}^{1,1}\right]\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+}$$
$$+ \left[{}^{\alpha}\mathrm{A}^{1,2}(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta) - {}^{\alpha}\mathrm{a}^{1,2}\right]\cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+}$$
$$+ \left[{}^{\alpha}\mathrm{A}(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta) - {}^{\alpha}\mathrm{a}\right]\cdot \nabla_{\eta,\theta} v^1|_{\eta=0^+}\Bigg),$$

$$\forall \eta < 0, \quad v^1_{BL}(\eta,\theta) = (\sigma_m - \sigma_0)\varepsilon^{2\beta}\Bigg(\kappa(\theta){}^{\alpha}\mathrm{A}^{1,1}(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta)\cdot \nabla_{\eta,\theta} v^0|_{\eta=0^+}$$
$$+ {}^{\alpha}\mathrm{A}^{1,2}(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta)\cdot \partial_\theta \nabla_{\eta,\theta} v^0|_{\eta=0^+} + {}^{\alpha}\mathrm{A}(\eta/\varepsilon^\beta, \theta/\varepsilon^\beta)\cdot \nabla_{\eta,\theta} v^1|_{\eta=0^+}\Bigg).$$

3 Error estimates for $\alpha \in (0,2)$

In the previous section we have formally derived the first order of the asymptotic expansion of v^ε, by defining appropriate boundary layer correctors. The aim of the section is to prove that this formal construction is indeed an approximation of v^ε at the first order.

Theorem 3.1 *Let $\alpha \in (0,2)$. Suppose the boundary data g is smooth*[1]*. Define W^1 by*

$$W^1 = v^\varepsilon - \left(v^0 + v^0_{BL} + \varepsilon^\beta(v^1 + v^1_{BL})\right).$$

Then W^1 is H^1–regular in each subdomain $\mathcal{O}_0$ and $\mathcal{O}_1$, and it satisfies

$$\|W^1\|_{H^1(\mathcal{O}_0)} + \|W^1\|_{H^1(\mathcal{O}_1)} = o(\varepsilon).$$

Remark 3.2 *The above theorem ensures that $v^0 + v^0_{BL} + \varepsilon^\beta(v^1 + v^1_{BL})$ gives an approximation of v^ε at the order $o(\varepsilon)$ in the whole domain Ω. In particular we have provided an accurate description of v^ε in a neighborhood of the rough thin layer, which was not obtained by the analysis derived by Vogelius et al.*

For $\alpha \leq 1$, more accurate estimates hold:

$$\|W^1\|_{H^1(\mathcal{O}_0)} + \|W^1\|_{H^1(\mathcal{O}_1)} = O(\varepsilon^2).$$

Proof 3 *According to the derivation of the first order terms, the function W^1 satisfies the following boundary value problem:*

$$\begin{aligned}
&\Delta_{\eta,\theta} W^1 = F_\varepsilon, \text{ in } \mathcal{O}^1 \cup \mathcal{O}^m_\varepsilon \cup \mathcal{O}^0_\varepsilon,\\
&(1 + d_0\kappa)\partial_\eta W^1|_{\eta=d_0} + \Lambda_0 W^1|_{\eta=d_0} = g^\varepsilon_{1+},\\
&(1 - d_0\kappa)\partial_\eta W^1|_{\eta=-d_0} - \Lambda_1 W^1|_{\eta=-d_0} = g^\varepsilon_{1-},
\end{aligned}$$

with the following transmission conditions:

$$\begin{aligned}
&\sigma_0 \partial^\Phi_n W^1|_{\gamma^+_\varepsilon} = \sigma_m \partial^\Phi_n W^1|_{\gamma^-_\varepsilon} + \varepsilon^{1+\beta} R^\varepsilon_1,\\
&\sigma_m \partial_\eta W^1|_{\eta=0^+} = \sigma_1 \partial_\eta W^1|_{\eta=0^-} + \varepsilon^{1+\beta} R^\varepsilon_2,
\end{aligned}$$

and

$$\begin{aligned}
&W^1|_{\gamma^+_\varepsilon} = W^1|_{\gamma^-_\varepsilon},\\
&W^1|_{\eta=0^+} = W^1|_{\eta=0^-} + \varepsilon^{1+\beta} R^\varepsilon_3.
\end{aligned}$$

Let $\mathfrak{L}(\gamma_\varepsilon)$ be the arc length of γ_ε, which equals:

$$\mathfrak{L}(\gamma_\varepsilon) = \sqrt{1 + \varepsilon^{1-\alpha} f'(\theta/\varepsilon^\alpha)}.$$

For $\alpha \leq 1$, since $\mathfrak{L}(\gamma_\varepsilon)$ is bounded with respect to ε, therefore for any smooth functions φ defined in $\mathcal{O}$, the following equality holds

$$\|\varphi|_{\gamma_\varepsilon}\|_{L^2(\gamma_\varepsilon)} = \int_0^1 |\varphi(\varepsilon f(\theta/\varepsilon^\alpha, \theta))|^2 \, \mathfrak{L}(\gamma_\varepsilon) \, \mathrm{d}\theta,$$

ensuring the boundedness of $\|\varphi|_{\gamma_\varepsilon}\|_{L^2(\gamma_\varepsilon)}$ by $\|\varphi\|_{H^1(\mathcal{O})}$ uniformly with respect to ε. Observe that according to estimates (12), the source terms R^ε_j, forj $= 1, 2$ are of order $\varepsilon^{1-\beta}$, hence following the proof of Theorem 4.1 of Ciuperca et al., we infer that

$$\|W^1\|_{H^1(\mathcal{O}_0)} + \|W^1\|_{H^1(\mathcal{O}_1)} = O(\varepsilon^2).$$

[1] This hypothesis can be weakened to $g \in H^s(\partial\Omega)$ for an index s large enough.

For $\alpha \in (1,2)$ the arc length $\mathfrak{L}(\gamma_\varepsilon)$ blows up like $\varepsilon^{1-\alpha}$ hence

$$\|\varphi_{|\gamma_\varepsilon}\|_{L^2(\gamma_\varepsilon)} \leq C\varepsilon^{1-\alpha}\|\varphi\|_{H^1(\mathcal{O})}.$$

Moreover since β equals 1 in this case, estimates (12) *do not provide any gain in power of ε, hence*

$$\|W^1\|_{H^1(\mathcal{O}_0)} + \|W^1\|_{H^1(\mathcal{O}_1)} = O(\varepsilon^{2+\beta-\alpha}) = o(\varepsilon),$$

since $\alpha < 2$.

Remark 3.3 *As soon as $\alpha \geq 2$, the estimate dramatically crashes since the arc length $\mathfrak{L}(\gamma_\varepsilon)$ cannot be compensated by the first order term of the asymptotic. It is therefore necessary to derive higher order terms in the expansion of v^ε in order to obtain appropriate results. This is quite technical, but we are confident that the reader has all the tools to push-forward the expansion at the desired order.*

3.1 Generalized Polarization Tensor

Generalized polarization tensor as described by Vogelius, Ammari *et al.* has a wide-range of application, in particular in the inverse problem research area (see Ammari *et al.*[4, 6] and references therein). In this section we provide an explicit characterization of the polarization tensor using the first order boundary layer corrector. Denote by $G(x,y)$ the Dirichlet solution for the Laplace operator defined in [5, 6] pp33 by

$$\begin{cases} \nabla_x \cdot \Big(\sigma \nabla_x G(x,y)\Big) = -\delta_y, \text{in } \Omega \\ G(x,y) = 0, \quad \forall x \in \partial\Omega. \end{cases}$$

Let K be a tubular neighborhood of $\partial\mathcal{O}^1$ and let ε_0 be such that for any $\varepsilon \in (0,\varepsilon)$, $\mathcal{O}^m_\varepsilon \subset K$. According to the definition of v^1 and to theorem 3.1, the following equality holds almost everywhere far from the layer:

$$v^\varepsilon(y) - v^0(y) = \varepsilon \int_\Gamma (\sigma_m - \sigma_0)\mathcal{M}_\alpha \begin{pmatrix} \partial_n u \\ \nabla_\Gamma u \end{pmatrix} \cdot \begin{pmatrix} \partial_n G \\ \nabla_\Gamma G \end{pmatrix} (s,y)\, \mathrm{d}_\Gamma(s) + o(\varepsilon), \quad \text{a.e. } \mathcal{O} \setminus K$$

where $\mathcal{M}_\alpha$ is the polarization tensor defined by

$$\mathcal{M}_\alpha = \varepsilon^{\beta-1} \begin{pmatrix} \sigma_0 \mathrm{a}^\beta_X & \sigma_0 \mathrm{a}^\beta_Y \\ D^2_X & D^2_Y \end{pmatrix}. \tag{30}$$

According to estimates (12) and using the definition (26) of D^2, the polarization tensor is of order $O(1)$. Property 2.3 ensures the symmetry of M, as proved in [10]. Therefore, using the boundary layer corrector $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ leads to define explicitely the polarization tensor for rough thin layer, for any roughness parameter $\alpha \in (0,2)$.

In addition, an accurate approximation at the order $o(\varepsilon)$ is obtained in a neighborhood of the rough thin layer, whereas the variational techniques used by Vogelius *et al.* provide estimates that are valid far from the layer.

Therefore we have provided a link between boundary layer correctors and polarization tensor for rough thin layer, for a roughness parameter $\alpha \in (0,2)$. Observe that as soon as $\alpha < 1$, the corrector $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ depends on $\varepsilon^{1-\alpha}$, while for $\alpha > 1$ $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ depends on $\varepsilon^{\alpha-1}$: in the next section we explicitly characterize the leading term of $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ for $\alpha \neq 1$.

4 Leading term of the corrector $({}^{\alpha}\mathrm{A}, {}^{\alpha}\mathrm{a})$ for $\beta \neq 1$

In the previous section we have derived the first order boundary layer correctors $({}^{\alpha}\mathrm{A}, {}^{\alpha}\mathrm{a})$, involving the parameter β, which is nothing but the minimum between α and 1. For $\alpha = 1$ the problem satsified by $({}^{\alpha}\mathrm{A}, {}^{\alpha}\mathrm{a})$ does not involve any small parameter, but as soon as $\alpha \neq 1$ it does. The aim of this section is to provide the leading term of $({}^{\alpha}\mathrm{A}, {}^{\alpha}\mathrm{a})$ in the two different cases $\alpha < 1$ and $\alpha > 1$.

4.1 The weakly oscillating case: $\alpha \in (0, 1)$

According to property 2.1, ${}^{\alpha}\mathrm{A}$ is bounded by $\varepsilon^{1-\beta}$. Therefore for $\alpha < 1$ it is possible to approximate $({}^{\alpha}\mathrm{A}, {}^{\alpha}\mathrm{a})$ by its leading term. This case is quite *simple*, since it deals with a smooth thin layer of non-constant thickness and has been previously described in [16]. More precisely the geometry is described by figure 4

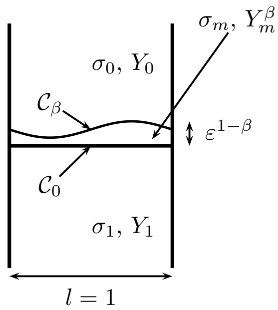

Figure 4: Geometry of the strip

Performing the rescaling $\Xi = X/\varepsilon^{1-\beta}$ in Y_m^β, and denoting by ${}^{\alpha}V$ the profile:

$$\forall (\Xi, Y) \in \{Y \in \mathbb{T},\, 0 < \Xi < f(Y)\}, \quad {}^{\alpha}V(\Xi, Y) = {}^{\alpha}\mathrm{A}(\varepsilon^{1-\beta}\Xi, Y)$$

then we infer

$$\begin{aligned}
&\Delta_{X,Y}{}^{\alpha}\mathrm{A} = 0, \text{ in } \{X < 0\} \cup \{X > \varepsilon^{1-\beta} f(Y)\},\\
&\partial_\Xi^2 {}^{\alpha}V + \left(\varepsilon^{1-\beta}\right)^2 \partial_Y {}^{\alpha}V = 0, \text{ in } \{Y \in \mathbb{T},\, 0 < \Xi < f(Y)\},\\
&\sigma_0 \nabla_{X,Y} {}^{\alpha}\mathrm{A}|_{\varepsilon^{1-\beta} f(Y), Y} \cdot n_{\mathcal{C}_{\alpha;1}} - \sigma_m \begin{pmatrix} \frac{1}{\varepsilon^{1-\beta}} \partial_\Xi {}^{\alpha}V \\ \partial_Y {}^{\alpha}V \end{pmatrix} |_{\Xi = f(Y)} \cdot n_{\mathcal{C}_{\alpha;1}} = n_{\mathcal{C}_{\alpha;1}},\\
&\sigma_1 \partial_X {}^{\alpha}\mathrm{A}|_{0,Y} - \sigma_m \frac{1}{\varepsilon^{1-\beta}} \partial_\Xi {}^{\alpha}V|_{(0,Y)} = -n_{\mathcal{C}_0},\\
&{}^{\alpha}\mathrm{A}|_{0,Y} = {}^{\alpha}V|_{(0,Y)}, \quad {}^{\alpha}\mathrm{A}|_{\varepsilon^{1-\beta} f(Y), Y} = {}^{\alpha}V|_{f(Y), Y}\\
&{}^{\alpha}\mathrm{A} \to_{X \to -\infty} 0,\ {}^{\alpha}\mathrm{A} \to_{X \to +\infty} {}^{\alpha}\mathrm{a}.
\end{aligned}$$

Then assuming the ansatz:

$$\begin{aligned}
{}^{\alpha}\mathrm{A} &= \sum_{n \geq 0} \left(\varepsilon^{1-\beta}\right)^n A_n,\\
{}^{\alpha}V &= \sum_{n \geq 0} \left(\varepsilon^{1-\beta}\right)^n V_n,
\end{aligned}$$

and identifying the term with the same power in $\varepsilon^{1-\beta}$ we infer that A_0 identically equals 0, and that A_1 equals:

$$A_1 = \begin{cases} 0, & \text{if } X < 0, \\ \dfrac{1}{\sigma_m}\begin{pmatrix} f \\ 0 \end{pmatrix}, & \text{if } X > 0, \end{cases}$$

from which we infer that

$$^{\alpha}\mathrm{a} = \varepsilon^{1-\beta}\frac{1}{\sigma_m}\begin{pmatrix} \int_{\mathbb{T}} f(Y)\,\mathrm{d}Y \\ 0 \end{pmatrix} + O(\varepsilon^{1-\beta}),$$

and

$$D^2 = \varepsilon^{1-\beta}\begin{pmatrix} 0 \\ \int_{\mathbb{T}} f(Y)\,\mathrm{d}Y \end{pmatrix} + O(\varepsilon^{1-\beta}),$$

and therefore setting $\tilde{v}^1 = v^1/\varepsilon^{1-\beta}$, we infer

$$\begin{aligned} &\Delta_{\eta,\theta}\tilde{v}^1 = 0, \text{ in } \mathcal{O}^1 \cup \mathcal{O}^0, \\ &(1 + d_0\kappa)\partial_\eta \tilde{v}^1|_{\eta=d_0} + \Lambda_0 \tilde{v}^1|_{\eta=d_0} = 0, \\ &(1 - d_0\kappa)\partial_\eta \tilde{v}^1|_{\eta=-d_0} - \Lambda_1 \tilde{v}^1|_{\eta=-d_0} = 0, \end{aligned}$$

with the following transmission conditions:

$$\begin{aligned} &\sigma_0\partial_\eta\tilde{v}^1|_{\eta=0^+} - \sigma_1\partial_\eta\tilde{v}^1|_{\eta=0^-} = (\sigma_m - \sigma_0)\overline{f}\partial_\theta^2 v|_{\eta=0^+}^0, \\ &\tilde{v}^1|_{\eta=0^+} - \tilde{v}^1|_{\eta=0^-} = \frac{\sigma_m - \sigma_0}{\sigma_m}\overline{f}\partial_\eta v^0|_{\eta=0^+}. \end{aligned}$$

where

$$\overline{f} = \int_{\mathbb{T}} f(Y)\,\mathrm{d}Y,$$

and according to theorem 3.1, we have shown that far from the layer:

$$v^\varepsilon - v^0 = \varepsilon\tilde{v}^1 + o(\varepsilon),$$

which means that the average of the roughness provided a first order approximation of v^ε, as previously described in [16], and the polarization tensor writes:

$$\mathcal{M}_\alpha = \overline{f}\begin{pmatrix} \dfrac{\sigma_0}{\sigma_m} & 0 \\ 0 & 1 \end{pmatrix}, \tag{31}$$

as obtained by Beretta *et al.* [8, 9] in the case of constant thickness ($\alpha = 0$).

Observe that as $\beta \in (0, 1)$ goes to 1 the convergence rate of $\varepsilon^{1-\beta}A_1$ to $^{\alpha}\mathrm{A}$ decreases meaning that the mean-value of the roughness is no more sufficient to approach v^ε: the roughness effect becomes as important as the layer thickness, and tangential component of the zeroth order potential v^0 are necessary to define v^1.

4.2 The very oscillating case $\alpha > 1$: two-scale limit of the boundary layer

We now focus on the case $\alpha > 1$. Then $\varepsilon^{\alpha-1}$ goes to zero, and we denote by δ this small parameter. Denote by $\mathcal{C}_0$ and $\mathcal{C}_\delta$ the closed curves defined by

$$\mathcal{C}_0 = \{0\} \times \mathbb{T}, \quad \mathcal{C}_\delta = \{(f(Y/\delta), Y), \, \forall Y \in \mathbb{T}\}, \quad \text{with } \delta = \varepsilon^{\alpha-1}.$$

The exterior normal to $\mathcal{C}_\delta$ is denoted by $n_{\mathcal{C}_\delta}$ and $n_{\mathcal{C}_0}$ is the exterior normal to $\mathcal{C}_0$:

$$n_{\mathcal{C}_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad n_{\mathcal{C}_\delta} = \frac{1}{\sqrt{1 + (f'(Y/\delta)/\delta)^2}} \begin{pmatrix} 1 \\ -f'(Y/\delta)/\delta \end{pmatrix}.$$

Let $\sigma^\sharp$ be the function defined by

$$\forall (X,\tau) \in \mathbb{R} \times \mathbb{T}, \quad \sigma^\sharp(X,\tau) = \begin{cases} \sigma_0, & \text{in } \{(X,\tau), \quad \tau \in \mathbb{T}, \, X > f(\tau)\}, \\ \sigma_m, & \text{in } \{(X,\tau), \quad \tau \in \mathbb{T}, \, 0 < x < f(\tau)\}, \\ \sigma_1, & \text{in } \{(X,\tau), \quad \tau \in \mathbb{T}, \, X < 0\}. \end{cases}$$

Observe

$$\forall (X,Y) \in \mathbb{R} \times \mathbb{T}, \quad \sigma^\flat(X,Y) = \sigma^\sharp(X, Y/\delta),$$

and to highlight the dependence on δ we rewrite $\sigma^\flat$ as $\sigma^\flat_\delta$. Since we focus on

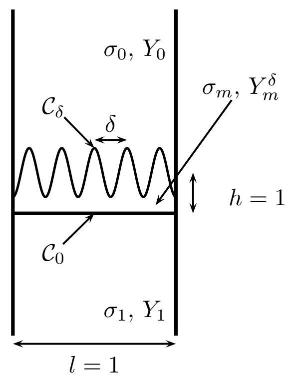

Figure 5: Rescaled strip

$\beta = 1$, we rewrite $({}^\alpha\mathrm{A}, {}^\alpha\mathrm{a})$ in $(\mathfrak{A}^\delta, \mathfrak{a}^\delta)$. We recall that $\mathfrak{A}^\delta$ is a continuous vector field and $\mathfrak{a}^\delta$ is a constant vector, which are given by the unique solution of the following problem

$$\Delta \mathfrak{A}^\delta = 0, \text{ in } \mathbb{R} \times \mathbb{T} \setminus (\mathcal{C}_0 \cup \mathcal{C}_\delta), \tag{32a}$$

$$\left[\sigma^\flat_\delta \partial_n \mathfrak{A}^\delta\right]_{\mathcal{C}_\delta} = n_{\mathcal{C}_\delta}, \tag{32b}$$

$$\left[\sigma^\flat_\delta \partial_n \mathfrak{A}^\delta\right]_{\mathcal{C}_0} = -n_{\mathcal{C}_0}, \tag{32c}$$

$$\mathfrak{A}^\delta \to_{x \to -\infty} 0, \, \mathfrak{A}^\delta \to_{x \to +\infty} \mathfrak{a}^\delta. \tag{32d}$$

Remark 4.1 *For all $\delta > 0$, the existence and the uniqueness of the couple $(\mathfrak{A}^\delta, \mathfrak{a}^\delta)$ is shown in property 2.1. We recall that the convergence rates are exponential as proved in [11]. We aim at deriving the limit corrector as δ goes to zero.*

Denote by $\mathscr{E}^\sharp$ be

$$\mathscr{E}^\sharp = \left\{ \begin{array}{l} \phi \in \left(H^1_{loc}(\mathbb{R}\times\mathbb{T}\times\mathbb{T})\right)^2 : \phi \text{ is } Y\text{–periodic and } \tau\text{–periodic,} \\ \displaystyle\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{T}} \|\nabla\phi(X,Y,\tau)\|^2 \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}\tau < +\infty; \quad \phi \to_{X\to-\infty} 0 \end{array} \right\},$$

Observe that the variational formulation of problem (32) is

$$\left\{ \begin{array}{l} \text{Find } \mathfrak{A}^\delta \in \mathscr{E} \text{ such that for all } \phi \in \mathscr{E} \\ \displaystyle\int_{\mathbb{R}\times\mathbb{T}} \sigma^\flat_\delta \nabla\mathfrak{A}^\delta\cdot\nabla\phi \,\mathrm{d}X\,\mathrm{d}Y = \int_0^1\int_0^{f(Y/\delta)} \nabla\cdot\phi(X,Y)\,\mathrm{d}X\,\mathrm{d}Y. \end{array} \right. \tag{33}$$

We remind that a straightforward application of Lax-Milgram theorem leads to existence and uniqueness of $\mathfrak{A}^\delta$, which were obtained previously in [11]. Moreover from the variational formulation (33) we derive straightforwardly the following estimate:

$$\int_{\mathbb{R}\times\mathbb{T}} \|\nabla\mathfrak{A}^\delta(X,Y)\|^2 \,\mathrm{d}X\,\mathrm{d}Y \le \frac{1}{\min(\sigma_0,\sigma_m,\sigma_1)}. \tag{34}$$

Therefore there exists a 2×2 matrix-function χ^0 defined on $\mathbb{R}\times\mathbb{T}\times\mathbb{T}$ such that $\chi^0(\cdot,\cdot,\tau)$ belongs to $\mathscr{E}$ for almost all $\tau\in\mathbb{T}$ and up to a subsequence, $\nabla\mathfrak{A}^\delta$ two-scale converges to χ^0.

Let (φ^0, φ^1) belong to $\mathscr{E}\times\mathscr{E}^\sharp$, and rewrite equality (33) with

$$\phi(X,Y) = \varphi^0(X,Y) + \delta\varphi^1(X,Y,Y/\delta),$$

and pass to the limit for δ tending to zero to obtain

$$\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{T}} \sigma^\sharp\chi^0\cdot\left(\nabla_{X,Y}\varphi^0 + \nabla_\tau\varphi^1\right)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}\tau = \int_{\mathbb{T}\times\mathbb{T}}\int_0^{f(\tau)}\left(\nabla_{X,Y}\cdot\varphi^0 + \partial_\tau\varphi^1_Y\right)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}\tau, \tag{35}$$

where for $\psi\in\mathscr{E}^\sharp$, $\nabla_\tau\psi$ denotes the following matrix

$$\nabla_\tau\psi = \begin{pmatrix} 0 & \partial_\tau\psi_X \\ 0 & \partial_\tau\psi_Y \end{pmatrix}. \tag{36}$$

Suppose that χ^0 writes

$$\forall (X,Y,\tau)\in\mathbb{R}\times\mathbb{T}\times\mathbb{T},\quad \chi^0(X,Y,\tau) = \nabla_{X,Y}\mathfrak{A}^0(X,Y) + \nabla_\tau\mathfrak{A}^1(X,Y,\tau).$$

4.2.a Problem satisfied by $\mathfrak{A}^1$

Denote by q the cumulative distribution function defined by

$$\forall X\in\mathbb{R},\quad q(X) = \int_{\mathbb{T}} \mathbb{1}_{\{0<X<f(\tau)\}}\,\mathrm{d}\tau.$$

From equality (35), by taking φ^0 identically equal to zero, we infer that for any $\psi \in \mathscr{E}^\sharp$

$$\int_{\mathbb{R}\times\mathbb{T}\times\mathbb{T}} \sigma^\sharp(X,\tau)\partial_\tau \mathfrak{A}^1 \cdot \partial_\tau \psi \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}\tau = \int_{\mathbb{R}\times\mathbb{T}\times\mathbb{T}} \begin{pmatrix} -\sigma^\sharp(X,\tau)\partial_Y \mathfrak{A}^0_X \\ -\sigma^\sharp(X,\tau)\partial_Y \mathfrak{A}^0_Y + \mathbb{1}_{\{0<X<f(\tau)\}} \end{pmatrix} \cdot \partial_\tau \psi \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}\tau,$$

hence the ordinary differential equation satisfied by $\mathfrak{A}^1$:

$$\partial_\tau \left(\sigma^\sharp \partial_\tau \mathfrak{A}^1\right) = \frac{\partial}{\partial\tau} \begin{pmatrix} -\sigma^\sharp \partial_Y \mathfrak{A}^0_X \\ -\sigma^\sharp \partial_Y \mathfrak{A}^0_Y + \mathbb{1}_{\{0<X<f(\tau)\}} \end{pmatrix}.$$

Since $\int_\mathbb{T} \partial_\tau \mathfrak{A}^1 \,\mathrm{d}\tau = 0$ we deduce

$$\sigma^\sharp \partial_\tau \mathfrak{A}^1 = \left(\frac{1}{\int_\mathbb{T} 1/\sigma^\sharp \,\mathrm{d}\tau} - \sigma^\sharp\right)\partial_Y \mathfrak{A}^0 + \begin{pmatrix} 0 \\ \mathbb{1}_{\{0<X<f(\tau)\}} - \dfrac{q(X)/\sigma_m}{\int_\mathbb{T} 1/\sigma^\sharp \,\mathrm{d}\tau} \end{pmatrix}. \quad (37)$$

Observe that

$$\forall X<0,\ \forall (Y,\tau)\in \mathbb{T}\times\mathbb{T}, \quad \sigma^\sharp \partial_\tau \mathfrak{A}^1(X,Y,\tau) = 0. \quad (38)$$

4.2.b Problem satisfied by $\mathfrak{A}^0$

From equality (35), by taking φ^1 independent on τ, we infer

$$\int_{\mathbb{R}\times\mathbb{T}} \left(\int_\mathbb{T} \sigma^\sharp(X,\tau)\left(\nabla_{X,Y}\mathfrak{A}^0(X,Y) + \nabla_\tau \mathfrak{A}^1(X,Y,\tau)\right)\,\mathrm{d}\tau\right)\cdot \nabla_{X,Y}\varphi^0 \,\mathrm{d}X\,\mathrm{d}Y = \int_{\mathbb{R}\times\mathbb{T}} q(X)\partial_X \varphi^0_X \,\mathrm{d}X\,\mathrm{d}Y,$$

hence and denoting by $\overline{\sigma}^\sharp$ the function

$$\forall X\in\mathbb{R}, \quad \overline{\sigma}^\sharp(X) = \int_\mathbb{T} \sigma^\sharp(X,\tau)\,\mathrm{d}\tau,$$

we infer that $\mathfrak{A}^0$ satisfies

$$\nabla\cdot\left(\overline{\sigma}^\sharp \nabla \mathfrak{A}^0\right) = \nabla_{X,Y}\, q - \nabla_{X,Y}\cdot\left(\int_\mathbb{T} \sigma^\sharp(.,\tau)\nabla_\tau \mathfrak{A}^1(.,.,\tau)\,\mathrm{d}\tau\right), \quad \text{in } \{X>0\}\times\mathbb{T},$$
$$\Delta \mathfrak{A}^0 = 0, \quad \text{in } \{X<0\}\times\mathbb{T},$$
$$\left[\overline{\sigma}^\sharp \partial_n \mathfrak{A}^0\right]_{\mathcal{C}_0} = n_{\mathcal{C}_0}$$

Therefore according to (36)–(38) we infer the strong formulation for $\mathfrak{A}^0$, which is continuous and satisfies:

$$\partial_X\left(\overline{\sigma}^\sharp \partial_X \mathfrak{A}^0\right) + \frac{1}{\int_\mathbb{T} 1/\sigma^\sharp \,\mathrm{d}\tau}\partial_Y^2 \mathfrak{A}^0 = \nabla q, \quad \text{in } \{X>0\}\times\mathbb{T}, \quad (39a)$$
$$\Delta \mathfrak{A}^0 = 0, \quad \text{in } \{X<0\}\times\mathbb{T}, \quad (39b)$$
$$\left[\overline{\sigma}^\sharp \partial_X \mathfrak{A}^0\right]_{\mathcal{C}_0} = n_{\mathcal{C}_0}, \quad (39c)$$
$$\mathfrak{A}^0 \to_{X\to-\infty} 0, \quad \mathfrak{A}^0 \to_{X\to+\infty} \mathfrak{a}^0 \quad (39d)$$

Remark 4.2 *We remind that*

$$n_{\mathcal{C}_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \quad \left[\overline{\sigma}^\sharp \partial_n \mathfrak{A}^0\right]_{\mathcal{C}_0} = \sigma_m \partial_X \mathfrak{A}^0|_{X=0^+} - \sigma_1 \partial_X \mathfrak{A}^0|_{X=0^-}.$$

We emphasize that since $q(0) = 1$ and $q(M) = 0$ for all $M > \|f\|_\infty$, the compatibility condition to ensure existence and uniqueness of $\mathfrak{A}^0$ is satisfied (see [11] Lemma 2.2).

Equation (39a) *can be rewritten as*

$$\nabla. \left(\begin{pmatrix} \overline{\sigma}^\sharp & 0 \\ 0 & 1/(\int_{\mathbb{T}} \sigma^\sharp \, \mathrm{d}\tau) \end{pmatrix} \nabla \mathfrak{A}^0 \right) = \nabla q,$$

and this has to be related with the well-known homogenization formula for laminate structures: in the laminate alignment (here the X-direction), the simple mean of the conductivities appears, while in the transverse direction (the Y-direction in our case) the harmonic mean holds.

4.2.c Computation of the coefficients $\mathfrak{a}^0$ and D^2.

Observe that the second component of $\mathfrak{a}^0$ is necessarily equal to 0, since all the source terms vanish according to (39a). To obtain the value of the first component $\mathfrak{a}^0_X$, multiply (39a) by a test function $v \in H^2([-M, M] \times \mathbb{T})$, which is independent on Y and integrate by parts two times to obtain

$$\sigma_0 \int_0^1 \mathfrak{A}^0_X(M, Y) \partial_X v|_M \, \mathrm{d}Y - \sigma_1 \int_0^1 \mathfrak{A}^0_X(-M, Y) \partial_X v|_{-M} \, \mathrm{d}Y - \sigma_0 \int_0^1 \partial_X \mathfrak{A}^0_X(M, y) v|_M \, \mathrm{d}Y$$
$$+ \sigma_1 \int_0^1 \partial_X \mathfrak{A}^0_X(-M, y) v|_{-M} \, \mathrm{d}Y - \int_{(-M,0)\times\mathbb{T}} \sigma_1 \mathfrak{A}^0_X \cdot \partial_X^2 v \, \mathrm{d}X \, \mathrm{d}Y - \int_{(0,M)\times\mathbb{T}} \mathfrak{A}^0_X \cdot \partial_X \left(\overline{\sigma}^\sharp \partial_X v\right) \, \mathrm{d}X \, \mathrm{d}Y$$
$$= - \int_0^{+\infty} \int_{\mathbb{T}} q(X) \partial_X v(X) \, \mathrm{d}X.$$

Now choose v as follows:

$$v(X) = \begin{cases} \displaystyle\int_0^X 1/\overline{\sigma}^\sharp(s) \, \mathrm{d}s, & \text{if } X > 0, \\ X/\sigma_1, & \text{if } X < 0, \end{cases}$$

to infer the value of $\mathfrak{a}^0_X$:

$$\mathfrak{a}^0_X = - \int_0^{+\infty} \frac{q(s)}{\overline{\sigma}^\sharp(s)} \, \mathrm{d}s. \tag{40}$$

According to property 2.3, and since $\mathfrak{a}^0_Y = 0$, we infer that $D^2_X = 0$. Moreover note that $\mathfrak{A}^0_Y$ is identically equal to 0 in the whole strip band $\mathbb{R}\times\mathbb{T}$ therefore using (37):

$$\sigma^\sharp \partial_\tau \mathfrak{A}^1_Y = \mathbb{1}_{\{0<X<f(\tau)\}} - \frac{q(X)/\sigma_m}{\int_{\mathbb{T}} 1/\sigma^\sharp \, \mathrm{d}\tau}.$$

Then passing to the two-scale limit in (25), we infer that D^2_Y converges to D_∞ equal

$$D_\infty = \int_0^{+\infty} (\sigma_m - \sigma_0) \frac{q(X)(1 - q(X))}{\sigma_0 q(X) + \sigma_m (1 - q(X))} \, \mathrm{d}X - \int_{\mathbb{T}} f(Y) \, \mathrm{d}Y. \tag{41}$$

The polarization tensor is then equal to

$$\mathcal{M}_\alpha = \begin{pmatrix} -\int_0^{+\infty} q(s)(\sigma_0/\overline{\sigma}^\sharp(s))\,\mathrm{d}s & 0 \\ 0 & D_\infty \end{pmatrix} \tag{42}$$

Remark 4.3 *Note that since*

$$\int_0^{+\infty} q(s)\,\mathrm{d}s = \int_{\mathbb{T}} f(Y)\,\mathrm{d}Y$$

then

$$\sigma_0 \mathfrak{a}_X^0 = (\sigma_0 - \sigma_m)\int_0^{+\infty} \frac{q^2(s)}{\overline{\sigma}^\sharp(s)}\,\mathrm{d}s - \int_{\mathbb{T}} f(Y)\,\mathrm{d}Y,$$

hence the formula of $\sigma_0 \mathfrak{a}_X^0$ and the coefficient $(\sigma_m - \sigma_0)r_1 - \tilde{f}$ of the polarization tensor given by Ciuperca et al. (Theorem 2.3, pp 6 [13]) coincide.

5 Conclusion

In this paper we have derived boundary layer correctors for the conductivity problem involving rough thin layers. We have provided a general framework that allow to treat simultaneaously soft, rough and very rough thin layer, and we have shown error estimates that are valid in the whole domain. More particularly our results lead to accurate description of the potential in the vicinity of the rougness. We also have explicitly described the generalized polarization tensor $\mathcal{M}_\alpha$. In particular, for $\alpha < 1$ the polarization tensor is a diagonal matrix given by (31). For $\alpha = 1$ the polarization tensor is a plain 2×2 matrix while for $\alpha > 1$ it is diagonal again, but the matrix coefficients involve a mixture of the conductivities (see (42)). Numerical results have been published in [12].

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