Critical phase transitions made self-organized : a dynamical system feedback mechanism for self-organized criticality

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Critical phase transitions made self-organized : a dynamical system feedback mechanism for self-organized criticality

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Abstract. — According to Kadanoff, self-organized criticality (SOC) implies the operation of a feedback mechanism that ensures a steady state in which the system is marginally stable against a disturbance. Here, we extend this idea and propose a picture according to which SOC relies on a non-linear feedback of the order parameter on the control parameter(s), the amplitude of this feedback being tuned by the spatial correlation length ξ. The self-organized nature of the criticality stems from the fact that the limit $\xi \to +\infty$ is attracting the non-linear feedback dynamics. It is applied to known self-organized critical systems such as « sandpile » models as well as to a simple dynamical generalization of the percolation model. Using this feedback mechanism, it is possible in principle to convert standard « unstable » critical phase transitions into self-organized critical dynamics, thereby enlarging considerably the number of models presenting SOC. These ideas are illustrated on the 2D Ising model and the values of the various « avalanche » exponents are expressed in terms of the static and dynamic Ising critical exponents.

1. Introduction.

Self-organized criticality (SOC) has been proposed as a unifying concept describing the dynamics of a vast class of open non-linear spatio-temporal systems, which evolve spontaneously towards a critical state characterized by (1) response functions obeying powerlaws and (2) a self-similar underlying geometry [1]. SOC thus corresponds to situations where the time evolution exhibit long tails and the correlation length is infinite, so that both temporal and spatial correlations are long-ranged [2].

Since this suggestion, many systems have been argued to exhibit SOC, such as real sandpiles [3, 4], earthquake dynamics [5-7] and plate tectonics [8], magnetic bubble systems [9, 10], breakdown dynamics in semi-conductors [11]... Connections between SOC and other models have been noted, among others with DLA fractal growth processes [12] (fluid imbibition as exemplified for instance by invasion percolation, dendritic growth, dielectric

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breakdown, rupture in random media), turbulence [13, 14], dynamics of biological populations [15], interacting economic systems [14]... The rules and constraints that a system must obey to qualify as SOC are not yet known in general. The minimal ingredients seem however to be that the system must possess many interacting spatial degrees of freedom, be open and coupled with the exterior, its dynamics must be spatio-temporal and non-linear and the flux of the « order parameter » must not be uniform in space, as induced for instance by inhomogeneous boundary conditions. Theoretical works have attempted to specify further the nature of the SOC phenomenon. To this end, correspondences with self-avoiding walks with branching [16] and with different diffusion equations with non-linear corrections driven by an external noise [17] have been proposed. However, inherent in these analyses is the assumption that the system is already at a critical point : indeed, the proposed condition of the existence of a conservation law [17] is not sufficient to ensure long-range spatial correlations even if long-time tails are present [18], therefore entailing the adjustment of a control parameter in order to attain criticality. A different mechanism has recently been suggested [19] according to which an open driven system may self-organize into a critical state because its continuum diffusion limit has singularities in the diffusion constants at the critical point. Another suggestion is that SOC may result from non-linear interactions between Goldstone modes [20].

Kadanoff has recently written [21] that « the phrase 'self-organized criticality' is intended to imply the operation of a feedback mechanism that ensures a steady state in which the system is marginally stable against a disturbance ». However, he adds « One would like to know the laws that govern this state ». It is the purpose of this paper to make some progress in this direction. We build on this general mechanism believed to produce self-organized criticality in extended open systems, which relies on the idea of a feedback of the order parameter onto the control parameter. In section 2, we explicit the basic idea and illustrate it on a simple dynamical generalization of the bond percolation problem and on the dynamical thermal fuse model recently introduced [22]. Then, we show in section 3 how it can apply to standard thermal critical phase transitions and illustrate the method on the 2D Ising model using an exact correspondence with percolation theory. The values of the various « avalanche » exponents are expressed in terms of the static and dynamic Ising critical exponents. § 4 discusses some possible developments and open problems.

2. The non-linear feedback of the order parameter on the control parameter.

2.1 A SIMPLE DYNAMICAL VARIANT OF THE BOND PERCOLATION MODEL : DILUTION BY HUNGRY ANTS. — Our version of the idea of the feedback mechanism can be illustrated on one of the simplest models presenting critical behaviour. Consider therefore the following variation of the problem of 2D-bond percolation on a discrete lattice [23]. Starting from an undeteriorated system with a fraction of present bonds $p = 1$, we suppose that the bonds are made of candy and that a finite density $n \sim \ell^{-2}$ (number per unit surface) of ants move around on the network following random walks. In the sequel, we are interested in the limit of vanishing ant density n, i.e. very large ℓ. For times $t \gg t_\ell$, where $t_\ell \sim \ell^{2d/d_s}$ (d being the space dimension and d_s the spectral dimension) is the typical time needed by an ant to cross the distance ℓ [24], each bond which belongs to the infinite cluster has the same uniform probability to be visited by an ant.

We then suppose that at a rate (number of events per unit time) much smaller than t_ℓ^{-1} each ant selects at random, and therefore with the same probability, one (among all atainable bonds) candy and eats it, thereby removing it. The slowness of this removal process ensures the relaxation of the ant density which implies that the probability to be eaten is the same for all bonds belonging to the infinite cluster. Continuing the process, the network is progressively deleted and the control parameter p decreases. As this process goes on, the ants begin to create finite clusters of undeteriorated bonds which are disconnected from the main infinite cluster. When a finite cluster appears, i.e. a finite number of connected bonds become disconnected from the infinite cluster, the ant which has just eaten its last connection can either stay on the finite cluster or move to the neighbouring bond belonging to the infinite cluster. If it does so, the finite cluster becomes unreachable by the ants and remains so for ever. Therefore, this process will not generate the standard distribution of cluster sizes, but tends to favor the existence of small clusters. Only the infinite cluster possesses a geometry which can be mapped onto that of the infinite cluster of the standard percolation model.

As the eating process goes on, one will eventually reach a threshold at which no infinite cluster exists anymore. The ants will then be trapped on finite clusters of typical size ℓ, some clusters of this size will not contain any ant and will survive for ever. We have thus introduced a model which goes to criticality when $\ell \to \infty$, i.e. in the limit of a very dilute ant population. The interesting feature of this model is that the dynamical rules make the system evolve spontaneously towards a critical point, characterized by arbitrary large finite clusters, each presenting the same fractal geometry of the critical percolation cluster (fractal dimension $D_f = 2 - \beta/\nu \approx 1.9$, where β and ν are two percolation critical exponents) [23]. Note that the geometry obtained is « critical » (i.e. self-similar) but there is no dynamics once it is reached. In order to introduce a dynamical evolution in the spirit of avalanches in the sandpile model [1], one must suppose in addition that the bonds can grow again very slowly after being eaten (« healing » process). Then, there will appear an interesting evolving dynamical geometry resulting from the competition between the slow eating and growth processes.

Our purpose here is not to study this model in detail but rather to focus on the qualitative idea which has been illustrated by this model : the presence of bonds forming an infinite cluster (corresponding to a non-zero percolation order parameter P_∞, which is the probability for a bond to belong to the infinite cluster [23]) allows ants to move and eat, thus decreasing the effective value of the control parameter p. For the infinite cluster, this dynamics stops when it becomes disconnected into very large clusters whose size goes to infinity in the limit $\ell \to +\infty$. In other words, when the system becomes disconnected ($\xi < +\infty$), P_∞ becomes zero and most of the clusters, as large as they are, remain forever. This illustrates the idea that the feedback of the order parameter on the control parameter is controlled by the value of the correlation length, here defined as the typical size of the largest cluster. Note that the above proposed dynamical rules make the percolation model self-organized into a critical state, only in the limit $n \sim \ell^{-2} \to 0$, i.e. we still need to tune a parameter. However, the proposed critical state now possesses a dynamical nature and is attracting the dynamics.

Let us now examine another dynamical model which evolves naturally to the bond percolation critical state, without any parameter tuning.

2.2 THE THERMAL FUSE MODEL [22]. — It consists in a simple dynamical generalization of the electrical random fuse model for rupture in random media [25]. One thus considers a network of fuses fed by a constant current source applied at its two busbar extremities. The electrical fuse resistances are distributed randomly according to a given probability distribution. The electric current in a given fuse is obtained by solving Kirchhoff's equation at each node (or Laplace equation for the voltage field). Then, it is assumed that each fuse is heated locally by a generalized Joule effect, i.e. The temperature T_n of the n-th fuse of specific heat C, resistance R_n and carrying the current i_n obeys the dynamical equation $C\, dT_n/dt = R_n\, i_n^b$. The term $R_n\, i_n^b$ accounts for a generalized Joule heat source. When T_n reaches a given threshold T_r, the fuse burns out irreversibly and becomes an insulator. Then, the current distribution is readjusted in all remaining bonds and the temperature field evolves obeying the above heat equation with the new electric current field. The process goes on until a continuous path of rupture bonds deconnects the two busbars.

For any fixed exponent $0 \leqslant b \leqslant +\infty$, rupture patterns are found self-similar with fractal dimensions depending continuously on b [22]. The limit $b \to 0$ is especially interesting since it provides a simple geometrical explanation for the observed fractal rupture patterns. Indeed in this limit, the heating rates become independent of the current field and the thermal field only depends on the initial quenched resistance disorder distribution. Since the resistances are assumed to be independent random variables, the temperature of each bond evolves independently. Therefore, the successive bond breakdowns are independent random events solely controlled by the distribution and spatial position of the electrical resistances. When a continuous path of ruptured fuses appear, the rupture process stops since the global network resistance becomes infinite and no current flows anymore within the network. At this point, the distribution and position of ruptured bonds are almost exactly given by the bond percolation model at its critical point $p = p_c$, apart from the fact that closed loops cannot exit. The geometry of the incipient infinite cluster is thus given by that of the percolation model with the small difference that loops decorating the backbone remain open. This difference has been found to be of negligible importance for the fractal dimension of the infinite cluster [22]. The rupture dynamics is thus spontaneously attracted to a critical state very close to that of the bond percolation model, thus providing a new example of self-organized critical growth [12]. For $b > 0$, these rupture processes constitute a new class of systems exhibiting SOC, with self-similar rupture patterns whose fractal dimensions decrease continuously from $D_f = 2 - \beta/\nu \approx 1.9$ for $b = 0$ to $D_f \approx 1$ for $b \to +\infty$ [22]. Note, again, that these SOC growth models belong to a different class of phenomena than sandpile models. In sandpiles, no irreversible « damage » occurs and the dynamics is still active in the critical state under external sollicitations. This distinction drops out if we allow for a slow healing of the bonds in the thermal rupture model, leading to the resetting of conduction paths. As the analog of the size of an avalanche, one can choose the electric energy drop at rupture, which is known to obey a multifractal distribution [23]), in the limit $b \to +\infty$ (this limit corresponds to the quasi-static random fuse [23]. We expect similar rich scaling behaviors for arbitrary b's as suggested from preliminary computations [22].

Let us now present the dynamical rupture process of this thermal fuse model from a slightly different view point, which is illuminating for our purpose. Before complete rupture, the fraction p of present bonds (or undamaged fuses) is larger than p_c and the percolation order parameter $P_\infty(p)$, equal to the probability for a bond to belong to the infinite cluster [23], is non-zero. As long as $P_\infty(p) \neq 0$, a current flows into the network and fuses continue to burn out one by one, as soon as their continuous heating give them a temperature higher than T_r. This burning process decreases steadily the percolation « control parameter » p. When p reaches p_c, $P_\infty(p)$ becomes zero and the process stops. Similarly to our interpretation outlined in section 2.1, we are led to consider this dynamical process as a feedback of the order parameter $P_\infty(p)$ onto the « control parameter » p, in the sense that the non-zero value of $P_\infty(p)$ determines the existence and the rate of the time evolution of p. This feedback stops when the percolation correlation length ξ becomes infinite (i.e. when an infinite cluster of connected ruptured bonds appear), at which the system barely disconnects into two pieces. Note that this mechanism underlines the importance, on the appearance of SOC, of the spatial coupling between many degrees of freedom.

2.3 APPLICATION TO THE SANDPILE MODEL. — It is straightforward to see that the proposed feedback mechanism applies to the sandpile models [1, 26] and thus transitively to all the models and experimental systems [3-11] which have been put into correspondence with sandpiles. In this case, the order parameter is the flux J of sand grains out of the system and the control parameter is the slope θ of the sandpile [27] : $J \sim (\theta - \theta_c)^\beta$ for $\theta \geqslant \theta_c$ and $J = 0$ for $\theta < \theta_c$, where θ_c is the slope of marginal stability of the sandpile. The usual picture according to which an initial slope larger than the critical one entails the existence of large avalanches which bring back the slope to its marginal critical value θ_c is a vivid illustration of the proposed principle of a non-linear feedback of the order parameter (grain flux) on the control parameter (slope). This feedback stops when the control parameter reaches its critical value for which the correlation length becomes infinite and spatial correlation functions exhibit long tails [26].

Note that we can interpret the attractiveness of the marginal critical slope as corresponding to the only state for which the spatial correlation length is infinite. This is very important because this gives a criterion for defining the attracting « fixed point » of the dynamics based solely on the universal condition that the correlation length be infinite (independently of the details of the system and especially of the precise nature of its control parameter). Note that the importance of a dynamical attraction to a state where the system is barely decorrelated at infinity has been stressed in the original work on SOC [1] where it is stated that « the system will become stable (dynamically) precisely at the point when the network of minimally stable clusters has been broken down to the level where the noise signal cannot be communicated through infinite distances. At this point, there will be no length scale and consequently no time scale ».

More generally, in a SOC system, there are by definition no tunable control parameters. Therefore, the non-linear dynamics which must attract the system to its critical state cannot be defined and controlled by tunable parameters. The only accessible universal physical quantity which provides a signature of the approach to criticality in its transient dynamics, and which is independent of specific details of the system, is the correlation length ξ. Thus, only ξ can control the intensity of the dynamical feedback of the order parameter onto the various control parameters.

3. The critical Ising phase transition made self-organized.

3.1 DESCRIPTION OF THE DYNAMICAL RULES PROPOSED TO MAKE THE ISING MODEL SELF-ORGANIZED. — Using the above principle for SOC, let us now show how to convert standard « unstable » critical phase transitions into self-organized critical dynamics. This can be considered as a game which may be useful to illustrate the feedback mechanism and outline their differences. The procedure which is described below may seem somewhat artificial but is in fact quite close in spirit to the recent idea [28] of directing chaotic trajectories to target states by suitable feedback perturbations. This idea has been made concrete by experiments which use a (somewhat artificial) electronic feedback [29].

Consider the simplest paradigm of a thermal critical phase transition, namely the Ising model in 2D. Generalization to higher dimensions for the order parameter and for the embedding space can be devised in the same spirit.

Our goal is thus to construct a dynamics whereby the order parameter acts on the control parameter in relation to the value of the correlation length, such that the critical state is the sole attractor of the dynamics. We thus imagine a system in which every spin is cooled down at a very small rate, independently of the state of the system. In addition, each spin can be heated by its own « heater », if there is a conducting path throughout the system so that for instance a macroscopic electric current can flow and supply the « heater ». If this is the case, we assume that all spin heaters will function in the same way, whether or not the spin belongs to the conduction path (democratic process). Note that the condition, that heaters function only when a connected path exists, will ensure that the dynamics is attracted to a state where the correlation length is infinite.

Following Coniglio and Klein [30], we use the correspondence between the thermal Ising problem and the geometrical bond percolation model. We thus define the connectivity in the sense of a physical « droplet », according to which two nearest neighbor spins are connected if they are both up and furthermore if the bond between them is active, a bond being active with a probability $p = 1 - e^{-2J/k_B T}$, where J is the exchange coupling constant between spins. For zero magnetic field, this definition of connectivity maps the Ising model exactly onto the percolation model [30].

Let us consider an Ising system whose initial temperature is larger than its critical value T_c, corresponding to a disordered case with zero magnetization M. Since only finite droplets exist, the heaters are not supplied by the external source of energy and they do not heat. Only the coolers are working and they slowly decrease the temperature, eventually making the system reach the critical temperature. Now imagine an Ising system initially prepared at a temperature smaller than T_c, with a non-zero magnetization. From our definition of the connectivity, there exists an infinite path connecting a finite fraction of the spin heaters to the energy source. Thus, according to our definition, all spins will begin to heat up. When approaching T_c, the connected path becomes less and less dense until it finally disappears just when the correlation length becomes infinite. Suddently, all heaters stop their action and the system is then cooled back to the critical state by the action of the coolers. Therefore, using a feedback of the order state on the temperature field controlled by the existence or not of an infinite correlation length, we have produced a self-organized critical system out of the « unstable » Ising model.

It is possible to treat similarly the presence of other control parameters such as an external applied magnetic field h, using the improved droplet definition proposed by Wang and Swendsen [31]. This definition is similar to that recalled above under zero magnetic field with this addition that a ghost spin is introduced which is parallel to the applied magnetic field. This ghost field is connected to any other spin in the lattice with probability $1 - e^{-2H/k_B T}$, where $2H$ is the energy to flip a spin from the direction parallel to h to the direction antiparallel to h. In order to create SOC, the magnetic field has again to depend self-consistently on the magnetization order parameter. The easiest procedure is to assume that the magnet which produces this field is precisely made of the same compound as the system under study (Ising spin system) and is at the same temperature. Using the above cooler and heater networks, the temperature is changed so that T adjusts itself to T_c at which the external magnetic field is exactly zero, thus qualifying the dynamics as leading to SOC.

Note that we deal here with a type of SOC reminiscent of the non-linear Langevin equation models [17], in which noise (coming here from the existence of a temperature) is acting on the system, in addition to the non-linear dynamics.

In the spirit of sandpiles viewed as cellular automata, it is also useful and interesting to define numerical algorithms which generalize the Glauber dynamics (non-conserved order parameter) or Monte Carlo dynamics, taking into account the evolution of temperature in the proposed dynamical version of the Ising model. Let us now outline what is maybe the simplest algorithm one can think of. Other more elaborate versions are of course possible. Suppose that one starts from a given arbitrary spin configuration and temperature T. Then, let us apply a standard Monte-Carlo procedure [32] : 1) select one spin at random in the system and flip it with the usual rules [32] defined for the given temperature T. 2) Then, one must analyze the connectivity of the new spin configuration using the Klein-Coniglio recipe. If an infinite cluster exists, change T into $T + \Delta T$, where ΔT is an infinitesimal increment. If only finite clusters exist (ξ is finite), change T into $T - \Delta T$. Then, start again from step 1) and apply again the standard Monte Carlo procedure for selecting a spin and flipping it. Then, follow step 2) and so on. Continuing this dynamical process for a long time, the temperature will be attracted to the critical Ising temperature T_c characterized by the existence in the spin system of a marginally stable infinite cluster.

3.2 Determination of the critical exponents for the SOC Ising model.

3.2.1 *Correlation exponent.* — In the model proposed above, the spin-spin correlation function $\langle \delta s(0)\, \delta s(x)\rangle \sim |x|^{-(d-2+\eta)}$ at criticality (with $\eta = 1/4$ in 2D for the Ising model) is the analog of the height-height correlation function of sandpiles [25]. The order of magnitude of the number $N(0)$ of spins which are identical to the spin at the origin is thus given by the space integral of this correlation function $N(0) \sim L^{2-\eta}$ in a system of size L, which is also equal to the susceptibility at criticality : $\chi \sim L^{\gamma/\nu}$ (since $\gamma/\nu = 2 - \eta$).

3.2.2 *Avalanche exponents.* — The susceptibility also describes the response of the spin system upon flipping upside-down a single spin. It thus gives the typical number of spins which will be affected by this upside-down flip, i.e. the typical size of the largest « avalanche » triggered by the motion of single spin (which corresponds to the addition of a single grain in sandpile models). We thus propose an analogy between the number of spins which are affected by the motion of a single spin and the avalanche triggered by the addition of a single grain in the sandpile. More generally, it is tempting to define an avalanche as being the process of readjustment of all spins due to the local perturbation brought on a single spin, in a way similar to the addition of a single grain in sandpile models which may trigger avalanches. However, in the Monte Carlo algorithm described above, the correspondence is not completely clear due to the fact that spins are selected at random and their flipping occur without any connectivity constraint. In other words, the problem comes from the fact that thermal fluctuations are present. In this sense, an avalanche and its size can only be defined in a thermal average sense. With this definition, the dynamical exponent z of the duration of a given avalanche, i.e. the typical duration of a relaxation following a flip of a single spin is simply given by the z-exponent of the Ising model : $t \sim \ell^z$.

In fact, an aesthetic way to circumvent the above problem is found in the new dynamics of spin relaxation introduced by Swendsen and Wang [31]. In their procedure, they use the Klein-Coniglio-Swendsen-Wang recipe to map the Ising model onto a percolation model, which thus allows the whole distribution of geometrical droplets covering the lattice to be defined unambiguously. Then, they propose to flip simultaneously all the spins in the same droplet according to a Metropolis rule. It is clear that this algorithm is ieal for defining avalanches : an avalanche corresponds naturally to the simultaneous flip of all spins in a selected droplet. If, in addition, we add the recipe that, if an infinite cluster exists, the temperature T is changed into $T + \Delta T$, whereas if only finite clusters exist (ξ is finite), T is changed into $T - \Delta T$, we obtain a SOC dynamical Ising model. The avalanche size distribution is then simply obtained from the distribution of droplets sizes in the Ising model mapped onto the percolation model. We thus get $P(s)\, ds \sim s^{-(2+1/\delta)}\, ds \sim s^{-2.05}\, ds$ with $\delta = 91/5$ in 2D percolation [23].

The fractal dimension of the avalanches is easily derived from the susceptibility exponent γ since the number of spins which are affected by the flip of a single spin is of order $L^{\gamma/\nu}$ in a system of size L. Alternatively, we can use the language of percolation and recall that the exponent γ can also be defined in terms of the mean cluster size [23] which is proportional to $L^{\gamma/\nu}$ Thus, the fractal dimension of spin avalanches is given by $D_f = \gamma/\nu = 7/4 = 1.75$ for 2D percolation.

To finish this section, we note that the droplet fluctuations, which have been discussed above, are reminiscent of the Goldstone modes discussed by Obukhov [20] in an original scenario of SOC. Indeed, similarly to the Goldstone modes of a system with a spontaneous broken *continuous* symmetry, droplet fluctuations can dominate long-distance and long-time correlation functions of ordered phases with discrete symmetries [33].

4. Final remarks.

We have discussed a general mechanism for Self-Organized Criticality (SOC), which proceeds by a non-linear dynamical feedback of the order parameter onto the control parameter, this dynamics being controlled by the value of the spatial correlation length. This mechanism has been illustrated on simple dynamical generalizations of the bond percolation model. We would like to conclude by proposing a different line of reasoning which allows us to strengthen further the relevance of the non-linear dynamical feedback to SOC.

The idea of feedback of the order parameter on the control parameter can be illustrated by considering the following dynamical equation $\partial U/\partial T = \mu U - U^3 = (\mu - U^2)\,U$, where U is the dynamical order parameter (i.e. the set of physical variables describing the state of a system). The effective control parameter $\mu \to \mu^* = \mu - U^2$ is indeed a function of U. Here, the feedback corresponds to a non-linear folding of the map. It is well known that such non-linear feedbacks are responsible for a wealth of irregular and chaotic dynamics in more general low-dimensional systems. However, this non-linear feedback is not sufficient to ensure the convergence of the dynamics towards a critical state. According to our discussion in this paper, we need furthermore to impose a kind of self-consistent condition, according to which the intensity of the non-linear feedback is controlled by a spatial correlation length. This control can thus only occur in spatially extended systems, made of many coupled sub-systems, for which a correlation length can be defined. However, how to exhibit spatio-temporal partial differential equations which leads to SOC is not known yet and is certainly an interesting problem for future research. This would enable to connect SOC with the rich phenomenology of non-linear partial differential equations, of which time-dependent Landau-Ginzburg equations provide a particular significant example.

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