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Γ-convergence of discrete functionals with non convex perturbation for image classification

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Abstract: The purpose of this report is to show the theoretical soundness of a variational method proposed in image processing for supervised classification. Based on works developed for phase transitions in fluid mechanics, the classification is obtained by minimizing a sequence of functionals. The method provides an image composed of homogeneous regions with regular boundaries, a region being defined as a set of pixels belonging to the same class. In this paper, we show the Γ-convergence of the sequence of functionals which differ from the ones proposed in fluid mechanics in the sense that the perturbation term is not quadratic but has a finite asymptote at infinity, corresponding to an edge preserving regularization term in image processing.

Key-words: Γ-convergence, image classification, edge-preserving regularization.

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Γ-convergence d'une suite de fonctionnelles discrètes avec une r'gularisation non quadratique pour la classification d'image

Résumé : Ce rapport contient la justification mathématique du modèle variationnel proposé en traitement d'image pour la classification supervisée. A partir des travaux effectués en mécanique des fluides pour les transitions de phase, nous avons développé un modèle de classification par minimisation d'une suite de fonctionnelles. Le résultat est une image de classes formée de régions homogènes séparées par des contours réguliers. Ce modèle diffère de ceux utilisés en mécanique des fluides car la perturbation utilisée n'est pas quadratique mais correspond à une fonction de régularisation d'image préservant les contours. La Γ-convergence de cette nouvelle suite de fonctionnelles est prouvée.

Mots-clés : Γ-convergence, classification d'images, régularisation non quadratique.

1 Introduction

Image classification consists of assigning a label to each site of an image to produce a partition of the image into homogeneous labelled areas. The classification problem concerns many applications as, for instance, land use management in remote sensing.

Based on results conducted in the Van der Waals-Cahn-Hilliard theory framework for phase transitions in fluid mechanics [2, 4, 11, 14, 16], we have recently proposed a sequence of functionals for image classification [15]. The soundness of such a method relies upon Γ-convergence theory. The purpose of this paper is to prove the Γ-convergence of the sequence of functionals we use, which differs from the one used in fluid mechanics in the sense that the perturbation term is not quadratic, but it is an edge preserving regularization term as defined in image processing.

Let Ω be an open bounded subset of $\mathbf{R}^2$, $I : \Omega \to \mathbf{R}$ the observed data to classify, $I \in L^\infty(\Omega)$. A class is characterized by parameters of the spatial distribution of intensity, i.e. the mean and standard deviation for Gaussian hypothesis. This work takes place in the general framework of supervised classification which means that the number n of classes and the parameters of the Gaussian distribution of the classes (a_i, σ_i) are a priori known. These values are either given by an expert or are pre-computed by using a fuzzy Cmeans algorithm with an entropy term (see [13] for instance). Knowing $(a_i, \sigma_i)\, i = 1, .., n$, the question is now to find a partition of Ω based on the observed image, where a component is the set of pixels in class i. We also add a regularity constraint on the partition. In order to assign a class i to each pixel x, we have proposed in [15] the sequence of functionals

$$F_\varepsilon(u) = \underbrace{\int_\Omega |u(x) - I(x)|^2\, dx}_{\text{data term}} + \varepsilon \underbrace{\int_\Omega \varphi(|\nabla u(x)|) dx}_{\text{restoration term}} + \frac{1}{\varepsilon} \underbrace{\int_\Omega W(u(x)) dx}_{\text{classification}}, \tag{1}$$

and the associated problem consists in finding u_0 such as:

$$u_0 = \lim_{\varepsilon \to 0^+} \Big[\arg\min_u F_\varepsilon(u) \Big]. \tag{2}$$

Let us first consider the functional with a fixed ε. The first two terms of (1) are standard for noisy image restoration by nonquadratic regularization [8, 3]. Function φ is a smoothing function that will be defined later.

The third term of (1) is a level constraint such that $W : \mathbf{R} \to \mathbf{R}^+$ attracts the values of $u(x)$ towards the mean a_i of class i, taking into account the standard deviation σ_i. W has n minima at a_i such that $W(a_i) = 0$, $\forall\, i = 1, .., n$. W is quadratic around each minima (from the Gaussian distribution hypothesis), i.e. around the a_i, $W(t) = (\frac{t - a_i}{\sigma_i})^2$, and is piecewise parabolic between the wells (see Figure 1).

Considering a sequence of energies F_ε when $\varepsilon \to 0$ is inspired from works conducted in the Van der Waals-Cahn-Hilliard theory framework for phase transitions in fluid mechanics

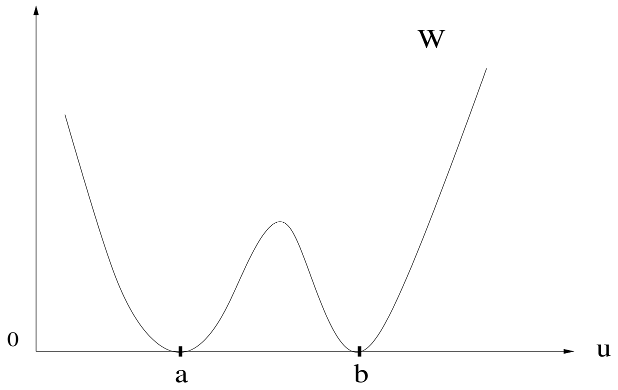


Figure 1: Example of double-well potential W, in the case of two classes with $a_1 = a, a_2 = b$.

[2, 4, 11, 14, 16] using Γ-convergence.

We recall the definition and some properties of Γ-convergence (see [10]). Let X be a metric space, and let $f_\varepsilon : X \to [0, +\infty]$ be a family of functions indexed by $\varepsilon > 0$. We say that f_ε Γ-converge as $\varepsilon \to 0^+$ to $f : X \to [0, +\infty]$ if the following two conditions

$$\forall x_\varepsilon \to x \quad \liminf_{\varepsilon \to 0^+} f_\varepsilon(x_\varepsilon) \geq f(x), \tag{3}$$

and

$$\exists x_\varepsilon \to x \quad \limsup_{\varepsilon \to 0^+} f_\varepsilon(x_\varepsilon) \leq f(x), \tag{4}$$

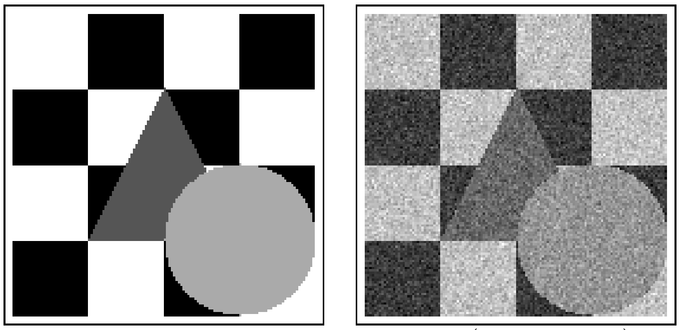
are fulfilled for every $x \in X$. The Γ-limit, if it exists, is unique and lower semicontinuous. The Γ-convergence is stable under continuous perturbations, that is, $(f_\varepsilon + v)$ Γ-converge to $(f + v)$ if f_ε Γ-converge to f and v is continuous. The most important property of Γ-convergence is the following: if $\{x_\varepsilon\}_\varepsilon$ is asymptotically minimizing, i.e.

$$\lim_{\varepsilon \to 0^+} \left(f_\varepsilon(x_\varepsilon) - \inf_X f_\varepsilon \right) = 0, \tag{5}$$

and if $\{x_{\varepsilon_h}\}_h$ converge to x for some sequence $\varepsilon_h \to 0$, then x minimizes f.

The minimization problem (2) relies upon Γ-convergence arguments. If $\varphi(t) = t^2$, then it can be shown from [4] that the sequence of functionals (1) Γ-converges to

$$F_0(u) = \begin{cases} \displaystyle\sum_{i=1}^{n} \int_{A_i} |a_i - I|^2 dx \quad + \sum_{i,l=1}^{n} \kappa_{i,l} \mathcal{H}^1(\partial^* A_i \cap \partial^* A_l \cap \Omega) \\ \qquad\qquad\qquad\qquad \text{if } u \in BV(\Omega; \{a_1, \ldots, a_n\}), \\ +\infty \qquad\qquad\qquad\quad \text{elsewhere in } L^2(\Omega). \end{cases} \tag{6}$$



original noisy (SNR=10 dB)

Figure 2: Synthetic “check” image.

where $BV(\Omega)$ is the space of functions of bounded variation [1], $\mathcal{H}^1$ is the one-dimensional Hausdorff measure, and $\partial^* A_i$ is the essential boundary of the subset A_i. For $u \in BV(\Omega; \{a_1, \ldots, a_n\})$, $A_i = \{x \in \Omega : u(x) = a_i\}$ for any $i = 1, \ldots, n$. Then the sets $A_1, \ldots, A_n$ define a partition of Ω into sets with finite perimeter. This partition is the classification result. The weight $\kappa_{i,l}$ is defined by

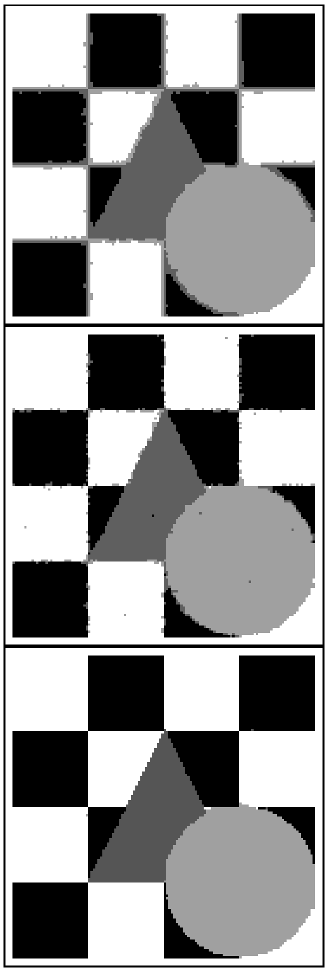
$$\kappa_{i,l} = \int_{a_i}^{a_l} \sqrt{W(t)}dt. \tag{7}$$

From Γ-convergence and compactness results, we know that the sequence of minimizers u_ε of $F_\varepsilon(u)$ converges (up to a subsequence) to a minimizer of F_0. So u_0 defines a partition of Ω according to the predefined classes, with minimal interfaces with respect to the weighted length (6), (7).

From the numerical point of view, when ε decreases, the functional turns from a restoration process (the third term in (1) is negligeable) into a classification process.

We do not use the quadratic function for φ but an edge-preserving regularizing function $\varphi(t) = \frac{t^2}{1+\mu t^2}$ because, numerically, it gives better results by preserving high gradients which represent edges [8]. This is illustrated on a synthetic image of size 128×128 pixels ("check" image), containing four classes.

The white Gaussian noise introduced is such that the Signal to Noise Ratio (SNR) given by $SNR = 10 \log_{10} \frac{non\ noisy\ signal\ variance}{noise\ variance}$ is 10 dB. Fig. 2 presents the synthetic image (non noisy and noisy). From the noisy one, we compute the classification as in (2) for different φ-functions (see [15] for the detailed algorithm). The results are presented in Fig. 3. For a Tikhonov regularization ($\varphi(t) = t^2$), edges are oversmoothed. With a convex φ, there are still many misclassified pixels on the boundaries. Best results are provided with the use of the nonconvex function $\varphi(t) = \frac{t^2}{1+\mu t^2}$, with $\mu > 0$.



$\varphi(t) = t^2$: convex (Tikhonov)

$\varphi(t) = \log(\cosh(t))$: convex (Green)

$\varphi(t) = \frac{t^2}{1+\mu t^2}$: nonconvex (Geman & McClure)

Figure 3: Classification of "check" image with different functions φ. Nonconvex functions provide better results than convex functions which lead to oversmooth results: we get damaged edges.

Before stating the result shown in this paper, let us show that the family of functionals F_ε in (1) does not Γ-converge to the limit F_0 given in (6) if $\varphi(t) = \frac{t^2}{1+\mu t^2}$ as it does when $\varphi(t) = t^2$. As the term $\int_\Omega |u(x) - I(x)|^2 \, dx$ is a continuous perturbation, it is sufficient to show that

$$E_\varepsilon^\mu(u) = \varepsilon \int_\Omega \frac{|\nabla u(x)|^2}{1+\mu|\nabla u(x)|^2} dx + \frac{1}{\varepsilon} \int_\Omega W(u(x)) dx \tag{8}$$

does not Γ-converge to E_0 defined by

$$E_0(u) = \begin{cases} \displaystyle\sum_{i,l=1}^{n} \kappa_{i,l} \mathcal{H}^1(\partial^* A_i \cap \partial^* A_l \cap \Omega) & \text{if } u \in BV(\Omega; \{a_1, \ldots, a_n\}), \\ \\ +\infty & \text{elsewhere in } L^2(\Omega). \end{cases}$$

Let $n = 2$ and $u_0 \in BV(\Omega; \{a_1, a_2\})$ with $a_1 < 0 < a_2$. Let A_ε be the tubular neighborhood of S_{u_0}, the set of jumps of u_0, defined by $A_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, S_{u_0}) < \lambda_\varepsilon\}$, with $\lambda_\varepsilon > 0$. Then, let $u_{\varepsilon,\mu}$ be defined as

$$u_{\varepsilon,\mu}(x) = \begin{cases} a_1 & \text{if } x \in \{u_0(x) = a_1\} \backslash A_\varepsilon \\ a_2 & \text{if } x \in \{u_0(x) = a_2\} \backslash A_\varepsilon \\ \frac{a_1}{\lambda_\varepsilon} \mathrm{dist}(x, S_{u_0}) & \text{if } x \in \{u_0(x) = a_1\} \cap A_\varepsilon \\ \frac{a_2}{\lambda_\varepsilon} \mathrm{dist}(x, S_{u_0}) & \text{if } x \in \{u_0(x) = a_2\} \cap A_\varepsilon. \end{cases}$$

The function $u_{\varepsilon,\mu}$ belongs to the Sobolev space $W^{1,2}(\Omega)$ so that the energy $E_\varepsilon^\mu(u_{\varepsilon,\mu})$ is finite for any ε.

Since $\frac{t^2}{1+\mu t^2} < \frac{1}{\mu}$ for any $t \geq 0$, it is easy to check that

$$E_\varepsilon^\mu(u_{\varepsilon,\mu}) \leq \frac{\varepsilon}{\mu} |A_\varepsilon| + \frac{1}{\varepsilon} W_0 |A_\varepsilon|, \tag{9}$$

where $|\cdot|$ denotes the Lebesgue measure and $W_0 = \max_{t \in (a_1, a_2)} W(t)$. As λ_ε can be chosen in such a way that $|A_\varepsilon| \to 0$ as fast as we want with ε, then the right-hand side of inequality (9) converges to 0 as $\varepsilon \to 0$. So $u_{\varepsilon,\mu}$ is a counter-example to the lower inequality of the Γ-convergence. This example shows that too sharp transitions make the proof of the Γ-convergence fail. In order to obtain the Γ-convergence with the considered function φ, we have to introduce the subspace of $W^{1,2}(\Omega)$ of finite elements as in Chambolle and Dal Maso [7]. The meshsize of the discretization will limit the sharpness of the transitions.

The paper is organized as follows. In Section 2 we define the sequence of functionals and we state the Γ-convergence result. The proof of Γ-convergence is given in Sections 3 and 4. The Section 5 is devoted to the compactness of the minimizers for the sequence of functionals. In Section 6 we show that the evaluation of the discrete functionals via the vertex quadrature rule does not affect the Γ-convergence and compactness results.

2 Mathematical preliminaries and statement of the result

In the following $|A|$ denotes the two-dimensional Lebesgue-measure of a set $A \subset \mathbf{R}^2$, and $\mathcal{H}^1(\partial A)$ denotes the one-dimensional Hausdorff measure of ∂A.

Let $\Omega \subset \mathbf{R}^2$ be a bounded open set. We will use standard notation for the Lebesgue and Sobolev spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$. We say that $u \in L^1(\Omega)$ is a function of bounded variation in Ω, and we write $u \in BV(\Omega)$, if the distributional derivative Du of u is a vector-valued Radon measure with finite total variation in Ω. We denote by $|Du|$ the total variation of Du, and by ∇u the density of the absolutely continuous part of Du with respect to the Lebesgue measure. It can be proved [1] that ∇u coincides almost everywhere with the approximate differential of u. We denote by $u^-(x)$, $u^+(x)$ the approximate lower and upper limit of u at the point x, and we denote by S_u the discontinuity set of u in an approximate sense, defined as

$$S_u = \{x \in \Omega : u^-(x) < u^+(x)\}.$$

We say that a Borel set $A \subset \mathbf{R}^2$ is a set with finite perimeter in Ω if $\chi_A \in BV(\Omega)$, where χ_A denotes the characteristic function of A. We denote by $\partial^* A$ the essential boundary of A, i.e., the set of points where A does not have density 0 or 1. The perimeter of A in Ω is then given by $|D\chi_A|(\Omega) = \mathcal{H}^1(\partial^* A \cap \Omega)$.

In the following $\Omega \subset \mathbf{R}^2$ will denote an open polygonal domain. Let θ_0 be an angle such that $0 < \theta_0 < \pi$, and let $\nu(h)$ be a function such that $\nu(h) \geq 6h$ for any $h > 0$ and $\nu(h) = O(h)$ as $h \to 0^+$. For any $h > 0$, we denote by $\tau_h(\Omega) = \tau_h(\Omega, \theta_0)$ the set of all triangulations of Ω made of triangles whose edges have length between h and $\nu(h)$, and whose angles are all greater than or equal to θ_0.

We denote by $V_h(\Omega) \subset W^{1,2}(\Omega) \cap C^0(\overline{\Omega})$ the linear finite element space

$$V_h(\Omega) = \{u : \Omega \to \mathbf{R} \ : \ \exists \mathbf{T}_h \in \tau_h(\Omega) : \ u \text{ continuous}, u|_T \in P_1(T) \ \forall T \in \mathbf{T}_h\},$$

where T denotes a triangle of $\mathbf{T}_h$, $u|_T$ denotes the restriction of u to T, and $P_1(T)$ denotes the space of polynomials of degree 1 on T. We denote by $\pi_h : C^0(\overline{\Omega}) \to V_h(\Omega)$ the Lagrange interpolation operator.

Let $\{a_1, \ldots, a_n\} \subset \mathbf{R}$ with $a_1 < a_2 < \cdots < a_n$. Let $W : \mathbf{R} \to \mathbf{R}$ be a function with the following properties:

(i) W is $\mathcal{C}^1(\mathbf{R})$ with Lipschitz continuous derivative;

(ii) W is $\mathcal{C}^2$ in a neighborhood of a_i for any $i \in \{1, \ldots, n\}$;

(iii) $W(t) > 0$ for any $t \notin \{a_1, \ldots, a_n\}$ and

$$W(a_i) = 0, \quad W'(a_i) = 0, \quad W''(a_i) > 0, \qquad \forall i \in \{1, \ldots, n\}; \tag{10}$$

(iv) $W(t)$ is monotone increasing for $t \geq a_n$, and monotone decreasing for $t \leq a_1$.

For any $i, l \in \{1, \ldots, n\}$ we set

$$\kappa_{i,l} = \int_{a_i}^{a_l} \sqrt{W(t)}dt.$$

For any $I \in L^\infty(\Omega)$ such that $\|I\|_{L^\infty(\Omega)} \leq K < +\infty$, we set

$$+\infty > M > \max\{|a_1|, |a_n|, K\}. \tag{11}$$

For any $h > 0$ and any $\varepsilon > 0$ we define the functional $E_{\varepsilon,h} : L^2(\Omega) \to [0, +\infty]$ by

$$E_{\varepsilon,h}(u) = \begin{cases} \varepsilon \displaystyle\int_\Omega \frac{|\nabla u|^2}{1 + \mu_{\varepsilon,h}|\nabla u|^2} dx + \frac{1}{\varepsilon} \int_\Omega W(u)dx & \text{if } u \in \mathcal{D}(E_{\varepsilon,h}), \\ +\infty & \text{elsewhere in } L^2(\Omega), \end{cases}$$

where $\mu_{\varepsilon,h} > 0$ and $\mathcal{D}(E_{\varepsilon,h}) = \{u \in V_h(\Omega) : \|u\|_{L^\infty(\Omega)} \leq M\}$.

We say that n Borel sets $A_1, \ldots, A_n$ define a partition of Ω if

$$A_i \cap A_l = \emptyset \quad \forall i, l \in \{1, \ldots, n\},\ i \neq l, \qquad |\Omega \setminus \cup_{i=1}^n A_i| = 0.$$

Let $u \in BV(\Omega; \{a_1, \ldots, a_n\})$ and let $A_i = \{x \in \Omega : u(x) = a_i\}$ for any $i = 1, \ldots, n$; then the sets $A_1, \ldots, A_n$ define a partition of Ω into sets with finite perimeter.

Then we define the functional $E_0 : L^2(\Omega) \to [0, +\infty]$ by

$$E_0(u) = \begin{cases} 2 \displaystyle\sum_{\substack{i,l=1 \\ i<l}}^n \kappa_{i,l} \mathcal{H}^1(\partial^* A_i \cap \partial^* A_l \cap \Omega) & \text{if } u \in BV(\Omega; \{a_1, \ldots, a_n\}), \\ +\infty & \text{elsewhere in } L^2(\Omega). \end{cases}$$

Finally we state the main result of the paper. We define

$$F_{\varepsilon,h}(u) = \int_\Omega (u - I)^2 dx + E_{\varepsilon,h}(u),$$

and we will prove the following theorems.

Theorem 2.1 *Assume that $h = o(\varepsilon|\log \varepsilon|^{-1})$ and that the following condition is satisfied:*

$$\mu_{\varepsilon,h} = \frac{\varepsilon h^\beta}{c}, \quad \textit{with } \beta > 1 \textit{ and } c \geq \frac{12\kappa_{1,n}}{\sin\theta_0}. \tag{12}$$

Then the family $\{F_{\varepsilon,h}\}_\varepsilon$ Γ-converges to the functional

$$\int_\Omega (u - I)^2 dx + E_0(u)$$

in the $L^2(\Omega)$-topology as $\varepsilon \to 0^+$.

Since the term $\int_\Omega (u-I)^2 dx$ is a continuous perturbation with respect to the strong-$L^2(\Omega)$ topology, in order to prove the theorem it will be enough to prove that the family of functionals $\{E_{\varepsilon,h}\}_\varepsilon$ Γ-converges to the functional E_0.

Theorem 2.2 *Assume that $h = o(\varepsilon|\log\varepsilon|^{-1})$ and that $\mu_{\varepsilon,h}$ satisfies (12). Then any family $\{u_{\varepsilon,h}\}_\varepsilon$ of absolute minimizers of $F_{\varepsilon,h}$ is relatively compact in $L^2(\Omega)$, and each of its limit points minimizes the functional*

$$\int_\Omega (u-I)^2 dx + E_0(u).$$

3 Lower inequality

In this section we investigate the Γ-convergence lower inequality (3) with $f_\varepsilon = E_{\varepsilon,h}$ and $f = E_0$.

Theorem 3.1 *Let $\mu_{\varepsilon,h}$ satisfy (12) and let $h = h(\varepsilon)$ with $\lim_{\varepsilon\to 0^+} h(\varepsilon) = 0$. Then, for every function $u_0 \in L^2(\Omega)$ and for every sequence $\{u_{\varepsilon,h}\}_\varepsilon \subset L^2(\Omega)$ converging to u_0 in $L^2(\Omega)$ as $\varepsilon \to 0^+$, we have*

$$\liminf_{\varepsilon\to 0^+} E_{\varepsilon,h}(u_{\varepsilon,h}) \geq E_0(u_0).$$

First we prove the following lemma.

Lemma 3.2 *Assume that $\mu_{\varepsilon,h}$ satisfies (12). Then, for every $\varepsilon > 0$ and for every $u \in V_h(\Omega)$ $\quad (0 < h < 1)$, there exists $v \in BV(\Omega)$ such that*

$$E_{\varepsilon,h}(u) \geq (1 - h^{\beta-1})\varepsilon \int_\Omega |\nabla v|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(v)dx + 2\kappa_{1,n}\mathcal{H}^1(S_v), \tag{13}$$

$$\frac{h}{c} E_{\varepsilon,h}(u) \geq |\{x \in \Omega : v(x) \neq u(x)\}|, \tag{14}$$

and $v(x) = a_1$ for any $x \in \Omega$ such that $v(x) \neq u(x)$.

Proof. We use a method developed by Chambolle and Dal Maso in [7], Section 3. For any $\delta \in (0,1)$ we have

$$\frac{t^2}{1 + (\varepsilon h^\beta / c)t^2} \geq \min\{\delta t^2, (1-\delta)\frac{c}{\varepsilon h^\beta}\} \qquad \forall t \geq 0. \tag{15}$$

Let $\mathbf{T}_h \in \tau_h(\Omega)$ be a triangulation such that $u|_T \in P_1(T)$ for any triangle $T \in \mathbf{T}_h$. Let ∇u_T denote the constant value of the gradient of u on each triangle $T \in \mathbf{T}_h$. Using the inequality (15) we have

$$\begin{aligned} E_{\varepsilon,h}(u) &\geq \varepsilon \sum_{T\in\mathbf{T}_h} |T| \frac{|\nabla u_T|^2}{1 + (\varepsilon h^\beta/c)|\nabla u_T|^2} + \frac{1}{\varepsilon}\int_\Omega W(u)dx \\ &\geq \varepsilon \sum_{T\in\mathbf{T}_h} |T| \min\{\delta|\nabla u_T|^2, (1-\delta)\frac{c}{\varepsilon h^\beta}\} + \frac{1}{\varepsilon}\int_\Omega W(u)dx. \end{aligned} \tag{16}$$

Let $\mathbf{T}_h^1 = \{T \in \mathbf{T}_h : \delta|\nabla u_T|^2 > (1-\delta)\frac{c}{\varepsilon h^\beta}\}$. We define v in the following way:

$$v(x) = \begin{cases} a_1 & \text{on every } T \in \mathbf{T}_h^1 \\ u(x) & \text{on every } T \in \mathbf{T}_h \setminus \mathbf{T}_h^1. \end{cases}$$

Clearly we have $v \in BV(\Omega)$ and

$$S_v \subseteq \bigcup_{T \in \mathbf{T}_h^1} \partial T. \tag{17}$$

Using inequality (16) we find

$$E_{\varepsilon,h}(u) \geq (1-\delta)\frac{c}{h^\beta} \sum_{T \in \mathbf{T}_h^1} |T|. \tag{18}$$

We now set $\delta = 1 - h^{\beta-1}$. The estimate (14) then follows by using (18).

By the assumptions on the triangulation, for any $T \in \mathbf{T}_h$ the following inequality holds (see [7], Section 3):

$$|T| \geq \frac{1}{6} \cdot h \sin\theta_0 \cdot \mathcal{H}^1(\partial T),$$

so that, using (17) we have

$$\sum_{T \in \mathbf{T}_h^1} |T| \geq \frac{1}{6} \cdot h \sin\theta_0 \sum_{T \in \mathbf{T}_h^1} \mathcal{H}^1(\partial T) \geq \frac{1}{6} \cdot h \sin\theta_0 \cdot \mathcal{H}^1(S_v). \tag{19}$$

Then, denoting by ∇v_T the constant value of the gradient of v on each triangle $T \in \mathbf{T}_h$, and using (16) we find

$$\begin{aligned} E_{\varepsilon,h}(u) &\geq \delta\varepsilon \sum_{T \in \mathbf{T}_h \setminus \mathbf{T}_h^1} |T||\nabla u_T|^2 + \frac{1}{\varepsilon}\int_\Omega W(u)dx + \sum_{T \in \mathbf{T}_h^1} |T|(1-\delta)\frac{c}{h^\beta} \\ &\geq (1-h^{\beta-1})\varepsilon \sum_{T \in \mathbf{T}_h} |T||\nabla v_T|^2 + \frac{1}{\varepsilon}\int_\Omega W(v)dx + \frac{c}{h}\sum_{T \in \mathbf{T}_h^1} |T|, \end{aligned} \tag{20}$$

and, using (19),

$$E_{\varepsilon,h}(u) \geq (1-h^{\beta-1})\varepsilon \int_\Omega |\nabla v|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(v)dx + \frac{c}{6} \cdot \sin\theta_0 \cdot \mathcal{H}^1(S_v).$$

The estimate (13) then follows by using (12). ■

We can now prove Theorem 3.1.

Proof of Theorem 3.1. Up to the extraction of a subsequence, we may assume that $\{u_{\varepsilon,h}\}_\varepsilon \subset \mathcal{D}(E_{\varepsilon,h})$, and

$$+\infty > \liminf_{\varepsilon \to 0^+} E_{\varepsilon,h}(u_{\varepsilon,h}) = \lim_{\varepsilon \to 0^+} E_{\varepsilon,h}(u_{\varepsilon,h}), \tag{21}$$

otherwise the result is trivial. To simplify the notation we set $u_\varepsilon = u_{\varepsilon,h(\varepsilon)}$ and we assume that u_ε converges a.e. to u_0 as $\varepsilon \to 0^+$. Using (21) we deduce that

$$\liminf_{\varepsilon\to 0^+} \int_\Omega W(u_\varepsilon)dx = 0,$$

and by Fatou's Lemma

$$0 \leq \int_\Omega W(u_0)dx = \int_\Omega \liminf_{\varepsilon\to 0^+} W(u_\varepsilon)dx \leq \liminf_{\varepsilon\to 0^+} \int_\Omega W(u_\varepsilon)dx = 0,$$

thus $W(u_0(x)) = 0$ a.e. in Ω. Then, using (10) there exists a partition $\{A_i\}_{i=1,\dots,n}$ of Ω (with A_i measurable for any i) such that

$$u_0(x) = \sum_{i=1}^n a_i \chi_{A_i}(x). \tag{22}$$

For any $\varepsilon > 0$, Lemma 3.2 provides a function $v_\varepsilon \in BV(\Omega)$ which satisfies (13) and (14). We have

$$\int_\Omega (v_\varepsilon - u_0)^2 dx = \int_{v_\varepsilon = u_\varepsilon} (u_\varepsilon - u_0)^2 dx + \int_{v_\varepsilon \neq u_\varepsilon} (v_\varepsilon - u_0)^2 dx. \tag{23}$$

The first term of the right-hand side of (23) tends to 0. Furthermore, we have $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq M$ so that, using (22), it follows $|v_\varepsilon - u_0| \leq 2M$ a.e. in Ω, and

$$\int_{v_\varepsilon \neq u_\varepsilon} (v_\varepsilon - u_0)^2 dx \leq 4M^2 |\{x \in \Omega : v_\varepsilon(x) \neq u_\varepsilon(x)\}|. \tag{24}$$

Using (14) and (21) we deduce that v_ε converges to u_0 in $L^2(\Omega)$ as $\varepsilon \to 0^+$. Let $\widehat{v}_\varepsilon$, with $a_1 \leq \widehat{v}_\varepsilon \leq a_n$, denote the truncated function

$$\widehat{v}_\varepsilon = \max\{a_1, \min\{v_\varepsilon, a_n\}\}.$$

We have that $\widehat{v}_\varepsilon$ converges to u_0 in $L^2(\Omega)$ as $\varepsilon \to 0^+$. Now we define the two functions

$$g(y) = \int_{a_1}^y \sqrt{W(t)}dt, \tag{25}$$

and $\psi_\varepsilon(x) = g(\widehat{v}_\varepsilon(x))$. Since g is Lipschitz continuous we have $\psi_\varepsilon \to g(u_0)$ in $L^2(\Omega)$ and, using (22),

$$g(u_0(x)) = \kappa_{1,l} \quad \text{if } u_0(x) = a_l, \quad \forall l \in \{1,\dots,n\}. \tag{26}$$

For any ε we have $\psi_\varepsilon \in BV(\Omega)$, $S_{\psi_\varepsilon} \subseteq S_{\widehat{v}_\varepsilon} = S_{v_\varepsilon}$, and the approximate gradient is given by

$$\nabla\psi_\varepsilon(x) = \sqrt{W(\widehat{v}_\varepsilon(x))}\nabla\widehat{v}_\varepsilon(x). \tag{27}$$

The following estimate then holds for the total variation

$$|D\psi_\varepsilon|(\Omega) = \int_\Omega |\nabla\psi_\varepsilon| dx + \int_{S_{\psi_\varepsilon}} |\psi_\varepsilon^+ - \psi_\varepsilon^-| d\mathcal{H}^1 \leq \int_\Omega |\nabla\psi_\varepsilon| dx + \kappa_{1,n}\mathcal{H}^1(S_{\widehat{v}_\varepsilon}). \tag{28}$$

Using (13), (27) and (28) it follows

$$\begin{aligned} E_{\varepsilon,h}(u_\varepsilon) &\geq (1-h^{\beta-1})\varepsilon \int_\Omega |\nabla v_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(v_\varepsilon)dx + 2\kappa_{1,n}\mathcal{H}^1(S_{v_\varepsilon}) \\ &\geq (1-h^{\beta-1})\varepsilon \int_\Omega |\nabla \widehat{v}_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(\widehat{v}_\varepsilon)dx + 2\kappa_{1,n}\mathcal{H}^1(S_{\widehat{v}_\varepsilon}) \\ &\geq 2(1-h^{\beta-1})^{1/2}\int_\Omega \sqrt{W(\widehat{v}_\varepsilon)}|\nabla\widehat{v}_\varepsilon| dx + 2\kappa_{1,n}\mathcal{H}^1(S_{\widehat{v}_\varepsilon}) \\ &\geq 2(1-h^{\beta-1})^{1/2}|D\psi_\varepsilon|(\Omega), \end{aligned} \tag{29}$$

so that, using (21), we find that the total variation $|D\psi_\varepsilon|(\Omega)$ is uniformly bounded with respect to ε, for ε small enough. Since

$$\|\psi_\varepsilon\|_{L^\infty(\Omega)} \leq \int_{a_1}^{a_n} \sqrt{W(t)}dt = \kappa_{1,n},$$

using the compactness theorem in BV [12] and the uniqueness of the limit, it follows that $g(u_0) \in BV(\Omega)$. Then, using (26), we find that the sets $A_i = \{x \in \Omega : u_0(x) = a_i\}$ have finite perimeter for any $i = 1,\ldots,n$, so that $u_0 \in BV(\Omega;\{a_1,\ldots,a_n\})$ and $E_0(u_0) < +\infty$.

Let us now consider the total variation

$$|Dg(u_0)|(\Omega) = \int_\Omega |\nabla g(u_0)| dx + \int_{S_{g(u_0)}} |g(u_0)^+ - g(u_0)^-| d\mathcal{H}^1.$$

We have $\int_\Omega |\nabla g(u_0)|dx = 0$, and

$$\begin{aligned} \int_{S_{g(u_0)}} |g(u_0)^+ - g(u_0)^-| d\mathcal{H}^1 &= \sum_{\substack{i,l=1 \\ i<l}}^n |g(a_i) - g(a_l)|\mathcal{H}^1(\partial^* A_i \cap \partial^* A_l \cap \Omega) \\ &= \sum_{\substack{i,l=1 \\ i<l}}^n \kappa_{i,l}\mathcal{H}^1(\partial^* A_i \cap \partial^* A_l \cap \Omega). \end{aligned}$$

Thus

$$|Dg(u_0)|(\Omega) = \sum_{\substack{i,l=1 \\ i<l}}^n \kappa_{i,l}\mathcal{H}^1(\partial^* A_i \cap \partial^* A_l \cap \Omega). \tag{30}$$

Finally, using (29), (30) and the lower semicontinuity of the total variation, we find

$$\begin{aligned}\liminf_{\varepsilon\to 0^+} E_{\varepsilon,h}(u_\varepsilon) &\geq 2\lim_{\varepsilon\to 0^+}(1-h^{\beta-1})^{1/2}\liminf_{\varepsilon\to 0^+}|D\psi_\varepsilon|(\Omega) \geq 2|Dg(u_0)|(\Omega) \\ &= 2\sum_{\substack{i,l=1\\ i<l}}^{n}\kappa_{i,l}\mathcal{H}^1(\partial^* A_i\cap\partial^* A_l\cap\Omega),\end{aligned}$$

which concludes the proof. ■

4 Upper inequality

In this section we prove the upper inequality of Γ-convergence of the family of functionals $E_{\varepsilon,h}$ to the functional E_0.

Theorem 4.1 *Let $h = o(\varepsilon|\log\varepsilon|^{-1})$. Then, for every function $u_0 \in L^2(\Omega)$ there exists a sequence $\{u_{\varepsilon,h}\}_\varepsilon \subset L^2(\Omega)$ converging to u_0 in $L^2(\Omega)$ as $\varepsilon \to 0^+$ such that*

$$\limsup_{\varepsilon\to 0^+} E_{\varepsilon,h}(u_{\varepsilon,h}) \leq E_0(u_0).$$

First we prove the following lemma.

Lemma 4.2 *For any $i,l \in \{1,\ldots,n\}$ with $i < l$, there exists a sequence of functions $\{\gamma_\varepsilon^{(i,l)}\}_\varepsilon \subset \mathcal{C}^1(\mathbf{R})$ with the following properties:*

(i) *$\gamma_\varepsilon^{(i,l)}(t) = a_l$ for $t \geq \rho_\varepsilon$, $\gamma_\varepsilon^{(i,l)}(t) = a_i$ for $t \leq -\rho_\varepsilon$, with $\rho_\varepsilon > 0$ independent of the pair (i,l) and $\rho_\varepsilon = O(\varepsilon|\log\varepsilon|)$;*

(ii)

$$\|d\gamma_\varepsilon^{(i,l)}/dt\|_{L^\infty(-\rho_\varepsilon,\rho_\varepsilon)} = O\left(\frac{1}{\varepsilon}\right), \qquad \|d^2\gamma_\varepsilon^{(i,l)}/dt^2\|_{L^\infty(-\rho_\varepsilon,\rho_\varepsilon)} = O\left(\frac{1}{\varepsilon^2}\right);$$

(iii)

$$\lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\left[\varepsilon\left(\frac{d\gamma_\varepsilon^{(i,l)}}{dt}\right)^2 + \frac{1}{\varepsilon}W(\gamma_\varepsilon^{(i,l)})\right]dt = 2\kappa_{i,l}.$$

Proof. Fix $m \in \{i,\ldots,l-1\}$ and consider the solution $\eta^{(m,m+1)}$ of the ordinary differential equation:

$$\frac{d\eta^{(m,m+1)}}{dt} = \sqrt{W(\eta^{(m,m+1)})}, \qquad \eta^{(m,m+1)}(0) = \frac{1}{2}(a_m + a_{m+1}). \tag{31}$$

Since $\sqrt{W(y)}$ is Lipschitz-continuous in a neighborhood of $\frac{1}{2}(a_m + a_{m+1})$, local existence is guaranteed. Moreover, by writing

$$\int_{\frac{1}{2}(a_m+a_{m+1})}^{\eta^{(m,m+1)}(t)} \frac{dy}{\sqrt{W(y)}} = t,$$

and using the properties of the function W, the local solution can be extended to all of $\mathbf{R}$. Then the solution $\eta^{(m,m+1)} : \mathbf{R} \to \mathbf{R}$ is an increasing function with the following properties:

$$a_m < \eta^{(m,m+1)}(t) < a_{m+1}, \qquad \forall t \in \mathbf{R},$$

$$\lim_{t\to+\infty} \eta^{(m,m+1)}(t) = a_{m+1}, \qquad \lim_{t\to-\infty} \eta^{(m,m+1)}(t) = a_m, \tag{32}$$

and, using (10) and arguing as in [16], Section 1-B, the following decay estimates hold:

$$\lim_{t\to+\infty} \frac{a_{m+1} - \eta^{(m,m+1)}(t)}{\exp(-\sqrt{\alpha_{m+1}}t)} = B_{m,m+1}, \qquad \lim_{t\to-\infty} \frac{\eta^{(m,m+1)}(t) - a_m}{\exp(\sqrt{\alpha_m}t)} = C_{m,m+1}, \tag{33}$$

where $2\alpha_m = W''(a_m)$, $2\alpha_{m+1} = W''(a_{m+1})$, and $B_{m,m+1}$, $C_{m,m+1}$ are positive constants. Now we set

$$t_{m,\varepsilon} = \frac{\varepsilon|\log\varepsilon|}{\sqrt{\alpha_m}}, \qquad \sigma_\varepsilon^{(m,m+1)} = t_{m+1,\varepsilon} - t_{m,\varepsilon},$$

and we define $\eta_\varepsilon^{(m,m+1)} : \mathbf{R} \to \mathbf{R}$ in the following way:

$$\eta_\varepsilon^{(m,m+1)}(t - \sigma_\varepsilon^{(m,m+1)}) = \begin{cases} \eta^{(m,m+1)}\left(\dfrac{t}{\varepsilon}\right) & \text{if } -t_{m,\varepsilon} \le t \le t_{m+1,\varepsilon}, \\ q_{m,\varepsilon}(t) & \text{if } -2t_{m,\varepsilon} \le t \le -t_{m,\varepsilon}, \\ a_m & \text{if } t \le -2t_{m,\varepsilon}, \\ p_{m+1,\varepsilon}(t) & \text{if } t_{m+1,\varepsilon} \le t \le 2t_{m+1,\varepsilon}, \\ a_{m+1} & \text{if } t \ge 2t_{m+1,\varepsilon}, \end{cases}$$

where $q_{m,\varepsilon}$, $p_{m+1,\varepsilon}$ are cubic polynomials chosen in such a way that $\eta_\varepsilon^{(m,m+1)} \in \mathcal{C}^1(\mathbf{R})$ for any $\varepsilon > 0$. Moreover we have $\eta_\varepsilon^{(m,m+1)}(t) = a_{m+1}$ for $t \ge \rho_\varepsilon^{(m,m+1)}$, $\eta_\varepsilon^{(m,m+1)}(t) = a_m$ for $t \le -\rho_\varepsilon^{(m,m+1)}$, with

$$\rho_\varepsilon^{(m,m+1)} = t_{m,\varepsilon} + t_{m+1,\varepsilon} = \varepsilon|\log\varepsilon|\left(\frac{1}{\sqrt{\alpha_m}} + \frac{1}{\sqrt{\alpha_{m+1}}}\right).$$

If we write the polynomial $q_{m,\varepsilon}$ in the form

$$q_{m,\varepsilon}(t) = a_m + q^{(1)}_{m,\varepsilon}(t + 2t_{m,\varepsilon})^2 + q^{(2)}_{m,\varepsilon}(t + 2t_{m,\varepsilon})^3,$$

using (33) one can check that

$$q^{(1)}_{m,\varepsilon} = O\left(\frac{1}{\varepsilon|\log\varepsilon|}\right), \qquad q^{(2)}_{m,\varepsilon} = O\left(\frac{1}{\varepsilon^2|\log\varepsilon|^2}\right).$$

Analogous relations hold for the polynomial $p_{m+1,\varepsilon}$. Hence we obtain the following estimates:

$$\begin{aligned}
|q_{m,\varepsilon}(t) - a_m| &\le Q^{(0)}_m \varepsilon|\log\varepsilon|, && \text{for } -2t_{m,\varepsilon} \le t \le -t_{m,\varepsilon},\\
|q'_{m,\varepsilon}(t)| \le Q^{(1)}_m, \quad |q''_{m,\varepsilon}(t)| &\le \frac{Q^{(2)}_m}{\varepsilon|\log\varepsilon|}, && \text{for } -2t_{m,\varepsilon} \le t \le -t_{m,\varepsilon},\\
|p_{m+1,\varepsilon}(t) - a_{m+1}| &\le P^{(0)}_{m+1} \varepsilon|\log\varepsilon|, && \text{for } t_{m+1,\varepsilon} \le t \le 2t_{m+1,\varepsilon},\\
|p'_{m+1,\varepsilon}(t)| \le P^{(1)}_{m+1}, \quad |p''_{m+1,\varepsilon}(t)| &\le \frac{P^{(2)}_{m+1}}{\varepsilon|\log\varepsilon|}, && \text{for } t_{m+1,\varepsilon} \le t \le 2t_{m+1,\varepsilon},
\end{aligned} \tag{34}$$

where $Q^{(j)}_m, P^{(j)}_{m+1}$, $j = 0, 1, 2$, are positive constants independent of ε. Since

$$\frac{d^2\eta^{(m,m+1)}}{dt^2} = \frac{1}{2}W'(\eta^{(m,m+1)}),$$

using (34), we find

$$\|d\eta^{(m,m+1)}_\varepsilon/dt\|_{L^\infty(-\rho^{(m,m+1)}_\varepsilon, \rho^{(m,m+1)}_\varepsilon)} = O\left(\frac{1}{\varepsilon}\right), \tag{35}$$

$$\|d^2\eta^{(m,m+1)}_\varepsilon/dt^2\|_{L^\infty(-\rho^{(m,m+1)}_\varepsilon, \rho^{(m,m+1)}_\varepsilon)} = O\left(\frac{1}{\varepsilon^2}\right). \tag{36}$$

Now we set

$$I^{(m,m+1)}_\varepsilon = \int_{-\infty}^{+\infty} \left[\varepsilon\left(\frac{d\eta^{(m,m+1)}_\varepsilon}{dt}\right)^2 + \frac{1}{\varepsilon}W(\eta^{(m,m+1)}_\varepsilon)\right] dt,$$

and we show that

$$\lim_{\varepsilon\to 0^+} I^{(m,m+1)}_\varepsilon = 2\kappa_{m,m+1}. \tag{37}$$

We have

$$
\begin{aligned}
I_\varepsilon^{(m,m+1)} &= \int_{-2t_{m,\varepsilon}}^{-t_{m,\varepsilon}} \left[\varepsilon(q'_{m,\varepsilon})^2 + \frac{1}{\varepsilon}W(q_{m,\varepsilon})\right] dt \\
&+ \int_{t_{m+1,\varepsilon}}^{2t_{m+1,\varepsilon}} \left[\varepsilon(p'_{m+1,\varepsilon})^2 + \frac{1}{\varepsilon}W(p_{m+1,\varepsilon})\right] dt \\
&+ \int_{-t_{m,\varepsilon}}^{t_{m+1,\varepsilon}} \left[\varepsilon\left(\frac{1}{\varepsilon}\frac{d\eta^{(m,m+1)}}{dt}(t/\varepsilon)\right)^2 + \frac{1}{\varepsilon}W(\eta^{(m,m+1)}(t/\varepsilon))\right] dt.
\end{aligned}
\tag{38}
$$

Using (31) we find that the third integral in the right-hand side of (38) is equal to

$$
\begin{aligned}
&\int_{-t_{m,\varepsilon}/\varepsilon}^{t_{m+1,\varepsilon}/\varepsilon} \left[\left(\frac{d\eta^{(m,m+1)}}{dt'}\right)^2 + W(\eta^{(m,m+1)}(t'))\right] dt' \\
&= 2\int_{-|\log\varepsilon|/\sqrt{\alpha_m}}^{|\log\varepsilon|/\sqrt{\alpha_{m+1}}} \left|\frac{d\eta^{(m,m+1)}}{dt'}\right| \sqrt{W(\eta^{(m,m+1)}(t'))}dt' \\
&= 2\int_{\eta^{(m,m+1)}(-|\log\varepsilon|/\sqrt{\alpha_m})}^{\eta^{(m,m+1)}(|\log\varepsilon|/\sqrt{\alpha_{m+1}})} \sqrt{W(y)}dy,
\end{aligned}
$$

and, using (32), this term converges to $2\kappa_{m,m+1}$ as $\varepsilon \to 0$. Using the estimate $|q'_{m,\varepsilon}(t)| \leq Q_m^{(1)}$ we have

$$
\lim_{\varepsilon\to 0^+} \int_{-2t_{m,\varepsilon}}^{-t_{m,\varepsilon}} \varepsilon(q'_{m,\varepsilon})^2 dt = 0.
$$

Since $W(q_{m,\varepsilon}(t)) = W'(\hat{a})(q_{m,\varepsilon}(t) - a_m)$ for $t \in (-2t_{m,\varepsilon}, -t_{m,\varepsilon})$ and for some $\hat{a}$ close to a_m, using the first estimate of (34) we have

$$
\frac{1}{\varepsilon}\int_{-2t_{m,\varepsilon}}^{-t_{m,\varepsilon}} W(q_{m,\varepsilon})dt = O(\varepsilon|\log\varepsilon|^2),
$$

so that this term converges to zero. Analogously we get

$$
\lim_{\varepsilon\to 0^+} \int_{t_{m+1,\varepsilon}}^{2t_{m+1,\varepsilon}} \left[\varepsilon(p'_{m+1,\varepsilon})^2 + \frac{1}{\varepsilon}W(p_{m+1,\varepsilon})\right] dt = 0,
$$

and (37) is then proved.

Now we construct the function $\gamma_\varepsilon^{(i,l)} : \mathbf{R} \to \mathbf{R}$ by means of translations of the functions $\eta_\varepsilon^{(m,m+1)}$ in such a way that the smooth transitions between the values a_m, a_{m+1} do not overlap. We set

$$
\rho_\varepsilon = \sum_{j=1}^{n-1} \rho_\varepsilon^{(j,j+1)},
$$

and we define a partition of the interval $[-\rho_\varepsilon, \rho_\varepsilon]$ into closed sub-intervals with disjoint interiors:

$$[-\rho_\varepsilon, \rho_\varepsilon] = \bigcup_{m=i}^{l-1} [\xi_{m,\varepsilon}, \xi_{m+1,\varepsilon}],$$

with

$$\xi_{i,\varepsilon} = -\rho_\varepsilon, \qquad \xi_{l,\varepsilon} = \rho_\varepsilon.$$

In each sub-interval $[\xi_{m,\varepsilon}, \xi_{m+1,\varepsilon}]$ we set $\gamma_\varepsilon^{(i,l)}$ equal to $\eta_\varepsilon^{(m,m+1)}$ translated in such a way that

$$\gamma_\varepsilon^{(i,l)}(\xi_{m,\varepsilon}) = a_m, \qquad \gamma_\varepsilon^{(i,l)}(\xi_{m+1,\varepsilon}) = a_{m+1}.$$

This is obtained by setting

$$\xi_{m,\varepsilon} = -\rho_\varepsilon + 2\sum_{j=i}^{m-1} \rho_\varepsilon^{(j,j+1)}, \qquad \text{for } i+1 \le m \le l-1,$$

and

$$\gamma_\varepsilon^{(i,l)}(t) = \eta_\varepsilon^{(m,m+1)}(t - \xi_{m,\varepsilon} - \rho_\varepsilon^{(m,m+1)}), \qquad \text{for } \xi_{m,\varepsilon} \le t \le \xi_{m+1,\varepsilon},$$

for $i \le m \le l-1$. Notice that

$$\gamma_\varepsilon^{(i,l)}(t) = a_l, \qquad \text{for } \xi_{l-1,\varepsilon} + 2\rho_\varepsilon^{(l-1,l)} \le t \le \xi_{l,\varepsilon} = \rho_\varepsilon.$$

The definitions above imply that $\gamma_\varepsilon^{(i,l)} \in \mathcal{C}^1(\mathbf{R})$ for any $\varepsilon > 0$, and $\gamma_\varepsilon^{(i,l)}(t) = a_l$ for $t \ge \rho_\varepsilon$, $\gamma_\varepsilon^{(i,l)}(t) = a_i$ for $t \le -\rho_\varepsilon$, and we have obtained the property (i) of the statement of the lemma. The property (ii) then follows from (35) and (36).

By the construction of $\gamma_\varepsilon^{(i,l)}$, using (37), we obtain

$$\begin{aligned}
\lim_{\varepsilon\to 0^+} \int_{-\infty}^{+\infty} \left[\varepsilon \left(\frac{d\gamma_\varepsilon^{(i,l)}}{dt} \right)^2 + \frac{1}{\varepsilon} W(\gamma_\varepsilon^{(i,l)}) \right] dt &= \lim_{\varepsilon\to 0^+} \sum_{m=i}^{l-1} I_\varepsilon^{(m,m+1)} \\
= \sum_{m=i}^{l-1} \lim_{\varepsilon\to 0^+} I_\varepsilon^{(m,m+1)} &= \sum_{m=i}^{l-1} 2\kappa_{m,m+1} = 2\kappa_{i,l},
\end{aligned}$$

and we have proved the property (iii). ■

In order to prove Theorem 4.1 we need the following density result.

Lemma 4.3 *Let $u_0 \in BV(\Omega; \{a_1, \ldots, a_n\})$; then there exists a sequence $\{u_\varepsilon\}_\varepsilon \subset BV(\Omega; \{a_1, \ldots, a_n\})$ such that*

(i) *the set $A_i^\varepsilon = \{x \in \Omega : u_\varepsilon(x) = a_i\}$ is polygonal and $\mathcal{H}^1(\partial A_i^\varepsilon \cap \partial\Omega) = 0$ for any $i = 1, \ldots n$, and for any $\varepsilon > 0$;*

(ii) *$u_\varepsilon \to u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0^+$;*

(iii)

$$\lim_{\varepsilon \to 0^+} E_0(u_\varepsilon) = E_0(u_0).$$

Proof. This approximation lemma is due to Baldo [4] (Lemma 3.1). Parts (i) and (ii) are stated exactly as above in Baldo [4]. Part (iii) needs some developments. We first set out Baldo's result and then we explain how we can use it in our context.
Let W be given as in (10) and let us define on $\mathbf{R}$ the metric

$$d(\xi_1, \xi_2) = \inf \left\{ \int_0^1 \sqrt{W(\gamma(t))} |\gamma'(t)| dt \quad ; \quad \gamma(0) = \xi_1,\, \gamma(1) = \xi_2,\, \gamma \in \mathcal{C}^1\left([0,1]; \mathbf{R}\right) \right\}.$$

Then let us set

$$g_i(\xi) = d(a_i, \xi),$$

and let us define the Borel measures

$$\mu_i(B) = \int_B |Dg_i(u_0)|,$$
$$\mu_i^\varepsilon(B) = \int_B |Dg_i(u_\varepsilon)|,$$

where B is a Borel set and u_ε is the sequence given in parts (i) and (ii) of Lemma 4.3. In [4], Baldo proved the following result:

$$\lim_{\varepsilon \to 0+} \left(\bigvee_{i=1}^n \mu_i^\varepsilon \right)(\Omega) = \left(\bigvee_{i=1}^n \mu_i \right)(\Omega) = \frac{1}{2} \sum_{i,j=1}^n d(a_i, a_j) \mathcal{H}^1(\partial^* A_i \cap \partial^* A_j \cap \Omega), \tag{39}$$

where the symbol $\bigvee$ denotes the supremum of a family of measures. In the sequel we show that (39) is nothing else than part (iii) of Lemma 4.3.

STEP 1. Let us first show that for any $i, j = 1, \ldots, n$, $i < j$, we have

$$d(a_i, a_j) = \kappa_{i,j} = \int_{a_i}^{a_j} \sqrt{W(y)} dy. \tag{40}$$

According to the definition of $d(a_i, a_j)$ we have

$$d(a_i, a_j) \leq \int_0^1 \sqrt{W(\gamma(t))} |\gamma'(t)| dt$$

for any $\mathcal{C}^1$-curve with $\gamma(0) = a_i$ and $\gamma(1) = a_j$. In particular if $\gamma'(t) > 0$ for all $t \in [0,1]$, by making the change of variable $y = \gamma(t)$, we get

$$d(a_i, a_j) \leq \int_{a_i}^{a_j} \sqrt{W(y)} dy = \kappa_{ij}.$$

Let us show the reverse inequality. Let us set

$$I(\gamma) = \int_0^1 \sqrt{W(\gamma(t))}|\gamma'(t)|dt.$$

Since $t \to \gamma(t)$ is continuous, it is bounded on $[0,1]$, i.e. there exist m_γ and M_γ in $\mathbf{R}$ such that

$$m_\gamma = \min_{[0,1]} \gamma(t) \leq \gamma(t) \leq M_\gamma = \max_{[0,1]} \gamma(t) \quad \forall t \in [0,1].$$

Moreover for all $\tau \in [m_\gamma, M_\gamma]$ there exists $t_\tau \in [0,1]$ such that $\gamma(t_\tau) = \tau$, which implies

$$\#\{t;\, \gamma(t) = \tau\} \geq 1. \tag{41}$$

Then, from the coarea-formula, we have

$$I(\gamma) = \int_{m_\gamma}^{M_\gamma} \left(\sqrt{W(\tau)} \int_{\gamma(t)=\tau} d\mathcal{H}^0 \right) d\tau = \int_{m_\gamma}^{M_\gamma} \sqrt{W(\tau)}\, \#\{t;\, \gamma(t) = \tau\} d\tau.$$

Therefore, with (40) and since a_i and a_j belong to $[m_\gamma, M_\gamma]$

$$I(\gamma) \geq \int_{a_i}^{a_j} \sqrt{W(\tau)}\, \#\{t;\, \gamma(t) = \tau\} d\tau \geq \int_{a_i}^{a_j} \sqrt{W(\tau)} d\tau = \kappa_{ij} \tag{42}$$

and we conclude that $d(a_i, a_j) \geq \kappa_{ij}$ by taking the infimum with respect to γ in the left hand-side of (42).

STEP 2. Let us now show that (39) is equivalent to part (iii) of Lemma 4.3. If $(\mu_\alpha)_{\alpha \in A}$ is a family of regular positive Borel measures, then we define the supremum of $(\mu_\alpha)_{\alpha \in A}$ as follows: let E be any subset of Ω, then

$$\left(\bigvee_{\alpha \in A} \mu_\alpha \right)(E) = \sup \left\{ \sum_{\alpha \in A'} \mu_\alpha(E_\alpha);\, E_\alpha \text{ disjoint open sets in } \Omega, E = \bigcup_{\alpha \in A'} E_\alpha \right\},$$

where A' is any finite or countable subfamily of A.
If we now set

$$\mu_i(E) = \int_E |Dg_i(u_0)|,$$

for any open subset $\Omega' \subset \Omega$ we have

$$\left(\bigvee_{i=1}^{n} \mu_i \right)(\Omega') = \sup \left\{ \sum_i \mu_i(\Omega_i);\, \Omega_i \text{ disjoint, } \Omega' = \bigcup \Omega_i \right\}.$$

But with the same computations we used to prove (30) we get

$$\begin{aligned}\sum_i \mu_i(\Omega_i) &= \sum_i \sum_{k<j} \kappa_{jk}\mathcal{H}^1(\partial^* A_k \cap \partial^* A_j \cap \Omega_i) \\ &= \sum_{k<j} \kappa_{jk} \sum_i \mathcal{H}^1(\partial^* A_k \cap \partial^* A_j \cap \Omega_i) \\ &= \sum_{k<j} \kappa_{jk}\mathcal{H}^1(\partial^* A_k \cap \partial^* A_j \cap \Omega').\end{aligned}$$

Thus for all $\Omega' \subset \Omega$ and for all $i = 1, \ldots n$, we obtain

$$\left(\bigvee_{i=1}^n \mu_i\right)(\Omega') = \sum_{k<j} \kappa_{jk}\mathcal{H}^1(\partial^* A_k \cap \partial^* A_j \cap \Omega'),$$

i.e. the supremum with respect to i "disappears". In particular for $\Omega' = \Omega$ Baldo's result (39) reads as

$$\lim_{\varepsilon\to 0^+} \sum_{k<j} \kappa_{jk}\mathcal{H}^1(\partial A_k^\varepsilon \cap \partial A_j^\varepsilon) = \sum_{k<j} \kappa_{jk}\mathcal{H}^1(\partial^* A_k \cap \partial^* A_j \cap \Omega),$$

which exactly means that

$$\lim_{\varepsilon\to 0^+} E_0(u_\varepsilon) = E_0(u_0),$$

i.e. part (iii) of Lemma 4.3. ■

We can now prove Theorem 4.1.

Proof of Theorem 4.1. We assume $u_0 \in BV(\Omega; \{a_1, \ldots, a_n\})$, otherwise the inequality is trivial. In the following, we use a method developed by Bellettini, Paolini and Verdi in [6]. Using Lemma 4.3 and a diagonal argument we can suppose that the set $A_i = \{x \in \Omega : u_0(x) = a_i\}$ is a polygonal domain with $\mathcal{H}^1(\partial A_i \cap \partial\Omega) = 0$ for any $i = 1, \ldots, n$. We have

$$S_{u_0} = \bigcup_{\substack{i,l=1 \\ i<l}}^n (\partial A_i \cap \partial A_l \cap \Omega).$$

We denote by $\mathcal{P}_i$ the set of the vertices of the polygon A_i and we set $\mathcal{P} = \cup_{i=1}^n \mathcal{P}_i$; hence $\mathcal{P}$ is a finite set of points. We denote by ω the minimum angle between the edges of S_{u_0}.

We introduce the following notations. Let $\mathbf{T}_h \in \tau_h(\Omega)$; we set

$$\begin{aligned}
(S_{u_0})_{\rho_\varepsilon} &= \{x \in \Omega : \mathrm{dist}(x, S_{u_0}) \le \rho_\varepsilon\};\\
(S_{u_0})_{\rho_\varepsilon,h} &= \bigcup\{T \in \mathbf{T}_h : T \cap (S_{u_0})_{\rho_\varepsilon} \ne \emptyset\};\\
\Pi_{S_{u_0}}(x) &= \{y \in S_{u_0} : |y-x| = \mathrm{dist}(x, S_{u_0})\};\\
Q_\varepsilon &= \{x \in (S_{u_0})_{\rho_\varepsilon} : \mathrm{dist}(\Pi_{S_{u_0}}(x), \mathcal{P}) \le \cot(\frac{\omega}{2})\rho_\varepsilon\};\\
Q_{\varepsilon,h} &= \bigcup\{T \in \mathbf{T}_h : T \cap Q_\varepsilon \ne \emptyset\}.
\end{aligned}$$

We denote by $\mathcal{M}$ the set of all the pairs of integers (i,l) such that $i, l \in \{1, \dots, n\}$, $i < l$, and $\mathcal{H}^1(\partial A_i \cap \partial A_l) > 0$. Then for any $(i,l) \in \mathcal{M}$ we set

$$\begin{aligned}
d_{i,l}(x) &= \begin{cases} \mathrm{dist}(x, \partial A_i \cap \partial A_l), & \text{if } x \in A_l,\\ -\mathrm{dist}(x, \partial A_i \cap \partial A_l), & \text{if } x \in A_i; \end{cases}\\
L_\varepsilon^{(i,l)} &= \{x \in \Omega : |d_{i,l}(x)| \le \rho_\varepsilon\};\\
L_{\varepsilon,h}^{(i,l)} &= \bigcup\{T \in \mathbf{T}_h : T \cap L_\varepsilon^{(i,l)} \ne \emptyset\}.
\end{aligned}$$

Using the above definitions it follows that

$$\left(\cup_{(i,l)\in\mathcal{M}} L_\varepsilon^{(i,l)} \setminus Q_\varepsilon\right) \cup Q_\varepsilon = (S_{u_0})_{\rho_\varepsilon}, \tag{43}$$

and, because of the dependence of the set Q_ε on the angle ω, we have for ε small enough

$$\left(L_\varepsilon^{(i,l)} \setminus Q_\varepsilon\right) \cap \left(L_\varepsilon^{(i',l')} \setminus Q_\varepsilon\right) = \emptyset, \tag{44}$$

if either $i \ne i'$ or $l \ne l'$, with $(i,l), (i',l') \in \mathcal{M}$. Moreover we have

$$|(S_{u_0})_{\rho_\varepsilon}| = O\left(\varepsilon|\log\varepsilon|\right), \qquad |(S_{u_0})_{\rho_\varepsilon,h}| = O\left(\varepsilon|\log\varepsilon|\right), \tag{45}$$

$$|Q_\varepsilon| = O\left(\varepsilon^2|\log\varepsilon|^2\right), \qquad |Q_{\varepsilon,h}| = O\left(\varepsilon^2|\log\varepsilon|^2\right), \tag{46}$$

$$|L_\varepsilon^{(i,l)}| = O\left(\varepsilon|\log\varepsilon|\right), \qquad |L_{\varepsilon,h}^{(i,l)}| = O\left(\varepsilon|\log\varepsilon|\right), \tag{47}$$

for any $(i,l) \in \mathcal{M}$.

Using the properties (43), (44) we define the following function u_ε on $\Omega \setminus Q_\varepsilon$:

$$u_\varepsilon(x) = \begin{cases} u_0(x), & \text{if } x \in \Omega \setminus (S_{u_0})_{\rho_\varepsilon}, \\ \\ \gamma_\varepsilon^{(i,l)}(d_{i,l}(x)), & \text{if } x \in L_\varepsilon^{(i,l)} \setminus Q_\varepsilon, \quad \forall\ (i,l) \in \mathcal{M}. \end{cases} \tag{48}$$

Since $|\nabla d_{i,l}(x)| = 1$ for a.e. $x \in \Omega$, using the properties (i) and (ii) of Lemma 4.2, we have that u_ε is Lipschitz continuous in $\Omega \setminus Q_\varepsilon$ with $\mathrm{Lip}(u_\varepsilon) = O(\varepsilon^{-1})$ and $a_1 + O(\varepsilon|\log\varepsilon|) \leq u_\varepsilon(x) \leq a_n + O(\varepsilon|\log\varepsilon|)$. Because of the particular shape of each connected component of Q_ε, following [6], by means of a standard extension theorem [1], u_ε can be extended on the whole Ω as a Lipschitz continuous function such that

$$a_1 + O(\varepsilon|\log\varepsilon|) \leq u_\varepsilon(x) \leq a_n + O(\varepsilon|\log\varepsilon|), \quad \text{a.e. } x \in \Omega, \tag{49}$$

with $\mathrm{Lip}(u_\varepsilon) = O(\varepsilon^{-1})$. Using (45) we find

$$\begin{aligned} \int_\Omega |u_\varepsilon - u_0| dx &= \int_{(S_{u_0})_{\rho_\varepsilon}} |u_\varepsilon - u_0| dx \leq (a_n - a_1 + O(\varepsilon|\log\varepsilon|))|(S_{u_0})_{\rho_\varepsilon}| \\ &= O\left(\varepsilon|\log\varepsilon|\right), \end{aligned}$$

so that $u_\varepsilon \to u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0^+$.

Now we define $u_{\varepsilon,h} = \pi_h(u_\varepsilon)$. Then, using (11) and (49), we have $u_{\varepsilon,h} \in \mathcal{D}(E_{\varepsilon,h})$ for any ε small enough. Moreover we have

$$\|\nabla u_{\varepsilon,h}\|_{L^\infty(\Omega)} \leq \|\nabla u_\varepsilon\|_{L^\infty(\Omega)} = O\left(\frac{1}{\varepsilon}\right). \tag{50}$$

Using the properties of the Lagrange interpolation operator [9], (45), (50) and the condition $h = o(\varepsilon|\log\varepsilon|^{-1})$, we find

$$\begin{aligned} \int_\Omega |u_\varepsilon - u_{\varepsilon,h}| dx &= \int_{(S_{u_0})_{\rho_{\varepsilon,h}}} |u_\varepsilon - u_{\varepsilon,h}| dx \leq Ch|(S_{u_0})_{\rho_\varepsilon,h}|\ \|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \\ &\leq Ch|\log\varepsilon| = o(\varepsilon), \end{aligned}$$

from which it follows that $u_{\varepsilon,h} \to u_0$ in $L^2(\Omega)$ as $\varepsilon \to 0^+$.

Now we write

$$\begin{aligned} E_{\varepsilon,h}(u_{\varepsilon,h}) &= \varepsilon \int_\Omega \frac{|\nabla u_{\varepsilon,h}|^2}{1 + \mu_{\varepsilon,h}|\nabla u_{\varepsilon,h}|^2} dx + \frac{1}{\varepsilon}\int_\Omega W(u_{\varepsilon,h}) dx \\ &\leq \varepsilon \int_\Omega |\nabla u_{\varepsilon,h}|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u_{\varepsilon,h}) dx, \end{aligned}$$

and we split the functional on the right-hand side as follows:

$$\begin{aligned} E_{\varepsilon,h}(u_{\varepsilon,h}) &\leq \varepsilon \int_\Omega |\nabla u_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_\Omega W(u_\varepsilon) dx \\ &+ \varepsilon \int_\Omega \left(|\nabla u_{\varepsilon,h}|^2 - |\nabla u_\varepsilon|^2\right) dx \\ &+ \frac{1}{\varepsilon}\int_\Omega \left(W(u_{\varepsilon,h}) - W(u_\varepsilon)\right) dx \\ &= I_{\varepsilon,h} + II_{\varepsilon,h} + III_{\varepsilon,h}. \end{aligned}$$

STEP 1. We prove that

$$\limsup_{\varepsilon\to 0^+} I_{\varepsilon,h} \leq E_0(u_0).$$

Using (43) we decompose the domain Ω as follows:

$$\Omega = (\Omega \setminus (S_{u_0})_{\rho_\varepsilon}) \cup \left(\cup_{(i,l)\in\mathcal{M}} L_\varepsilon^{(i,l)} \setminus Q_\varepsilon\right) \cup Q_\varepsilon. \tag{51}$$

Using (48) and the properties of W we have

$$\varepsilon \int_{\Omega\setminus(S_{u_0})_{\rho_\varepsilon}} |\nabla u_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_{\Omega\setminus(S_{u_0})_{\rho_\varepsilon}} W(u_\varepsilon) dx = 0. \tag{52}$$

Using (49) and (50), we have

$$\varepsilon \int_{Q_\varepsilon} |\nabla u_\varepsilon|^2 dx \leq \frac{C_1}{\varepsilon}|Q_\varepsilon|, \qquad \frac{1}{\varepsilon}\int_{Q_\varepsilon} W(u_\varepsilon) dx \leq \frac{C_2}{\varepsilon}|Q_\varepsilon|,$$

so that, using (46), we obtain

$$\varepsilon \int_{Q_\varepsilon} |\nabla u_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_{Q_\varepsilon} W(u_\varepsilon) dx = O\left(\varepsilon|\log\varepsilon|^2\right). \tag{53}$$

For any $(i,l) \in \mathcal{M}$ we set

$$J_\varepsilon^{(i,l)} = \varepsilon \int_{L_\varepsilon^{(i,l)}\setminus Q_\varepsilon} |\nabla u_\varepsilon|^2 dx + \frac{1}{\varepsilon}\int_{L_\varepsilon^{(i,l)}\setminus Q_\varepsilon} W(u_\varepsilon) dx.$$

Using (48), the property $|\nabla d_{i,l}(x)| = 1$ for a.e. $x \in \Omega$, and the coarea formula [1], we can write

$$\begin{aligned} J_\varepsilon^{(i,l)} &= \int_{L_\varepsilon^{(i,l)}\setminus Q_\varepsilon} \left[\varepsilon\left(\frac{d\gamma_\varepsilon^{(i,l)}}{dt}(d_{i,l}(x))\right)^2 + \frac{1}{\varepsilon}W(\gamma_\varepsilon^{(i,l)}(d_{i,l}(x)))\right] |\nabla d_{i,l}(x)| dx \\ &= \int_{-\rho_\varepsilon}^{+\rho_\varepsilon} \left[\varepsilon\left(\frac{d\gamma_\varepsilon^{(i,l)}}{dt}\right)^2 + \frac{1}{\varepsilon}W(\gamma_\varepsilon^{(i,l)}(t))\right] \mathcal{H}^1\left(\{x\in\Omega : d_{i,l}(x) = t\} \setminus Q_\varepsilon\right) dt. \end{aligned}$$

Since [1]

$$\lim_{t\to 0}\mathcal{H}^1\left(\{x\in\Omega : d_{i,l}(x)=t\}\right)=\mathcal{H}^1(\partial A_i\cap\partial A_l\cap\Omega),$$

we have

$$J_\varepsilon^{(i,l)}\leq[\mathcal{H}^1(\partial A_i\cap\partial A_l\cap\Omega)+O(\varepsilon|\log\varepsilon|)]\int_{-\rho_\varepsilon}^{+\rho_\varepsilon}\left[\varepsilon\left(\frac{d\gamma_\varepsilon^{(i,l)}}{dt}\right)^2+\frac{1}{\varepsilon}W(\gamma_\varepsilon^{(i,l)}(t))\right]dt,$$

and using the property (iii) of Lemma 4.2 we find

$$\begin{aligned}\limsup_{\varepsilon\to 0^+}J_\varepsilon^{(i,l)} &\leq \mathcal{H}^1(\partial A_i\cap\partial A_l\cap\Omega)\lim_{\varepsilon\to 0^+}\int_{-\infty}^{+\infty}\left[\varepsilon\left(\frac{d\gamma_\varepsilon^{(i,l)}}{dt}\right)^2+\frac{1}{\varepsilon}W(\gamma_\varepsilon^{(i,l)})\right]dt\\ &\leq 2\kappa_{i,l}\mathcal{H}^1(\partial A_i\cap\partial A_l\cap\Omega).\end{aligned}$$

Then we obtain

$$\limsup_{\varepsilon\to 0^+}\sum_{(i,l)\in\mathcal{M}}J_\varepsilon^{(i,l)}\leq 2\sum_{\substack{i,l=1\\ i<l}}^{n}\kappa_{i,l}\mathcal{H}^1(\partial A_i\cap\partial A_l\cap\Omega). \tag{54}$$

Using (44) and (51), and collecting (52), (53) and (54), the proof of STEP 1 is concluded.

STEP 2. We prove that

$$\lim_{\varepsilon\to 0^+}II_{\varepsilon,h}=0.$$

Using (48) we have

$$u_{\varepsilon,h}=u_\varepsilon\quad\text{on}\quad\Omega\setminus\left(\cup_{(i,l)\in\mathcal{M}}L_{\varepsilon,h}^{(i,l)}\right).$$

Then, using (50) we find

$$\begin{aligned}II_{\varepsilon,h} &= \varepsilon\int_{\cup_{(i,l)\in\mathcal{M}}L_{\varepsilon,h}^{(i,l)}}\left(|\nabla u_{\varepsilon,h}|^2-|\nabla u_\varepsilon|^2\right)dx\\ &\leq \varepsilon\int_{\cup_{(i,l)\in\mathcal{M}}L_{\varepsilon,h}^{(i,l)}}|\nabla(u_\varepsilon+u_{\varepsilon,h})||\nabla(u_\varepsilon-u_{\varepsilon,h})|dx\\ &\leq C\int_{\cup_{(i,l)\in\mathcal{M}}L_{\varepsilon,h}^{(i,l)}}|\nabla(u_\varepsilon-u_{\varepsilon,h})|dx\\ &\leq C\int_{Q_{\varepsilon,h}}|\nabla(u_\varepsilon-u_{\varepsilon,h})|dx+C\sum_{(i,l)\in\mathcal{M}}\int_{L_{\varepsilon,h}^{(i,l)}\setminus Q_{\varepsilon,h}}|\nabla(u_\varepsilon-u_{\varepsilon,h})|dx.\end{aligned}$$

Using (46) and (50) it follows

$$\int_{Q_{\varepsilon,h}}|\nabla(u_\varepsilon-u_{\varepsilon,h})|dx\leq\frac{C}{\varepsilon}|Q_{\varepsilon,h}|=O\left(\varepsilon|\log\varepsilon|^2\right). \tag{55}$$

Using the properties of the Lagrange interpolation operator (see Theorem 3.1.5 of [9]), for any $(i,l) \in \mathcal{M}$ we have

$$\int_{L_{\varepsilon,h}^{(i,l)} \setminus Q_{\varepsilon,h}} |\nabla(u_\varepsilon - u_{\varepsilon,h})| dx \leq Ch |L_{\varepsilon,h}^{(i,l)} \setminus Q_{\varepsilon,h}| \, \|\nabla^2 u_\varepsilon\|_{L^\infty(L_\varepsilon^{(i,l)} \setminus Q_\varepsilon)} \, . \tag{56}$$

Since $d_{i,l}$ is a distance function from polygonal boundaries, it follows that

$$\nabla^2 u_\varepsilon = \frac{d^2 \gamma_\varepsilon^{(i,l)}}{dt^2} \nabla d_{i,l} \otimes \nabla d_{i,l} + \frac{d\gamma_\varepsilon^{(i,l)}}{dt} \nabla^2 d_{i,l} = \frac{d^2 \gamma_\varepsilon^{(i,l)}}{dt^2} \nabla d_{i,l} \otimes \nabla d_{i,l}$$

on $L_\varepsilon^{(i,l)} \setminus Q_\varepsilon$, and using the property (ii) of Lemma 4.2 we have

$$\|\nabla^2 u_\varepsilon\|_{L^\infty(L_\varepsilon^{(i,l)} \setminus Q_\varepsilon)} \leq \|d^2 \gamma_\varepsilon^{(i,l)} / dt^2\|_{L^\infty(-\rho_\varepsilon, \rho_\varepsilon)} = O\left(\frac{1}{\varepsilon^2}\right).$$

Then, using (47), (56) and the condition $h = o(\varepsilon |\log \varepsilon|^{-1})$, we get

$$\lim_{\varepsilon \to 0^+} \sum_{(i,l) \in \mathcal{M}} \int_{L_{\varepsilon,h}^{(i,l)} \setminus Q_{\varepsilon,h}} |\nabla(u_\varepsilon - u_{\varepsilon,h})| dx \leq C \lim_{\varepsilon \to 0^+} \frac{h}{\varepsilon} |\log \varepsilon| = 0. \tag{57}$$

The proof of STEP 2 then follows from (55) and (57).

STEP 3. We prove that

$$\lim_{\varepsilon \to 0^+} III_{\varepsilon,h} = 0.$$

Using (48) we have

$$W(u_{\varepsilon,h}) = W(u_\varepsilon) = 0 \quad \text{on} \quad \Omega \setminus \left(\cup_{(i,l) \in \mathcal{M}} L_{\varepsilon,h}^{(i,l)} \right),$$

from which it follows

$$\begin{aligned} III_{\varepsilon,h} &\leq \frac{1}{\varepsilon} \int_\Omega |W(u_{\varepsilon,h}) - W(u_\varepsilon)| \, dx \leq \frac{1}{\varepsilon} \sum_{(i,l) \in \mathcal{M}} \int_{L_{\varepsilon,h}^{(i,l)}} |W(u_{\varepsilon,h}) - W(u_\varepsilon)| \, dx \\ &\leq \frac{\mathrm{Lip}(W)}{\varepsilon} \sum_{(i,l) \in \mathcal{M}} \int_{L_{\varepsilon,h}^{(i,l)}} |u_{\varepsilon,h} - u_\varepsilon| \, dx, \end{aligned} \tag{58}$$

where $\mathrm{Lip}(W)$ denotes the Lipschitz constant of the function W in the interval $[a_1 - \delta, a_n + \delta]$, with $\delta > 0$ and ε small enough. Using the properties of the Lagrange interpolation operator we find

$$\int_{L_{\varepsilon,h}^{(i,l)}} |u_{\varepsilon,h} - u_\varepsilon| \, dx \leq Ch |L_{\varepsilon,h}^{(i,l)}| \, \|\nabla u_\varepsilon\|_{L^\infty(\Omega)}.$$

Then, using (47), (50), (58) and the condition $h = o(\varepsilon |\log \varepsilon|^{-1})$, we get

$$\lim_{\varepsilon \to 0^+} III_{\varepsilon,h} \leq C \cdot \mathrm{Lip}(W) \lim_{\varepsilon \to 0^+} \frac{h}{\varepsilon} |\log \varepsilon| = 0,$$

from which the proof of STEP 3 follows.

The proof of the theorem then follows from STEPS 1–3. ■

Theorem 2.1 then follows from Theorem 3.1, Theorem 4.1, and the fact that the term $\int_\Omega |u(x) - I(x)|^2 \, dx$ is a continuous perturbation.

5 Convergence of minimizers

In this section we prove Theorem 2.2 stated in Section 2.

Proof of Theorem 2.2. The existence of a minimizer u_ε of $F_{\varepsilon,h(\varepsilon)}$ is obtained easily since in fact we search for a minimizer in a compact subset of the space V_h which is of finite dimension. Moreover there exists a constant $c > 0$ such that

$$F_{\varepsilon,h(\varepsilon)}(u_\varepsilon) \leq c. \tag{59}$$

For any $\varepsilon > 0$, Lemma 3.2 provides a function $v_\varepsilon \in BV(\Omega)$ which satisfies (13) and (14). Let $\psi_\varepsilon(x) = g(v_\varepsilon(x))$, where g is the function defined by (25). We have $\psi_\varepsilon \in BV(\Omega)$ and $S_{\psi_\varepsilon} \subseteq S_{v_\varepsilon}$.

Set $c_M = \int_{-M}^{M} \sqrt{W(t)} dt$. Since $\|v_\varepsilon\|_{L^\infty(\Omega)} \leq M$, the following estimates hold for any $\varepsilon > 0$:

$$\|\psi_\varepsilon\|_{L^\infty(\Omega)} < c_M, \tag{60}$$

$$|D\psi_\varepsilon|(\Omega) \leq \int_\Omega |\nabla \psi_\varepsilon| dx + c_M \mathcal{H}^1(S_{v_\varepsilon}). \tag{61}$$

Arguing as in the proof of Theorem 3.1, see (29), and using (59) and (13), we find

$$\begin{aligned} c \geq E_{\varepsilon,h}(u_\varepsilon) \quad &\geq \quad (1 - h^{\beta-1})\varepsilon \int_\Omega |\nabla v_\varepsilon|^2 dx + \frac{1}{\varepsilon} \int_\Omega W(v_\varepsilon) dx + 2\kappa_{1,n} \mathcal{H}^1(S_{v_\varepsilon}) \\ &\geq \quad 2(1 - h^{\beta-1})^{1/2} \left[\int_\Omega |\nabla \psi_\varepsilon| dx + \kappa_{1,n} \mathcal{H}^1(S_{v_\varepsilon}) \right] \\ &\geq \quad 2\frac{\kappa_{1,n}}{c_M}(1 - h^{\beta-1})^{1/2} \left[\int_\Omega |\nabla \psi_\varepsilon| dx + c_M \mathcal{H}^1(S_{v_\varepsilon}) \right], \end{aligned}$$

where we have used $c_M \geq \kappa_{1,n}$, see (11). Using (60) and (61), it follows that ψ_ε is uniformly bounded in $BV(\Omega)$ with respect to ε, for ε small enough. Then, using the compactness theorem in BV [12], there exists a subsequence $\{\psi_{\varepsilon_j}\}_j$ converging in $L^2(\Omega)$ to a function ψ_0.

Set $u_0 = g^{-1}(\psi_0)$. Since the function g is monotone increasing, using the properties of the potential W it follows that the inverse function g^{-1} is bounded and uniformly continuous on compact subsets of $\mathbf{R}$. Then

$$v_{\varepsilon_j} = g^{-1} \circ \psi_{\varepsilon_j} \to u_0 \quad \text{in } L^2(\Omega) \text{ as } \varepsilon_j \to 0^+,$$

see also [14], proof of Proposition 3. Reasoning as in (23), (24) and using Lemma 3.2, we find that $u_{\varepsilon_j} \to u_0$ in $L^2(\Omega)$ as $\varepsilon_j \to 0^+$.

Hence, the statement of Theorem 2.2 follows from Theorem 2.1 and the property (5) of Γ-convergence. ■

6 Numerical integration

In this section we show that the numerical approximation of the lower order terms in the energy via the vertex quadrature rule does not change the results previously obtained (see [5, 6]). More precisely, for any $\varepsilon > 0$ let $I_\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ approximate the function $I \in L^\infty(\Omega)$ so that [5]

$$I_\varepsilon \to I \text{ in } L^2(\Omega), \quad \|I_\varepsilon\|_{L^\infty(\Omega)} \le \|I\|_{L^\infty(\Omega)}, \quad \|\nabla I_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}. \tag{62}$$

For any $h > 0$ and any $\varepsilon > 0$ we define the functional $\widehat{E}_{\varepsilon,h}$ by

$$\widehat{E}_{\varepsilon,h}(u) = \varepsilon \int_\Omega \frac{|\nabla u|^2}{1 + \mu_{\varepsilon,h}|\nabla u|^2} dx + \frac{1}{\varepsilon}\int_\Omega \pi_h(W(u))dx,$$

and the functional $\widehat{F}_{\varepsilon,h} : L^2(\Omega) \to [0, +\infty]$ by

$$\widehat{F}_{\varepsilon,h}(u) = \begin{cases} \widehat{E}_{\varepsilon,h}(u) + \displaystyle\int_\Omega \pi_h((u - I_\varepsilon)^2)dx & \text{if } u \in \mathcal{D}(\widehat{F}_{\varepsilon,h}), \\ +\infty & \text{elsewhere in } L^2(\Omega), \end{cases}$$

where $\mathcal{D}(\widehat{F}_{\varepsilon,h}) = \{u \in V_h(\Omega) : \|u\|_{L^\infty(\Omega)} \le M\}$. The integrals in $\widehat{F}_{\varepsilon,h}$ can be evaluated via the vertex quadrature rule, which is exact for piecewise linear functions.

Let $u \in V_h(\Omega)$ and let $\mathcal{N}_h$ denote the set of all nodes of a triangulation $\mathbf{T}_h$. Define the function $\tilde{u} \in V_h(\Omega)$ in the following way: for any $q \in \mathcal{N}_h$ set $\tilde{u}(q) = u(q)$ if $|u(q)| \le M$, $\tilde{u}(q) = M$ if $u(q) > M$, and $\tilde{u}(q) = -M$ if $u(q) < -M$. Since the function $\varphi(t) = \frac{t^2}{1+\mu t^2}$ is monotone increasing for $t \ge 0$, by using the property (iv) of the potential W, we have $\widehat{F}_{\varepsilon,h}(\tilde{u}) \le \widehat{F}_{\varepsilon,h}(u)$. It follows that any absolute minimizer $u_{\varepsilon,h}$ of $\widehat{F}_{\varepsilon,h}$ belongs to the domain $\mathcal{D}(\widehat{F}_{\varepsilon,h})$.

We prove the following proposition.

Proposition 6.1 *Assume that* $h = o(\varepsilon|\log\varepsilon|^{-1})$ *and that* $\mu_{\varepsilon,h}$ *satisfies* (12). *Then the family* $\{\widehat{F}_{\varepsilon,h}\}_\varepsilon$ Γ*-converges to the functional*

$$\int_\Omega (u - I)^2 dx + E_0(u)$$

in the $L^2(\Omega)$*-topology as* $\varepsilon \to 0^+$.

Proof. We prove first the lower inequality: for every function $u_0 \in L^2(\Omega)$ and for every sequence $\{u_{\varepsilon,h}\}_\varepsilon \subset L^2(\Omega)$ converging to u_0 in $L^2(\Omega)$ we have

$$\liminf_{\varepsilon\to 0^+} \widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) \geq \int_\Omega (u_0 - I)^2 dx + E_0(u_0). \tag{63}$$

We can suppose, possibly extracting a subsequence, that $\{u_{\varepsilon,h}\}_\varepsilon \subset \mathcal{D}(\widehat{F}_{\varepsilon,h})$ and $\liminf_{\varepsilon\to 0^+} \widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) = \lim_{\varepsilon\to 0^+} \widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) = L < +\infty$, otherwise (63) is obvious. Following the method of [5, 6] we split $\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h})$ as follows:

$$\begin{aligned}
\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) &= F_{\varepsilon,h}(u_{\varepsilon,h}) \\
&+ \frac{1}{\varepsilon}\int_\Omega [\pi_h(W(u_{\varepsilon,h})) - W(u_{\varepsilon,h})] dx \\
&+ \int_\Omega [\pi_h((u_{\varepsilon,h} - I_\varepsilon)^2) - (u_{\varepsilon,h} - I)^2] dx \\
&= F_{\varepsilon,h}(u_{\varepsilon,h}) + I_{\varepsilon,h} + II_{\varepsilon,h}.
\end{aligned} \tag{64}$$

In view of Theorem 2.1, in order to show (63) it will be enough to prove that $\lim_{\varepsilon\to 0^+} I_{\varepsilon,h} = \lim_{\varepsilon\to 0^+} II_{\varepsilon,h} = 0$. Using the notations and the results of the proof of Lemma 3.2, see (20), for a fixed $\delta \in (0,1)$ we have

$$\begin{aligned}
\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) &\geq \delta\varepsilon \sum_{T\in \mathbf{T}_h\setminus\mathbf{T}_h^1} |T||\nabla u_{\varepsilon,h}|^2 + \sum_{T\in\mathbf{T}_h^1} |T|(1-\delta)\frac{c}{h^\beta} \\
&= \delta\varepsilon \int_{\mathcal{A}_{\varepsilon,h}} |\nabla u_{\varepsilon,h}|^2 dx + (1-\delta)\frac{c}{h^\beta}|\Omega\setminus\mathcal{A}_{\varepsilon,h}|,
\end{aligned} \tag{65}$$

where $\mathcal{A}_{\varepsilon,h} = \bigcup_{T\in\mathbf{T}_h\setminus\mathbf{T}_h^1} T$.

Using Theorem 3.1.5 of [9] we have

$$\begin{aligned}
|I_{\varepsilon,h}| &\leq \frac{1}{\varepsilon}\sum_{T\in\mathbf{T}_h}\int_T |\pi_h(W(u_{\varepsilon,h})) - W(u_{\varepsilon,h})| dx \\
&\leq C_1\frac{h^2}{\varepsilon}\sum_{T\in\mathbf{T}_h\setminus\mathbf{T}_h^1} |T|\|\nabla^2 W(u_{\varepsilon,h})\|_{L^\infty(T)} + \frac{C_2}{\varepsilon}\sum_{T\in\mathbf{T}_h^1}|T| \\
&\leq C_1 \mathrm{Lip}(W')\frac{h^2}{\varepsilon}\sum_{T\in\mathbf{T}_h\setminus\mathbf{T}_h^1} |T||\nabla u_{\varepsilon,h}|^2 + \frac{C_2}{\varepsilon}\sum_{T\in\mathbf{T}_h^1}|T| \\
&= C_1 \mathrm{Lip}(W')\frac{h^2}{\varepsilon}\int_{\mathcal{A}_{\varepsilon,h}} |\nabla u_{\varepsilon,h}|^2 dx + \frac{C_2}{\varepsilon}|\Omega\setminus\mathcal{A}_{\varepsilon,h}|,
\end{aligned}$$

where we have used $\|\nabla^2 W(u_{\varepsilon,h})\|_{L^\infty(T)} \leq \mathrm{Lip}(W')\|\nabla u_{\varepsilon,h}\otimes\nabla u_{\varepsilon,h}\|_{L^\infty(T)} \leq \mathrm{Lip}(W')|\nabla u_{\varepsilon,h}|^2$ on T. Since (65) yields for small enough ε

$$\int_{\mathcal{A}_{\varepsilon,h}} |\nabla u_{\varepsilon,h}|^2 dx \leq \frac{L+1}{\delta\varepsilon}, \qquad |\Omega\setminus\mathcal{A}_{\varepsilon,h}| \leq \frac{L+1}{(1-\delta)c}h^\beta, \tag{66}$$

it follows

$$|I_{\varepsilon,h}| \leq C_1 \frac{h^2}{\varepsilon^2} + C_2 \frac{h^\beta}{\varepsilon}.$$

Since $h = o(\varepsilon|\log\varepsilon|^{-1})$ and $\beta > 1$ we have $\lim_{\varepsilon\to 0^+} I_{\varepsilon,h} = 0$.

Let us prove that $\lim_{\varepsilon\to 0^+} II_{\varepsilon,h} = 0$. We have

$$\begin{aligned} |II_{\varepsilon,h}| &\leq \sum_{T\in\mathbf{T}_h} \int_T |\pi_h((u_{\varepsilon,h} - I_\varepsilon)^2) - (u_{\varepsilon,h} - I)^2| dx \\ &\leq \sum_{T\in\mathbf{T}_h\setminus\mathbf{T}_h^1} \int_T |\pi_h((u_{\varepsilon,h} - I_\varepsilon)^2) - (u_{\varepsilon,h} - I)^2| dx + C_2|\Omega\setminus\mathcal{A}_{\varepsilon,h}|. \end{aligned}$$

In [5], proof of Theorem 4.1, the following estimate has been proved:

$$\begin{aligned} &\sum_{T\in\mathbf{T}_h\setminus\mathbf{T}_h^1} \int_T |\pi_h((u_{\varepsilon,h} - I_\varepsilon)^2) - (u_{\varepsilon,h} - I)^2| dx \leq C_{1,1} h^2 \int_{\mathcal{A}_{\varepsilon,h}} |\nabla u_{\varepsilon,h}|^2 dx \\ + \quad & C_{1,2} h \|\nabla I_\varepsilon\|_{L^\infty(\Omega)} + C_{1,3} \frac{h^2}{\varepsilon} \left(\int_{\mathcal{A}_{\varepsilon,h}} |\nabla u_{\varepsilon,h}|^2 dx \right)^{\frac{1}{2}} + C_{1,4} \frac{h}{\varepsilon} \int_{\mathcal{A}_{\varepsilon,h}} |u_{\varepsilon,h}| dx \\ + \quad & C_{1,5} \left(\int_{\mathcal{A}_{\varepsilon,h}} (I_\varepsilon - I)^2 dx \right)^{\frac{1}{2}}. \end{aligned}$$

Collecting all the above estimates, using (62) and (66), we obtain

$$|II_{\varepsilon,h}| \leq C_{1,1} \frac{h^2}{\varepsilon} + C_{1,2} \frac{h}{\varepsilon} + C_{1,3} \frac{h^2}{\varepsilon\sqrt{\varepsilon}} + C_{1,4} \frac{h}{\varepsilon} + C_{1,5} \left(\int_\Omega (I_\varepsilon - I)^2 dx \right)^{\frac{1}{2}} + C_2 h^\beta,$$

which proves that $\lim_{\varepsilon\to 0^+} II_{\varepsilon,h} = 0$ and concludes the proof of (63).

We now prove the upper inequality: for every function $u_0 \in L^2(\Omega)$ there exists a sequence $\{u_{\varepsilon,h}\}_\varepsilon \subset L^2(\Omega)$ converging to u_0 in $L^2(\Omega)$ such that

$$\limsup_{\varepsilon\to 0^+} \widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) \leq \int_\Omega (u_0 - I)^2 dx + E_0(u_0). \tag{67}$$

We assume $u_0 \in BV(\Omega; \{a_1, \ldots, a_n\})$, otherwise the inequality is obvious. Let $\{u_{\varepsilon,h}\}_\varepsilon$ be the sequence converging to u_0 constructed in the proof of Theorem 4.1 and such that

$$\limsup_{\varepsilon\to 0^+} E_{\varepsilon,h}(u_{\varepsilon,h}) \leq E_0(u_0).$$

The proof of Theorem 4.1 shows that the following estimate holds:

$$\int_\Omega |\nabla u_{\varepsilon,h}|^2 dx \leq \frac{C}{\varepsilon}. \tag{68}$$

Let us split $\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h})$ as in (64):

$$\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) = F_{\varepsilon,h}(u_{\varepsilon,h}) + I_{\varepsilon,h} + II_{\varepsilon,h}.$$

Then the results obtained in the proof of the lower inequality and the estimate (68) guarantee that

$$\lim_{\varepsilon\to 0^+} I_{\varepsilon,h} = \lim_{\varepsilon\to 0^+} II_{\varepsilon,h} = 0$$

holds true. It follows that

$$\limsup_{\varepsilon\to 0^+} \widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) = \limsup_{\varepsilon\to 0^+} F_{\varepsilon,h}(u_{\varepsilon,h}) \leq \int_\Omega (u_0 - I)^2 dx + E_0(u_0),$$

which yields (67). This concludes the proof of the proposition. ■

We now prove the convergence of the discrete minimizers.

Proposition 6.2 *Assume that* $h = o(\varepsilon|\log\varepsilon|^{-1})$ *and that* $\mu_{\varepsilon,h}$ *satisfies* (12). *Then any family* $\{u_{\varepsilon,h}\}_\varepsilon$ *of absolute minimizers of* $\widehat{F}_{\varepsilon,h}$ *is relatively compact in* $L^2(\Omega)$, *and each of its limit points minimizes the functional*

$$\int_\Omega (u - I)^2 dx + E_0(u).$$

Proof. Let $\{u_{\varepsilon,h}\}_\varepsilon$ be a sequence of absolute minimizers of $\widehat{F}_{\varepsilon,h}$: then there exists a constant $c > 0$ such that $\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) \leq c$.

Let us split again $\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h})$ as in (64):

$$\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h}) = F_{\varepsilon,h}(u_{\varepsilon,h}) + I_{\varepsilon,h} + II_{\varepsilon,h}.$$

Since $\widehat{F}_{\varepsilon,h}(u_{\varepsilon,h})$ is uniformly bounded, from the proof of the lower inequality in Proposition 6.1 we deduce that $\lim_{\varepsilon\to 0^+} I_{\varepsilon,h} = \lim_{\varepsilon\to 0^+} II_{\varepsilon,h} = 0$. Hence also $F_{\varepsilon,h}(u_{\varepsilon,h})$ is uniformly bounded. The proof of Theorem 2.2 then shows that there exists a subsequence $\{u_{\varepsilon_j,h_j}\}_j$ converging in $L^2(\Omega)$ to a function u_0 as $\varepsilon_j \to 0^+$.

Hence, the statement of the proposition follows from Proposition 6.1 and the property (5) of Γ-convergence. ■

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