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LINEAR THEORY OF DISLOCATIONS IN A SMETIC A

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Résumé. — On décrit l'état de distorsion d'un smectique à partir de la structure de référence planaire par un vecteur déplacement de couches **u** (et les contraintes associées σ_{ij}), ou par la rotation du directeur en chaque point $\boldsymbol{\omega}$ (et les couples associés). En principe, le choix de la variable indépendante $\boldsymbol{\omega}$ est justifié dans l'étude des dislocations de rotation, celui de **u** dans l'étude des dislocations de translation. Dans le cas de ces dernières, et s'en tenant aux petites déformations, on étend différents résultats établis en théorie des dislocations dans les solides au cas des smectiques. En introduisant le champ de contraintes dû à une force unité localisée (fonction de Green), on exprime le champ de déplacements d'une ligne comme une intégrale de surface, la tension de ligne et l'énergie d'interaction avec d'autres lignes sous forme d'intégrales de ligne. En particulier, on montre que deux segments de ligne perpendiculaires n'ont pas d'énergie d'interaction, on établit le champ de déplacement d'une boucle coin, on obtient la fonction de Green d'un échantillon homéotrope ayant un bord libre et un bord ancré, on introduit la méthode des forces images.

Abstract. — The distortions of a smectic phase with respect to a planar state of reference are described either by a displacement **u** of the layers (and the conjugated stresses σ_{ij}), or by a rotation $\boldsymbol{\omega}$ of the director (and the conjugated torques). In principle, the choice of $\boldsymbol{\omega}$ as an independent variable is justified for the study of disclinations, and the choice of **u** for the study of dislocations of translation. In this latter case, and restricting the theory to small distortions, one extends various results in dislocation theory in solids to the case of smectics. With the help of the stress field due to an unit point force (Green's function), one is able to express the displacement field of a line as a surface integral, the line tension and the interaction energy with other lines as line integrals. In particular, one shows as a result that two mutually perpendicular line segments do not interact; one establishes the displacement field of an edge loop; one obtains the Green's function of a homeotropic sample with one free boundary and one anchored boundary; the force image method is also introduced.

1. **Introduction.** — Smectics are characterized in their ground state by a stacking of plane layers of constant thickness d [1]. In smectics A, these layers consist of molecules perpendicular to the layer plane and distributed at random. This configuration explains the well-known elastic properties of smectics A, intermediate between those of nematics and crystalline solids : they act as fluids for any translation of the molecules (or of the layers) parallel to the layers, but respond like solids to any stress perpendicular to the layers. This elasticity has been given a theoretical description by Martin *et al.* [2], where the equations are set up for any small deformation with respect to the ground state; the deformation is described by a continuous *scalar* variable $u(\mathbf{r}, t)$ which measures the displacement of the layer along the normal Oz. The molecules stay perpendicular to the layers in any deformation. Permeation, i. e. the activated process by which molecules can jump from one layer to another [3], is not taken into account. Hence the director $\mathbf{n}(\mathbf{r}, t)$, which is the unit vector along the molecules, does not appear as an independent parameter variable.

In the frame of such a theory, to any virtual displacement of the scalar $u(\mathbf{r}, t)$, viz. $\delta u(\mathbf{r}, t)$ corresponds a generalized scalar force $\mathcal{F}$ such that the variation in the total free energy reads

$$\delta w = - \int \mathcal{F} \, \delta u \, \mathrm{d}V . \tag{1}$$

It may be more convenient for the application to some particular problems to introduce a set of variables for which the conjugated forces have a more direct and intuitive meaning; this is certainly the case if one wants to deal with dislocations in smectics, and take advantage of the conceptual frame which has been worked out for dislocations in solid crystals (see, for example, [4], Chap. II). We have therefore re-established the theory of elasticity of smectics

A using as a variable a *vectorial* displacement **u**. The corresponding forces (conjugated to the $u_{i,j}$) are stresses. The director **n** is also an interesting independent variable : to its variation would correspond the work of torques, whose consideration might be useful in order to describe the elastic effects of disclinations.

We therefore first establish in this article the elastic theory of smectics in terms of layer displacements and directors, torques and stresses. Since we restrict to the case of elastostatics, we shall assume that the directors are perpendicular to the layers, which introduces a constraint in the form of a Lagrange multiplier. Permeation is not taken into account. These concepts will afterwards be applied to the theory of dislocations.

2. **Torques and stresses in a distorted smectic A.** — We start from a free energy density [5]

$$F = \frac{1}{2} K(\operatorname{div} \mathbf{n})^2 + \frac{1}{2} B\left(\frac{\partial u}{\partial n}\right)^2 - \frac{1}{2} \chi(\mathbf{H}.\mathbf{n})^2 \quad (2)$$

where K is the Frank-Oseen elastic constant relative to splay [6] and B Young's modulus. u is the projection of the displacement **u** on **n**

$$u = \mathbf{u}.\mathbf{n}\,; \quad \frac{\partial u}{\partial n} = n_i \frac{\partial u}{\partial x_i} = u_{i,j}\, n_i\, n_j + u_i\, n_j\, n_{i,j}\,.$$

We take into account the action of an applied field **H** by introducing the third term. The condition of perpendicularity of $\mathbf{n} + \delta\mathbf{n}$ to the displaced layer reads :

$$\delta\mathbf{n} = \omega \wedge \mathbf{n} \quad (3)$$

where

$$\omega_i = \varepsilon_{ikl}\, \delta u_{p,k}\, n_p\, n_l$$

is the small rotation suffered by **n** when the molecule moves by a distance $\delta\mathbf{u}$. Eq. (3) therefore reads :

$$\delta n_k - \delta u_{j,i}\, n_i\, n_j\, n_k + \delta u_{j,k}\, n_j = 0 \quad (4)$$

and we take into account this condition by introducing a Lagrangian multiplier μ_k.

Another condition comes from the liquid-like behaviour of the molecules inside the layers. This can be written as a Lagrange condition, which expresses the fact that the component of the displacement parallel to the layers corresponds to an incompressible deformation :

$$- p \operatorname{div}(\mathbf{u} - u\mathbf{n}) = 0 \quad (5)$$

where p is the corresponding Lagrange multiplier.

Instead of the torques corresponding to $\boldsymbol{\omega}$, we use the so-called molecular field **h** introduced by de Gennes [7] and conjugated to $\delta\mathbf{n}$. The development now follows that already proposed by Ericksen [8].

The virtual variation of the total energy reads

$$\delta W = \delta \int F\, \mathrm{d}V = \int [- f_i\, \delta u_i - \sigma_{ij,i}\, \delta u_j - h_k\, \delta n_k + \lambda n_k\, \delta n_k]\, \mathrm{d}V + \int (\sigma_{ij}\, \delta u_j + \pi_{ij}\, \delta n_j)\, \mathrm{d}S_i$$

where we have separated in the usual manner body forces and surface forces. The notations are the following

$$\pi_{ij} = \frac{\partial F}{\partial n_{j,i}}$$

$$f_i = - \frac{\partial F}{\partial u_i}$$

$$\sigma_{ij} = - \pi_{ik}\, n_{k,j} + \frac{\partial F}{\partial u_{j,i}}$$

$$h_k = \pi_{ik,i} - \frac{\partial F}{\partial n_k}$$

λ is a Lagrange multiplier taking into account the condition $\mathbf{n}^2 = 1$.

The equilibrium equations read

$$\sigma_{ij,i} + f_j = 0\,; \quad (6)$$

$$\mathbf{h} \wedge \mathbf{n} = 0\,. \quad (7)$$

The algebra makes use of the following points : the Lagrange multipliers μ_k are obtained from eq. (7) and substituted in the expression of σ_{ij}. We are therefore left with 3 equations of equilibrium (eq. (6)), for 6 unknowns (p, 2 angular variables for **n**, and **u**). We also must satisfy eq. (5) and the condition of existence of surfaces normal to **n**. (Eq. (3), equivalent to two independent conditions). Since the only physical component of **u** is along **n**, we satisfy eq. (5) by letting $\mathbf{u} = u\mathbf{n}$. We get finally the following result :

$$\begin{aligned}\sigma_{ij} = &- K \operatorname{div} \mathbf{n}.n_{i,j} + B \frac{\partial u}{\partial n} n_i\, n_j \\ &+ K n_j \left[\frac{\partial}{\partial x_i} \operatorname{div} \mathbf{n} - n_i (\mathbf{n}\nabla \operatorname{div} \mathbf{n})\right] \\ &+ \chi n_j (\mathbf{H}.\mathbf{n})\, [H_i - (\mathbf{H}.\mathbf{n})\, n_i] - p(\delta_{ij} - n_i\, n_j) \\ &+ p u n_j\, n_k\, n_{i,k} - p u n_{i,j} \\ &+ u n_j (p_{,i} - n_i\, n_k\, p_{,k})\,. \end{aligned} \quad (8)$$

The problem simplifies greatly if we specialize to the case in which eq. (6) is linear with respect to u and its derivatives. Remembering we must find a solution in which the only independent variable is u, we first express **n** as a function of u and its derivatives. The equation

$$\mathbf{n} = \left(- \frac{\partial u}{\partial x}, - \frac{\partial u}{\partial y}, + 1\right)$$

expresses the fact that $\mathbf{n}$ is normal to the distorted layers $z = z_0 + u(\mathbf{r})$ and satisfies $\mathbf{n}^2 = 1$, $\mathbf{n}.\text{curl}\ \mathbf{n} = 0$ [1]; with such a value of $\mathbf{n}$, it can be shown that p can be taken equal to zero; the only components of σ_{ij} which are non-vanishing are therefore σ_{11}, σ_{21}, σ_{31}, and eq. (6) reduces to only one equation for the variable u

$$\sigma_{13,1} + \sigma_{23,2} + \sigma_{33,3} = 0$$

with the following definition

$$\begin{aligned}\sigma_{13} &= -K\frac{\partial}{\partial x}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \chi H_3^2 \frac{\partial u}{\partial x} \\ &\quad + \chi\left(H_3 - H_1\frac{\partial u}{\partial x} - H_2\frac{\partial u}{\partial y}\right)H_1 \\ \sigma_{23} &= -K\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \chi H_3^2 \frac{\partial u}{\partial y} \\ &\quad + \chi\left(H_3 - H_1\frac{\partial u}{\partial x} - H_2\frac{\partial u}{\partial y}\right)H_2 \\ \sigma_{33} &= B\frac{\partial u}{\partial z} + \chi H_3\left(\frac{\partial u}{\partial x}H_1 + \frac{\partial u}{\partial y}H_2\right).\end{aligned} \tag{9}$$

It is in the framework defined by eq. (9) that we develop the theory of dislocations in the following paragraphs.

3. **Dislocation theory.** — The essential reason why we can apply most of the theorems of dislocation theory is that we limit ourselves to the case where eq. (9) is linear in u. This implies immediately that the superposition principle holds true.

Hence the distinction between internal stresses and applied stresses is useful. Internal stresses are those due to an internal defect with boundary conditions similar to those in the *free* medium. These boundary conditions can concern either the surface forces $\sigma_{ij}\,\mathrm{d}S_i$ or the displacements u_i, or the surface torques $\varepsilon_{ijk}\,\pi_{li}\,n_k\,\mathrm{d}S_l$. Applied stresses are generally due to imposed forces on the boundaries. For any given sample, the distinction we make will be valid if there is no interaction energy between these two kinds of stresses; we consider this assertion as part of the definition of internal and external stresses, since we have here to be more cautious than in the case of solids, because of the presence of possible surface torques.

Consider an unit force directed along the z direction, to which the medium is subjected at $\mathbf{r}_0$. It gives rise at any point $\mathbf{r}$ to a displacement field $U(\mathbf{r}_0/\mathbf{r})$ which obeys the equation

$$\sigma_{i3,i} + \delta(\mathbf{r} - \mathbf{r}_0) = 0\,. \tag{10}$$

[1] $\mathbf{n}\ \text{curl}\ \mathbf{n} = 0$ is the scalar condition which expresses the fact that the lines of force of $\mathbf{n}$ form a congruence of normals. This condition replaces eq. (3), which expresses the same property in the hypothesis that $\mathbf{u}$ and $\mathbf{n}$ are independent variables.

In an infinite medium, the solution of eq. (10) in terms of a Fourier expansion is straight forward. If we assume that there are no applied fields, we get explicitly

$$U(\mathbf{r}_0/\mathbf{r}) = \frac{1}{(2\pi)^3}\int_{-\infty}^{+\infty} \frac{\exp - i\mathbf{k}(\mathbf{r} - \mathbf{r}_0)}{Bk_z^2 + Kq^4}\, d_3 k \tag{11}$$

$$q^2 = k_x^2 + k_y^2$$

$U(\mathbf{r}_0/\mathbf{r})$ is a typical Green's function, even in $\mathbf{r}$-$\mathbf{r}_0$. The corresponding stresses constitute evidently a set of applied stresses. Such a Green's function also exists in a finite medium. The formulae we are now deriving assume only that the Green's function can be chosen in such a way that the corresponding stresses are applied stresses, and these formulae are then valid in any medium.

Let us introduce the stress fields $\Sigma_{i3}(\mathbf{r}/\mathbf{r}')$ of a point source in $\mathbf{r}$, and consider a line of dislocation L, Burgers' vector d_j. Since there is no interaction energy, the work $u(\mathbf{r}) \times 1$ performed by the point force when the line L is introduced (giving rise to a displacement $u(\mathbf{r})$) is equal and opposite to the work of the stresses Σ_{i3} on the cut surface (cf. for example [9], Chap. IV or [4], Chap. II); hence :

$$u(\mathbf{r}) = -d_j \int \Sigma_{ij}(\mathbf{r}/\mathbf{r}')\,\mathrm{d}S_i' \tag{12}$$

which reduces to, since

$$\begin{aligned} & d_j = (0, 0, d) \\ u(\mathbf{r}) &= -d\int (\Sigma_{13}(\mathbf{r}/\mathbf{r}')\,\mathrm{d}y'\,\mathrm{d}z' + \\ &\qquad + \Sigma_{23}(\mathbf{r}/\mathbf{r}')\,\mathrm{d}x'\,\mathrm{d}z' + \Sigma_{33}(\mathbf{r}/\mathbf{r}')\,\mathrm{d}x'\,\mathrm{d}y') \\ \Sigma_{13}(\mathbf{r}/\mathbf{r}') &= -K\frac{\partial}{\partial x'}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right)U(\mathbf{r}/\mathbf{r}') \\ \Sigma_{23}(\mathbf{r}/\mathbf{r}') &= -K\frac{\partial}{\partial y'}\left(\frac{\partial^2}{\partial x'^2} + \frac{\partial^2}{\partial y'^2}\right)U(\mathbf{r}/\mathbf{r}') \\ \Sigma_{33}(\mathbf{r}/\mathbf{r}') &= B\frac{\partial}{\partial z'}\,U(\mathbf{r}/\mathbf{r}')\,.\end{aligned} \tag{13}$$

3.1 Straight screw dislocation in an infinite medium. — Assume the line to be located along the z-axis, and select the $y - z$ half plane ($y > 0$) as the cut surface. (13) reduces to

$$\begin{aligned} u(\mathbf{r}) &= -d\int \Sigma_{13}(\mathbf{r}/\mathbf{r}')\,\mathrm{d}y'\,\mathrm{d}z' \\ &= +\frac{\mathrm{d}K}{(2\pi)^3}\int \frac{ik_x\,q^2}{Kq^4 + Bk_z^2}\exp - i\mathbf{k}(\mathbf{r}-\mathbf{r}')\,d_3\,\mathbf{k}\,\mathrm{d}y'\,\mathrm{d}z'\end{aligned}$$

Integrating first with respect to z' leads to

$$\begin{aligned} u(\mathbf{r}) &= \frac{d}{(2\pi)^2}\int \frac{ik_x}{q^2}\exp - ik_x\,x \\ &\qquad \times \exp - ik_y(y - y')\,\mathrm{d}y'\,\mathrm{d}k_x\,\mathrm{d}k_y\,.\end{aligned}$$

The integration proceeds with the change of variables

$$k_x = q\cos\theta \qquad k_y = q\sin\theta$$
$$x = \rho\cos\varphi \qquad y - y' = \rho\sin\varphi\,.$$

Using the relation

$$J_1(t) = \frac{1}{2\pi}\oint \exp(it\sin\theta)\exp(-i\theta)\,d\theta$$

where $J_1(t)$ is a Bessel function, we find

$$u(\mathbf{r}) = \frac{d}{2\pi}\int \frac{x}{\rho} J_1(q\rho)\,dq\,dy'$$

and finally, since

$$\int_0^\infty J_1(z)\,dz = 1$$

$$u(\mathbf{r}) = \frac{d}{2\pi}\psi \tag{14}$$

where ψ is the polar angle of $\mathbf{r}$ in a plane perpendicular to z. Eq. (14) could of course have been inferred directly.

3.2 STRAIGHT EDGE DISLOCATION IN AN INFINITE MEDIUM. — The calculation uses eq. (13) where only an integration in the x-y plane is necessary. We obtain de Gennes' result [10]

$$u(\mathbf{r}) = \varepsilon\frac{d}{4} + \varepsilon\frac{d}{4\pi}\int_{-\infty}^{+\infty}\frac{dq}{iq}\exp(-\lambda q^2\mid z\mid + iqx)\,. \tag{15}$$

3.3 CIRCULAR PRISMATIC LOOP. — The same process of integration on a circle of radius R in the plane $z = 0$ leads to

$$u(\mathbf{r}) = \varepsilon\frac{dR}{2}\int_0^\infty \exp(-\varepsilon\lambda q^2 z)\,J_0(q\rho)\,J_1(qR)\,dq \tag{16}$$

where

$$\varepsilon = 1\ (z > 0)$$
$$= -1\ (z < 0)\,.$$

3.4 SELF-ENERGY AND INTERACTION ENERGY. — Let us consider two loops L and L', Burgers' vectors d and d'. The interaction energy can be calculated according to a formula well-known in dislocation theory as a surface integral over one of the loops

$$W_I = d\iint_{S_L}\sigma'_{i3}\,dS_i = d'\iint_{S_L}\sigma_{i3}\,dS'_i \tag{17}$$

where σ'_{i3}, for instance, are the stresses created on S_L by the loop L'. S_L is any surface bounded by L. The integration does not depend indeed on S_L, since σ'_{i3} is divergence-free.

A similar formula has to be used for a self-energy, but with a factor $\frac{1}{2}$

$$W = \frac{d}{2}\iint_{S_L}\sigma_{i3}\,dS_i\,. \tag{18}$$

Since these formulae do not depend on S_L, but only on L, it is possible to reduce them to line integrals. Such a procedure has already been used in dislocation theory by Blin [11] and Kröner [12] in the case of isotropic elasticity. Obviously it is also possible to transform eq. (13) to a line integral, and such an effort has also been already made in dislocation theory. In this latter case, a difficulty arises from the fact that the displacement is a discontinuous function on the cut surface, whereas a line integral is not (cf. [13], p. 64). We have not attempted here to derive line integrals for eq. (15). But it appears that trying to get such line integrals in the evaluation of the energies lead to a very simple and interpretable result.

Let us introduce the operators of differentiation

$$\Sigma_{13} = -K\frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$$
$$\Sigma_{23} = -K\frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) \tag{19}$$
$$\Sigma_{33} = B\frac{\partial}{\partial z}\,.$$

We therefore from now on write

$$\Sigma_{13}(\mathbf{r}/\mathbf{r}') = \Sigma'_{13}\,.\,U(\mathbf{r}/\mathbf{r}')$$

and, using eq. (13)

$$\sigma'_{i3}(\mathbf{r}) = -d'\iint_{S_L}\Sigma_{j3}\,\Sigma'_{i3}\,U(\mathbf{r}/\mathbf{r}')\,dS_i\,dS'_j\,. \tag{20}$$

With this notation, the interaction energy reads

$$W_I = -dd'\iint_{S_L}\iint_{S_{L'}}\Sigma_{j3}\,\Sigma'_{i3}\,U(\mathbf{r}/\mathbf{r}')\,dS_i\,dS'_j \tag{21}$$

and the self energy

$$W_S = -\frac{d^2}{2}\iint_{S_L}\iint_{S_{L'}}\Sigma_{j3}\,\Sigma'_{i3}\,U(\mathbf{r}/\mathbf{r}')\,dS_i\,dS'_j \tag{22}$$

where $S_L = S_{L'}$ in eq. (22).

In these equations the operator $\Sigma_{j3}\,\Sigma'_{i3}\,dS_i\,dS'_j$ contains 9 terms. By a general use of Riemann's theorem, which allows to transform surface integrals to line integrals according to the formula

$$\iint_{S_L}\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)dx\,dy = \int_L P\,dy - Q\,dx$$

and noting that $\frac{\partial}{\partial x_i} U + \frac{\partial}{\partial x'_i} U = 0$, it is a lengthy but easy calculation to obtain the following results [1]

$$W_{\mathrm{I}} = - dd' \, KB \oint\oint_{LL'} \Delta_{\parallel} \, U(\mathbf{r}/\mathbf{r}') \, (\mathrm{d}x \, \mathrm{d}x' + \mathrm{d}y \, \mathrm{d}y')$$

$$W_{\mathrm{S}} = - \frac{d^2}{2} KB \oint\oint_{LL'} \Delta_{\parallel} \, U(\mathbf{r}/\mathbf{r}') \, (\mathrm{d}x \, \mathrm{d}x' + \mathrm{d}y \, \mathrm{d}y') \tag{23}$$

where

$$\Delta_{\parallel} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} .$$

We immediately notice that a *screw* dislocation has zero self-energy. This is straightforward if we use eq. (14) and substitute in the free energy density. In the frame of linear elasticity a screw dislocation has core energy only.

More generally, it appears in eq. (23) that only the edge components of the lines have non-vanishing energies.

Also, two straight and perpendicular edge dislocations have zero interaction energies.

Applying eq. (23) to the case of an isolated edge dislocation in an infinite medium, for which we know the Green function $U(\mathbf{r}/\mathbf{r}')$ (eq. 11), we get, per unit length of line

$$W_{\mathrm{S}} = \frac{Kd^2}{2 \lambda \xi} \tag{24}$$

where $\lambda = \sqrt{K/B}$ is a penetration length and ξ the core radius. The quantity ξ appears in the integration by imposing a cut-off $q_c = 2 \pi/\xi$ on the variation of the wave vector. This result is similar to that one obtained by Kléman and Williams [14] by a direct integration. The interaction energies calculated in the same paper are also obtained from eq. (23).

We also obtain the self-energy of a planar loop as

$$W = \frac{K}{\lambda} \frac{\pi}{2} R^2 \int_0^{q_c} J_1^2(qR) \, q \, \mathrm{d}q$$

for the energy of the total loop. It is easy to check that this quantity increases with R.

[1] An easy way of using Riemann's theorem is to write it with the notation of exterior calculus $\int \mathrm{d}\omega = \iint \mathrm{dd}\omega$ where, if $\mathrm{d}\omega = A \, \mathrm{d}x + \cdots$, we have

$$\mathrm{dd}\omega = \frac{\partial A}{\partial y} \mathrm{d}y \wedge \mathrm{d}x + \frac{\partial A}{\partial z} \mathrm{d}z \wedge \mathrm{d}x + \cdots ;$$

$\mathrm{d}x \wedge \mathrm{d}y$, $\mathrm{d}x \wedge \mathrm{d}z$, etc... are related to the surface differential elements $\mathrm{d}x \, \mathrm{d}y$, $\mathrm{d}x \, \mathrm{d}z$, etc... by

$$\begin{aligned} \mathrm{d}x \wedge \mathrm{d}y &= - \mathrm{d}y \wedge \mathrm{d}x = \mathrm{d}x \, \mathrm{d}y \\ \mathrm{d}y \wedge \mathrm{d}z &= - \mathrm{d}z \wedge \mathrm{d}y = \mathrm{d}y \, \mathrm{d}z \\ \mathrm{d}z \wedge \mathrm{d}x &= - \mathrm{d}x \wedge \mathrm{d}z = \mathrm{d}z \, \mathrm{d}x \\ \mathrm{d}x \wedge \mathrm{d}x &= \mathrm{d}y \wedge \mathrm{d}y = \mathrm{d}z \wedge \mathrm{d}z = 0 . \end{aligned}$$

3.5 FINITE MEDIUM : METHOD OF IMAGES. — In a finite medium, the Green function we have derived in section 3 is no longer useful. However, the boundary conditions are generally simple enough, in experimental situations, to allow for an easy guess of correct images.

Let us consider the case when the boundary is along a smectic plane. Two conditions are possible, whether the surface is free or anchored.

3.5.1 On the free surface, it is reasonable to assume that the forces $\sigma_{i3} \, \mathrm{d}S_i$ vanish, i. e. $\partial u/\partial z = 0$. According to eq. (15), this means that the image of an edge dislocation line is a dislocation line of *opposite* sign. There is therefore an attraction towards the free surface for any edge dislocation.

3.5.2 If the molecules are anchored on the surface, the boundary condition reads $u = 0$. This means that an edge dislocation is repelled by the surface.

In the language of Green's functions the image of a point force with respect to a free surface is a point force of the *same* sign, and with respect to the anchoring surface a point force of *opposite* sign. This appears clearly if we look at formula (13), where $u(\mathbf{r})$ is obtained by integration of $\frac{\partial U}{\partial z} (\mathbf{r}/\mathbf{r}')$ on the $x - y$ plane for an edge dislocation.

Consider for example a sample of constant thickness, situated between an anchoring surface at $z = 0$ and a free surface at $z = D$. The Green's function $U(\mathbf{r}'/\mathbf{r})$ corresponds to the effect of point forces of positive sign in z', $- z'$, $z' + 4 D$, $z' - 4 D$, etc..., and of point forces of negative sign in z_1, $- z_1$, $z_1 + 4 D$, $z_1 - 4 D$, etc... (Fig. 1) with $z_1 = 2 D - z'$

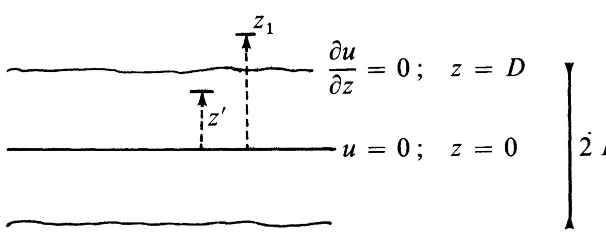

FIG. 1.

We have, expanding

$$\delta(z - z') + \delta(z + z') + \cdots + \delta(z - z_1) - - \delta(z + z_1) + \cdots$$

in a Fourier series

$$\Sigma \pm \delta(z - z_i) = = \frac{1}{D} \sum_{n=0}^{\infty} \cos \frac{\pi n z}{2 D} \cos \frac{\pi n}{2 D} z' \; \{ 1 - (-)^n \}$$

and the Green's function we obtain reads :

$$U(\mathbf{r}/\mathbf{r}') = \frac{1}{D}\,\frac{1}{(2\,\pi)^2}\int \exp - i\mathbf{k}(\mathbf{r}' - \mathbf{r}) \sum_{n=0}^{\infty} \times$$

$$\times \frac{\cos(\pi n z'/2\,D)\cos(\pi n z/2\,D)\,(1 - (-)^n)}{(B\pi^2\, n^2/4\,D^2) + Kq^4}\, d^3 k\,. \tag{25}$$

From this expression it is possible by the methods we have outlined before (eq. (13) and (23)) to obtain any quantity relative to a dislocation in the sample.

4. **Conclusion.** — We have not tried here anything else than to apply classical results of the theory of dislocations in solids to the case of smectics. The first task has been to rewrite the physical quantities in terms of stresses. The application we make are restricted to the approximation of linear elasticity and to the case where the state of reference is the perfect smectic (planar layers). It would be of interest to use the set of eq. (8), where no assumption is made concerning the state of reference, in the vicinity of confocal domains. This certainly would first require a study and definition of states of reference which are not planar (1). Another development would be to apply the Peierls-Nabarro method in order to study the core splitting of a dislocation and the anchoring of smectics on surfaces perpendicular to the layers. A direct application of the Peierls equation to our case leads to the integrodifferential equation

$$\frac{K}{8\sqrt{\pi\lambda^3}}\int_{-\infty} \frac{\partial u}{\partial z'}\,\frac{\mathrm{d}z'}{(z - z')\sqrt{|\,z - z'\,|}} = -\,\frac{\partial \gamma}{\partial u} \tag{26}$$

where γ is the surface energy. The presence of the $(z - z')^{-3/2}$ factor indicates the difficulty of a complete study. The core splitting is certainly small (very few layers broken to create the dislocation) and the Peierls-Nabarro force large.

But to summarize the results of the present study, let us be reminded that we have been able to express the energies as line integrals, with the help of a Green's function, in a very simple expression. We get also the results, among others, that the screw parts have zero energy and that perpendicular edge dislocations do not interact. These results are certainly true far from the core where elastic theory breaks down. In the core region either a Peierls-Nabarro calculation or use of a more general theory of smectics, such as that suggested by Parodi, would be necessary (1).

(1) Parodi, O., private communication.

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(1) A similar study of dislocations in smectics, using a Green's-function technique, has been independently performed by P. S. Pershan, who kindly sent me a preprint. This paper insists more on applications to various cases than on a general formulation.

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